A New Extended Alpha Power Transformed Family of Distributions: Properties and Applications

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Abstract

In this paper, a new method has been proposed to incorporate an extra parameter to a family of lifetime distributions for more flexibility. A special sub-case has been considered in details namely; two parameter Weibull distribution. Various mathematical properties of the proposed distribution, including explicit expressions for the moments, quantiles, moment generating function, residual life, mean residual life and order statistics are derived. The maximum likelihood estimators of unknown parameters cannot be obtained in explicit forms, and they have to be obtained by solving non-linear equations only. A simulation study is conducted to evaluate the performances of these estimators. For the illustrative purposes, two data sets have been analyzed to show how the proposed model work in practice.

Keywords: Family of distributions; Alpha power transformation; Weibull distribution; Moments; Order statistic; Residual life function; Maximum likelihood estimation.

1. Introduction

Introducing an extra parameter to an existing family of distribution functions, is a well-known technique in the statistical distribution theory. Often adding an extra parameter brings more flexibility to a class of distribution functions, and it can be very useful for statistical modeling. For example, Marshal and Olkin (1997) introduced the Marshal-Olkin generated (MO-G) family by introducing an extra parameter to the weibull distributions to bring more flexibility to the Weibull model takes the following form

$$G(x;\sigma,\xi)=\frac{F(x;\xi)}{1-(1-\sigma)(1-F(x;\xi))}, \qquad \sigma,\xi>0,\ x\in\mathbb{R}. \tag{1}$$

Mudholkar and Srivastava (1993) proposed a method to introduce an extra parameter to a two-parameter Weibull distribution. The cumulative distribution function (cdf) of the Mudholkar and Srivastava (1993)'s proposed exponentiated Weibull (EW) model has the following form

$$G(x;a,\xi)=\left(1-e^{-\gamma x^{\theta}}\right)^{a}, \qquad x\geq 0,\ a,\xi>0. \tag{2}$$

The model (2) has two shape parameters and one scale parameter. Due to presence of an extra shape parameter, the proposed EW distribution is more flexible than the traditional two-parameter Weibull model.

Eugene et al. (2002) introduced the beta generated method that uses the beta distribution with parameters *a* and *b* as the generator to develop the beta generated distributions. The distribution of a beta-generated random variable *X* is defined as

$$G(x;a,b,\xi)=\int_{0}^{F(x;\xi)} r(t)\,dt, \qquad a,b,\xi>0. \tag{3}$$

where *r(t)* is the probability density function (pdf) of a beta random variable and $F(x;\xi)$ is the cdf of any random variable *X*.

Alzaatreh et al. (2013) proposed a new method for generating families of continuous distributions called *T-X* family by replacing the beta pdf with a pdf, *b(t)*, of a continuous random variable and applying a function $W\left\{F\left(x;\xi\right)\right\}$ that satisfies some specific conditions. Other families belong to *T-X* family including: Gamma-G Type-1 of Zografos and Balakrishnan (2009), Gamma-G Type-2 of Risti´c and Balakrishnan (2012), Gamma-G Type-3 of Torabi and Montazeri (2012), McDonald-G (Mc-G) of Alexander et al. (2012), Logistic-G of Torabi and Montazeri (2014) and Logistic-*X* Family of Tahir et al. (2016). Recently Aljarrah et al. (2014) used quantile functions to generate *T-X* family of distributions.

Shaw and Buckley (2009) proposed another useful approach known as the quadratic rank transmutation map defined by

$$G\left(x;\lambda,\xi\right)=\left(1+\lambda\right)F\left(x;\xi\right)-\lambda F\left(x;\xi\right)^{2} \qquad \xi>0,\ \left|\lambda\right|\leq 1,\ x\in\mathbb{R}. \tag{4}$$

where λ is a transmuted parameter.

Cordeiro and de Castro (2010) proposed another prominent approach known as the Kumaraswamy-G (Ku-G) family of distributions by

$$G\left(x;a,b,\xi\right)=1-\left\{1-\left(1-F\left(x;\xi\right)^{a}\right)\right\}^{b} \qquad a,b,\xi>0,\ x\in\mathbb{R}. \tag{5}$$

where $a>0$ and $b>0$ are the additional shape parameters.

Mahdavi and Kundu (2017) proposed a method for introducing new lifetime distributions by the cdf given by

$$G\left(x;\alpha,\xi\right)=\frac{\alpha^{F(x;\xi)}-1}{\alpha-1}, \qquad \alpha,\xi>0,\ \alpha\neq 1,\ x\in\mathbb{R}. \tag{6}$$

Using (1), Mahdavi and Kundu (2017) and Dey et al. (2017) introduced the alpha power exponential (APE) and alpha power transformedWeibull (APTW) distributions, respectively.

Elbatal et al. (2018) proposed a new power transformation to extend the existing distributions. The distribution function of the Elbatal et al. (2018)'s new alpha power transformed family of distributions is given by

$$G\left(x;\alpha,\xi\right)=\frac{F\left(x;\xi\right)\alpha^{F(x;\xi)}}{\alpha}, \qquad \alpha,\xi>0,\ \alpha\neq 1,\ x\in\mathbb{R}. \tag{7}$$

Using $F\left(x;\xi\right)$ as the cdf of the Weibull model, Elbatal et al. (2018) proposed a three-parameter extension of the Weibull model.

Recently, Ahmad (2018) proposed a new family of life distributions, called the Zubair-G family by the cdf

$$G\left(x;\alpha,\xi\right)=\frac{e^{\alpha F(x;\xi)^{2}}-1}{e^{\alpha}-1}, \qquad \alpha,\xi>0,\ x\in\mathbb{R}. \tag{8}$$

For more survey about methods to generating distributions see Lee et al. (2013) and Jones (2015). The key aim of this research is to introduce an extra parameter to a family of lifeimte distribution functions to bring more flexibility to the given family. We call this new method as a new extended alpha power transformation (NEAPT) method. The proposed APT method is very easy to use, hence it can be used quite effectively for data analysis purposes.

This rest of this article is organized as follows. In section 2, we define the proposed method. A special sub-model of the proposed family along with the graphical skecting of its pdf and cdf is discussed in section 3. Some mathematical properties are obtained in section 4. Maximu likelihood estimates of the model parameters are obtained in section 5. A simulation study is conducted in section 6. Section 7, is devoted to analyze two real life applications. Finally, concluding remarks are provided in section 8.

2. Proposed Method and Motivation

In this section, we define the propose class, new extended alpha power transformed (NEAPT) family. Let $F(x;\xi)$ be the cdf of a continuous random variable X depending upon the vector parameter $(\xi)^T$, then the new extended alpha power transformation of $F(x;\xi)$ for $x \in \mathbb{R}$, is defined as follows

$$G(x;\alpha,\xi) = \frac{\alpha^{F(x;\xi)} - e^{\alpha F(x;\xi)}}{\alpha - e^{\alpha}}, \qquad \alpha,\xi > 0,\ \alpha \neq e,\ x \in \mathbb{R}, \tag{9}$$

where, $F(x;\xi)$ is cdf of the baseline random variable depending on the vector parameter ξ and α is an additional parameter. The probability density function (pdf), survival function (sf), hazard rate function (hrf), reverse hazard rate function (rhrf) and cumulative hazard rate function (chrf) of the NEAPT family are given by (10)-(13), respectively.

$$g(x;\alpha,\xi) = \frac{f(x;\xi)\left((\log\alpha)\alpha^{F(x;\xi)} - \alpha e^{\alpha F(x;\xi)}\right)}{\alpha - e^{\alpha}}, \qquad x \in \mathbb{R}, \tag{10}$$

$$S(x;\alpha,\xi) = \frac{\alpha - e^{\alpha} - \alpha^{F(x;\xi)} + e^{\alpha F(x;\xi)}}{\alpha - e^{\alpha}}, \qquad x \in \mathbb{R}, \tag{11}$$

$$h(x;\alpha,\xi) = \frac{f(x;\xi)\left((\log\alpha)\alpha^{F(x;\xi)} - \alpha e^{\alpha F(x;\xi)}\right)}{\alpha - e^{\alpha} - \alpha^{F(x;\xi)} + e^{\alpha F(x;\xi)}}, \quad x \in \mathbb{R}, \tag{12}$$

$$H(x;\alpha,\xi) = -\log\left(\frac{\alpha - e^{\alpha} - \alpha^{F(x;\xi)} + e^{\alpha F(x;\xi)}}{\alpha - e^{\alpha}}\right), \qquad x \in \mathbb{R}. \tag{13}$$

The new pdf is most tractable when $F(x,\xi)$ and $f(x,\xi)$ have simple analytical expressions. Henceforth, a random variable X with pdf (10) is denoted by $X \sim NEAPT(x;\alpha,\xi)$. Furthermore, for the sake of simplicity, the dependence on the vector of the parameters is omitted and simply $G(x) = G(x;\alpha,\xi)$ will be used. Moreover, the key motivations for using the NEAPT family in practice are the following:

- A very simple and convienent method of adding additional parameters to modify the existing distributions.
- To improve the characteristics and flexibility of the existing distributions.
- To introduce the extended version of the baseline distribution having closed form for cdf, sf as well as hrf.
- To provide better fits than the other modified models.

3. Sub-Model Description

Let X be the Weibull random variable with cdf $F(x;\xi) = 1 - e^{-\gamma x^{\theta}}, \quad x \geq 0,\ \gamma,\theta > 0$, where $\xi = (\gamma,\theta)$. Then, the cdf of the NEAPTW distribution has the following expression

$$G(x;\alpha,\xi) = \frac{\alpha^{\left(1-e^{-\gamma x^{\theta}}\right)} - e^{\alpha\left(1-e^{-\gamma x^{\theta}}\right)}}{\alpha - e^{\alpha}}, \qquad \alpha,\xi > 0,\ \alpha \neq e,\ x \geq 0. \tag{14}$$

The pdf, sf and hrf of the NEAPTW distribution are given, respectively, by

$$g(x;\alpha,\xi) = \frac{\gamma\theta x^{\theta-1} e^{-\gamma x^{\theta}}}{\alpha - e^{\alpha}}\left((\log\alpha)\alpha^{\left(1-e^{-\gamma x^{\theta}}\right)} - \alpha e^{\alpha\left(1-e^{-\gamma x^{\theta}}\right)}\right), \qquad x \geq 0, \tag{15}$$

$$S(x;\alpha,\xi)=\frac{\alpha-e^{\alpha}-\alpha^{\left(1-e^{-\gamma x^{\theta}}\right)}+e^{\alpha\left(1-e^{-\gamma x^{\theta}}\right)}}{\alpha-e^{\alpha}}, \qquad x\geq 0, \tag{16}$$

and

$$h(x;\alpha,\xi)=\frac{\gamma\theta x^{\theta-1}e^{-\gamma x^{\theta}}\left((\log\alpha)\alpha^{\left(1-e^{-\gamma x^{\theta}}\right)}-\alpha e^{\alpha\left(1-e^{-\gamma x^{\theta}}\right)}\right)}{\alpha-e^{\alpha}-\alpha^{\left(1-e^{-\gamma x^{\theta}}\right)}+e^{\alpha\left(1-e^{-\gamma x^{\theta}}\right)}}, \qquad x\geq 0. \tag{17}$$

For different values of the model parameters, plots of the pdf of the NEAPTW distribution are sketched in Figure 1.

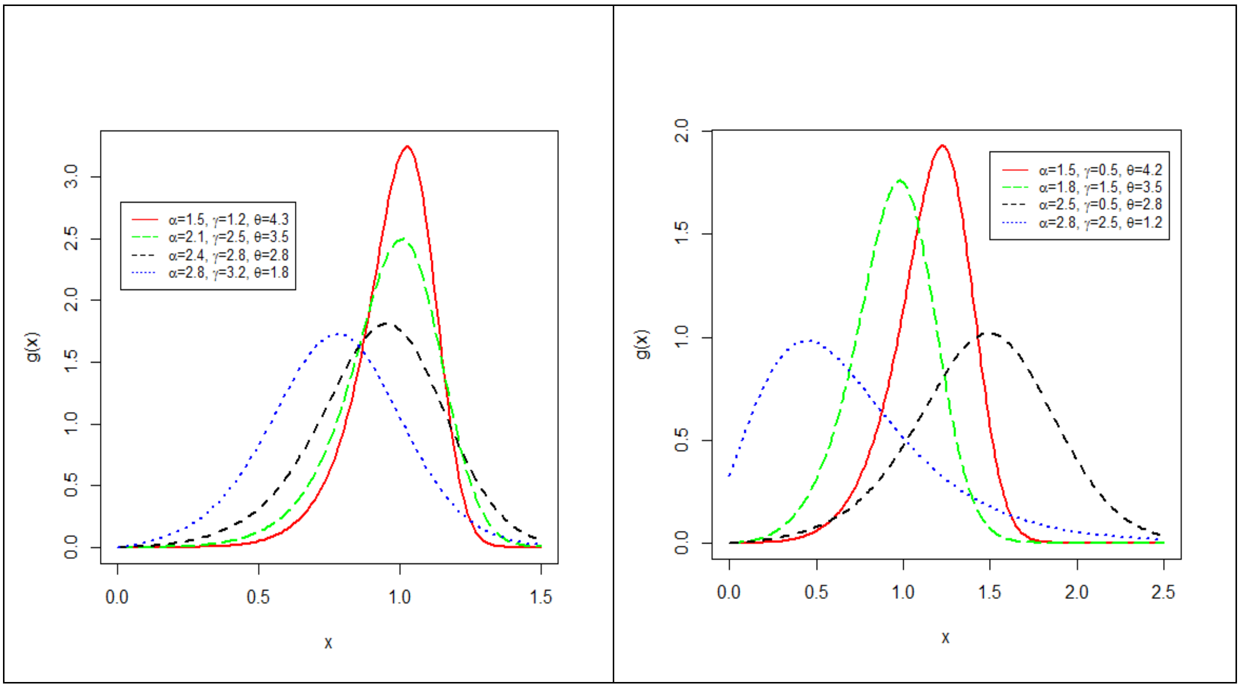

Figure 1. Different plots for the pdf of the NEAPTW distribution.

For the selected values of parameters, some possible shapes for the hrf of the NEAPTW model are drawn in Figure 2.

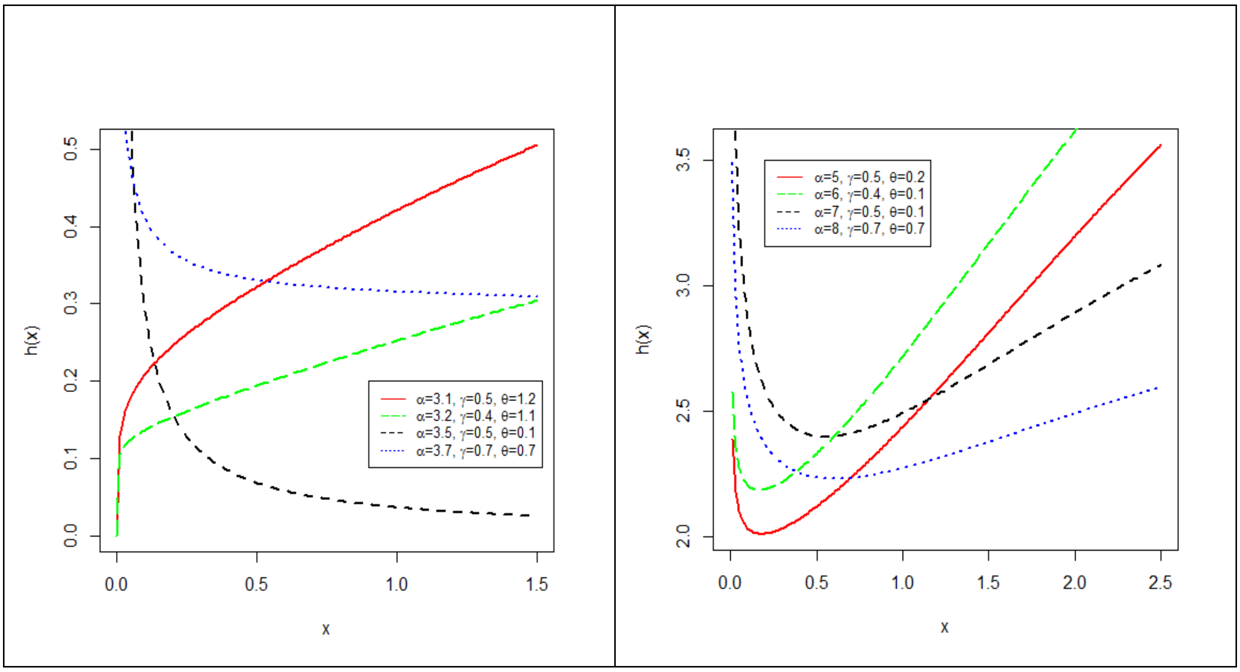

Figure 2. Different plots for the hrf of the NEAPTW distribution.

4. Basic Mathematical Properties

In this section, we derive some general properties of the new model, such as thequantile function, moments, moment generating function, residual, reverse residual life and order statistic.

4.1. Quantile function

Let *X* be the NEAPTW random variable with pdf (10), the quantile function of *X*, say *Q(u)*, is given by

$$Q(u)=\alpha F(x;\xi)-\log\left(\alpha^{F(x;\xi)}-u\left(\alpha-e^{\alpha}\right)\right), \qquad u(0,1). \tag{18}$$

Hence, generating random numbers form the NEAPT distributions can be done by setting *X* = *Q(U)*, where *U* follows the standard uniform distribution.

4.2. Moments

Moments are very important and helps to describe the important characteristics of the distribution (e.g., central tendency, dispersion, skewness and kurtosis). The r^{th} moment of the NEAPT family of distributions are derived as follows

$$\mu_r^{/}=\frac{1}{\alpha-e^{\alpha}}\left(\sum_{i=0}^{\infty}\frac{(\log\alpha)^{i+1}}{i!}\int_{-\infty}^{\infty}x^r f(x;\xi)F(x;\xi)^i\,dx-\sum_{i=0}^{\infty}\frac{\alpha^i}{i!}\int_{-\infty}^{\infty}x^r f(x;\xi)F(x;\xi)^i\,dx\right), \tag{19}$$

$$\mu_r^{/}=\frac{1}{\alpha-e^{\alpha}}\left(\sum_{i=0}^{\infty}\frac{(\log\alpha)^{i+1}}{i!}\eta_{r,i}-\sum_{i=0}^{\infty}\frac{\alpha^i}{i!}\eta_{r,i}\right),$$

where,

$$\eta_{r,i}=\int_{-\infty}^{\infty}x^r f(x;\xi)F(x;\xi)^i\,dx.$$

Furthermore, a general expression for moment generating function (mgf) of the NEAPT random variable X is

$$M_x(t)=\frac{1}{\alpha-e^{\alpha}}\left(\sum_{i,r=0}^{\infty}\frac{(\log\alpha)^{i+1}t^r}{r!i!}\eta_{r,i}-\sum_{i,r=0}^{\infty}\frac{\alpha^i t^r}{r!i!}\eta_{r,i}\right). \tag{20}$$

4.3. Residual and Reverse Residual Life

The residual life offer wider applications in reliability theory and risk management. The residual lifetime of NEAPT random variable X denoted by $R_{(t)}$ is derived as

$$R_{(t)}(x)=\frac{S(x+t)}{S(t)},$$

$$R_{(t)}(x)=\frac{\alpha-e^{\alpha}-\alpha^{F(x+t;\xi)}+e^{\alpha F(x+t;\xi)}}{\alpha-e^{\alpha}-\alpha^{F(t;\xi)}+e^{\alpha F(t;\xi)}}. \tag{21}$$

Additionally, the reverse residual lifetime of the NEAPT random variable denoted by $\bar{R}_{(t)}$ is

$$\bar{R}_{(t)}=\frac{S(x-t)}{S(t)},$$

$$\bar{R}_{(t)}(x)=\frac{\alpha-e^{\alpha}-\alpha^{F(x-t;\xi)}+e^{\alpha F(x-t;\xi)}}{\alpha-e^{\alpha}-\alpha^{F(t;\xi)}+e^{\alpha F(t;\xi)}}. \tag{22}$$

4.4. Order statistics

Let $X_1, X_2, \ldots, X_k$ be a random sample of size k taken independently from the NEAPT distribution with parameters α and ξ. Let $X_{1:k}, X_{2:k}, \ldots, X_{k:k}$ be the corresponding order statistics. Then, from David (1981), the density of $X_{r:k}$ for ($r=1, 2, \ldots, k$) is gi ven by

$$g_{r:k}(x)=\frac{g(x;\alpha,\xi)}{B(r,\ k\ -r+1)}\sum_{i=0}^{k-r}\binom{k-r}{i}(-1)^i\left[G(x;\alpha,\xi)\right]^{i+r-1}. \tag{23}$$

Using (21) and (22), in (23), we get the density of the r^{th} order statistic.

5. Estimation

In this section, the maximum likelihood estimators of the parameters α and ξ of NEAPT family from complete samples are derived. Let $X_1, X_2, \ldots, X_k$ be a simple random sample from NEAPT family with observed values $x_1, x_2, \ldots, x_k$. The log-likelihood function of this sample is

$$\log L(x;\alpha,\xi)=-k\log\left(\alpha-e^{\alpha}\right)+\sum_{i=1}^{k}\log\left[f(x_i;\xi)\right]+\sum_{i=1}^{k}\log\left[(\log\alpha)\alpha^{F(x_i;\xi)}-\alpha e^{\alpha F(x_i;\xi)}\right], \tag{24}$$

Obtaining the partial derivatives of (24), we get

$$\frac{\partial}{\partial\alpha}\log L(x;\alpha,\xi)=-\frac{k\left(1-e^{\alpha}\right)}{\alpha-e^{\alpha}}+\sum_{i=1}^{k}\frac{\left((\log\alpha)F(x_i;\xi)+1\right)\alpha^{F(x_i;\xi)-1}-\alpha e^{\alpha F(x_i;\xi)}\left(1+F(x_i;\xi)\right)}{(\log\alpha)\alpha^{F(x_i;\xi)}-e^{F(x_i;\xi)}}, \tag{25}$$

$$\frac{\partial}{\partial \xi}\log L(x;\alpha,\xi)=\sum_{i=1}^{k}\frac{\partial f(x_i;\xi)/\partial \xi}{f(x_i;\xi)}+\sum_{i=1}^{k}\frac{(\log\alpha)^2\alpha^{F(x_i;\xi)}\partial F(x_i;\xi)/\partial\xi-\alpha^2 e^{\alpha F(x_i;\xi)}\partial F(x_i;\xi)/\partial\xi}{(\log\alpha)\alpha^{F(x_i;\xi)}-\alpha e^{\alpha F(x_i;\xi)}}. \quad (26)$$

Setting $\frac{\partial}{\partial \alpha}\log L(x;\alpha,\xi)$ and $\frac{\partial}{\partial \xi}\log L(x;\alpha,\xi)$ equal to zero and solving numerically these expressions simultaneously, yields the maximum likelihood estimates (MLEs) of (α,ξ).

6. Simulation Study

In this section, we assess the performance of the maximum likelihood estimators in terms of the sample size *n*. A numerical evaluation is carried out to examine the performance of maximum likelihood estimators for NEAPTW model (as particular case from the family). The evaluation of estimates is performed based on the following quantities for each sample size; the Biases and the empirical man square errors (MSEs) using the Mathematic package. The numerical steps are listed as follows:

1. A random sample $X_1, X_2, \ldots, X_n$ of sizes; $n=30$ and 50 are considered, these random samples are generated from the NEAPTW distribution by using inversion method.
2. Six set of the parameters are considered. The MLEs of NEAPTW model are evaluated for each parameter value and for each sample size.
3. 3000 repetitions are made to calculate the Biases and mean square error (MSE) of these estimators.
4. Formulas used for calculating bias and MSE are given by $Bias(\hat{\alpha})=\frac{1}{1000}\sum_{i=1}^{1000}(\hat{\alpha}-\alpha)$ and $MSE(\hat{\alpha})=\frac{1}{1000}\sum_{i=1}^{1000}(\hat{\alpha}-\alpha)^2$, respectively.
5. Step (iii) is also repeated for the other parameters (θ,γ).

Empirical results are reported in Table (1). We can detect from these tables that the estimates are quite stable and are close to the true value of the parameters as the sample sizes increase.

Table (1): Simulation Results: MLEs, Biases and MSEs

n	Parameters	Set 1 (α =0.5, γ=0.5, θ=0.5)			Set 2 (α =0.5, γ=0.5, θ=1)		
		MLE	Bias	MSE	MLE	Bias	MSE
30	α	0.3173	-0.1827	0.0400	0.3458	-0.1542	0.0395
	γ	0.5775	0.0775	0.1840	0.6468	0.1468	0.1472
	θ	1.4039	0.9039	1.0334	3.3607	2.3607	6.1295
50	α	0.4214	-0.0786	0.0107	0.3923	-0.1077	0.0271
	γ	0.5690	0.0690	0.0187	0.6522	0.1522	0.1021
	θ	0.9435	0.4435	0.2094	2.0076	1.0076	1.6355

Continued of Table (1)

n	Parameter	Set 3 (α =0.5, γ=0.5, θ=0.2)			Set 4 (α =0.5, γ=1, θ=0.5)		
		MLE	Bias	MSE	MLE	Bias	MSE
30	α	0.4139	-0.0861	0.0080	0.3788	-0.1212	0.0149
	γ	0.3796	-0.1204	0.0206	0.9839	-0.0161	0.0113
	θ	1.1505	0.9505	0.9371	1.7008	1.2008	1.4631
50	α	0.5453	0.0453	0.0052	0.4526	-0.0474	0.0024
	γ	0.3656	-0.1344	0.0193	0.9933	-0.0067	0.0064
	θ	0.7862	0.5862	0.4112	1.0624	0.5624	0.3292

Continued of Table (1)

n	Parameter	Set 3 (α =0.5, γ=1.5, θ=0.5)			Set 4 (α =0.5, γ=2, θ=0.5)		
		MLE	Bias	MSE	MLE	Bias	MSE
30	α	0.3823	-0.1177	0.0140	0.3835	-0.1165	0.0138
	γ	1.4461	-0.0539	0.0216	2.0013	0.0013	0.0636
	θ	1.6326	1.1326	1.3134	1.7260	1.2260	1.5385
50	α	0.4600	-0.0400	0.0018	0.4526	-0.0474	0.0027
	γ	1.4300	-0.0700	0.0174	1.9423	-0.0577	0.0138
	θ	0.9927	0.4927	0.2490	1.0227	0.5227	0.2860

Continued of Table (1)

n	Parameter	Set 3 (α =1, γ=0.5, θ=0.5)			Set 4 (α =0.7, γ=0.5, θ=0.5)		
		MLE	Bias	MSE	MLE	Bias	MSE
30	α	0.3805	-0.6195	0.3854	0.3617	-0.3383	0.1185
	γ	0.5353	0.0353	0.0193	0.5744	0.0744	0.0362
	θ	1.3352	0.8352	0.8101	1.4132	0.9132	0.9261
50	α	0.4796	-0.5204	0.2709	0.4819	-0.2181	0.0482
	γ	0.5005	0.0005	0.0028	0.4916	-0.0084	0.0012
	θ	0.8493	0.3493	0.1277	0.9416	0.4416	0.2124

7. Applications

To prove the flexibility of the proposed family two applications to real data sets are analyzed. The goodness of fits of the NEAPTW distribution have been compared with the other lifetime models such as exponentiated Weibull, Marshall-Olkin Weibull and Kumaraswamy Weibull distributions. The distibutions functions of the competing distribution are as:

- The exponentiated Weibull is given by

$$G(x,a,\theta,\gamma)=\left(1-e^{-\gamma x^{\theta}}\right)^{a}, \qquad x,a,\theta,\gamma>0.$$

- The Marshall-Olkin Weibull is

$$G(x;\sigma,\xi)=\frac{\left(1-e^{-\gamma x^{\theta}}\right)^{a}}{1-(1-\sigma)\left(1-\left(1-e^{-\gamma x^{\theta}}\right)^{a}\right)}, \qquad \sigma,\xi>0.$$

- The Kumaraswsamy Weibull i given by

$$G(x,a,b,\theta,\gamma)=1-\left(1-\left(1-e^{-\gamma x^{\theta}}\right)^{a}\right)^{b}, \qquad x,a,b,\theta,\gamma>0.$$

The analytical measures of goodness of fit including the Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC) and Kolmogorov-Smirnov (KS) statistic are considered to compare the proposed method with the other fitted models. In general, a model with smaller values of these analytical measure indicate better fit to the data. All the required computations have been carried out in the R-language using "SANN" algorithm.

Data 1: The data set represents survival times of guinea pigs injected with the different amount of tubercle bacilli studied by Bjerkedal (1960), see appendix A. The MLEs and the considered statistics are shown in Tables 2 and 3, respectively.

Table 2. Maximum likelihood estimates of the fitted distributions using data 1.

Dist.	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\sigma}$	$\hat{a}$	$\hat{b}$
NEAPTW	0.017	0.074	1.989			
MOW		0.210	0.698	1.770		
EW		0.708	1.171		1.994	
Ku-W		0.641	1.062		2.310	1.432

Table 3. The statistics of the fitted models using data 1.

Dist.	KS	AIC	BIC	CIAC	HQIC
NEAPTW	0.094	209.53	216.36	209.88	212.25
MOW	0.106	213.45	220.28	213.80	216.17
EW	0.100	211.62	218.45	211.97	214.341
Ku-W	0.097	213.63	222.73	214.22	217.25

Corresponding to data 1, the estimated pdf and cdf are provided in Figure 3, the PP plot and Kaplan Meier survival plot are sketched in Figure 4, whereas, the estimated hazard function and scale-TTT are given in Figure 5.

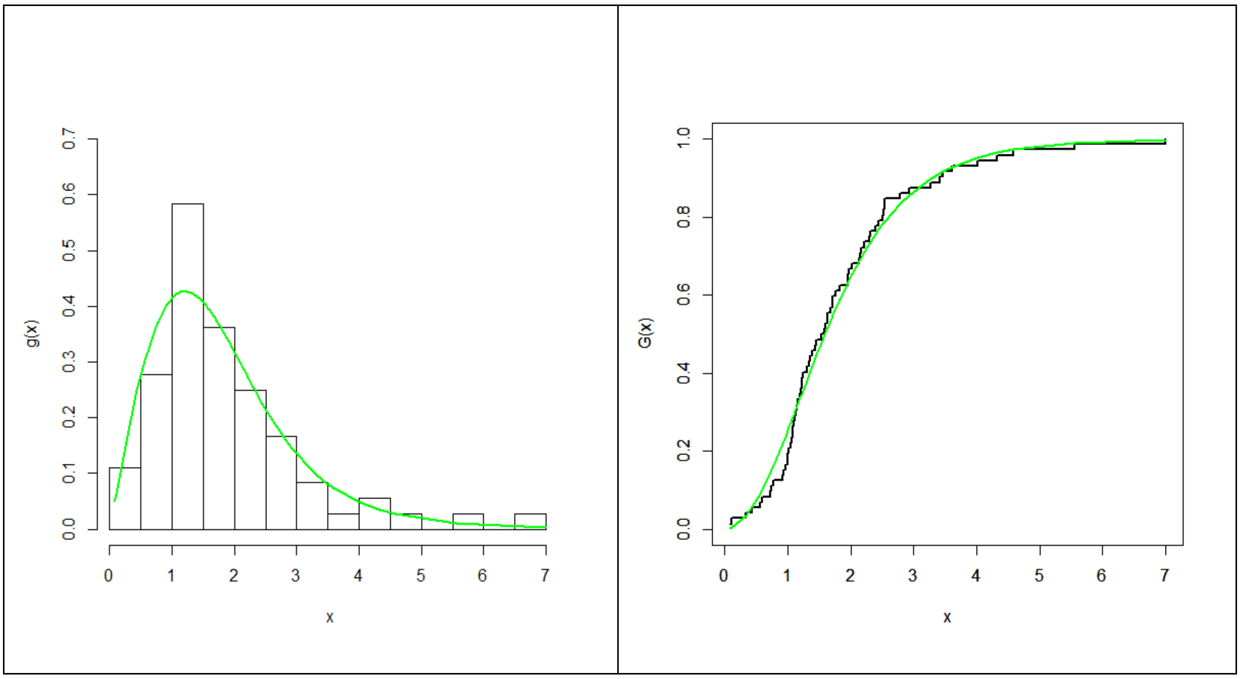

Figure 3. Plots of the estimated pdf and cdf of the NEAPTW distribution corresponding to data 1.

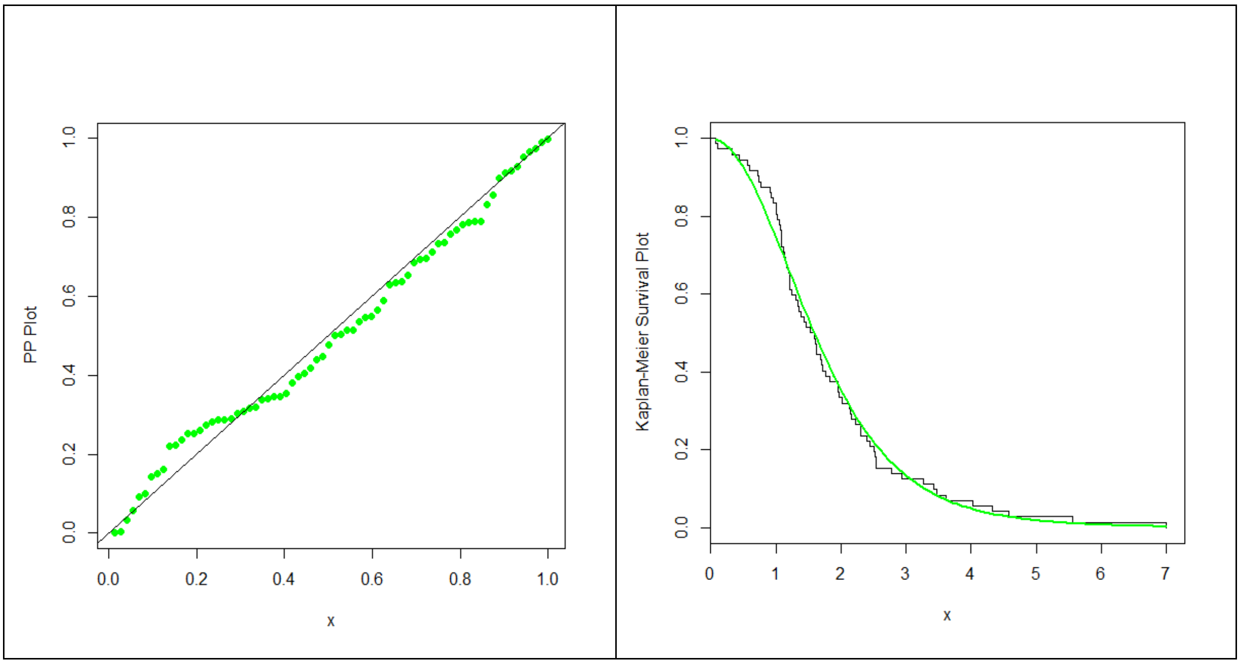

Figure 4. PP and Kaplan-Meir survival plots of the NEAPTW distribution corresponding to data 1.

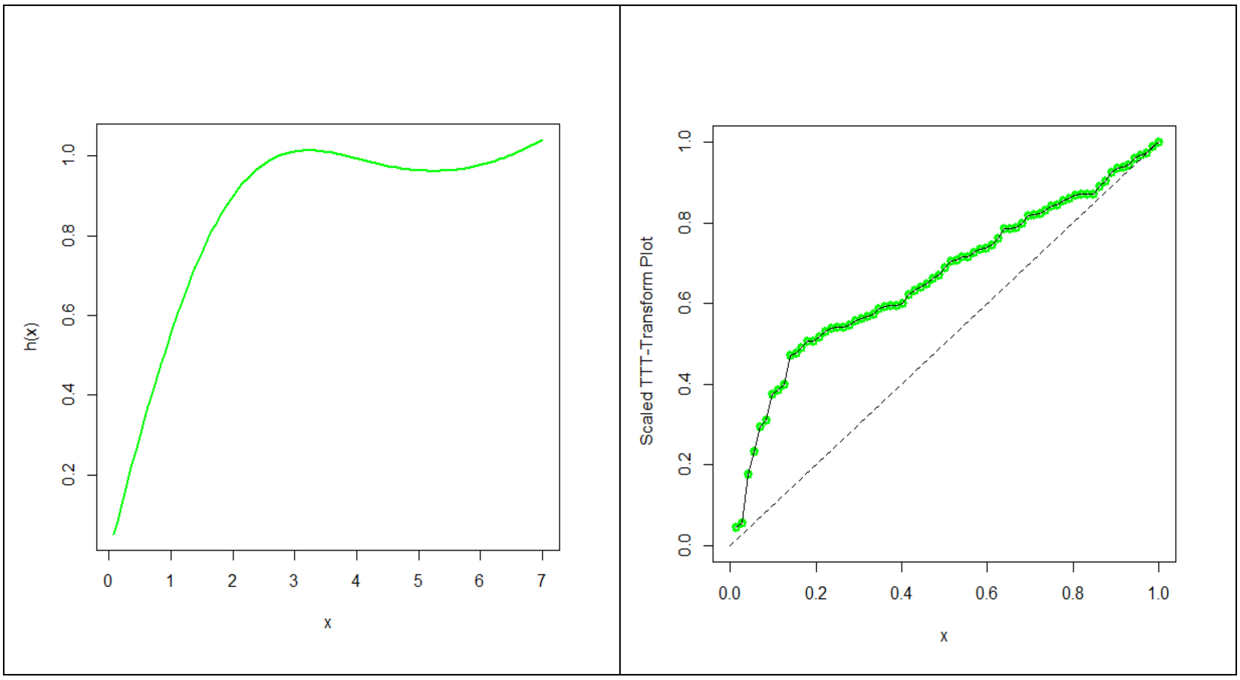

Figure 5. Scale TTT-transform plot and estimated hazard rate plot of the NEAPTW distribution corresponding to data 1.

Data 2: The second data set representing the remission times (in months) of a random sample of 128 bladder cancer patients taken from Lee and Wang (2003), see appendix A. The MLEs and the considered statistics are shown in Tables 4 and 5, respectively.

Table 4. Maximum likelihood estimates of the fitted distributions using data 2.

Dist.	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\sigma}$	$\hat{a}$	$\hat{b}$
NEAPTW	6.984	0.901	0.511			
MOW		0.877	0.564	11.829		
EW		0.720	0.541		4.332	
Ku-W		0.487	0.520		3.712	1.988

Table 5. Analytical measures of the fitted distributions using data 2.

Dist.	KS	AIC	BIC	CIAC	HQIC
NEAPTW	0.034	825.58	834.14	825.78	829.06
MOW	0.075	834.98	843.54	835.18	838.46
EW	0.046	828.21	836.77	828.41	831.69
Ku-W	0.041	829.20	840.61	829.53	833.84

Corresponding to data 1, the estimated pdf and cdf are proved in Figure 6, the PP plot and Kaplan Meier survival plot are sketched in Figure 7, whereas, the estimated hazard function and scale-TTT are given in Figure 8.

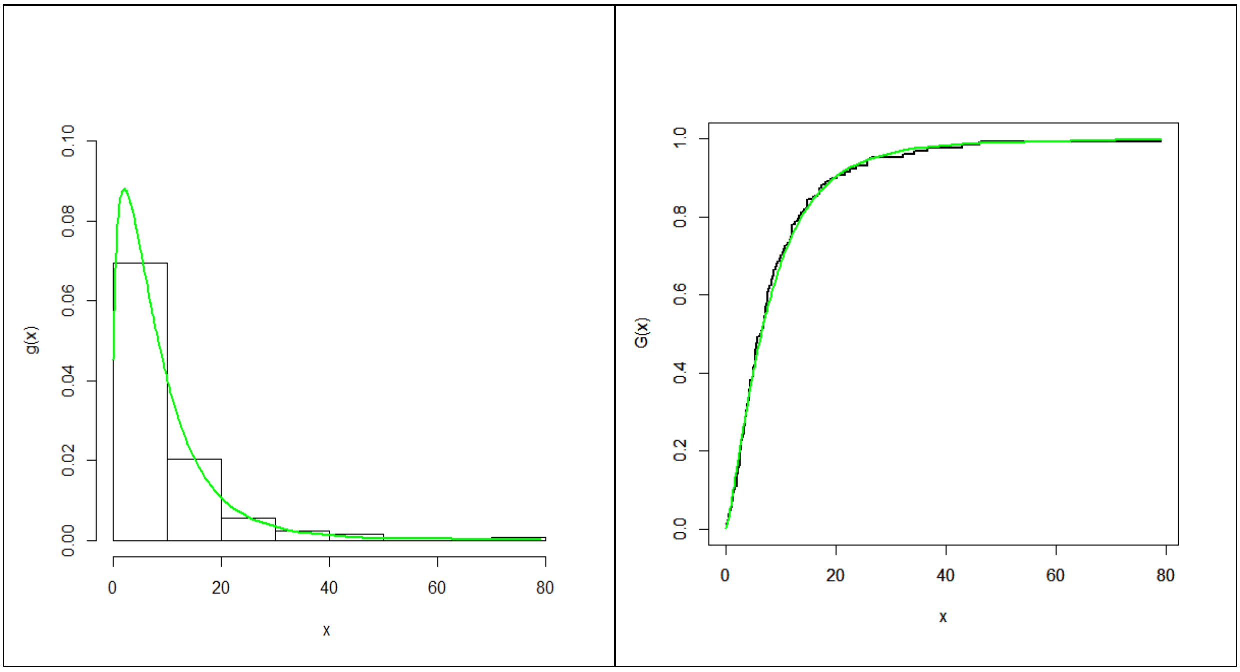

Figure 6. Plots of the estimated pdf and cdf of the NEAPTW distribution for data 2.

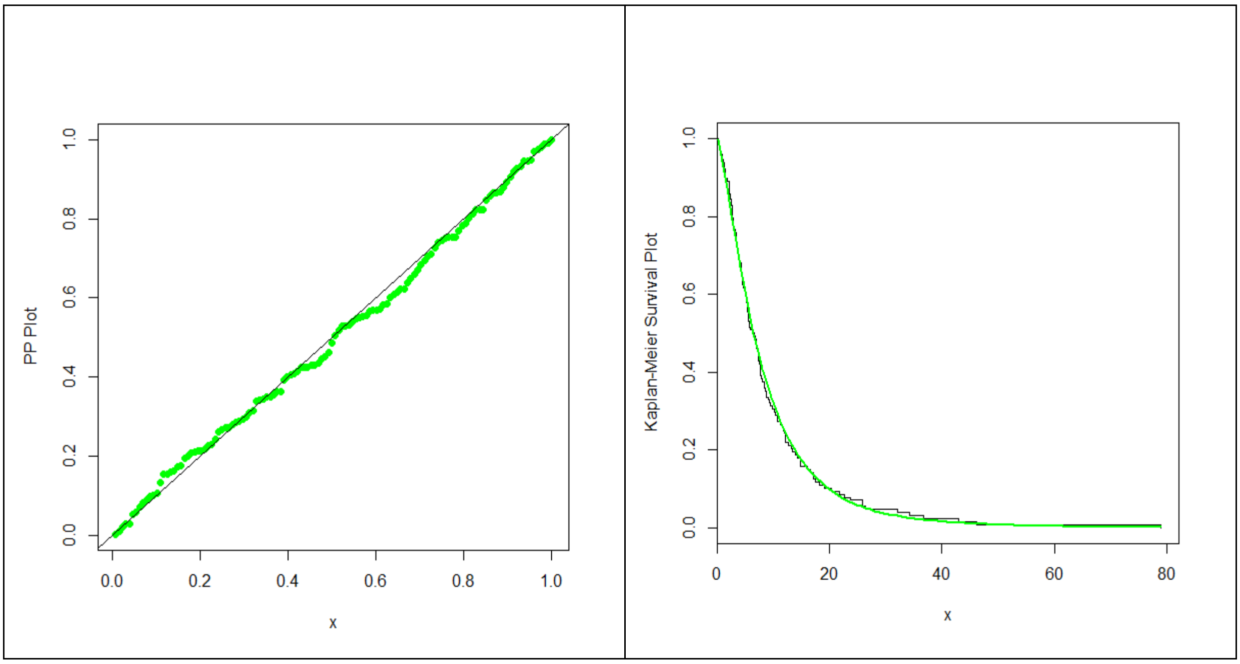

Figure 7. PP and Kaplan-Meir survival plots of the NEAPTW distribution for data 2.

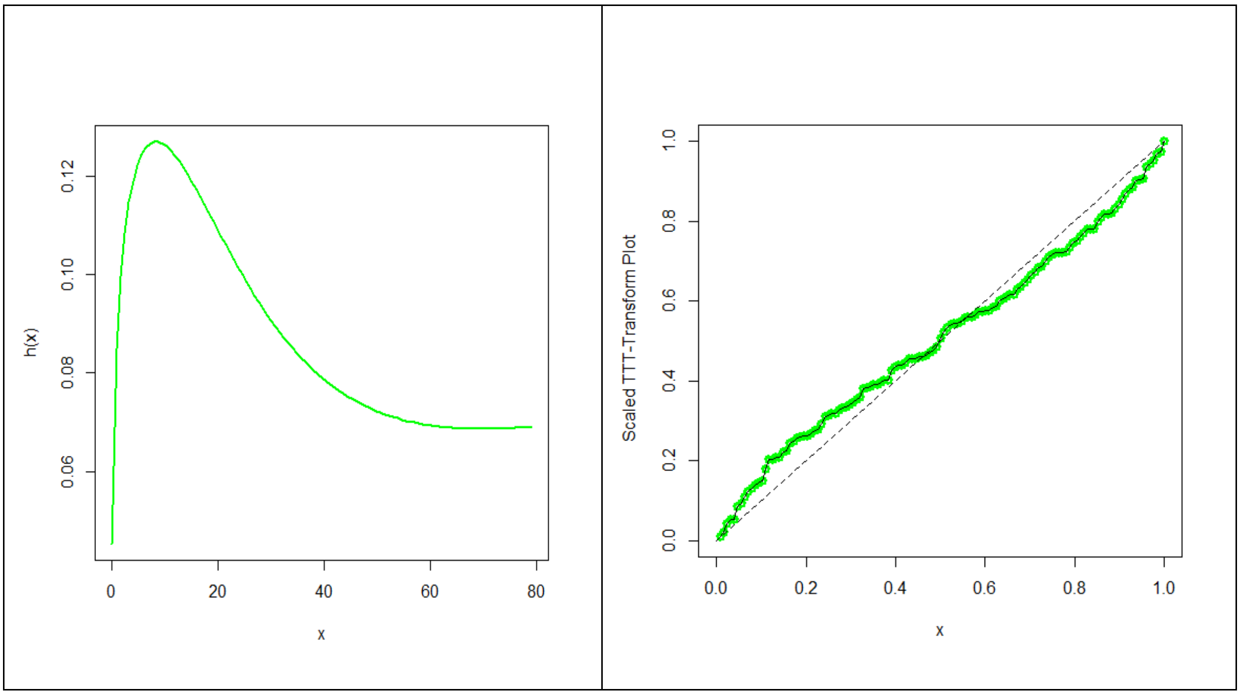

Figure 8. Scale TTT-transform plot and estimated hazard rate plot of the NEAPTW distribution for data 2.

8. Conclusions

In this article, a new method is adopted to add an additional parameter to the existing distributions. This effort leads to a new family of lifetime distributions, called the NEAPT family of distributions. General expressions for some of mathematical properties of the new family are derived. The estimation of the of model parameters through maximum likelihood method is discussed and a simulation study is carried out. There are certain advantages of using the proposed method like its cdf has a closed form solution and facilitating data modeling with monotonic and non-monotonic failure rates. A special sub-model of the new family, called NEAPTW distribution is considered and two real applications are analyzed. The practical applications of the proposed model reveal better fits to real-life data than the other well-known existing distributions.

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Appendix

Availability of data and material

The idea of the power transformation has been used to propose the new family of distributions.

Data 1: The first data set taken from Bjerkedal (1960) as follows: 0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 07, 0.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

Data 2: The second data set taken from Lee and Wang (2003) are as follows: 0.08, 0.20, 0.40, 0.50, 0.51, 0.81, 0.90, 1.05, 1.19, 1.26, 1.35, 1.40, 1.46, 1.76, 2.02, 2.02, 2.07, 2.09, 2.23, 2.26, 2.46, 2.54, 2.62, 2.64, 2.69, 2.69, 2.75, 2.83, 2.87, 3.02, 3.25, 3.31, 3.36, 3.36, 3.48, 3.52, 3.57, 3.64, 3.70, 3.82, 3.88, 4.18, 4.23, 4.26, 4.33, 4.34, 4.40, 4.50, 4.51, 4.87, 4.98, 5.06, 5.09, 5.17, 5.32, 5.32, 5.34, 5.41, 5.41, 5.49, 5.62, 5.71, 5.85, 6.25, 6.54, 6.76, 6.93, 6.94, 6.97, 7.09, 7.26, 7.28, 7.32, 7.39, 7.59, 7.62, 7.63, 7.66, 7.87, 7.93, 8.26, 8.37, 8.53, 8.65, 8.66, 9.02, 9.22, 9.47, 9.74, 10.06, 10.34, 10.66, 10.75, 11.25, 11.64, 11.79, 11.98, 12.02, 12.03, 12.07, 12.63, 13.11, 13.29, 13.80, 14.24, 14.76, 14.77, 14.83, 15.96, 16.62, 17.12, 17.14, 17.36, 18.10, 19.13, 20.28, 21.73, 22.69, 23.63, 25.74, 25.82, 26.31, 32.15, 34.26, 36.66, 43.01, 46.12, 79.05.