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Recurrence relations for moments of progressively Type-II censored order statistics from a Pareto distribution with fixed and binomial removals

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Abstract: In this paper, some recurrence relations are presented for the single and product moments of progressively Type-II right censored order statistics from a Pareto distribution. These relations are obtained for a progressively censored sample from Pareto distribution with fixed and random removals, where in the random case, the number of units removed at each failure time follows a binomial distribution. In addition, Thomas-Wilson's Mixture Formula for Moments are obtained with with fixed and random removals. Finally, a numerical study is carried out to compare real and simulation results based on biases and MSEs of the expected termination time.

Keywords: Binomial removal; Monte Carlo simulation; Product moments; Progressive Type-II right-censored order statistics; Recurrence relations; Single moments.
Mathematics Subject Classification (2010): 62G30, 62E99, 62F10.

1 Introduction

In lifetime testing and analysis of reliability data, it is common to use different censoring scheme in order to reduce costs and time. In some life-testing experiments, the experimenter seek to remove units at different stages in the study for various reasons. This would lead to progressive censoring. In particular, a progressive Type-II censoring scheme is taken account of an important scheme in life-testing experiments.

The progressive Type-II censoring can be described as follows, Suppose n units are placed on a life test. Immediately following the first failure, r_1 surviving units are removed from the test at random. Then, immediately following the second failure, r_2 surviving units are removed from the test at random. This process continues until, at the time of the m-th failure, all the remaining $r_m = n - r_1 - r_2 - \cdots - r_{m-1} - m$ units

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are removed from the experiment. Therefore, the joint probability density function $X_{1:m:n}, \cdots, X_{m:m:n}$ from a random sample of size n with PDF $f(.)$ and CDF $F(.)$ under progressive Type-II censored sampling is given by:

$$f(x_{1:m:n}, \cdots, x_{m:m:n}) = A(n, m-1) \prod_{i=1}^{m} f(x_{i:m:n}) \left[1 - F(x_{i:m:n})\right]^{r_i}, \tag{1}$$

where $x_{1:m:n} < \cdots < x_{m:m:n}$ and $A(n, m-1) = n(n-1-r_1)(n-2-r_1-r_2)\cdots(n-m+1-r_1\cdots-r_{m-1})$. Note that $\sum_{i=1}^{m} r_i = n-m$, $r_i \geq 0$, $i = 1, \cdots, m$; see Balakrishnan and Aggarwala (2000).

In this scheme, $r_1, r_2, \cdots, r_m$ are all pre-fixed. However, these numbers may occur at random in some practical situations. In some reliability experiments, an experimenter may distinguish that it is inappropriate or too dangerous to continue the testing on some of the tested units even though these units have not failed; see Yuen and Tse (1996) and Zeinab (2008). In such cases, the pattern of removal at each failure is random. This leads to progressive censoring with random removals (PCR); see Table 1.

Table 1: A schematic representation of the progressive Type-II censoring

The numbers in life testing	Binomial removals	Remains
n	$R_1 \sim B(n-m, p)$	$n-1-r_1$
$n-1-r_1$	$R_2 \vert r_1 \sim B(n-m-r_1, p)$	$n-2-r_1-r_2$
$\vdots$	$\vdots$	$\vdots$
$n-(m-2)-\sum_{j=1}^{m-2} r_j$	$R_{m-1} \vert r_1, \cdots, r_{m-2}$	$n-(m-1)$
	$\sim B(n-m-\sum_{j=1}^{m-2} r_j, p)$	$-\sum_{j=1}^{m-1} r_j$
$n-(m-1)-\sum_{j=1}^{m-1} r_j$	$R_m = n-m-\sum_{j=1}^{m-1} r_j$	0

In particular, the joint probability distribution of $\boldsymbol{R} = (R_1, \cdots, R_m)$ is given by

$$f(\boldsymbol{r}; p) = P(\boldsymbol{R} = \boldsymbol{r}; p) = bp^d(1-p)^e$$

where $b = \dfrac{(n-m)!}{\left(n-m-\sum_{j=1}^{m-1} r_j\right)! \prod_{j=1}^{m-1} r_j!}$, $\quad d = \sum_{j=1}^{m-1} r_j$, $\quad e = (m-1)(n-m) - \sum_{j=1}^{m-1}(m-j)r_j$; see Weian et al. (2011).

The statistical inference on lifetime distributions under progressive censoring with random removals is discussed by several authors. Tse et al. (2000) presented the maximum likelihood estimations of Weibull distribution under progressive censoring with binomial removals. Wu and Chang (2003) and Wu et al. (2007) discussed the estimation of the Burr Type-XII distribution and Pareto distribution based on progressively censored samples with random removals, where the number of removed units has a discrete uniform removal pattern and binomial distribution. Xiang and Tse (2005) discussed the maximum likelihood estimation of the model parameters and derived the corresponding asymptotic variances based on Type-II progressive interval censoring with binomial removals. Wu et al. (2006) obtained MLE and the estimated expected test time for the two-parameter Gompertz distribution under progressive censoring with binomial removals. Recently, Weian et al. (2011) presented statistical analysis of generalized exponential distribution under progressive censoring with binomial removals.

In Section 2, we derive some recurrence relations for a progressively censored sample with fixed and random removals. In Section 3, we present Thomas-Wilson's mixture formula for moments and generalize it for binomial removals. Then, in Section 4, a numerical example to obtain the real values of moments and a Monte Carlo simulation study are carried out to compare real and simulation results. Finally, we state some results.

2 Moments of progressively type-II censored order statistics

The idea of obtaining moments of usual order statistics in a recursive manner has been discussed by many authors for a wide array of distributions; for example, see Arnold and Balakrishnan (1989) and Balakrishnan and Sultan (1998).

2.1 Single moments of progressively type-II censored order statistics with fixed removals

In this section, we prove recurrence relations for the single moments of progressively Type-II right censored order statistics from a Pareto distribution with fixed removals.

$$\begin{aligned} \mu_{i:m:n}^{(r_1,\cdots,r_m)^{(k)}} &= E\left[X_{i:m:n}^{(r_1,\cdots,r_m)^{k}}\right] \\ &= A(n,m-1)\int\int\cdots\int_{1<x_1<x_2<\cdots<x_m<\infty} x_i^k f(x_1)\left[1-F(x_1)\right]^{r_1} \\ &\quad \times f(x_2)\left[1-F(x_2)\right]^{r_2}\cdots f(x_m)\left[1-F(x_m)\right]^{r_m} dx_1\cdots dx_m. \end{aligned} \quad (2)$$

We have the following recurrence relations for these single moments.

Theorem 2.1. *For $2 \le m \le n$ and $k \ge 0$,*

$$\mu_{1:m:n}^{(r_1,\cdots,r_m)^{(k)}} = \frac{\theta}{k-\theta(r_1+1)}\left\{(n-r_1-1)\,\mu_{1:m-1:n}^{(r_1+r_2+1,\cdots,r_m)^{(k)}} - \frac{A(n,\,m-1)}{A(n,\,m-2)}\right\}$$

and for $m=1,\ n=1,2,\cdots,\ \theta>k$ and $k\ge 0$,

$$\mu_{1:1:n}^{(r_1)^{(k)}} = \frac{n\theta}{1-n\theta(r_1+1)}$$

Proof.

$$\begin{aligned}\mu_{1:m:n}^{(r_1,\cdots,r_m)^{(k)}} &= A(n,\, m-1) \int\int_{1<x_1<x_2<\cdots<x_m<\infty}\cdots\int U(x_2)\\ &\times f(x_2)[1-F(x_2)]^{r_2}\cdots f(x_m)[1-F(x_m)]^{r_m}dx_2\cdots dx_m, \qquad (3)\end{aligned}$$

where

$$\begin{aligned}U(x_2) &= \int_1^{x_2} {x_1}^k f(x_1)[1-F(x_1)]^{r_1}dx_1\\ &= \int_1^{x_2} \theta {x_1}^{k-1}[1-F(x_1)]^{r_1+1}dx_1\\ &= \frac{\theta}{k}\left\{{x_2}^k[1-F(x_1)]^{r_1+1} - 1 + (r_1+1)\int_1^{x_2} {x_1}^k f(x_1)[1-F(x_1)]^{r_1}dx_1\right\}.\end{aligned}$$

By substituting the above expression into (3), we have

$$\begin{aligned}\mu_{1:m:n}^{(r_1,\cdots,r_m)^{(k)}} &= \frac{\theta}{k}\left\{(n-r_1-1)\mu_{1:m-1:n}^{(r_1+r_2+1,\cdots,r_m)^{(k)}} - \frac{A(n,\, m-1)}{A(n,\, m-2)}\right.\\ &\left. +(r_1+1)\mu_{1:m:n}^{(r_1,\cdots,r_m)^{(k)}}\right\},\end{aligned}$$

so

$$\mu_{1:m:n}^{(r_1,\cdots,r_m)^{(k)}} = \frac{\theta}{k-\theta(r_1+1)}\left\{(n-r_1-1)\mu_{1:m-1:n}^{(r_1+r_2+1,\cdots,r_m)^{(k)}} - \frac{A(n,\, m-1)}{A(n,\, m-2)}\right\}.$$

Next, for $m=1,\ n=1,2,\cdots,\ \theta>k$ and $k\geq 0$ (with $r_1=n-1$ in this case), we have

$$\begin{aligned}\mu_{1:1:n}^{(r_1)^{(k)}} &= A(n,\, 0)\int_1^{\infty} {x_1}^k f(x_1)[1-F(x_1)]^{r_1}dx_1\\ &= n\theta\int_1^{\infty} {x_1}^{k-1}[1-F(x_1)]^{r_1+1}dx_1\\ &= n\theta + n\theta(r_1+1)\int_1^{\infty} {x_1}^k f(x_1)[1-F(x_1)]^{r_1}dx_1\\ &= n\theta + n\theta(r_1+1)\mu_{1:1:n}^{(r_1)^{(k)}},\end{aligned}$$

so

$$\mu_{1:1:n}^{(r_1)^{(k)}} = \frac{n\theta}{1-n\theta(r_1+1)}.$$

□

Theorem 2.2. *For $2\leq i\leq m-1,\ m\leq n$ and $k\geq 0$,*

$$\begin{aligned}\mu_{i:m:n}^{(r_1,\cdots,r_m)^{(k)}} &= \frac{\theta}{k-\theta(r_i+1)}\left\{(n-r_1-\cdots-r_i-i)\ \mu_{i:m-1:n}^{(r_1,\cdots,r_i+r_{i+1}+1,\cdots,r_m)^{(k)}}\right.\\ &\left. -(n-r_1-\cdots-r_i-i+1)\ \mu_{i-1:m-1:n}^{(r_1,\cdots,r_{i-1}+r_i+1,\cdots,r_m)^{(k)}}\right\}.\end{aligned}$$

Proof.

$$\begin{aligned}\mu_{i:m:n}^{(r_1,\cdots,r_m)^{(k)}} &= A(n,\, m-1) \int\int_{1<x_1<\cdots<x_{i-1}<x_{i+1}<\cdots<x_m<\infty}\cdots\int U(x_{i-1},\, x_{i+1}) \\ &\quad \times f(x_1)[1-F(x_1)]^{r_1}\cdots f(x_m)[1-F(x_m)]^{r_m} \\ &\quad dx_1\cdots dx_{i-1}dx_{i+1}\cdots dx_m, \end{aligned} \tag{4}$$

where

$$\begin{aligned} U(x_{i-1},\, x_{i+1}) &= \int_{x_{i-1}}^{x_{i+1}} {x_i}^k f(x_i)[1-F(x_i)]^{r_i}dx_i \\ &= \int_{x_{i-1}}^{x_{i+1}} \theta {x_i}^{k-1}[1-F(x_i)]^{r_i+1}dx_i \\ &= \frac{\theta}{k}\Big\{ {x_{i+1}}^k[1-F(x_{i+1})]^{r_i+1} - {x_{i-1}}^k[1-F(x_{i-1})]^{r_i+1} \\ &\quad +(r_i+1)\int_{(x_{i-1}}^{x_{i+1}} {x_i}^k f(x_i)[1-F(x_i)]^{r_i}dx_i \Big\}. \end{aligned}$$

By substituting the above expression into (4), we have

$$\begin{aligned} \mu_{i:m:n}^{(r_1,\cdots,r_m)^{(k)}} &= \frac{\theta}{k}\Big\{(n-r_1-\cdots-r_i-i)\mu_{i:m-1:n}^{(r_1,\cdots,r_i+r_{i+1}+1,\cdots,r_m)^{(k)}} \\ &\quad -(n-r_1-\cdots-r_i-i+1)\mu_{i-1:m-1:n}^{(r_1,\cdots,r_{i-1}+r_i+1,\cdots,r_m)^{(k)}} \\ &\quad +(r_i+1)\mu_{i:m:n}^{(r_1,\cdots,r_m)^{(k)}}\Big\}, \end{aligned}$$

in which

$$\begin{aligned} \mu_{i:m:n}^{(r_1,\cdots,r_m)^{(k)}} &= \frac{\theta}{k-\theta(r_i+1)}\Big\{(n-r_1-\cdots-r_i-i)\ \mu_{i:m-1:n}^{(r_1,\cdots,r_i+r_{i+1}+1,\cdots,r_m)^{(k)}} \\ &\quad -(n-r_1-\cdots-r_i-i+1)\ \mu_{i-1:m-1:n}^{(r_1,\cdots,r_{i-1}+r_i+1,\cdots,r_m)^{(k)}}\Big\}. \end{aligned}$$

□

Theorem 2.3. *For $2 \le m \le n$, $k \ge 0$ and $k \le \theta$,*

$$\mu_{m:m:n}^{(r_1,\cdots,r_m)^{(k)}} = \frac{\theta(r_m+1)}{\theta(r_m+1)-k}\ \mu_{m-1:m-1:n}^{(r_1,\cdots,r_{m-1}+r_m+1)^{(k)}}.$$

Proof.

$$\begin{aligned} \mu_{m:m:n}^{(r_1,\cdots,r_m)^{(k)}} &= A(n,\, m-1)\int\int_{1<x_1<\cdots<x_{m-1}<\infty}\cdots\int U(x_{m-1}) \\ &\quad \times f(x_1)[1-F(x_1)]^{r_1}\cdots f(x_{m-1})[1-F(x_{m-1})]^{r_{m-1}}dx_1\cdots dx_{m-1}, \end{aligned} \tag{5}$$

where

$$\begin{aligned} U(x_{m-1}) &= \int_{x_{m-1}}^{\infty} {x_m}^k f(x_m)[1-F(x_m)]^{r_m} dx_m \\ &= \int_{x_{m-1}}^{\infty} \theta {x_m}^{k-1}[1-F(x_m)]^{r_m+1} dx_m \\ &= \frac{\theta}{k}\Big\{ - {x_{m-1}}^k [1-F(x_{m-1})]^{r_m+1} \\ &\quad +(r_m+1)\int_{x_{m-1}}^{\infty} {x_m}^k f(x_m)[1-F(x_m)]^{r_m} dx_m \Big\}. \end{aligned}$$

By substituting the above expression into (5), we have

$$\mu_{m:m:n}^{(r_1,\cdots,r_m)^{(k)}} = \frac{\theta}{k}\left\{-(r_m+1)\mu_{m-1:m-1:n}^{(r_1,\cdots,r_{m-1}+r_m+1)^{(k)}} + (r_m+1)\mu_{m:m:n}^{(r_1,\cdots,r_m)^{(k)}}\right\},$$

so

$$\mu_{m:m:n}^{(r_1,\cdots,r_m)^{(k)}} = \frac{\theta(r_m+1)}{\theta(r_m+1)-k}\, \mu_{m-1:m-1:n}^{(r_1,\cdots,r_{m-1}+r_m+1)^{(k)}}.$$

□

Note that $\mu_{m:m:n}$ is the expected termination time of an experiment, in fact it shows that the time required to complete the experiment. In practical applications, this information is important to choose an appropriate sampling plan because the time required to complete the experiment is directly related to costs and time.

2.2 Product moments of progressively type-II censored order statistics with fixed removals

In this section, we prove some recurrence relations for the product moments of progressively Type-II censored order statistics with fixed removal from the Pareto distribution. By using (1), we can obtain the product moment of the i-th and j-th progressively Type-II censored order statistics as

$$\begin{aligned} \mu_{i,j:m:n}^{(r_1,\cdots,r_m)^{(k,t)}} &= E\left[X_{i:m:n}^{(r_1,\cdots,r_m)^k}\ X_{j:m:n}^{(r_1,\cdots,r_m)^t}\right] \\ &= A(n,m-1)\int\int\cdots\int_{1<x_1<x_2<\cdots<x_m<\infty} x_i^k x_j^t f(x_1)\left[1-F(x_1)\right]^{r_1} \\ &\quad \times f(x_2)\left[1-F(x_2)\right]^{r_2}\cdots f(x_m)\left[1-F(x_m)\right]^{r_m} dx_1\cdots dx_m. \end{aligned} \tag{6}$$

We have the following relations for these product moments.

Theorem 2.4. *For* $1 \le i < j \le m-1$, $j-i \ge 2$, $m \le n$ *and* $k,t \ge 0$,

$$\begin{aligned} \mu_{i,j:m:n}^{(r_1,\cdots,r_m)^{(k,l)}} &= \frac{\theta}{k-\theta(r_j+1)}\Big\{(n-r_1-\cdots-r_j-j)\ \mu_{i,j:m-1:n}^{(r_1,\cdots,r_j+r_{j+1}+1,\cdots,r_m)^{(k,t)}} \\ &\quad -(n-r_1-\cdots-r_{j-1}-j+1)\ \mu_{i,j-1:m-1:n}^{(r_1,\cdots,r_{j-1}+r_j+1,\cdots,r_m)^{(k,t)}}\Big\}. \end{aligned}$$

Proof.

$$\mu_{i,j:m:n}^{(r_1,\cdots,r_m)^{(k,t)}} = A(n,\, m-1) \underset{1<x_1<\cdots<x_{j-1}<x_{j+1}<\cdots<x_m<\infty}{\int\cdots\int} x_i^k f(x_1)[1-F(x_1)]^{r_1}\cdots$$
$$\times\left\{\int_{x_{j-1}}^{x_{j+1}} {x_j}^t f(x_j)[1-F(x_j)]^{r_j}dx_j\right\} f(x_m)[1-F(x_m)]^{r_m} \qquad (7)$$
$$dx_1\cdots dx_{j-1}dx_{j+1}\cdots dx_m,$$

where

$$\int_{x_{j-1}}^{x_{j+1}} {x_j}^t f(x_j)[1-F(x_j)]^{r_j}dx_j = \int_{x_{j-1}}^{x_{j+1}} \theta {x_j}^{t-1}[1-F(x_j)]^{r_j+1}dx_j$$
$$= \frac{\theta}{t}\Big\{ {x_{j+1}}^t[1-F(x_{j+1})]^{r_j+1}$$
$$-{x_{j-1}}^t[1-F(x_{j-1})]^{r_j+1} + (r_j+1)$$
$$\times\int_{x_{j-1}}^{x_{j+1}} {x_j}^t f(x_j)[1-F(x_j)]^{r_j}dx_j\Big\}.$$

By substituting the above expression into (7), we have

$$\mu_{i,j:m:n}^{(r_1,\cdots,r_m)^{(k,t)}} = \frac{\theta}{t}\Big\{(n-r_1-\cdots-r_j-j)\mu_{i,j:m-1:n}^{(r_1,\cdots,r_j+r_{j+1}+1,\cdots,r_m)^{(k,t)}}$$
$$-(n-r_1-\cdots-r_{j-1}-j+1)\mu_{i,j-1:m-1:n}^{(r_1,\cdots,r_{j-1}+r_j+1,\cdots,r_m)^{(k.t)}}$$
$$+(r_j+1)\mu_{i,j:m:n}^{(r_1,\cdots,r_m)^{(k,t)}}\Big\},$$

so

$$\mu_{i,j:m:n}^{(r_1,\cdots,r_m)^{(k,t)}} = \frac{\theta}{k-\theta(r_j+1)}\Big\{(n-r_1-\cdots-r_j-j)\ \mu_{i,j:m-1:n}^{(r_1,\cdots,r_j+r_{j+1}+1,\cdots,r_m)^{(k,t)}}$$
$$-(n-r_1-\cdots-r_{j-1}-j+1)\ \mu_{i,j-1:m-1:n}^{(r_1,\cdots,r_{j-1}+r_j+1,\cdots,r_m)^{(k,t)}}\Big\}.$$

□

Theorem 2.5. *For $1\le i\le m-1$, $m\le n$, $k,t\ge 0$ and $t\le\theta$,*

$$\mu_{i,m:m:n}^{(r_1,\cdots,r_m)^{(k,t)}} = \frac{\theta(n-r_1-\cdots r_{m-1}-m+1)}{\theta(r_m+1)-t}\ \mu_{i,m-1:m-1:n}^{(r_1,\cdots,r_{m-1}+r_m+1)^{(k,t)}}.$$

Proof.

$$\mu_{i,m:m:n}^{(r_1,\cdots,r_m)^{(k,t)}} = A(n,\, m-1)\underset{1<x_1<\cdots<x_{m-1}<\infty}{\int\cdots\int} x_i^k\Big\{\int_{x_{m-1}}^{\infty} {x_m}^t f(x_m)[1-F(x_m)]^{r_m}dx_m\Big\}$$
$$\times f(x_1)[1-F(x_1)]^{r_1}\cdots f(x_{m-1})[1-F(x_{m-1})]^{r_{m-1}}dx_1\cdots dx_{m-1}, \quad (8)$$

where

$$\int_{x_{m-1}}^{\infty} {x_m}^t f(x_m)[1-F(x_m)]^{r_m} dx_m = \int_{x_{m-1}}^{\infty} \theta {x_m}^{t-1}[1-F(x_m)]^{r_m+1} dx_m$$
$$= \frac{\theta}{t}\Big\{(r_m+1)\int_{x_{m-1}}^{\infty} {x_m}^t f(x_m)[1-F(x_m)]^{r_m} dx_m$$
$$-{x_{m-1}}^t[1-F(x_{m-1})]^{r_m+1}\Big\}.$$

By substituting the above expression into (8), we have

$$\mu_{i,m:m:n}^{(r_1,\cdots,r_m)^{(k,t)}} = \frac{\theta}{t}\Big\{-(n-r_1-\cdots r_{m-1}-m+1)\mu_{i,m-1:m-1:n}^{(r_1,\cdots,r_{m-1}+r_m+1)^{(k,t)}}$$
$$+(r_m+1)\mu_{i,m:m:n}^{(r_1,\cdots,r_m)^{(k,t)}}\Big\},$$

so

$$\mu_{i,m:m:n}^{(r_1,\cdots,r_m)^{(k,t)}} = \frac{\theta(n-r_1-\cdots r_{m-1}-m+1)}{\theta(r_m+1)-t}\;\mu_{i,m-1:m-1:n}^{(r_1,\cdots,r_{m-1}+r_m+1)^{(k,t)}}.$$

□

Theorem 2.6. *For $2 \le i < j \le m-1$, $j-i \ge 2$, $m \le n$ and $k,t \ge 0$*

$$\mu_{i,j:m:n}^{(r_1,\cdots,r_m)^{(k,t)}} = \frac{\theta}{k-\theta(r_i+1)}\Big\{(n-r_1-\cdots-r_i-i)\;\mu_{i,j:m-1:n}^{(r_1,\cdots,r_i+r_{i+1}+1,\cdots,r_m)^{(k,t)}}$$
$$-(n-r_1-\cdots-r_{i-1}-i+1)\;\mu_{i-1,j:m-1:n}^{(r_1,\cdots,r_{i-1}+r_i+1,\cdots,r_m)^{(k.t)}}\Big\}.$$

Proof.

$$\mu_{i,j:m:n}^{(r_1,\cdots,r_m)^{(k,t)}} = A(n,\,m-1)\int\cdots\int_{1<x_1<\cdots<x_{i-1}<x_{i+1}<\cdots<x_m<\infty} {x_j}^t\left\{\int_{x_{i-1}}^{x_{i+1}} x_i^k f(x_i)[1-F(x_i)]^{r_i}dx_i\right\}$$
$$\times f(x_1)[1-F(x_1)]^{r_1}\cdots f(x_m)[1-F(x_m)]^{r_m} \tag{9}$$
$$\times dx_1\cdots dx_{i-1}dx_{i+1}\cdots dx_m,$$

where

$$\int_{x_{i-1}}^{x_{i+1}} {x_i}^k f(x_i)[1-F(x_j)]^{r_i}dx_i = \int_{x_{i-1}}^{x_{i+1}} \theta {x_i}^{k-1}[1-F(x_i)]^{r_i+1}dx_i$$
$$= \frac{\theta}{k}\Big\{{x_{i+1}}^k[1-F(x_{i+1})]^{r_i+1} - {x_{i-1}}^k[1-F(x_{i-1})]^{r_i+1}$$
$$+(r_i+1)\int_{x_{i-1}}^{x_{i+1}} {x_i}^k f(x_i)[1-F(x_i)]^{r_i}dx_i\Big\}.$$

By substituting the above expression into (9), we have

$$\begin{aligned}\mu_{i,j:m:n}^{(r_1,\cdots,r_m)^{(k,t)}} &= \frac{\theta}{k}\Big\{(n-r_1-\cdots-r_i-i)\mu_{i,j:m-1:n}^{(r_1,\cdots,r_i+r_{i+1}+1,\cdots,r_m)^{(k,t)}} \\ &\quad -(n-r_1-\cdots-r_{i-1}-i+1)\mu_{i-1,j:m-1:n}^{(r_1,\cdots,r_{i-1}+r_i+1,\cdots,r_m)^{(k,t)}} \\ &\quad +(r_i+1)\mu_{i,j:m:n}^{(r_1,\cdots,r_m)^{(k,t)}}\Big\},\end{aligned}$$

so

$$\begin{aligned}\mu_{i,j:m:n}^{(r_1,\cdots,r_m)^{(k,t)}} &= \frac{\theta}{k-\theta(r_i+1)}\Big\{(n-r_1-\cdots-r_i-i)\ \mu_{i,j:m-1:n}^{(r_1,\cdots,r_i+r_{i+1}+1,\cdots,r_m)^{(k,t)}} \\ &\quad -(n-r_1-\cdots-r_{i-1}-i+1)\ \mu_{i-1,j:m-1:n}^{(r_1,\cdots,r_{i-1}+r_i+1,\cdots,r_m)^{(k.t)}}\Big\}.\end{aligned}$$

□

Theorem 2.7. *For $2 \le i \le m-1$, $m \le n$ and $k,t \ge 0$,*

$$\begin{aligned}\mu_{i,i+1:m:n}^{(r_1,\cdots,r_m)^{(k,t)}} &= \frac{\theta}{k-\theta(r_i+1)}\Big\{(n-r_1-\cdots-r_i-i)\mu_{i:m-1:n}^{(r_1,\cdots,r_i+r_{i+1}+1,\cdots,r_m)^{(k+t)}} \\ &\quad -(n-r_1-\cdots-r_{i-1}-i+1)\mu_{i-1,i:m-1:n}^{(r_1,\cdots,r_{i-1}+r_i+1,\cdots,r_m)^{(k,t)}}\Big\}.\end{aligned}$$

Proof.

$$\begin{aligned}\mu_{i,i+1:m:n}^{(r_1,\cdots,r_m)^{(k,t)}} &= A(n,\,m-1)\int\cdots\int_{1<x_1<\cdots<x_{i-1}<x_{i+1}<\cdots<x_m<\infty} {x_{i+1}}^t \\ &\quad \times\left\{\int_{x_{i-1}}^{x_{i+1}} x_i^k f(x_i)[1-F(x_i)]^{r_i}dx_i\right\} f(x_1)[1-F(x_1)]^{r_1} \\ &\quad \times\cdots f(x_m)[1-F(x_m)]^{r_m}dx_1\cdots dx_{i-1}dx_{i+1}\cdots dx_m, \qquad (10)\end{aligned}$$

where

$$\begin{aligned}\int_{x_{i-1}}^{x_{i+1}} {x_i}^k f(x_i)[1-F(x_i)]^{r_i}dx_i &= \int_{x_{i-1}}^{x_{i+1}} \theta {x_i}^{k-1}[1-F(x_i)]^{r_i+1}dx_i \\ &= \frac{\theta}{k}\Big\{{x_{i+1}}^k[1-F(x_{i+1})]^{r_i+1} - {x_{i-1}}^k[1-F(x_{i-1})]^{r_i+1} \\ &\quad +(r_i+1)\int_{x_{i-1}}^{x_{i+1}} {x_i}^k f(x_i)[1-F(x_i)]^{r_i}dx_i\Big\}.\end{aligned}$$

By substituting the above expression into (10), we have

$$\begin{aligned}\mu_{i,i+1:m:n}^{(r_1,\cdots,r_m)^{(k,t)}} &= \frac{\theta}{k}\Big\{(n-r_1-\cdots-r_i-i)\mu_{i:m-1:n}^{(r_1,\cdots,r_i+r_{i+1}+1,\cdots,r_m)^{(k+t)}} \\ &\quad -(n-r_1-\cdots-r_{i-1}-i+1)\mu_{i-1,i:m-1:n}^{(r_1,\cdots,r_{i-1}+r_i+1,\cdots,r_m)^{(k,t)}} \\ &\quad +(r_i+1)\mu_{i,i+1:m:n}^{(r_1,\cdots,r_m)^{(k,t)}}\Big\},\end{aligned}$$

so

$$\begin{aligned}\mu_{i,i+1:m:n}^{(r_1,\cdots,r_m)^{(k,t)}} &= \frac{\theta}{k-\theta(r_i+1)}\Big\{(n-r_1-\cdots-r_i-i)\mu_{i:m-1:n}^{(r_1,\cdots,r_i+r_{i+1}+1,\cdots,r_m)^{(k+t)}} \\ &\quad -(n-r_1-\cdots-r_{i-1}-i+1)\mu_{i-1,i:m-1:n}^{(r_1,\cdots,r_{i-1}+r_i+1,\cdots,r_m)^{(k,t)}}\Big\}.\end{aligned}$$

□

Theorem 2.8. *For $m \geq 2$, $m \leq n$ and $k, t \geq 0$,*

$$\begin{aligned}\mu_{1,2:m:n}^{(r_1,\cdots,r_m)^{(k,t)}} &= \frac{\theta}{k-\theta(r_1+1)}\Big\{(n-r_1-1)\;\mu_{1:m-1:n}^{(r_1+r_2+1,\cdots,r_m)^{(k+t)}} \\ &\quad -\frac{A(n,\,m-1)}{A(n-1,\,m-2)}\;\mu_{1:m-1:n-1}^{(r_2,\cdots,r_m)^{(t)}}\Big\}.\end{aligned}$$

Proof.

$$\begin{aligned}\mu_{1,2:m:n}^{(r_1,\cdots,r_m)^{(k,t)}} &= A(n,\,m-1)\underset{1<x_2<\cdots<x_m<\infty}{\int\cdots\int} x_2^t\left\{\int_1^{x_2} x_1^k f(x_1)[1-F(x_1)]^{r_1}dx_1\right\} \\ &\quad \times f(x_2)[1-F(x_2)]^{r_2}\cdots f(x_m)[1-F(x_m)]^{r_m}dx_2\cdots dx_m, \qquad (11)\end{aligned}$$

where

$$\begin{aligned}\int_1^{x_2} {x_1}^k f(x_1)[1-F(x_1)]^{r_1}dx_1 &= \int_1^{x_2}\theta {x_1}^{k-1}[1-F(x_1)]^{r_1+1}dx_1 \\ &= \frac{\theta}{k}\Big\{{x_2}^k[1-F(x_2)]^{r_1+1}-1 \\ &\quad +(r_1+1)\int_1^{x_2}{x_1}^k f(x_1)[1-F(x_1)]^{r_1}dx_1\Big\}.\end{aligned}$$

By substituting the above expression into (11), we have

$$\begin{aligned}\mu_{1,2:m:n}^{(r_1,\cdots,r_m)^{(k,t)}} &= \frac{\theta}{k}\Big\{(n-r_1-1)\mu_{1:m-1:n}^{(r_1+r_2+1,\cdots,r_m)^{(k+t)}} \\ &\quad -\frac{A(n,\,m-1)}{A(n-1,\,m-2)}\mu_{1:m-1:n-1}^{(r_2,\cdots,r_m)^{(t)}}+(r_1+1)\mu_{1,2:m:n}^{(r_1,\cdots,r_m)^{(k,t)}}\Big\},\end{aligned}$$

so

$$\begin{aligned}\mu_{1,2:m:n}^{(r_1,\cdots,r_m)^{(k,t)}} &= \frac{\theta}{k-\theta(r_1+1)}\Big\{(n-r_1-1)\;\mu_{1:m-1:n}^{(r_1+r_2+1,\cdots,r_m)^{(k+t)}} \\ &\quad -\frac{A(n,\,m-1)}{A(n-1,\,m-2)}\;\mu_{1:m-1:n-1}^{(r_2,\cdots,r_m)^{(t)}}\Big\}.\end{aligned}$$

□

Theorem 2.9. *For $1 \leq i \leq m-2$, $m \leq n$ and $k, t \geq 0$,*

$$\mu_{i,i+1:m:n}^{(r_1,\cdots,r_m)^{(k,t)}} = \frac{\theta}{t-\theta(r_{i+1}+1)} \Big\{ (n-r_1-\cdots-r_{i+1}-i-1)\mu_{i,i+1:m-1:n}^{(r_1,\cdots,r_{i+1}+r_{i+2}+1,\cdots,r_m)^{(k,t)}} - (n-r_1-\cdots-r_i-i)\mu_{i:m-1:n}^{(r_1,\cdots,r_i+r_{i+1}+1,\cdots,r_m)^{(k+t)}} \Big\}.$$

Proof.

$$\begin{aligned} \mu_{i,i+1:m:n}^{(r_1,\cdots,r_m)^{(k,t)}} &= A(n,m-1) \int \cdots \int_{1<x_1<\cdots<x_i<x_{i+2}<\cdots<x_m<\infty} {x_i}^k \\ &\times \left\{ \int_{x_i}^{x_{i+2}} x_{i+1}^t f(x_{i+1})[1-F(x_{i+1})]^{r_{i+1}} dx_{i+1} \right\} \\ &\times f(x_1)[1-F(x_1)]^{r_1} f(x_2)[1-F(x_2)]^{r_2} \cdots f(x_m)[1-F(x_m)]^{r_m} \\ &\times dx_1 \cdots dx_i dx_{i+2} \cdots dx_m, \end{aligned} \tag{12}$$

where

$$\begin{aligned} \int_{x_i}^{x_{i+2}} x_{i+1}^t f(x_{i+1})[1-F(x_{i+1})]^{r_{i+1}} dx_{i+1} &= \int_{x_i}^{x_{i+2}} \theta x_{i+1}^{t-1}[1-F(x_{i+1})]^{r_{i+1}+1} dx_{i+1} \\ &= \frac{\theta}{t}\Big\{ {x_{i+2}}^t [1-F(x_{i+2})]^{r_{i+1}+1} - {x_i}^t [1-F(x_i)]^{r_{i+1}+1} \\ &\quad + (r_{i+1}+1)\int_{x_i}^{x_{i+2}} {x_{i+1}}^t f(x_{i+1})[1-F(x_{i+1})]^{r_{i+1}} dx_{i+1} \Big\}. \end{aligned}$$

By substituting the above expression into (12), we have

$$\begin{aligned} \mu_{i,i+1:m:n}^{(r_1,\cdots,r_m)^{(k,t)}} &= \frac{\theta}{t}\Big\{ (n-r_1-\cdots-r_{i+1}-i-1)\mu_{i,i+1:m-1:n}^{(r_1,\cdots,r_{i+1}+r_{i+2}+1,\cdots,r_m)^{(k,t)}} \\ &\quad - (n-r_1-\cdots-r_i-i)\mu_{i:m-1:n}^{(r_1,\cdots,r_i+r_{i+1}+1,\cdots,r_m)^{(k+t)}} \\ &\quad + (r_{i+1}+1)\mu_{i,i+1:m:n}^{(r_1,\cdots,r_m)^{(k,t)}} \Big\}, \end{aligned}$$

so

$$\mu_{i,i+1:m:n}^{(r_1,\cdots,r_m)^{(k,t)}} = \frac{\theta}{t-\theta(r_{i+1}+1)} \Big\{ (n-r_1-\cdots-r_{i+1}-i-1)\mu_{i,i+1:m-1:n}^{(r_1,\cdots,r_{i+1}+r_{i+2}+1,\cdots,r_m)^{(k,t)}} - (n-r_1-\cdots-r_i-i)\mu_{i:m-1:n}^{(r_1,\cdots,r_i+r_{i+1}+1,\cdots,r_m)^{(k+t)}} \Big\}.$$

□

Theorem 2.10. *For $m \geq 2$, $m \leq n$, $k, t \geq 0$ and $t \leq \theta$,*

$$\mu_{m-1,m:m:n}^{(r_1,\cdots,r_m)^{(k,t)}} = \frac{\theta(n-r_1-\cdots-r_{m-1}-m+1)}{\theta(r_m+1)-t}\, \mu_{m-1:m-1:n}^{(r_1,\cdots,r_{m-1}+r_m+1)^{(k+t)}}.$$

Proof.

$$\mu_{m-1,m:m:n}^{(r_1,\cdots,r_m)^{(k,t)}} = A(n,\, m-1)\int\cdots\int_{1<x_1<\cdots<x_{m-1}<\infty} {x_{m-1}}^k \left\{\int_{x_{m-1}}^{\infty} x_m^t f(x_m)(1-F(x_m))^{r_m}dx_m\right\} f(x_1)[1-F(x_1)]^{r_1}\cdots \times f(x_{m-1})[1-F(x_{m-1})]^{r_{m-1}}dx_1\cdots dx_{m-1}, \quad (13)$$

where

$$\int_{x_{m-1}}^{\infty} x_m^t f(x_m)[1-F(x_m)]^{r_m}dx_m = \int_{x_{m-1}}^{\infty} \theta x_m^{t-1}[1-F(x_m)]^{r_m+1}dx_m = \frac{\theta}{t}\left\{-{x_{m-1}}^t[1-F(x_{m-1})]^{r_m+1} + (r_m+1)\int_{x_{m-1}}^{\infty} {x_m}^t f(x_m)[1-F(x_m)]^{r_m}dx_m\right\}.$$

By substituting the above expression into (13), we have

$$\mu_{m-1,m:m:n}^{(r_1,\cdots,r_m)^{(k,t)}} = \frac{\theta}{t}\left\{-(n-r_1-\cdots-r_{m-1}-m+1)\mu_{m-1:m-1:n}^{(r_1,\cdots,r_{m-1}+r_m+1)^{(k+t)}} + (r_m+1)\mu_{m-1,m:m:n}^{(r_1,\cdots,r_m)^{(k,t)}}\right\},$$

so

$$\mu_{m-1,m:m:n}^{(r_1,\cdots,r_m)^{(k,t)}} = \frac{\theta(n-r_1-\cdots-r_{m-1}-m+1)}{\theta(r_m+1)-t}\,\mu_{m-1:m-1:n}^{(r_1,\cdots,r_{m-1}+r_m+1)^{(k+t)}}.$$

□

2.3 Single moments of progressively type-II censored order statistics with binomial removals

In this section, we prove some relations for the single moments of progressively Type-II right censored order statistics from the Pareto distribution with binomial removals. Let $\boldsymbol{R} = (R_1, \cdots, R_m)$ and $\boldsymbol{r} = (r_1, \cdots, r_m)$ be the random and fixed removals, respectively. From (2), we have the following recurrence relations for these product moments.

Theorem 2.11. *For $2 \le m \le n$ and $k \ge 0$,*

$$\mu_{1:m:n}^{*\,(R_1,\cdots,R_m)^{(k)}} = \sum_{r_1=0}^{n-m}\sum_{r_2=0}^{n-m-r_1}\cdots\sum_{r_m=0}^{n-m-\sum_{l=0}^{m-1} r_l} \frac{\theta}{k-\theta(r_1+1)} \times\left\{(n-r_1-1)\mu_{1:m-1:n}^{(r_1+r_2+1,\cdots,r_m)^{(k)}} - \frac{A(n,m-1)}{A(n,m-2)}\right\} f(\boldsymbol{r};p).$$

Proof. Using the iteration formula for the expectation, we have

$$\begin{aligned}\mu^{*(R_1,\cdots,R_m)^{(k)}}_{1:m:n} &= E\left[X^{(R_1,\cdots,R_m)^k}_{1:m:n}\right] = E\left[E\left[X^{(R_1,\cdots,R_m)^k}_{1:m:n}|\boldsymbol{R}\right]\right] \\ &= \sum_{r_1=0}^{n-m}\sum_{r_2=0}^{n-m-r_1}\cdots\sum_{r_m=0}^{n-m-\sum_{l=0}^{m-1}r_l} E\left[X^{(r_1,\cdots,r_m)^k}_{1:m:n}|\boldsymbol{R}=\boldsymbol{r}\right]P(\boldsymbol{R}=\boldsymbol{r};p).\end{aligned}$$

Thus using Theorem 2.1 in the fixed removals case, we have

$$\begin{aligned}E\left[X^{(r_1,\cdots,r_m)^k}_{1:m:n}|\boldsymbol{R}=\boldsymbol{r}\right] &= \mu^{(r_1,r_2,\cdots,r_m)^{(k)}}_{1:m:n} \\ &= \frac{\theta}{k-\theta(r_1+1)}\Bigg\{(n-r_1-1)\,\mu^{(r_1+r_2+1,\cdots,r_m)^{(k)}}_{1:m-1:n} \\ &\quad -\frac{A(n,m-1)}{A(n,m-2)}\Bigg\}f(\boldsymbol{r};p).\end{aligned}$$

Hence, we conclude that

$$\begin{aligned}\mu^{*(R_1,\cdots,R_m)^{(k)}}_{1:m:n} &= \sum_{r_1=0}^{n-m}\sum_{r_2=0}^{n-m-r_1}\cdots\sum_{r_m=0}^{n-m-\sum_{l=0}^{m-1}r_l}\frac{\theta}{k-\theta(r_1+1)} \\ &\quad \Bigg\{(n-r_1-1)\,\mu^{(r_1+r_2+1,\cdots,r_m)^{(k)}}_{1:m-1:n} - \frac{A(n,m-1)}{A(n,m-2)}\Bigg\}f(\boldsymbol{r};p).\end{aligned}$$

□

Theorem 2.12. *For $2 \le i \le m-1$, $m \le n$ and $k \ge 0$,*

$$\begin{aligned}\mu^{*(R_1,\cdots,R_m)^{(k)}}_{i:m:n} &= \sum_{r_1=0}^{n-m}\sum_{r_2=0}^{n-m-r_1}\cdots\sum_{r_m=0}^{n-m-\sum_{l=0}^{m-1}r_l}\frac{\theta}{k-\theta(r_i+1)}f(\boldsymbol{r};p) \\ &\quad \times\Bigg\{(n-r_1-\cdots-r_i-i)\,\mu^{(r_1,\cdots,r_i+r_{i+1}+1,\cdots,r_m)^{(k)}}_{i:m-1:n} \\ &\quad -(n-r_1-\cdots-r_{i-1}-i+1)\,\mu^{(r_1,\cdots,r_{i-1}+r_i+1,\cdots,r_m)^{(k)}}_{i-1:m-1:n}\Bigg\}.\end{aligned}$$

Proof.

$$\begin{aligned}\mu^{*(R_1,\cdots,R_m)^{(k)}}_{i:m:n} &= E\left[X^{(R_1,\cdots,R_m)^k}_{i:m:n}\right] = E\left[E\left[X^{(R_1,\cdots,R_m)^k}_{i:m:n}|\boldsymbol{R}\right]\right] \\ &= \sum_{r_1=0}^{n-m}\sum_{r_2=0}^{n-m-r_1}\cdots\sum_{r_m=0}^{n-m-\sum_{l=0}^{m-1}r_l} E\left[X^{(r_1,\cdots,r_m)^k}_{i:m:n}|\boldsymbol{R}=\boldsymbol{r}\right]P(\boldsymbol{R}=\boldsymbol{r};p).\end{aligned}$$

So from Theorem 2.2, we have

$$E\left[X_{i:m:n}^{(r_1,\cdots,r_m)^k}|\boldsymbol{R}=\boldsymbol{r}\right]=\mu_{i:m:n}^{(r_1,r_2,\cdots,r_m)^{(k)}}$$
$$=\frac{\theta}{k-\theta(r_i+1)}\Bigg\{(n-r_1-\cdots-r_i-i)\mu_{i:m-1:n}^{(r_1,\cdots,r_i+r_{i+1}+1,\cdots,r_m)^{(k)}}$$
$$-(n-r_1-\cdots-r_{i-1}-i+1)\mu_{i-1:m-1:n}^{(r_1,\cdots,r_{i-1}+r_i+1,\cdots,r_m)^{(k)}}\Bigg\}.$$

Hence

$$\mu_{i:m:n}^{*\,(R_1,\cdots,R_m)^{(k)}} = \sum_{r_1=0}^{n-m}\sum_{r_2=0}^{n-m-r_1}\cdots\sum_{r_m=0}^{n-m-\sum_{l=0}^{m-1}r_l}\frac{\theta}{k-\theta(r_i+1)}f(\boldsymbol{r};p)$$
$$\times\Bigg\{(n-r_1-\cdots-r_i-i)\,\mu_{i:m-1:n}^{(r_1,\cdots,r_i+r_{i+1}+1,\cdots,r_m)^{(k)}}$$
$$-(n-r_1-\cdots-r_{i-1}-i+1)\,\mu_{i-1:m-1:n}^{(r_1,\cdots,r_{i-1}+r_i+1,\cdots,r_m)^{(k)}}\Bigg\}.$$

□

Theorem 2.13. *For* $2\leq m\leq n$ *and* $0\leq k\leq\theta$,

$$\mu_{m:m:n}^{*(R_1,\cdots,R_m)^{(k)}}=\sum_{r_1=0}^{n-m}\sum_{r_2=0}^{n-m-r_1}\cdots\sum_{r_m=0}^{n-m-\sum_{l=0}^{m-1}r_l}\frac{\theta(r_m+1)}{\theta(r_m+1)-k}\mu_{m-1:m-1:n}^{(r_1,\cdots,r_{m-1}+r_m+1)^{(k)}}f(\boldsymbol{r};p).$$

Proof. We have

$$\mu_{m:m:n}^{*(R_1,\cdots,R_m)^{(k)}}=E\left[X_{m:m:n}^{(R_1,\cdots,R_m)^k}\right]=E\left[E\left[X_{m:m:n}^{(R_1,\cdots,R_m)^k}|\boldsymbol{R}\right]\right]$$
$$=\sum_{r_1=0}^{n-m}\sum_{r_2=0}^{n-m-r_1}\cdots\sum_{r_m=0}^{n-m-\sum_{l=0}^{m-1}r_l}E\left[X_{m:m:n}^{(r_1,\cdots,r_m)^k}|\boldsymbol{R}=\boldsymbol{r}\right]P(\boldsymbol{R}=\boldsymbol{r};p).$$

Therefore, by Theorem 2.3, we have

$$E\left[X_{m:m:n}^{(r_1,\cdots,r_m)^k}|\boldsymbol{R}=\boldsymbol{r}\right]=\mu_{m:m:n}^{(r_1,r_2,\cdots,r_m)^{(k)}}=\frac{\theta(r_m+1)}{\theta(r_m+1)-k}\,\mu_{m-1:m-1:n}^{(r_1,\cdots,r_{m-1}+r_m+1)^{(k)}}.$$

Hence, the proof is completed. □

2.4 Product moments of progressively type-II censored order statistics with binomial removals

In this section, we prove some relations for the product moments of progressively Type-II censored order statistics with binomial random from the Pareto distribution. From (6), we have the following relations for these product moments.

Theorem 2.14. *For $2 \le i < j \le m-1$, $j-i \le 2$, $m \le n$ and $k,t \ge 0$,*

$$\begin{aligned}\mu^{*(R_1,\cdots,R_m)^{(k,t)}}_{i,j:m:n} &= \sum_{r_1=0}^{n-m}\sum_{r_2=0}^{n-m-r_1}\cdots\sum_{r_m=0}^{n-m-\sum_{l=0}^{m-1}r_l}\frac{\theta}{k-\theta(r_j+1)}f(\boldsymbol{r};p)\\ &\quad\times\Big\{(n-r_1-\cdots-r_j-j)\mu^{(r_1,\cdots,r_j+r_{j+1}+1,\cdots,r_m)^{(k,t)}}_{i,j:m-1:n}\\ &\quad-(n-r_1-\cdots-r_{j-1}-j+1)\,\mu^{(r_1,\cdots,r_{j-1}+r_j+1,\cdots,r_m)^{(k,t)}}_{i,j-1:m-1:n}\Big\}.\end{aligned}$$

Proof.

$$\begin{aligned}\mu^{*(R_1,\cdots,R_m)^{(k,t)}}_{i,j:m:n} &= E\left[X^{(R_1,\cdots,R_m)^k}_{i:m:n}\,X^{(R_1,\cdots,R_m)^t}_{j:m:n}\right]\\ &= E\left[E\left[X^{(R_1,\cdots,R_m)^k}_{i:m:n}\,X^{(R_1,\cdots,R_m)^t}_{j:m:n}|\boldsymbol{R}\right]\right]\\ &= \sum_{r_1=0}^{n-m}\sum_{r_2=0}^{n-m-r_1}\cdots\sum_{r_m=0}^{n-m-\sum_{l=0}^{m-1}r_l} E\left[X^{(R_1,\cdots,R_m)^k}_{i:m:n}\,X^{(R_1,\cdots,R_m)^t}_{j:m:n}|\boldsymbol{R}=\boldsymbol{r}\right]\\ &\quad\times P(\boldsymbol{R}=\boldsymbol{r};p),\end{aligned}$$

from Theorem 2.4, we have

$$\begin{aligned}E\big[X^{(R_1,\cdots,R_m)^k}_{i:m:n}\,X^{(R_1,\cdots,R_m)^t}_{j:m:n}|\boldsymbol{R}=\boldsymbol{r}\big] &= \mu^{(r_1,r_2,\cdots,r_m)^{(k,t)}}_{i,j:m:n}\\ &= \frac{\theta}{k-\theta(r_j+1)}\Big\{(n-r_1-\cdots-r_j-j)\mu^{(r_1,\cdots,r_j+r_{j+1}+1,\cdots,r_m)^{(k,t)}}_{i,j:m-1:n}\\ &\quad-(n-r_1-\cdots-r_{j-1}-j+1)\mu^{(r_1,\cdots,r_{j-1}+r_j+1,\cdots,r_m)^{(k,t)}}_{i,j-1:m-1:n}\Big\},\end{aligned}$$

hence, we reach

$$\begin{aligned}\mu^{*(R_1,\cdots,R_m)^{(k,t)}}_{i,j:m:n} &= \sum_{r_1=0}^{n-m}\sum_{r_2=0}^{n-m-r_1}\cdots\sum_{r_m=0}^{n-m-\sum_{l=0}^{m-1}r_l}\frac{\theta}{k-\theta(r_j+1)}f(\boldsymbol{r};p)\\ &\quad\times\Big\{(n-r_1-\cdots-r_j-j)\mu^{(r_1,\cdots,r_j+r_{j+1}+1,\cdots,r_m)^{(k,t)}}_{i,j:m-1:n}\\ &\quad-(n-r_1-\cdots-r_{j-1}-j+1)\,\mu^{(r_1,\cdots,r_{j-1}+r_j+1,\cdots,r_m)^{(k,t)}}_{i,j-1:m-1:n}\Big\}.\end{aligned}$$

□

Theorem 2.15. *For $1 \le i \le m-1$, $m \le n$, $k \ge 0$ and $0 \le t \le \theta$,*

$$\begin{aligned}\mu^{*(R_1,\cdots,R_m)^{(k,t)}}_{i,m:m:n} &= \sum_{r_1=0}^{n-m}\sum_{r_2=0}^{n-m-r_1}\cdots\sum_{r_m=0}^{n-m-\sum_{l=0}^{m-1}r_l}\frac{\theta(n-r_1-\cdots-r_{m-1}-m+1)}{\theta(r_m+1)-t}\\ &\quad\times\mu^{(r_1,\cdots,r_{m-1}+r_m+1)^{(k,t)}}_{i,m-1:m-1:n}\;f(\boldsymbol{r};p).\end{aligned}$$

Proof.

$$\begin{aligned}
\mu^{*(R_1,\cdots,R_m)^{(k,t)}}_{i,m:m:n} &= E\left[X^{(R_1,\cdots,R_m)^k}_{i:m:n}\, X^{(R_1,\cdots,R_m)^t}_{m:m:n}\right]\\
&= E\left[E\left[X^{(R_1,\cdots,R_m)^k}_{i:m:n}\, X^{(R_1,\cdots,R_m)^t}_{m:m:n}|\boldsymbol{R}\right]\right]\\
&= \sum_{r_1=0}^{n-m}\sum_{r_2=0}^{n-m-r_1}\cdots\sum_{r_m=0}^{n-m-\sum_{l=0}^{m-1} r_l} E\left[X^{(R_1,\cdots,R_m)^k}_{i:m:n} X^{(R_1,\cdots,R_m)^t}_{m:m:n}|\boldsymbol{R}=\boldsymbol{r}\right]\\
&\qquad\times P(\boldsymbol{R}=\boldsymbol{r};p),
\end{aligned}$$

from Theorem 2.5, we have

$$\begin{aligned}
E\big[X^{(R_1,\cdots,R_m)^k}_{i:m:n}\, X^{(R_1,\cdots,R_m)^t}_{m:m:n}|\boldsymbol{R}=\boldsymbol{r}\big] &= \mu^{(r_1,r_2,\cdots,r_m)^{(k,t)}}_{i,m:m:n}\\
&= \frac{\theta(n-r_1-\cdots-r_{m-1}-m+1)}{\theta(r_m+1)-t}\ \mu^{(r_1,\cdots,r_{m-1}+r_m+1)^{(k,t)}}_{i,m-1:m-1:n},
\end{aligned}$$

hence, we reach

$$\begin{aligned}
\mu^{*(R_1,\cdots,R_m)^{(k,t)}}_{i,m:m:n} &= \sum_{r_1=0}^{n-m}\sum_{r_2=0}^{n-m-r_1}\cdots\sum_{r_m=0}^{n-m-\sum_{l=0}^{m-1} r_l} \frac{\theta(n-r_1-\cdots-r_{m-1}-m+1)}{\theta(r_m+1)-t}\\
&\quad\times\mu^{(r_1,\cdots,r_{m-1}+r_m+1)^{(k,t)}}_{i,m-1:m-1:n}\ f(\boldsymbol{r};p).
\end{aligned}$$

□

Theorem 2.16. *For $2\le i<j\le m-1$, $j-i\le 2$, $m\le n$ and $k,t\ge 0$,*

$$\begin{aligned}
\mu^{*(R_1,\cdots,R_m)^{(k,t)}}_{i,j:m:n} \quad &= \quad \sum_{r_1=0}^{n-m}\sum_{r_2=0}^{n-m-r_1}\cdots\sum_{r_m=0}^{n-m-\sum_{l=0}^{m-1} r_l} \frac{\theta}{k-\theta(r_i+1)} f(\boldsymbol{r};p)\\
&\quad\times\Big\{(n-r_1-\cdots-r_i-i)\mu^{(r_1,\cdots,r_i+r_{i+1}+1,\cdots,r_m)^{(k,t)}}_{i,j:m-1:n}\\
&\quad-(n-r_1-\cdots-r_{i-1}-i+1)\ \mu^{(r_1,\cdots,r_{i-1}+r_i+1,\cdots,r_m)^{(k,t)}}_{i-1,j:m-1:n}\Big\}.
\end{aligned}$$

Proof.

$$\begin{aligned}
\mu^{*(R_1,\cdots,R_m)^{(k,t)}}_{i,j:m:n} &= E\left[X^{(R_1,\cdots,R_m)^k}_{i:m:n}\, X^{(R_1,\cdots,R_m)^t}_{j:m:n}\right]\\
&= E\left[E\left[X^{(R_1,\cdots,R_m)^k}_{i:m:n}\, X^{(R_1,\cdots,R_m)^t}_{j:m:n}|\boldsymbol{R}\right]\right]\\
&= \sum_{r_1=0}^{n-m}\sum_{r_2=0}^{n-m-r_1}\cdots\sum_{r_m=0}^{n-m-\sum_{l=0}^{m-1} r_l} E\left[X^{(R_1,\cdots,R_m)^k}_{i:m:n}\, X^{(R_1,\cdots,R_m)^t}_{j:m:n}|\boldsymbol{R}=\boldsymbol{r}\right]\\
&\qquad\times P(\boldsymbol{R}=\boldsymbol{r};p),
\end{aligned}$$

from Theoerm 2.6, we have

$$
\begin{aligned}
E\big[X_{i:m:n}^{(R_1,\cdots,R_m)^k}\ X_{j:m:n}^{(R_1,\cdots,R_m)^t}|\boldsymbol{R}=\boldsymbol{r}\big] &= \mu_{i,j:m:n}^{(r_1,r_2,\cdots,r_m)^{(k,t)}} \\
&= \frac{\theta}{k-\theta(r_i+1)}\Bigg\{(n-r_1-\cdots-r_i-i)\mu_{i,j:m-1:n}^{(r_1,\cdots,r_i+r_{i+1}+1,\cdots,r_m)^{(k,t)}} \\
&-(n-r_1-\cdots-r_{i-1}-i+1)\mu_{i-1,j:m-1:n}^{(r_1,\cdots,r_{i-1}+r_i+1,\cdots,r_m)^{(k,t)}}\Bigg\},
\end{aligned}
$$

hence, we reach

$$
\begin{aligned}
\mu_{i,j:m:n}^{*\,(R_1,\cdots,R_m)^{(k,t)}} &= \sum_{r_1=0}^{n-m}\sum_{r_2=0}^{n-m-r_1}\cdots\sum_{r_m=0}^{n-m-\sum_{l=0}^{m-1}r_l}\frac{\theta}{k-\theta(r_i+1)}f(\boldsymbol{r};p) \\
&\times\Bigg\{(n-r_1-\cdots-r_i-i)\mu_{i,j:m-1:n}^{(r_1,\cdots,r_i+r_{i+1}+1,\cdots,r_m)^{(k,t)}} \\
&-(n-r_1-\cdots-r_{i-1}-i+1)\ \mu_{i-1,j:m-1:n}^{(r_1,\cdots,r_{i-1}+r_i+1,\cdots,r_m)^{(k,t)}}\Bigg\}.
\end{aligned}
$$

□

Theorem 2.17. *For $2\le i\le m-1$, $m\le n$ and $k,t\ge 0$,*

$$
\begin{aligned}
\mu_{i,i+1:m:n}^{*\,(R_1,\cdots,R_m)^{(k,t)}} &= \sum_{r_1=0}^{n-m}\sum_{r_2=0}^{n-m-r_1}\cdots\sum_{r_m=0}^{n-m-\sum_{l=0}^{m-1}r_l}\frac{\theta}{k-\theta(r_i+1)}f(\boldsymbol{r};p) \\
&\times\Bigg\{(n-r_1-\cdots-r_i-i)\mu_{i:m-1:n}^{(r_1,\cdots,r_i+r_{i+1}+1,\cdots,r_m)^{(k+t)}} \\
&-(n-r_1-\cdots-r_{i-1}-i+1)\ \mu_{i-1,i:m-1:n}^{(r_1,\cdots,r_{i-1}+r_i+1,\cdots,r_m)^{(k,t)}}\Bigg\}.
\end{aligned}
$$

Proof.

$$
\begin{aligned}
\mu_{i,i+1:m:n}^{*(R_1,\cdots,R_m)^{(k,t)}} &= E\left[X_{i:m:n}^{(R_1,\cdots,R_m)^k}X_{i+1:m:n}^{(R_1,\cdots,R_m)^t}\right] \\
&= E\left[E\left[X_{i:m:n}^{(R_1,\cdots,R_m)^k}X_{i+1:m:n}^{(R_1,\cdots,R_m)^t}|\boldsymbol{R}\right]\right] \\
&= \sum_{r_1=0}^{n-m}\sum_{r_2=0}^{n-m-r_1}\cdots\sum_{r_m=0}^{n-m-\sum_{l=0}^{m-1}r_l} E\left[X_{i:m:n}^{(R_1,\cdots,R_m)^k}X_{i+1:m:n}^{(R_1,\cdots,R_m)^t}|\boldsymbol{R}=\boldsymbol{r}\right] \\
&\times P(\boldsymbol{R}=\boldsymbol{r};p),
\end{aligned}
$$

from Theorem 2.7, we have

$$E\big[X_{i:m:n}^{(R_1,\cdots,R_m)^k}\ X_{i+1:m:n}^{(R_1,\cdots,R_m)^t}|\boldsymbol{R}=\boldsymbol{r}\big]=\mu_{i,i+1:m:n}^{(r_1,r_2,\cdots,r_m)^{(k,t)}}$$
$$=\frac{\theta}{k-\theta(r_i+1)}\Big\{(n-r_1-\cdots-r_i-i)\mu_{i:m-1:n}^{(r_1,\cdots,r_i+r_{i+1}+1,\cdots,r_m)^{(k+t)}}$$
$$-(n-r_1-\cdots-r_{i-1}-i+1)\mu_{i-1,i:m-1:n}^{(r_1,\cdots,r_{i-1}+r_i+1,\cdots,r_m)^{(k,t)}}\Big\},$$

hence, we reach

$$\mu_{i,i+1:m:n}^{*\,(R_1,\cdots,R_m)^{(k,t)}} = \sum_{r_1=0}^{n-m}\sum_{r_2=0}^{n-m-r_1}\cdots\sum_{r_m=0}^{n-m-\sum_{l=0}^{m-1}r_l}\frac{\theta}{k-\theta(r_i+1)}f(\boldsymbol{r};p)$$
$$\times\Big\{(n-r_1-\cdots-r_i-i)\mu_{i:m-1:n}^{(r_1,\cdots,r_i+r_{i+1}+1,\cdots,r_m)^{(k+t)}}$$
$$-(n-r_1-\cdots-r_{i-1}-i+1)\ \mu_{i-1,i:m-1:n}^{(r_1,\cdots,r_{i-1}+r_i+1,\cdots,r_m)^{(k,t)}}\Big\}.$$

□

Theorem 2.18. *For $m\geq 2$, $m\leq n$ and $k,t\geq 0$,*

$$\mu_{1,2:m:n}^{*\,(R_1,\cdots,R_m)^{(k,t)}} = \sum_{r_1=0}^{n-m}\sum_{r_2=0}^{n-m-r_1}\cdots\sum_{r_m=0}^{n-m-\sum_{l=0}^{m-1}r_l}\frac{\theta}{k-\theta(r_1+1)}\Big\{(n-r_1-1)$$
$$\times\mu_{1:m-1:n}^{(r_1+r_2+1,\cdots,r_m)^{(k+t)}}-\frac{A(n,m-1)}{A(n,m-2)}\ \mu_{1:m-1:n-1}^{(r_2,\cdots,r_m)^{(t)}}\Big\}f(\boldsymbol{r};p).$$

Proof.

$$\mu_{1,2:m:n}^{*\,(R_1,\cdots,R_m)^{(k,t)}} = E\left[X_{1:m:n}^{(R_1,\cdots,R_m)^k}\ X_{2:m:n}^{(R_1,\cdots,R_m)^t}\right]$$
$$=E\left[E\left[X_{1:m:n}^{(R_1,\cdots,R_m)^k}\ X_{2:m:n}^{(R_1,\cdots,R_m)^t}|\boldsymbol{R}\right]\right]$$
$$=\sum_{r_1=0}^{n-m}\sum_{r_2=0}^{n-m-r_1}\cdots\sum_{r_m=0}^{n-m-\sum_{l=0}^{m-1}r_l}E\left[X_{1:m:n}^{(R_1,\cdots,R_m)^k}X_{2:m:n}^{(R_1,\cdots,R_m)^t}|\boldsymbol{R}=\boldsymbol{r}\right]$$
$$\times P(\boldsymbol{R}=\boldsymbol{r};p),$$

from Theorem 2.8, we have

$$E\big[X_{1:m:n}^{(R_1,\cdots,R_m)^k}\quad X_{2:m:n}^{(R_1,\cdots,R_m)^t}|\boldsymbol{R}=\boldsymbol{r}\big]=\mu_{1,2:m:n}^{(r_1,r_2,\cdots,r_m)^{(k,t)}}$$
$$=\frac{\theta}{k-\theta(r_1+1)}\Big\{(n-r_1-1)\mu_{1:m-1:n}^{(r_1+r_2+1,\cdots,r_m)^{(k+t)}}$$
$$-\frac{A(n,m-1)}{A(n,m-2)}\ \mu_{1:m-1:n-1}^{(r_2,\cdots,r_m)^{(t)}}\Big\},$$

hence, we reach

$$\mu^{*\,(R_1,\cdots,R_m)^{(k,t)}}_{1,2:m:n} = \sum_{r_1=0}^{n-m}\sum_{r_2=0}^{n-m-r_1}\cdots\sum_{r_m=0}^{n-m-\sum_{l=0}^{m-1} r_l} \frac{\theta}{k-\theta(r_1+1)}\Big\{(n-r_1-1) \times\mu^{(r_1+r_2+1,\cdots,r_m)^{(k+t)}}_{1:m-1:n} - \frac{A(n,m-1)}{A(n,m-2)}\,\mu^{(r_2,\cdots,r_m)^{(t)}}_{1:m-1:n-1}\Big\}f(\boldsymbol{r};p).$$

□

Theorem 2.19. *For $2\leq i\leq m-1$, $m\leq n$ and $k,t\geq 0$,*

$$\mu^{*\,(R_1,\cdots,R_m)^{(k,t)}}_{i,i+1:m:n} = \sum_{r_1=0}^{n-m}\sum_{r_2=0}^{n-m-r_1}\cdots\sum_{r_m=0}^{n-m-\sum_{l=0}^{m-1} r_l} \frac{\theta}{t-\theta(r_{i+1}+1)}f(\boldsymbol{r};p) \times\Big\{(n-r_1-\cdots-r_{i+1}-i-1)\,\mu^{(r_1,\cdots,r_{i+1}+r_{i+2}+1,\cdots,r_m)^{(k,t)}}_{i,i+1:m-1:n} -(n-r_1-\cdots-r_i-i)\,\mu^{(r_1,\cdots,r_i+r_{i+1}+1,\cdots,r_m)^{(k+t)}}_{i:m-1:n}\Big\}.$$

Proof.

$$\begin{aligned}\mu^{*\,(R_1,\cdots,R_m)^{(k,t)}}_{i,i+1:m:n} &= E\Big[X^{(R_1,\cdots,R_m)^k}_{i:m:n}\,X^{(R_1,\cdots,R_m)^t}_{i+1:m:n}\Big]\\ &= E\Big[E\Big[X^{(R_1,\cdots,R_m)^k}_{i:m:n}\,X^{(R_1,\cdots,R_m)^t}_{i+1:m:n}|\boldsymbol{R}\Big]\Big]\\ &= \sum_{r_1=0}^{n-m}\sum_{r_2=0}^{n-m-r_1}\cdots\sum_{r_m=0}^{n-m-\sum_{l=0}^{m-1} r_l} E\Big[X^{(R_1,\cdots,R_m)^k}_{i:m:n}\,X^{(R_1,\cdots,R_m)^t}_{i+1:m:n}|\boldsymbol{R}=\boldsymbol{r}\Big]\\ &\qquad\times P(\boldsymbol{R}=\boldsymbol{r};p),\end{aligned}$$

from Theorem 2.9, we have

$$E\big[X^{(R_1,\cdots,R_m)^k}_{i:m:n}\,X^{(R_1,\cdots,R_m)^t}_{i+1:m:n}\,|\,\boldsymbol{R}=\boldsymbol{r}\big] = \mu^{(r_1,r_2,\cdots,r_m)^{(k,t)}}_{i,i+1:m:n} = \frac{\theta}{t-\theta(r_{i+1}+1)} \times\Big\{(n-r_1-\cdots-r_{i+1}-i-1)\mu^{(r_1,\cdots,r_{i+1}+r_{i+2}+1,\cdots,r_m)^{(k,t)}}_{i,i+1:m-1:n} -(n-r_1-\cdots-r_i-i)\mu^{(r_1,\cdots,r_i+r_{i+1}+1,\cdots,r_m)^{(k+t)}}_{i:m-1:n}\Big\},$$

hence, we reach

$$\mu^{*\,(R_1,\cdots,R_m)^{(k,t)}}_{i,i+1:m:n} = \sum_{r_1=0}^{n-m}\sum_{r_2=0}^{n-m-r_1}\cdots\sum_{r_m=0}^{n-m-\sum_{l=0}^{m-1} r_l} \frac{\theta}{t-\theta(r_{i+1}+1)}f(\boldsymbol{r};p) \times\Big\{(n-r_1-\cdots-r_{i+1}-i-1)\,\mu^{(r_1,\cdots,r_{i+1}+r_{i+2}+1,\cdots,r_m)^{(k,t)}}_{i,i+1:m-1:n} -(n-r_1-\cdots-r_i-i)\,\mu^{(r_1,\cdots,r_i+r_{i+1}+1,\cdots,r_m)^{(k+t)}}_{i:m-1:n}\Big\}.$$

□

Theorem 2.20. *For $m \leq 2$, $m \leq n$, $k \geq 0$ and $0 \leq t \leq \theta$,*

$$\mu^{*\,(R_1,\cdots,R_m)^{(k,t)}}_{m-1,m:m:n} = \sum_{r_1=0}^{n-m} \sum_{r_2=0}^{n-m-r_1} \cdots \sum_{r_m=0}^{n-m-\sum_{l=0}^{m-1} r_l} \frac{\theta(n-r_1-\cdots-r_{m-1}-m+1)}{\theta(r_m+1)-t} \times \mu^{(r_1,\cdots,r_{m-1}+r_m+1)^{(k+t)}}_{m-1:m-1:n} f(\boldsymbol{r};p).$$

Proof.

$$\begin{aligned} \mu^{*\,(R_1,\cdots,R_m)^{(k,t)}}_{m-1,m:m:n} &= E\left[X^{(R_1,\cdots,R_m)^k}_{m-1:m:n} X^{(R_1,\cdots,R_m)^t}_{m:m:n}\right] \\ &= E\left[E\left[X^{(R_1,\cdots,R_m)^k}_{m-1:m:n} X^{(R_1,\cdots,R_m)^t}_{m:m:n} | \boldsymbol{R}\right]\right] \\ &= \sum_{r_1=0}^{n-m} \sum_{r_2=0}^{n-m-r_1} \cdots \sum_{r_m=0}^{n-m-\sum_{l=0}^{m-1} r_l} E\left[X^{(R_1,\cdots,R_m)^k}_{m-1:m:n} X^{(R_1,\cdots,R_m)^t}_{m:m:n} | \boldsymbol{R}=\boldsymbol{r}\right] \\ &\qquad \times P(\boldsymbol{R}=\boldsymbol{r};p). \end{aligned}$$

From Theorem2.10, we have

$$\begin{aligned} E[X^{(R_1,\cdots,R_m)^k}_{m-1:m:n} X^{(R_1,\cdots,R_m)^t}_{m:m:n} \mid \boldsymbol{R}=\boldsymbol{r}] &= \mu^{(r_1,r_2,\cdots,r_m)^{(k,t)}}_{m-1,m:m:n} \\ &= \frac{\theta(n-r_1-\cdots-r_{m-1}-m+1)}{\theta(r_m+1)-t} \mu^{(r_1,\cdots,r_{m-1}+r_m+1)^{(k+t)}}_{m-1:m-1:n}. \end{aligned}$$

Therefore,

$$\mu^{*\,(R_1,\cdots,R_m)^{(k,t)}}_{m-1,m:m:n} = \sum_{r_1=0}^{n-m} \sum_{r_2=0}^{n-m-r_1} \cdots \sum_{r_m=0}^{n-m-\sum_{l=0}^{m-1} r_l} \frac{\theta(n-r_1-\cdots-r_{m-1}-m+1)}{\theta(r_m+1)-t} \times \mu^{(r_1,\cdots,r_{m-1}+r_m+1)^{(k+t)}}_{m-1:m-1:n} f(\boldsymbol{r};p).$$

□

3 Thomas-Wilson's mixture formula for moments

Thomas and Wilson (1972) presented a procedure for calculating the means, variances, and covariance of order statistics for progressively Type-II censored samples from the corresponding moments of order statistics from complete samples.

Assume $X_{j:m:n}, j = 1, \cdots, m$ represent the progressively Type-II censored order statistics with censoring scheme $(r_1, \cdots, r_m)$ and the order statistics from complete sample denote by $X_{i:n}, i = 1, \cdots, n$. Let K_j be the rank of $X_{j:m:n}$, i.e., $X_{j:m:n} = X_{K_j}$ for some $j = 1, \cdots, m$. Note that $K_1 = 1$ and K_j can take on values $k_{j-1}+1, k_{j-1}+2, \cdots, j+r_1+\cdots+r_{j-1}$ for $j = 2, \cdots, m$, given $K_{j-1} = k_{j-1}$. Thomas and Wilson (1972) obtained the joint probability function of rank vector $(K_1, \cdots, K_m)$ as

$$P(K_1 = k_1, \cdots, K_m = k_m) = P(K_1 = k_1) \prod_{i=2}^{m} P(K_i = k_i | K_{i-1} = k_{i-1}, \cdots, K_1 = k_1)$$

with $P(K_1 = 1) = 1$, and showed that the conditional probabilities $P(K_i = k_i | K_{i-1} = k_{i-1}, \cdots, K_1 = k_1)$ are given by

$$P(K_i = k_i | K_{i-1} = k_{i-1}, \cdots, K_1 = k_1) = \frac{\binom{n-k_i}{\sum_{j=1}^{i-1}(r_j+1)-k_i+1}}{\binom{n-k_{i-1}}{\sum_{j=1}^{i-1}(r_j+1)-k_{i-1}}}, \qquad i = 2, \cdots, m.$$

Therefore, if M is the number of all observed vectors of $(k_1, \cdots, k_m)$, then $m \times n$ matrices $D_l,\ l = 1, \cdots, M$ can be defined with (i, j)-general element

$$d_{ij}^l = \begin{cases} 1 & j = r_i \\ 0 & j \neq r_i, \end{cases}$$

for $i = 1, \cdots, m$ and $j = 1, \cdots, n$. So by writing $\boldsymbol{X}^{ps} = \boldsymbol{D}\boldsymbol{X}^u$ where $\boldsymbol{X}^{ps}$ is the $m \times 1$ vector of progressively Type-II censored order statistics and $\boldsymbol{X}^u$ is the $n \times 1$ vector of order statistics from complete samples, we can obtain the vector of the mean and the variance-covariance matrix for progressive Type-II censoring with fixed and binomial removals.

3.1 Progressively type-II censoring with fixed removals

By denoting $E[\boldsymbol{X}^u]$ by $\boldsymbol{\mu}^u$, the mean vector of an order statistic for progressive Type-II censoring with fixed removals is easily obtained as follows:

$$E[\boldsymbol{X}^{ps}] = E[E[\boldsymbol{X}^{ps}|\boldsymbol{D}]] = \sum_{l=1}^{M} E[D_l \boldsymbol{X}^u | \boldsymbol{D} = D_l]\, p_l = \left(\sum_{l=1}^{M} D_l p_l\right) \boldsymbol{\mu}^u,$$

where p_l is the probability of the rank vector corresponding to $D_l,\ l = 1, \cdots, M$. Similarly, by denoting $Var[\boldsymbol{X}^u]$ by $\boldsymbol{\Sigma}^u$, the variance-covariance matrix $\boldsymbol{X}^{ps}$ is given by:

$$\begin{aligned} Var[\boldsymbol{X}^{ps}] &= E\left[\boldsymbol{X}^{ps}(\boldsymbol{X}^{ps})^T\right] - \boldsymbol{\mu}^u(\boldsymbol{\mu}^u)^T = E\left[\boldsymbol{X}^{ps}(\boldsymbol{X}^{ps})^T|\boldsymbol{D}\right] - \boldsymbol{\mu}^u(\boldsymbol{\mu}^u)^T \\ &= \sum_{l=1}^{M} E\left[D_l\, \boldsymbol{X}^u(\boldsymbol{X}^u)^T D_l^T | \boldsymbol{D} = D_l\right]\ p_l - \boldsymbol{\mu}^u(\boldsymbol{\mu}^u)^T \\ &= \sum_{l=1}^{M} D_l(\boldsymbol{\Sigma}^u + \boldsymbol{\mu}^u(\boldsymbol{\mu}^u)^T)D_l^T\, p_l. \end{aligned}$$

3.2 Progressively type-II censoring with binomial removals

In this section, the pattern of removal at each failure time is random from the binomial distribution. Let $\boldsymbol{R} = (R_1, \cdots, R_m)$ and $\boldsymbol{r} = (r_1, \cdots, r_m)$ be the random and fixed removals, respectively.
Therefore, by denoting $E[\boldsymbol{X}^u]$ by $\boldsymbol{\mu}^u$, the means vector of order statistic for progressive Type-II censoring with Binomial removals is easily obtained as follows:

$$E[\boldsymbol{X}^{ps}] = E[E[\boldsymbol{X}^{ps}|\boldsymbol{R}]] = \sum_{r_1=0}^{n-m} \sum_{r_2=0}^{n-m-r_1} \cdots \sum_{r_m=0}^{n-m-\sum_{i=0}^{m-1} r_i} E[\boldsymbol{X}^{ps}|\boldsymbol{R} = \boldsymbol{r}]\, P(\boldsymbol{R} = \boldsymbol{r}; p),$$

where

$$E\left[\boldsymbol{X}^{ps}|\boldsymbol{R}=\boldsymbol{r}\right]=E_{\boldsymbol{r}}\left[\boldsymbol{X}^{ps}\right]=E\left[E_{\boldsymbol{r}}\left[\boldsymbol{X}^{ps}|\boldsymbol{D}\right]\right]=\sum_{l=1}^{M}E_{\boldsymbol{r}}\left[D_l\boldsymbol{X}^{u}|\boldsymbol{D}=D_l\right]\ p_l.$$

Hence,

$$E\left[\boldsymbol{X}^{ps}\right]=\sum_{r_1=0}^{n-m}\ \sum_{r_2=0}^{n-m-r_1}\cdots\sum_{r_m=0}^{n-m-\sum_{i=0}^{m-1}r_i}\ \sum_{l=1}^{M}p_l\ f(\boldsymbol{r};p)\ D_l\ \boldsymbol{\mu}^{u},$$

where p_l is the probability function of the rank vector corresponding to D_l, $l=1,\cdots,M$ and $f(\boldsymbol{r};p)$ is the joint probability distribution of $\boldsymbol{r}$. Similarly, by denoting $Var[\boldsymbol{X}^u]$ by $\boldsymbol{\Sigma}^u$, the variance-covariance matrix $\boldsymbol{X}^{ps}$ is given by:

$$\begin{aligned}Var(\boldsymbol{X}^{ps})&=E\left[\boldsymbol{X}^{ps}\left(\boldsymbol{X}^{ps}\right)^T\right]-\boldsymbol{\mu}^{u}\left(\boldsymbol{\mu}^{u}\right)^T=E\left[\boldsymbol{X}^{ps}\left(\boldsymbol{X}^{ps}\right)^T|\boldsymbol{R}\right]-\boldsymbol{\mu}^{u}\left(\boldsymbol{\mu}^{u}\right)^T\\&=\sum_{r_1=0}^{n-m}\sum_{r_2=0}^{n-m-r_1}\cdots\sum_{r_m=0}^{n-m-\sum_{i=0}^{m-1}r_i}E[\boldsymbol{X}^{ps}(\boldsymbol{X}^{ps})^T|\boldsymbol{R}=\boldsymbol{r}]P(\boldsymbol{R}=\boldsymbol{r};p)-\boldsymbol{\mu}^{u}(\boldsymbol{\mu}^{u})^T,\end{aligned}$$

where

$$\begin{aligned}E\left[\boldsymbol{X}^{ps}\left(\boldsymbol{X}^{ps}\right)^T|\boldsymbol{R}=\boldsymbol{r}\right]&=E_{\boldsymbol{r}}\left[\boldsymbol{X}^{ps}\left(\boldsymbol{X}^{ps}\right)^T\right]=E\left[E_{\boldsymbol{r}}\left[\boldsymbol{X}^{ps}\left(\boldsymbol{X}^{ps}\right)^T|\boldsymbol{D}\right]\right]\\&=\sum_{l=1}^{M}E_{\boldsymbol{r}}\left[D_l\boldsymbol{X}^{u}\left(\boldsymbol{X}^{u}\right)^T D_l^T|\boldsymbol{D}=D_l\right]\ p_l.\end{aligned}$$

Hence,

$$Var[\boldsymbol{X}^{ps}]=\sum_{r_1=0}^{n-m}\sum_{r_2=0}^{n-m-r_1}\cdots\sum_{r_m=0}^{n-m-\sum_{i=0}^{m-1}r_i}\ \sum_{l=1}^{M}p_l f(\boldsymbol{r};p)D_l(\boldsymbol{\Sigma}^u+\boldsymbol{\mu}^{u}(\boldsymbol{\mu}^{u})^T)D_l^T-\boldsymbol{\mu}^{u}(\boldsymbol{\mu}^{u})^T.$$

Note that this computation would be really exhausted when m and n are large, but it can be useful to obtain some missing moments.

4 Numerical study

In this section, we present the results of numerical experiment of the real values of moments investigated in the previous section. Also, we present a Monte Carlo simulation to compare real values of moments with corresponding simulated values based on their biases and mean square errors (MSEs). Consider the binomial distribution as distribution of random removals and the joint probability distribution for $\boldsymbol{R}$ as

$$f(\boldsymbol{r};p)=b\,p^d(1-p)^e$$

where $b=\dfrac{(n-m)!}{\left(n-m-\sum_{j=1}^{m-1}r_j\right)!\prod_{j=1}^{m-1}r_j!}$, $d=\sum\limits_{j=1}^{m-1}r_j$, $e=(m-1)(n-m)-\sum\limits_{j=1}^{m-1}(m-j)r_j$.

4.1 Numerical example

Here, we obtain the real values of moments for progressive Type-II censoring with binomial removals from the exact formulas investigated in Subsection 3.2, directly. The results for $n = 5, 7$, $m = 2, 3$ and $\theta = 4, 6$ with the binomial removal corresponding to $p = 0.2, 0.8$ are displayed in Table 2.

4.2 Simulation study

A Monte Carlo simulation study is used to obtain the moments for progressive Type-II censoring with binomial removals and evaluate the biases and MSEs to compare with real values of moments. We randomly generated 1000 progressively censored samples from the Pareto distribution with $n = 5, 7$, $m = 2, 3$ and $\theta = 4, 6$. Note that we worked with the binomial removal corresponding to $p = 0.2, 0.8$. Table 3 displays the results of simulation, and also *biases and MSEs of the expected termination time*. All the computations are performed using R software.

From Tables 2 and 3, we conclude:

(i) When m, p and θ are fixed, the values of the expected termination time under progressive censoring with binomial removals decrease as n increases.

(ii) When n, p and θ are fixed, the values of expected termination time under progressive censoring with binomial removals increase as m increases.

(iii) For the effect of the removal probability p, when n, m and θ are fixed, the values of expected termination time of the progressive censoring with binomial removals are increasing as p increases. Increasing of the removal probability p means more items are removed at the early stage of the study. This would lead to the collection of observations much closer to the tail of the life-time distribution. Therefore, the experiment time under progressive censoring with binomial removals is an increasing function in p.

(iv) When n, m and p are fixed, the values of the expected termination time under progressive censoring with binomial removals increase as θ decreases.

Table 2: Real values of moments from exact formulas for $n = 5, 7$, $m = 2, 3$, $\theta = 4, 6$ and $p = 0.2, 0.8$.

θ	n	m	p	$E[X_{i:m:n}]$	$E[X^2_{i:m:n}]$	$E[X^3_{i:m:n}]$	$E[X_{i:m:n}X_{j:m:n}]$
4	5	2	0.2	1.05263	1.11111	1.17647	1.20603($i=1, j=2$)
				1.14255	1.32216	1.56206	
			0.8	1.05263	1.11111	1.17647	1.37198($i=1, j=2$)
				1.29977	1.84482	3.29441	
		3	0.2	1.05263	1.11111	1.17647	1.19719($i=1, j=2$)
				1.13418	1.29862	1.50395	1.49086($i=2, j=3$)
				1.30095	1.78022	2.75878	1.37323($i=1, j=3$)
			0.8	1.05263	1.11111	1.17647	1.24798($i=1, j=2$)
				1.18230	1.18231	1.76458	1.88097($i=2, j=3$)
				1.55923	2.78187	6.76674	1.64585($i=1, j=3$)
	7	2	0.2	1.03704	1.07692	1.12000	1.13630($i=1, j=2$)
				1.09421	1.20319	1.33081	
			0.8	1.03704	1.07692	1.12000	1.28167($i=1, j=2$)
				1.23420	1.62963	2.58726	
		3	0.2	1.03704	1.07692	1.12000	1.13310($i=1, j=2$)
				1.09113	1.19575	1.31682	1.29879($i=2, j=3$)
				1.18482	1.42823	1.77970	1.23039($i=1, j=3$)
			0.8	1.03704	1.07692	1.12000	1.19259($i=1, j=2$)
				1.14842	1.33903	1.59318	1.74860($i=2, j=3$)
				1.49861	2.54991	5.85046	1.55625($i=1, j=3$)
6	5	2	0.2	1.03448	1.07143	1.11111	1.13055($i=1, j=2$)
				1.09156	1.19758	1.32216	
			0.8	1.03448	1.07143	1.11111	1.22497($i=1, j=2$)
				1.18273	1.44164	1.84482	
		3	0.2	1.03448	1.07143	1.11111	1.12530($i=1, j=2$)
				1.08650	1.18519	1.29862	1.29487($i=2, j=3$)
				1.18668	1.43387	1.78022	1.22906($i=1, j=3$)
			0.8	1.03448	1.07143	1.11111	1.15580($i=1, j=2$)
				1.11594	1.25532	1.42560	1.49608($i=2, j=3$)
				1.32973	1.85465	2.78187	1.37722($i=1, j=3$)
	7	2	0.2	1.02439	1.05000	1.07692	1.08785($i=1, j=2$)
				1.06132	1.12874	1.20319	
			0.8	1.02439	1.05000	1.07692	1.17314($i=1, j=2$)
				1.14453	1.34031	1.62963	
		3	0.2	1.02439	1.05000	1.07692	1.08586($i=1, j=2$)
				1.05938	1.12438	1.19575	1.18662($i=2, j=3$)
				1.11791	1.25808	1.42823	1.14586($i=1, j=3$)
			0.8	1.02439	1.05000	1.07692	1.12238($i=1, j=2$)
				1.09501	1.20646	1.33903	1.42830($i=2, j=3$)
				1.29600	1.75633	2.54991	1.32840($i=1, j=3$)

Table 3: Simulated values of moments with Binomial removals for $n = 5, 7$, $m = 2, 3$, $\theta = 4, 6$ and $p = 0.2, 0.8$.

θ	n	m	p	$E[X_{i:m:n}]$ (Bais, MSE)	$E[X^2_{i:m:n}]$	$E[X^3_{i:m:n}]$	$E[X_{i:m:n}X_{j:m:n}]$
4	5	2	0.2	1.05373	1.11320	1.17932	1.20106
				1.13679	1.30440	1.51285	$(i = 1, j = 2)$
				(0.00576, 0.00174)			
			0.8	1.05380	1.11413	1.18243	1.20619
				1.14054	1.31834	1.55252	$(i = 1, j = 2)$
				(0.15923, 0.03009)			
		3	0.2	1.04945	1.10420	1.16524	1.19032
							$(i = 1, j = 2)$
				1.13142	1.29131	1.48859	1.46806
							$(i = 2, j = 3)$
				1.28550	1.48859	2.46507	1.35212
				(0.01546, 0.04828)			$(i = 1, j = 3)$
			0.8	1.05178	1.10915	1.17304	1.24684
							$(i = 1, j = 2)$
				1.18251	1.425921	1.76253	1.86012
							$(i = 2, j = 3)$
				1.54258	2.64086	5.25338	1.62649
				(0.01665, 0.01632)			$(i = 1, j = 3)$
	7	2	0.2	1.03367	1.06969	1.10831	1.12397
				1.08617	1.18458	1.29786	$(i = 1, j = 2)$
				(0.00804, 0.00158)			
			0.8	1.03754	1.07814	1.12224	1.13110
				1.08858	1.18946	1.30498	$(i = 1, j = 2)$
				(0.14562, 0.02367)			
		3	0.2	1.03748	1.07783	1.12142	1.13289
							$(i = 1, j = 2)$
				1.09047	1.19420	1.31393	1.29982
							$(i = 2, j = 3)$
				1.18612	1.43013	1.76352	1.23227
				(-0.00130, 0.00537)			$(i = 1, j = 3)$
			0.8	1.03641	1.07556	1.11774	1.19630
							$(i = 1, j = 2)$
				1.15250	1.34756	1.60327	1.77627
							$(i = 2, j = 3)$
				1.51931	2.68306	6.56234	1.57681
				(-0.02070, 0.23772)			$(i = 1, j = 3)$

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Table 3: Continued.

θ	n	m	p	$E[X_{i:m:n}]$ (Bais, MSE)	$E[X^2_{i:m:n}]$	$E[X^3_{i:m:n}]$	$E[X_{i:m:n}X_{j:m:n}]$
6	5	2	0.2	1.03380	1.06996	1.10872	1.12687
				1.08880	1.19061	1.30820	$(i=1, j=2)$
				(0.00276, 0.00108)			
			0.8	1.03492	1.07245	1.11287	1.12918
				1.08958	1.19304	1.31365	$(i=1, j=2)$
				(0.09314, 0.00896)			
		3	0.2	1.03526	1.07304	1.11362	1.12751
							$(i=1, j=2)$
				1.08778	1.18795	1.30305	1.29837
							$(i=2, j=3)$
				1.18888	1.44193	1.80263	1.23240
				(-0.00221, 0.00420)			$(i=1, j=3)$
			0.8	1.03389	1.07021	1.10928	1.15434
							$(i=1, j=2)$
				1.11506	1.25317	1.42121	1.49811
							$(i=2, j=3)$
				1.33302	1.85511	2.73377	1.37954
				(-0.00329, 0.00673)			$(i=1, j=3)$
	7	2	0.2	1.02512	1.05151	1.07929	1.08767
				1.06036	1.12642	1.19892	$(i=1, j=2)$
				(0.00096, 0.00057)			
			0.8	1.02460	1.05045	1.07766	1.08596
				1.05922	1.12412	1.19555	$(i=1, j=2)$
				(0.08530, 0.00772)			
		3	0.2	1.02640	1.05429	1.08381	1.09211
							$(i=1, j=2)$
				1.06322	1.13302	1.21040	1.19707
							$(i=2, j=3)$
				1.12308	1.27052	1.44977	1.15367
				(-0.00516, 0.00075)			$(i=1, j=3)$
			0.8	1.02290	1.04687	1.07201	1.11651
							$(i=1, j=2)$
				1.09108	1.19791	1.32573	1.40266
							$(i=2, j=3)$
				1.27787	1.68688	2.32127	1.30750
				(0.01813, 0.02233)			$(i=1, j=3)$

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