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Characterizations on the basis of cumulative residual entropy of sequential order statistics

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Abstract: This article deals with the problem of characterizing the parent distribution on the basis of the cumulative residual entropy of sequential order statistics under a conditional proportional hazard rates model. It is shown that the equality of the cumulative residual entropy in the first sequential order statistics determines uniquely the parent distribution. Subsequently, we characterize the Weibull distribution on the basis of the ratio of the cumulative residual entropy of first sequential order statistics to the corresponding mean. Also, we consider characterizations based on the dynamic cumulative residual entropy and derive some bounds for the cumulative residual entropy of residual lifetime of the sequential order statistics.

Keywords: Cumulative residual entropy; Sequential order statistics; Residual lifetime.
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1 Introduction

Let $X_1, \cdots, X_n$ be independent and identically distributed (IID) random variables coming from the cumulative density function (CDF) $F(x)$ and the probability density function (PDF) $f(x)$. The order statistics of the sample is defined by the arrangement of $X_1, \cdots, X_n$ by magnitude from the smallest to the largest, and denoted by $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$. These statistics are applied in a wide range of problems, including characterization of probability distributions, entropy estimation, detection of outliers, analysis of censored samples, quality control and strength of materials; For

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more details, see Arnold et al. (2008) and references therein. In reliability theory, sequential order statistics are used for statistical modelling. The sequential k-th order statistics in a sample of size n represents the lifetime of a sequential $(n-k+1)$-out-of-n system. Several entropy and information indices were developed and used in various disciplines and contexts. Sunoj and Linu (2012) defined cumulative residual Renyi entropy of order statistics and its dynamic version. Baratpour et al. (2007); Baratpour et al. (2008) considered Shannon entropy and Renyi entropy of order statistics and record values. Recently, Kumar and Taneja (2011) defined generalized cumulative residual information measure and its dynamic version based on Varmas entropy function. Ebrahimi (1996) defined the concept of dynamic Shannon entropy and obtained some its properties.

In the probabilistic modelling and analysis of engineering systems, an important method for improving the system reliability is to use redundants. A common scheme for redundancy is the r-out-of-n systems. A r-out-of-n system consists of n components which start working simultaneously. The system operates if at least $n-r+1$ components function, and then it falls if r or more components fall. Hence, there are some redundants in order to raise the system reliability.

Kamps (1995b) introduced the concept of "*sequential order statistics*" (SOS) as an extension of order statistics and used for modelling lifetimes of (sequential) r-out-of-n systems. Specifically, consider a given r-out-of-n system consisting of n components and $X_1, \cdots, X_n$ denote component lifetimes. Then, the system lifetime (T) coincides to the r-th order statistics among $X_1, \cdots, X_n$, that is $T = X_{r:n}$. In (usual) r-out-of-n systems, it is assumed that $X_1, \cdots, X_n$ are IID with a CDF, say F. Notice that failing a component does not change lifetimes of surviving components; For more details, see, David and Nagaraja (2003), Arnold et al. (2008), Hashempour and Doostparast (2016a) and references therein. Motivated by Cramer and Kamps (1996, 2001a), in practice, the failure of a component may results in a higher load on remaining components and hence causes the distribution of the surviving components change. In these cases, system lifetimes may be modelled by SOSs. To see this, suppose that F_j, for $j = 1, \cdots, n$, denotes the common CDF of the lifetime components when $n-j+1$ components are working. Components begin to work at time $t = 0$ independently with the common CDF F_1. When at time x_1, the first failure occurs, the remaining $n-1$ components work independently with the common CDF F_2. This process continues to $n-r+1$ components independently work with the common CDF F_r until the r-th failure occurs at time x_r and hence the whole system fails. The mentioned system is called "*sequential r-out-of-n system*" and the system lifetime coincides to the r-th component failure time, denoted by $X^{\star}_{(r)}$. In the literature, $(X^{\star}_{(1)}, \cdots, X^{\star}_{(n)})$ is called SOSs; See, Kamps (1995a,b), Hashempour and Doostparast (2016b) and references therein. The problem of estimating parameters on the basis of SOS has been considered in the literature. For example, Cramer and Kamps (1996) considered the problem of estimating the parameters on the basis of s independent SOSs samples under a conditional proportional hazard rates (CPHR) model, defined by $\bar{F}_j(t) = \bar{F}_0^{\alpha_j}(t)$ for $t > 0$ and $j = 1, \cdots, r$. In this case, the hazard rate function of the CDF F_j, defined by $h_j(t) = f_j(t)/\bar{F}_j(t)$ for $t > 0$ and $j = 1, \cdots, n$, is proportional to the hazard rate function of the baseline CDF F_0, i.e. $h_j(t) = \alpha_j h_0(t)$ where $\alpha_j > 0$; See, also Balakrishnan et al. (2012) and Hashempour and Doostparast (2016a,b), Hashempour (2017), Esmailian and Doostparast (2014).

A series system is particular case of r-out-of-n systems corresponding to $r = 1$. We consider series systems in this paper. Specifically, suppose $X_1, \cdots, X_n$ denote lifetimes of n components of a given series system. We assume that $X_i s$ are IID random variables with the common CDF F and the survival function (SF) $\bar{F}$. Also, suppose that $X^\star_{1:n}, \cdots, X^\star_{n:n}$ be the corresponding SOS of the lifetimes of the components. Then, $X^\star_{1:n}$ represents the lifetime of the series system with the SF $\bar{F}_{X^\star_{1:n}}(x) = \bar{F}^{n\alpha_1}(x)$, $x > 0$ and positive real number α_1. The SF of $X^\star_{1:n} - t$ given $X^\star_{1:n} > t$, is $\bar{F}_{X^\star_{1:n,t}}(x) = \big(\bar{F}(t+x)/\bar{F}(t)\big)^{n\alpha_1}$, where $X^\star_{1:n} - t$ is called the "*residual lifetime of system*". In this article, we consider the problem of characterizing the parent distributions based on the cumulative residual entropy (CRE) of first SOS under the above-mentioned conditional proportional hazard rates model. It is shown that the equality of the CRE in the first SOS determines uniquely the baseline distribution. Therefore, the rest of this article is organized as follows. Section 2 contains a review on the concept of entropy. In Section 3, we provide some characterizations based on the first-SOS $X^\star_{1:n}$. Also, the Weibull distribution is determined on the basis of the CRE of $X^\star_{1:n}$ divided by $E(X^\star_{1:n})$, the expectation of the first-SOS. We construct some bounds for the CRE of the residual lifetimes of series systems. Moreover, characterizations based on the CRE of residual lifetimes of series systems are studied in Section 4. Section 5 concludes.

2 Entropy

Shannon (1948) introduced the concept of entropy which is widely used in the fields of physics, probability, statistics, communication theory, information theory, economics, and so forth. In information theory, entropy is a measure of the uncertainty associated with a random variable. Shannon entropy represents the absolute limit on the best possible lossless compression of any communication; For more details, see Cover and Thomas (2006). Shannon entropy of a continuous random variable X with the PDF $f(x)$ is defined as

$$H(X) = -E(\log f(X)) = -\int_{-\infty}^{+\infty} f(x)\log f(x)dx,$$

where "log" means the natural logarithm. In this paper, we suppose that X is a positive and continuous random variable. We focus on the CRE, introduced by Rao et al. (2004), as

$$\begin{aligned} \mathrm{CRE}(X) &= -\int_0^{+\infty} \bar{F}(x)\log \bar{F}(x)dx \\ &= -\int_0^1 \frac{u\log u}{f\big(F^{-1}(1-u)\big)}du, \end{aligned} \tag{1}$$

where $\bar{F}(x) = 1 - F(x)$ is the SF of X. Rao et al. (2004) showed that it is more general than Shannon entropy and possesses more general mathematical properties. The CRE (1) can be easily estimated, see Rao (2005). Its estimator converges asymptotically to the true value. The CRE has applications in reliability engineering and computer vision; For more details, see Rao (2005). Asadi and Zohrevand (2007) defined the dynamic measure of CRE and obtained some of its properties. The dynamic CRE (DCRE) for the residual lifetime distribution of a system is

$$\mathrm{DCRE}(X_t) = -\int_t^{\infty} \bar{F}_t(x) \log \bar{F}_t(x) dx, \tag{2}$$

where $\bar{F}_t(x) = P(X - t > x \big| X > t) = \bar{F}(x+t)/\bar{F}(t)$ for $t > 0$. The DCRE is a measure of the information in the residual life distribution. They showed that the CRE and the DCRE are connected with some well-known reliability measures such as the mean residual lifetime and the hazard rate function. Also, they proved that if the DCRE (2) is a non-decreasing function on t, then it characterizes the underlying distribution function. For more details, see Navarro et al. (2010) and references therein. Note that CRE(X) = DCRE(X_0).

3 CRE and CPHR model

Let $X^{\star}_{1:n}$ be the first SOS under the CPHR model with the baseline CDF F and the corresponding PDF f. Then, the CDF of $X^{\star}_{1:n}$ is

$$F_{X^{\star}_{1:n}}(x) = 1 - \bar{F}^{n\alpha_1}(x), \quad \forall x \in \mathbb{R}. \tag{3}$$

Equations (1) and (3) yield

$$\begin{aligned} \mathrm{CRE}(X^{\star}_{1:n}) &= -n\alpha_1 \int_0^{\infty} \bar{F}^{n\alpha_1}(x) \log \bar{F}(x) dx \\ &= -n\alpha_1 \int_0^{1} u^{n\alpha_1} \log u \; dF^{-1}(u) du \\ &= -n\alpha_1 \int_0^{1} \frac{u^{n\alpha_1} \log u}{f(F^{-1}(1-u))} du, \end{aligned} \tag{4}$$

since $dF^{-1}(u)/du = [f(F^{-1}(u))]^{-1}$. Now we consider three well-known distributions.

3.1 Exponential model

A random variable X follows the exponential distribution, if

$$F(x) = 1 - \exp\{-\lambda x\}, \quad x > 0, \quad \lambda > 0, \tag{5}$$

where λ is the scale parameter. Notice that $E(X) = \lambda^{-1}$ and $\mathrm{CRE}(X) = \lambda^{-1}$. Thus,

$$\Lambda_1 = \frac{\mathrm{CRE}(X)}{E(X)} = 1. \tag{6}$$

Substituting Equation (5) into Equation (4), we have

$$\begin{aligned} \mathrm{CRE}(X^{\star}_{1:n}) &= \frac{n\alpha_1}{\lambda} \int_0^1 u^{n\alpha_1 - 1} (-\log u) du \\ &= (n\alpha_1 \lambda)^{-1}. \end{aligned} \tag{7}$$

By $E(X^\star_{1:n}) = (n\alpha_1\lambda)^{-1}$, and Equation (7), we obtain

$$\Lambda^\star_1 = \frac{\text{CRE}(X^\star_{1:n})}{E(X^\star_{1:n})} = 1. \tag{8}$$

For all n, the ratio (8) is constant under the CPHR model with a baseline exponential distribution. Since uncertainty of X is greater than $X^\star_{1:n}$, then $\Phi_1 = \text{CRE}(X) - \text{CRE}(X^\star_{1:n}) > 0$, for $n\alpha_1 > 0$. Notice that

$$\frac{\text{CRE}(X)}{\text{CRE}(X^\star_{1:n})} = n\alpha_1, \tag{9}$$

does not depend on λ.

3.2 Pareto model

Suppose that X has the Pareto distribution, with shape parameter $\gamma > 0$ and scale parameter $\beta > 0$; that is, $F(x) = 1 - (\beta/x)^\gamma$ for $x \geq \beta$. From Equations (1) and (4), the CRE(X) and the CRE($X^\star_{1:n}$) are

$$\text{CRE}(X) = \frac{\gamma\beta}{(\gamma-1)^2}, \quad \gamma > 1,$$

and

$$\text{CRE}(X^\star_{1:n}) = \frac{n\alpha_1\gamma\beta}{(n\alpha_1\gamma-1)^2}, \quad \gamma > \frac{1}{n\alpha_1},$$

and otherwise CREs are infinity. For $\gamma > 1$, $\Phi_2 = \text{CRE}(X) - \text{CRE}(X^\star_{1:n}) \geq 0$, then uncertainty of X is more than $X^\star_{1:n}$. Also, Φ_2 is an increasing function of n for $n > (\gamma\alpha_1)^{-1}$. Notice that

$$\frac{\text{CRE}(X)}{\text{CRE}(X^\star_{1:n})} = \frac{(n\alpha_1\gamma-1)^2}{n\alpha_1(\gamma-1)^2}, \tag{10}$$

does not depend on β.

3.3 Weibull model

A random variable X follows the Weibull distributed, if its CDF is

$$F(x) = 1 - \exp\{-(x\lambda)^q\}, \quad x > 0, \quad q > 0, \quad \lambda > 0,$$

where q and λ are shape and scale parameters, respectively. One can show that $E(X) = \lambda^{-1}\Gamma(1+q^{-1})$ and $\text{CRE}(X) = (\lambda q)^{-1}\,\Gamma(1+q^{-1})$, where $\Gamma(x) = \int_0^\infty t^{x-1}\exp\{-t\}\,dt$ stands for the complete gamma function. Thus, $\text{CRE}(X)/E(X) = q^{-1}$. The following lemma is used in the proof of Theorem 3.2.

Lemma 3.1. *(Kamps (1998)) For any increasing sequence of positive integers $\{m_i,\ i \geq 1\}$, $\sum_{i=1}^{\infty} {m_i}^{-1}$ is infinite, if and only if the sequence of polynomials $\{x^{m_i}\}$ is complete on $L(0,1)$.*

Theorem 3.2. *Suppose that $X_1, \cdots, X_n$ are independent and identically distributed observations from an absolutely continuous CDF $F(x)$ and PDF $f(x)$. Then F belongs to the Weibull family, if and only if $\mathrm{CRE}(X^{\star}_{1:n})/E(X^{\star}_{1:n}) = k$, for all $n = n_j,\ j \geq 1$, such that $\sum_{j=1}^{+\infty} {n_j}^{-1}$ is infinite.*

Proof. By (4), we have

$$\begin{aligned}\mathrm{CRE}(X^{\star}_{1:n}) &= \frac{n\alpha_1}{\lambda q}\int_0^1 u^{n\alpha_1 - 1}(-\log u)^{\frac{1}{q}}du \\ &= \frac{1}{\lambda q \sqrt[q]{n\alpha_1}}\Gamma\big(1+q^{-1}\big).\end{aligned}$$

Also $E(X^{\star}_{1:n}) = \lambda^{-1}(n\alpha_1)^{\frac{-1}{q}}\Gamma\big(1+q^{-1}\big)$. Thus, $\mathrm{CRE}(X^{\star}_{1:n})/E(X^{\star}_{1:n}) = q^{-1}$. This result shows that for all n, in the Weibull family, the ratio $\mathrm{CRE}(X^{\star}_{1:n})/E(X^{\star}_{1:n})$ is constant. Thus, the prove of the necessity part is completed. To prove the sufficiency part, note that

$$\begin{aligned}E(X^{\star}_{1:n}) &= \int_0^{\infty} n\alpha_1 x f(x)\bar{F}^{n\alpha_1 - 1}(x)dx \\ &= n\alpha_1\int_0^1 u^{n\alpha_1 - 1}F^{-1}(1-u)du, \qquad (11)\end{aligned}$$

by changing variable $u = 1 - F(x)$. Using Equations (4) and (11), and by the assumption we have

$$k = \frac{\mathrm{CRE}(X^{\star}_{1:n})}{E(X^{\star}_{1:n})} = -\frac{\int_0^1 \frac{u^{n\alpha_1}\log u}{f(F^{-1}(1-u))}du}{\int_0^1 u^{n\alpha_1 - 1}F^{-1}(1-u)du}, \qquad (12)$$

holds for $j \geq 1$, $n = n_j$, such that $\sum_{j=1}^{+\infty} {n_j}^{-1}$ is infinite or equivalently

$$\int_0^1 \Big[kF^{-1}(1-u) + \frac{u\log u}{f(F^{-1}(1-u))}\Big]u^{n\alpha_1 - 1}du = 0. \qquad (13)$$

By Lemma (3.1) and Equation (13), we have

$$kF^{-1}(v) + \frac{(1-v)\log(1-v)}{f(F^{-1}(v))} = 0 \quad \text{a.e},\ v \in (0,1).$$

Since $dF^{-1}(v)/dv = \big((F^{-1}(v))\big)^{-1}$, it follow that

$$kF^{-1}(v) + (1-v)\log(1-v)\frac{d}{dv}F^{-1}(v) = 0 \quad \text{a.e},\ v \in (0,1).$$

After some algebraic manipulations, we conclude that $F^{-1}(v) = k_1[-\log(1-v)]^k$, $v \in (0,1)$, and then $F(x) = 1 - \exp\big(-(x/k_1)^{\frac{1}{k}}\big)$, $x > 0$. This means that F belongs to the Weibull family of distribution. □

Remark 3.3. *For all n,*
$\Phi_3 = \text{CRE}(X) - \text{CRE}(X^\star_{1:n}) = \big(1-(n\alpha_1)^{\frac{-1}{q}}\big)(\lambda q)^{-1}\Gamma\big(1+q^{-1}\big) \geq 0$. *Then uncertainty of X is more than $X^\star_{1:n}$ and is increasing in n. Notice that*

$$\frac{\text{CRE}(X)}{\text{CRE}(X^\star_{1:n})} = (n\alpha_1)^{\frac{1}{q}}, \tag{14}$$

does not depend on λ.

Theorem 3.4. *Suppose Y and Z be two positive random variables with PDFs $f(x)$ and $g(x)$ and absolutely continuous CDFs $F(x)$ and $G(x)$, respectively. Then $\text{CRE}(Y^\star_{1:n}) = \text{CRE}(Z^\star_{1:n})$, for $j \geq 1$, $n = n_j$, such that $\sum_{j=1}^{+\infty} {n_j}^{-1} = \infty$, if and only if F and G belong to the same family of distributions, but for a possible location shift.*

Proof. First assume that $\text{CRE}(Y^\star_{1:n}) = \text{CRE}(Z^\star_{1:n})$. Then Equation (4) implies that

$$\int_0^1 u^{n\alpha_1} \log(u) \left(\frac{1}{f(F^{-1}(1-u))} - \frac{1}{g(G^{-1}(1-u))} \right) du = 0, \tag{15}$$

holds for $j \geq 1$, $n = n_j$, such that $\sum_{j=1}^{+\infty} n_j^{-1} = \infty$. Lemma (3.1) concludes that $f(F^{-1}(t)) = g(G^{-1}(t))$, for $t \in (0,1)$. On the other hand, since $dF^{-1}(t)/dv = (f(F^{-1}(t)))^{-1}$, we have $dF^{-1}(t)/dv = dG^{-1}(t)/dv$, $t \in (0,1)$. It then follows that $F^{-1}(t) = G^{-1}(t) + k$, $t \in (0,1)$. This means that F and G belong to the same family of distributions, but for a possible change in location. The necessity is trivial. □

4 DCRE in CPHR model

The CRE for the residual lifetime distribution of $X^\star_{1:n}$, that is $\text{DCRE}(X^\star_{1:n,t})$, is

$$\begin{aligned}\text{CRE}(X^\star_{1:n,t}) &= -\big(\bar{F}(t)\big)^{-n\alpha_1} \int_t^\infty \big(\bar{F}(x)\big)^{n\alpha_1} \log\big(\bar{F}(x)\big)^{n\alpha_1} dx \\ &+ n\alpha_1 M_{X^\star_{1:n}}(t) \log \bar{F}(t),\end{aligned} \tag{16}$$

where $M_{X^\star_{1:n}}(t) = E(X^\star_{1:n} - t \big| X^\star_{1:n} > t)$ is the mean residual lifetime of the system. To see this, note that

$$\begin{aligned}\text{CRE}(X^\star_{1:n,t}) &= -\int_0^\infty \bar{F}_{X^\star_{1:n,t}}(x) \log \bar{F}_{X^\star_{1:n,t}}(x) dx \\ &= -\int_t^\infty \left(\frac{\bar{F}(x)}{\bar{F}(t)}\right)^{n\alpha_1} \log \left(\frac{\bar{F}(x)}{\bar{F}(t)}\right)^{n\alpha_1} dx \\ &= -(\bar{F}(t))^{-n\alpha_1} \int_t^\infty \big(\bar{F}(x)\big)^{n\alpha_1} \log\big(\bar{F}(x)\big)^{n\alpha_1} dx \\ &+ n\alpha_1 \log \bar{F}(t) \int_t^\infty \Big(\frac{\bar{F}(x)}{\bar{F}(t)}\Big)^{n\alpha_1} dx \\ &= -\big(\bar{F}(t)\big)^{-n\alpha_1} \int_t^\infty \big(\bar{F}(x)\big)^{n\alpha_1} \log\big(\bar{F}(x)\big)^{n\alpha_1} dx \\ &+ n\alpha_1 M_{X^\star_{1:n}}(t) \log \bar{F}(t).\end{aligned}$$

Remark 4.1. *For* $t = 0$, $CRE(X^{\star}_{1:n,0}) = CRE(X^{\star}_{1:n})$.

In sequel, we provide a lower bound for the DCRE($X^{\star}_{1:n,t}$).

Proposition 4.2. *For all* t,

$$n\alpha_1 M_{X^{\star}_{1:n}}(t)|\log \bar{F}(t)|\big(\bar{F}(t)\big)^{n\alpha_1} \leq \text{CRE}(X^{\star}_{1:n}). \tag{17}$$

Proof. By Equation (16) and noting that $\log(1 - F(t)) \leq 0$, we conclude

$$\begin{aligned}
\text{CRE}(X^{\star}_{1:n,t}) &\leq -\big(\bar{F}(t)\big)^{-n\alpha_1} \int_t^{\infty} \big(\bar{F}(x)\big)^{n\alpha_1} \log\big(\bar{F}(x)\big)^{n\alpha_1} dx \\
&\leq -\big(\bar{F}(t)\big)^{-n\alpha_1} \int_0^{\infty} \big(\bar{F}(x)\big)^{n\alpha_1} \log\big(\bar{F}(x)\big)^{n\alpha_1} dx \\
&= \big(\bar{F}(t)\big)^{-n\alpha_1} \text{CRE}(X^{\star}_{1:n}).
\end{aligned}$$

Thus,

$$\text{CRE}(X^{\star}_{1:n,t})\big(\bar{F}(t)\big)^{n\alpha_1} \leq \text{CRE}(X^{\star}_{1:n}), \quad \forall t. \tag{18}$$

Equation (16), concludes that

$$\begin{aligned}
M_{X^{\star}_{1:n}}(t) &\leq -\frac{1}{n\alpha_1|\log \bar{F}(t)|\big(\bar{F}(t)\big)^{n\alpha_1}} \int_t^{\infty} \big(\bar{F}(x)\big)^{n\alpha_1} \log\big(\bar{F}(x)\big)^{n\alpha_1} dx \\
&\leq -\frac{1}{n\alpha_1|\log \bar{F}(t)|\big(\bar{F}(t)\big)^{n\alpha_1}} \int_0^{\infty} \big(\bar{F}(x)\big)^{n\alpha_1} \log\big(\bar{F}(x)\big)^{n\alpha_1} dx \\
&= \frac{1}{n\alpha_1 \log \bar{F}(t)|\big(\bar{F}(t)\big)^{n\alpha_1}} \text{CRE}(X^{\star}_{1:n}),
\end{aligned}$$

and the proof completes. □

Proposition 4.3. *Suppose that* Y *and* Z *be two positive random variables with PDFs* $g(x)$ *and* $h(x)$ *and absolutely continuous CDFs* $G(x)$ *and* $H(x)$*, respectively. Then* G *and* H *belong to the same family of distributions, but for a possible change in location and scale, if and only if for* $t > 0$, $\text{CRE}(Y_{1:n}, t) = \text{CRE}(Z_{1:n}, t)$, *for* $n = n_j$, $j \geq 1$ *such that* $\sum_{j=1}^{+\infty} n_j^{-1}$ *is infinite.*

Proof. Let $\forall\, n = n_j$, $j \geq 1$ such that $\sum_{j=1}^{+\infty} n_j^{-1}$ is infinite, $\text{CRE}(Y_{1:n}, t) = \text{CRE}(Z_{1:n}, t)$, then by Theorem (3.4), $Y\big|Y > t$ and $Z\big|Z > t$ have the same distribution but many be differ for a change in location parameter. That is $g_t(x) = h_t(x + c)$, where g_t and h_t are, respectively, PDFs of $Y\big|Y > t$ and $Z\big|Z > t$. Thus, $g(x) = h(x + c)\bar{G}(t)/H(t)$. Therefore, G and H belong to the same family of distributions, but for a possible change in scale and location parameters. □

Discussion and Conclusion

In this paper, we considered the problem of characterizing the parent distributions based on the CRE of first sequential order statistics under a conditional proportional hazard rate model. It was shown that the equality of the CRE in first sequential order statistics can determine uniquely the parent baseline distribution. Subsequently, we characterized the Weibull distribution based on the ratio of the CRE of first sequential order statistics to the mean of the first SOS. Also, we considered characterizations based on the DCRE and derived a lower bound for the CRE of residual lifetime of the first SOS.

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