

## Comparisons for series and parallel systems with discrete Weibull components via separate comparisons of parameters

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**Abstract:** In this paper, we obtain the usual stochastic order of series and parallel systems comprising heterogeneous discrete Weibull (DW) components. Suppose  $X_1, \dots, X_n$  and  $Y_1, \dots, Y_n$  denote the independent component's lifetimes of two systems such that  $X_i \sim DW(\beta_i, p_i)$  and  $Y_i \sim DW(\beta_i^*, p_i^*)$ ,  $i = 1, \dots, n$ . We obtain the usual stochastic order between series systems when the vector  $\beta$  is switched to the vector  $\beta^*$  with respect to the majorization order, and when the vector  $-\log(\mathbf{1} - \mathbf{p})$  is switched to the vector  $-\log(\mathbf{1} - \mathbf{p}^*)$  in the sense of the weak supermajorization order. We also discuss the usual stochastic order between series systems by using the unordered majorization between the vectors  $-\log(\mathbf{1} - \mathbf{p})$  and  $-\log(\mathbf{1} - \mathbf{p}^*)$ , and the  $p$ -majorization order between the parameters  $\beta$  and  $\beta^*$ . It is also shown that the usual stochastic order between parallel systems comprising heterogeneous discrete Weibull components when the vector  $-\log \mathbf{p}$  is switched to the vector  $-\log \mathbf{p}^*$  in the sense of the weak supermajorization order. These results enable us to find some lower bounds for the survival functions of a series and parallel systems consisting of independent heterogeneous discrete Weibull components.

**Keywords:** Discrete Weibull distribution;  $P$ -majorization order; Unordered majorization order; Weak submajorization order; Weak supermajorization order.

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## 1 Introduction

A random variable  $X$  is said to have geometric distribution with parameter  $p$  (denoted by  $X \sim Ge(p)$ ) if its probability mass function is

$$P(X = x) = p(1 - p)^{x-1}, \quad x = 1, 2, \dots \quad (1)$$

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The survival function of this distribution is given by:

$$\bar{F}(x) = P(X > x) = (1 - p)^x, \quad x = 1, 2, \dots \quad (2)$$

The geometric distribution is one of the most fundamental distributions in statistics and has found widely applications in engineering, game theory, regression, population selection, and reliability theory. It is also closely related to many well-known distributions such as binomial, Poisson and Gamma distributions. Margolin and Winokur (1967) studied the order statistics from geometric distribution in the context of inverse sampling. Jeske and Blessinger (2004) analyzed the maximum order statistics from heterogeneous geometric distributions and gave approximation formulas for expected value and variance of the maximum order statistics for the large sample size. They also pointed out that the maximum order statistics could be used in the real-world engineering models, such as wireless broadcast transmission systems.

In practice, we come across situations where lifetimes are recorded on a discrete scale. For example, the on/off switching devices, bulb of photocopier machine, to and fro motion of spring devices, etc. are some obvious such situations. In the last two decades, several discrete distributions such as the geometric, negative binomial (NB), generalized Poisson (GP) and Poisson-inverse Gaussian (PIG) have been employed to model lifetime data. However, there is a need to find more flexible discrete distributions to fit various types of data because some available discrete distributions are only appropriate for modelling over-or under-dispersed data.

The failure-time distribution is use to describe mathematically the life of a device, a material, or a structure. The exponential, gamma, Weibull and lognormal distributions are well-known in failure analysis. All these distributions have a continuous-time random variable. In failure studies, the time to failure is often measured in the number of cycles to failure and theretofore becomes a discrete random variable. It is well-known that the geometric and negative binomial distributions (with discrete random variables) correspond to the exponential and gamma distributions (with continuous random variables) respectively. As a discrete alternative to the Weibull distribution, Nakagawa and Osaki (1975) introduced a discrete Weibull distribution which it is useful in the failure data in failure studies such as cycles, blows, shocks, or revolutions.

A random variable  $X$  is said to have discrete Weibull distribution with parameters  $\beta$  and  $p$  (denoted by  $X \sim DW(\beta, p)$ ) if its survival function is

$$\bar{F}(x) = P(X > x) = (1 - p)^{x^\beta}, \quad \beta > 0, x = 1, 2, \dots \quad (3)$$

The probability mass function and hazard rate function of this distribution are, respectively, as follow:

$$P(X = x) = (1 - p)^{(x-1)^\beta} - (1 - p)^{x^\beta}, \quad \beta > 0, x = 1, 2, \dots,$$

$$r_X(x) = \frac{P(X = x)}{P(X \geq x)} = 1 - (1 - p)^{x^\beta - (x-1)^\beta}, \quad \beta > 0, x = 1, 2, \dots$$

It is important to note that discrete Weibull random variable has DFR property if  $\beta \leq 1$  and has IFR property if  $\beta \geq 1$  (Nakagawa and Osaki, 1975).

A random variable  $X$  is said to have Weibull distribution with shape parameter  $\beta$  and scale parameter  $\lambda$  (denoted by  $X \sim W(\beta, \lambda)$ ) if its survival function is given by

$$\bar{F}(x; \beta, \lambda) = e^{-(\lambda x)^\beta}, \quad x \in \mathbb{R}^+, \beta \in \mathbb{R}^+, \lambda \in \mathbb{R}^+.$$

It is well-known that the hazard rate of Weibull distribution is decreasing for  $\beta < 1$ , constant for  $\beta = 1$ , and increasing for  $\beta > 1$ . One may refer to Johnson et al. (1994) and Murthy et al. (2004) for comprehensive discussions on various properties and applications of the Weibull distribution. It is of interest to note that if we make the transformation  $-\log(1-p) = \lambda^\beta$  in (3), thus the survival function changes to survival function of Weibull distribution.

The Weibull distribution has been used to analyze failure in electronic components, ball bearings, etc. Failures of some devices often depend more on the total number of cycles than on the total time that they have been used. Such examples are switching devices, railroad tracks, and tires of automobiles. In this case, the discrete Weibull distribution will be a good approximation for such devices, materials, or structures (see Nakagawa and Osaki, 1975).

Since discrete Weibull variable is the discrete counterpart of Weibull variable, thus it is natural to investigate whether some typical stochastic comparison results with Weibull variables can be extended to the cases of discrete Weibull variables. For a comprehensive discussion on various stochastic orderings with Weibull variables, one may refer to Fang and Zhang (2012), Zhao et al. (2016) and Balakrishnan et al. (2018).

Various ordering results have been established for some classical discrete distributions such as Bernoulli, binomial, Poisson, geometric and negative binomial. Elaborate reviews of the available results dealing with many different forms of ordering can be found in the books by Müller and Stoyan (2002), Lai and Xie (2006), and Shaked and Shanthikumar (2007).

One of the most commonly used systems in reliability is an  $r$ -out-of- $n$  system. This system comprising of  $n$  components, works iff at least  $r$  components work, and it includes parallel and series systems all as special cases corresponding to  $r = 1$  and  $r = n$ , respectively. Let  $X_1, \dots, X_n$  denote the lifetimes of components of a system and  $X_{1:n} \leq \dots \leq X_{n:n}$  represent the corresponding order statistics. Then,  $X_{n-r+1:n}$  corresponds to the lifetime of a  $r$ -out-of- $n$  system. Due to this direct connection, the theory of order statistics becomes quite important in studying  $(n-r+1)$ -out-of- $n$  systems and in characterizing their important properties. The comparison of important characteristics associated with lifetimes of technical systems is an interesting topic in reliability theory, since it usually enables us to approximate complex systems with simpler systems and subsequently obtaining various bounds for important ageing characteristics of the complex system. A convenient tool for this purpose is the theory of stochastic orderings.

In this paper, we focus on the series and parallel systems with their component's lifetimes follow the discrete Weibull distribution and obtain some ordering results. Suppose  $X_1, \dots, X_n$  and  $Y_1, \dots, Y_n$  denote the independent component's lifetimes of two systems such that  $X_i \sim DW(\beta_i, p_i)$  and  $Y_i \sim DW(\beta_i^*, p_i^*)$ ,  $i = 1, \dots, n$ . The lifetimes of the two series systems are  $X_{1:n}$  and  $Y_{1:n}$ , respectively. Let us set  $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)$ ,  $\boldsymbol{\beta}^* = (\beta_1^*, \dots, \beta_n^*)$ ,  $-\mathbf{log}(\mathbf{1} - \mathbf{p}) = (-\log(1-p_1), \dots, -\log(1-p_n))$  and  $-\mathbf{log}(\mathbf{1} - \mathbf{p}^*) = (-\log(1-p_1^*), \dots, -\log(1-p_n^*))$ . We will show under some restrictions on the involved parameters that  $X_{1:n}$  is greater than  $Y_{1:n}$  with respect to the usual stochastic order, when the vector  $\boldsymbol{\beta}$  is switched to the vector  $\boldsymbol{\beta}^*$  with respect to the majorization order, and the vector  $-\mathbf{log}(\mathbf{1} - \mathbf{p})$  is switched to the vector  $-\mathbf{log}(\mathbf{1} - \mathbf{p}^*)$  in the sense of the weak supermajorization order. We also establish

the usual stochastic order between  $X_{1:n}$  and  $Y_{1:n}$  by using the unordered majorization between the vectors  $-\mathbf{log}(\mathbf{1} - \mathbf{p})$  and  $-\mathbf{log}(\mathbf{1} - \mathbf{p}^*)$ , and the  $p$ -majorization order between the parameters  $\boldsymbol{\beta}$  and  $\boldsymbol{\beta}^*$ . It is also shown that the usual stochastic order between parallel systems comprising heterogeneous discrete Weibull components when the vector  $-\mathbf{log} \mathbf{p}$  is switched to the vector  $-\mathbf{log} \mathbf{p}^*$  in the sense of the weak supermajorization order.

## 2 Preliminaries

In this section, we review some concepts on stochastic orders and majorization principle that will be used in the sequel. Throughout, the terms “increasing” and “decreasing” are used for “non-decreasing” and “non-increasing”, respectively.

**Definition 2.1.** *Suppose  $X$  and  $Y$  are two positive discrete random variables with probability mass functions  $P(X = x)$  and  $P(Y = y)$ , distribution functions  $F_X(x) = P(X \leq x)$  and  $F_Y(y) = P(Y \leq y)$ , survival functions  $\bar{F}_X(x)$  and  $\bar{F}_Y(y)$ , respectively. Then, it is said that  $X$  is greater than  $Y$  in the usual stochastic order (denoted by  $X \geq_{st} Y$ ) if  $\bar{F}_X(x) \geq \bar{F}_Y(x)$  for all  $x \in \mathbb{N} = \{1, 2, \dots\}$ .*

See Müller and Stoyan (2002), Shaked and Shanthikumar (2007), and Belzunce et al. (2016) for elaborate discussions on the theory of stochastic orders and their applications.

Next, we present some notions of vector majorization and the other related orders, which have found significant applications in different branches of mathematics such as matrix analysis, operator theory, probability theory and statistics.

**Definition 2.2.** *For two vectors  $\mathbf{a} = (a_1, \dots, a_n)$  and  $\mathbf{b} = (b_1, \dots, b_n)$ , suppose  $a_{(1)} \leq \dots \leq a_{(n)}$  and  $b_{(1)} \leq \dots \leq b_{(n)}$  denote, respectively, the components of  $\mathbf{a}$  and  $\mathbf{b}$  in increasing order.*

- (i)  $\mathbf{a}$  is said to majorize  $\mathbf{b}$  (denoted as  $\mathbf{a} \succ^m \mathbf{b}$ ) if  $\sum_{j=1}^i a_{(j)} \leq \sum_{j=1}^i b_{(j)}$  for  $i = 1, \dots, n - 1$ , and  $\sum_{j=1}^n a_{(j)} = \sum_{j=1}^n b_{(j)}$ ;
- (ii)  $\mathbf{a}$  is said to weakly supermajorizes  $\mathbf{b}$  (denoted as  $\mathbf{a} \succ^w \mathbf{b}$ ) if  $\sum_{j=1}^i a_{(j)} \leq \sum_{j=1}^i b_{(j)}$  for  $i = 1, \dots, n$ ;
- (iii)  $\mathbf{a}$  is said to weakly submajorize  $\mathbf{b}$  (denoted as  $\mathbf{a} \succ_w \mathbf{b}$ ) if  $\sum_{j=i}^n a_{(j)} \geq \sum_{j=i}^n b_{(j)}$  for  $i = 1, \dots, n$ ;

Clearly, the majorization order implies both the weak submajorization and supermajorization orders. Interested readers may refer to Marshall et al. (2011) for comprehensive discussions on the theoretical properties as well as the applications of the above vector orders.

Let us define

$$\begin{aligned} \mathcal{D}_n &= \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \geq \dots \geq x_n\}, \\ \mathcal{D}_n^+ &= \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \geq \dots \geq x_n > 0\}, \\ \mathcal{E}_n &= \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \leq \dots \leq x_n\}, \\ \mathcal{E}_n^+ &= \{(x_1, \dots, x_n) \in \mathbb{R}^n : 0 < x_1 \leq \dots \leq x_n\}. \end{aligned}$$

**Definition 2.3.** A function  $\phi : \mathcal{A} \rightarrow \mathbb{R}$  is said to be Schur-convex on  $\mathcal{A} \subset \mathbb{R}^n$  if for all  $\mathbf{a}, \mathbf{b} \in \mathcal{A}$  such that  $\mathbf{a} \succ^m \mathbf{b}$ , it holds that  $\phi(\mathbf{a}) \geq \phi(\mathbf{b})$ . Further, a function  $\phi : \mathcal{A} \rightarrow \mathbb{R}$  is said to be Schur-concave on  $\mathcal{A}$  if  $-\phi$  is Schur-convex on  $\mathcal{A}$ .

The next lemma gives the necessary and sufficient conditions for determining Schur-convex and Schur-concave functions on the spaces  $\mathcal{D}_n$  and  $\mathcal{E}_n$ .

**Lemma 2.4.**

- (i) Suppose the function  $\phi : \mathcal{D}_n \rightarrow \mathbb{R}$  is continuous on  $\mathcal{D}_n$  and continuously differentiable on the interior of  $\mathcal{D}_n$ . Then,  $\phi$  is Schur-convex on  $\mathcal{D}_n$  iff  $\phi_{(k)}(\mathbf{x})$  is decreasing in  $k \in \{1, \dots, n\}$ , for all  $\mathbf{x}$  in the interior of  $\mathcal{D}_n$ , where  $\phi_{(k)}(\mathbf{x}) = \partial\phi(\mathbf{x})/\partial a_k$  (Marshall et al., 2011, p. 83);
- (ii) Suppose the function  $\varphi : \mathcal{E}_n \rightarrow \mathbb{R}$  is continuous on  $\mathcal{E}_n$  and continuously differentiable on the interior of  $\mathcal{E}_n$ . Then,  $\varphi$  is Schur-convex on  $\mathcal{E}_n$  iff  $\varphi_{(k)}(\mathbf{x})$  is increasing in  $k \in \{1, \dots, n\}$ , for all  $\mathbf{x}$  in the interior of  $\mathcal{E}_n$  (Kundu et al., 2016).

**Remark 2.5.** In view of the proof of Lemma 2.4, it follows that its results remain true if the spaces  $\mathcal{D}_n$  and  $\mathcal{E}_n$  replace by the spaces  $\mathcal{D}_n^+$  and  $\mathcal{E}_n^+$ , respectively.

The next lemma given by Haidari et al. (2019) characterizes the functions that preserve the weak submajorization and supermajorization orders on the spaces  $\mathcal{D}_n^+$  and  $\mathcal{E}_n^+$ .

**Lemma 2.6.** Suppose the function  $\phi : \mathcal{D}_n^+(\mathcal{E}_n^+) \rightarrow \mathbb{R}$  is continuous on  $\mathcal{D}_n^+(\mathcal{E}_n^+)$  and continuously differentiable on the interior of  $\mathcal{D}_n^+(\mathcal{E}_n^+)$ . Then,

- (i)  $\phi$  preserves the weak submajorization order on  $\mathcal{D}_n^+(\mathcal{E}_n^+)$  iff  $\phi_{(k)}(\mathbf{x})$  is a non-negative decreasing (non-negative increasing) function in  $k \in \{1, \dots, n\}$ , for all  $\mathbf{x}$  in the interior of  $\mathcal{D}_n^+(\mathcal{E}_n^+)$ ;
- (ii)  $\phi$  preserves the weak supermajorization order on  $\mathcal{D}_n^+(\mathcal{E}_n^+)$  iff  $\phi_{(k)}(\mathbf{x})$  is a non-positive decreasing (non-positive increasing) function of  $k \in \{1, \dots, n\}$ , for all  $\mathbf{x}$  in the interior of  $\mathcal{D}_n^+(\mathcal{E}_n^+)$ .

Suppose  $\boldsymbol{\pi} = (\pi_1, \dots, \pi_n)$  is an element of the set  $P$ , the set of all permutation of  $\{1, 2, \dots, n\}$ . Let us define

$$\begin{aligned} \mathcal{D}_n^{\boldsymbol{\pi}} &= \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_{\pi_1} \geq \dots \geq x_{\pi_n}\}, \\ \mathcal{D}_n^{+\boldsymbol{\pi}} &= \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_{\pi_1} \geq \dots \geq x_{\pi_n} > 0\}. \end{aligned}$$

**Definition 2.7.** Consider vectors  $\mathbf{u} = (u_1, \dots, u_n)$  and  $\mathbf{v} = (v_1, \dots, v_n)$  in  $\mathbb{R}^n$  and suppose that  $\mathbf{p} = (p_1, \dots, p_n)$  is a vector with positive components. Then,  $\mathbf{u}$  is said to be  $p$ -majorize  $\mathbf{v}$  on  $\mathcal{D}_n^{\boldsymbol{\pi}}$ , denoted by  $\mathbf{u} \succ_{\mathbf{p}}^m \mathbf{v}$  on  $\mathcal{D}_n^{\boldsymbol{\pi}}$ , if  $\mathbf{u}, \mathbf{v} \in \mathcal{D}_n^{\boldsymbol{\pi}}$ ,  $\sum_{i=1}^k p_{\pi_i} u_{\pi_i} \geq \sum_{i=1}^k p_{\pi_i} v_{\pi_i}$  for  $k = 1, \dots, n-1$ , and  $\sum_{i=1}^n p_{\pi_i} u_{\pi_i} = \sum_{i=1}^n p_{\pi_i} v_{\pi_i}$ .

Interested readers may refer to Cheng (1977) for a comprehensive discussion on the  $p$ -majorization orders and their properties.

It is interesting to find specific conditions under which the functions preserving the weighted majorization order can be determined. In the following, we discuss this problem by recalling a general statement. Consider function  $\phi : \mathbb{R}^n \times \mathbb{R}^{+n} \rightarrow \mathbb{R}$  such that

$$\phi(\mathbf{u}; \mathbf{p}) = \phi(\mathbf{u}^\pi; \mathbf{p}^\pi) \quad \text{for all } \mathbf{u} \in \mathbb{R}^n, \mathbf{p} \in \mathbb{R}^{+n} \text{ and } \pi \in P, \tag{4}$$

and

$$\mathbf{u} \prec_p \mathbf{v} \text{ on } \mathcal{D}_n \Rightarrow \phi(\mathbf{u}; \mathbf{p}) \leq \phi(\mathbf{v}; \mathbf{p}). \tag{5}$$

If  $\mathbf{u} \prec_p \mathbf{v}$  on  $\mathcal{D}_n^\pi$ , it then easily follows that  $\mathbf{u}^\pi \prec_p \mathbf{v}^\pi$  on  $\mathcal{D}_n$ , and hence by (4) and (5) we have

$$\phi(\mathbf{u}; \mathbf{p}) = \phi(\mathbf{u}^\pi; \mathbf{p}^\pi) \leq \phi(\mathbf{v}^\pi; \mathbf{p}^\pi) = \phi(\mathbf{v}; \mathbf{p}).$$

Thus, if the function  $\phi$  is permutation invariant (the property given in (4)) and preserves the  $p$ -majorization order on  $\mathcal{D}_n$ , then it also preserves the  $p$ -majorization order on  $\mathcal{D}_n^\pi$  for all  $\pi \in P$ . This statement implies the preserving property of the permutation invariant's functions only on the space  $\mathcal{D}_n$ .

It should be noted here that the  $p$ -majorization order compares only the similarly ordered vectors. When the two vectors are ordered in different direction, then there exists a class of functions that do not preserve the  $p$ -majorization order; see page 17 of Cheng (1977) for more details. In  $p$ -majorization, the parameters vector are compared in weight, whereas this is not the case in the majorization. Also, in the  $p$ -majorization order, we can enter another additional parameter from the parameters into calculation.

In the case of differentiable functions, we have the following lemma to check the preservation property of the weighted majorization order.

**Lemma 2.8.** (Cheng, 1977, p. 25) Consider differentiable function  $\phi : \mathbb{R}^n \times \mathbb{R}^{+n} \rightarrow \mathbb{R}$  satisfying (4). Then, (5) is satisfied iff for all  $\mathbf{u} \in \mathbb{R}^n$  and all  $i, j = 1, \dots, n$

$$(u_i - u_j) \left( \frac{1}{p_i} \frac{\partial \phi(\mathbf{u}, \mathbf{p})}{\partial u_i} - \frac{1}{p_j} \frac{\partial \phi(\mathbf{u}, \mathbf{p})}{\partial u_j} \right) \geq 0. \tag{6}$$

**Definition 2.9.**  $\mathbf{u}$  is said to be unordered majorize  $\mathbf{v}$ , denoted by  $\mathbf{u} \succ^{uo} \mathbf{v}$ , if  $\sum_{i=1}^k u_i \geq \sum_{i=1}^k v_i$  for  $k = 1, \dots, n - 1$ , and  $\sum_{i=1}^n u_i = \sum_{i=1}^n v_i$ .

It is of important to note that in general, there is no correspondence between majorization and unordered majorization. But checking the condition of unordered is easier than majorization.

For more details on the unordered majorization and its applications, see Parker and Ram (1977). In the case of differentiable functions, we have the following lemma to check the preservation property of the unordered majorization order.

**Lemma 2.10.** (Parker and Ram, 1977) Suppose  $J \subset \mathbb{R}^{+n}$  and  $\phi : \mathbb{R}^{+n} \rightarrow \mathbb{R}^+$  is a differentiable function. Then,

$$\mathbf{x} \preceq^{uo} \mathbf{y} \text{ on } J \iff \phi(\mathbf{x}) \leq \phi(\mathbf{y})$$

iff  $\phi_{(k)}(\mathbf{z}) = \partial \phi(\mathbf{z}) / \partial z_k$  is decreasing in  $k \in \{1, \dots, n\}$ .

### 3 Stochastic comparisons for series systems

In this section, the usual stochastic order between series systems with discrete Weibull components is discussed. Let us define

$$\mathcal{S}_n = \left\{ (\mathbf{a}, \mathbf{b}) = \begin{pmatrix} a_1, \dots, a_n \\ b_1, \dots, b_n \end{pmatrix} : a_i, b_j > 0 \text{ and } (a_i - a_j)(b_i - b_j) \leq 0, i, j = 1, \dots, n \right\}.$$

The next theorem provides the usual stochastic between two series systems when components follow discrete Weibull distribution.

**Theorem 3.1.** *Suppose  $X_1, \dots, X_n$  and  $Y_1, \dots, Y_n$  are two sets of independent non-negative random variables with  $X_i \sim DW(\beta_i, p_i)$  and  $Y_i \sim DW(\beta_i^*, p_i^*)$ ,  $i = 1, \dots, n$ . If  $\beta \succ^m \beta^*$  on  $\mathcal{D}_n^+$  and  $-\log(\mathbf{1} - \mathbf{p}) \succ^w -\log(\mathbf{1} - \mathbf{p}^*)$  on  $\mathcal{E}_n^+$ , then we have  $X_{1:n} \geq_{st} Y_{1:n}$ .*

*Proof.* Suppose  $Z_1, \dots, Z_n$  is another set of independent non-negative random variables with  $Z_i \sim DW(\beta_i^*, p_i)$ ,  $i = 1, \dots, n$ . First, we establish  $X_{1:n} \geq_{st} Z_{1:n}$  when  $\beta \succ^m \beta^*$ . The survival function of  $X_{1:n}$  is as follows:

$$\bar{F}_{X_{1:n}}(x; \beta, \mathbf{p}) = P(X_{1:n} > x) = \prod_{i=1}^n (1 - p_i)^{x^{\beta_i}}, \quad x = 1, 2, \dots$$

In order to obtain the required result, it suffices to show that  $\bar{F}_{X_{1:n}}(x; \beta, \mathbf{p})$  is Schur-convex when  $\beta$  belongs to  $\mathcal{D}_n^+$  and  $-\log(\mathbf{1} - \mathbf{p})$  in  $\mathcal{E}_n^+$ . For this purpose, we have

$$\frac{\partial \bar{F}_{X_{1:n}}(x; \beta, \mathbf{p})}{\partial \beta_i} = \log(x) x^{\beta_i} \log(1 - p_i) \bar{F}_{X_{1:n}}(x; \beta, \mathbf{p}),$$

and then

$$\begin{aligned} \frac{\partial \bar{F}_{X_{1:n}}(x; \beta, \mathbf{p})}{\partial \beta_i} &= \log(x) x^{\beta_i} \log(1 - p_i) \bar{F}_{X_{1:n}}(x; \beta, \mathbf{p}) \\ &\geq \log(x) x^{\beta_{i+1}} \log(1 - p_i) \bar{F}_{X_{1:n}}(x; \beta, \mathbf{p}), \quad (\text{since } \beta \in \mathcal{D}_n^+) \\ &\geq \log(x) x^{\beta_{i+1}} \log(1 - p_{i+1}) \bar{F}_{X_{1:n}}(x; \beta, \mathbf{p}), \quad (\text{since } -\log(\mathbf{1} - \mathbf{p}) \in \mathcal{E}_n^+) \\ &= \frac{\partial \bar{F}_{X_{1:n}}(x; \beta, \mathbf{p})}{\partial \beta_{i+1}}. \end{aligned}$$

Then, this observation, it readily follows that  $\bar{F}_{X_{1:n}}(x; \beta, \mathbf{p})$  is Schur-convex with respect to  $\beta$ .

Now, we prove that  $Z_{1:n} \geq_{st} Y_{1:n}$  when  $-\log(\mathbf{1} - \mathbf{p}) \succ^w -\log(\mathbf{1} - \mathbf{p}^*)$ . Let us define the function  $\phi: \mathbb{R}^{+n} \times \mathbb{R}^{+n} \rightarrow \mathbb{R}^+$  as

$$\phi(\beta^*, \mathbf{s}) = \bar{F}_{Z_{1:n}}(x) = \prod_{i=1}^n e^{-s_i x^{\beta_i^*}},$$

where  $s = -\log(1 - p_i)$ ,  $i = 1, \dots, n$ . For obtaining the desired result, according to Part (ii) of Lemma 2, we must show that  $\partial \phi(\beta^*, \mathbf{s}) / \partial s_k$  is non-positive and increasing on  $\mathcal{E}_n^+$ . Therefore, we have

$$\frac{\partial \phi(\beta^*, \mathbf{s})}{\partial s_k} = -x^{\beta_k^*} \phi(\beta^*, \mathbf{s}),$$

which shows that  $\phi(\boldsymbol{\beta}^*, \mathbf{s})$  is decreasing. On the other hand,

$$\frac{\partial \phi(\boldsymbol{\beta}^*, \mathbf{s})}{\partial s_k} = -x^{\beta_k^*} \phi(\boldsymbol{\beta}^*, \mathbf{s}) \leq -x^{\beta_{k+1}^*} \phi(\boldsymbol{\beta}^*, \mathbf{s}) = \frac{\partial \phi(\boldsymbol{\beta}^*, \mathbf{s})}{\partial s_{k+1}}$$

which shows that  $\partial \phi(\boldsymbol{\beta}^*, \mathbf{s}) / \partial s_k$  is increasing on  $\mathcal{E}_n^+$ . Now, by combining  $X_{1:n} \geq_{st} Z_{1:n}$  and then  $Z_{1:n} \geq_{st} Y_{1:n}$ , we conclude  $X_{1:n} \geq_{st} Y_{1:n}$ . □

It is important to note that when  $\beta_1 = \dots = \beta_n = 1$  and  $\beta_1^* = \dots = \beta_n^* = 1$ , the DW distribution becomes the geometric distribution with parameters  $p_i$  and  $p_i^*$ . Therefore, the next corollary follows immediately from Theorem 3.1, which has been verified by Zu and Hu (2011).

**Corollary 3.1.** *Suppose  $X_1, \dots, X_n$  and  $Y_1, \dots, Y_n$  are two sets of independent non-negative random variables with  $X_i \sim Ge(p_i)$  and  $Y_i \sim Ge(p_i^*)$ ,  $i = 1, \dots, n$ . If  $-\log(\mathbf{1} - \mathbf{p}) \succ^w -\log(\mathbf{1} - \mathbf{p}^*)$  then, we have  $X_{1:n} \geq_{st} Y_{1:n}$ .*

**Remark 3.2.** *Theorem 3.1 can be used to compute a lower bound for the survival function of series systems based on a heterogeneous sample in terms of the survival function of series systems based on a homogeneous sample. More precisely, setting  $\bar{\beta} = \frac{1}{n} \sum_{i=1}^n \beta_i$  and  $\tilde{p} = 1 - (\prod_{i=1}^n (1 - p_i))^{1/n}$ . Then it can be easily seen that*

$$\boldsymbol{\beta} \succ^m \bar{\boldsymbol{\beta}} \quad \text{and} \quad -\log(\mathbf{1} - \mathbf{p}) \succ^w -\log(\mathbf{1} - \tilde{\mathbf{p}}).$$

Thus, by using these observations and Theorem 3.1, we can obtain the following lower bound for the distribution function of  $X_{1:n}$  based on that of  $Y_{1:n}$ :

$$\bar{F}_{X_{1:n}}(x; \boldsymbol{\beta}, \mathbf{p}) \geq (1 - \tilde{p})^{n x^{\bar{\beta}}}, \quad x = 1, 2, \dots$$

We may question whether the result of Theorem 3.1 could hold if  $(\boldsymbol{\beta}, -\log(\mathbf{1} - \mathbf{p})) \notin \mathcal{S}_n$  and  $(\boldsymbol{\beta}^*, -\log(\mathbf{1} - \mathbf{p}^*)) \notin \mathcal{S}_n$ . The following discussion provides a counterexample.

**Example 3.3.** *Suppose  $X_1, X_2, X_3$  and  $Y_1, Y_2, Y_3$  are two sets of independent non-negative random variables with  $X_i \sim DW(\beta_i, p_i)$  and  $Y_i \sim DW(\beta_i^*, p_i^*)$ ,  $i = 1, 2, 3$ . Assume that  $(\beta_1, \beta_2, \beta_3) = (1, 2, 3)$ ,  $(\beta_1^*, \beta_2^*, \beta_3^*) = (1.1, 2, 2.9)$ ,  $(p_1, p_2, p_3) = (1 - e^{-1}, 1 - e^{-2}, 1 - e^{-3})$  and  $(p_1^*, p_2^*, p_3^*) = (1 - e^{-1}, 1 - e^{-2.1}, 1 - e^{-3.2})$ . It is obvious to observe that  $\boldsymbol{\beta} \succ^m \boldsymbol{\beta}^*$ ,  $-\log(\mathbf{1} - \mathbf{p}) \succ^w -\log(\mathbf{1} - \mathbf{p}^*)$ , but  $(\boldsymbol{\beta}, -\log(\mathbf{1} - \mathbf{p}))$  and  $(\boldsymbol{\beta}^*, -\log(\mathbf{1} - \mathbf{p}^*))$  are not in  $\mathcal{S}_n$ . Now, by using the above matrices in survival function of  $X_{1:3}$  and  $Y_{1:3}$ , we observe that*

$$\begin{aligned} \bar{F}_{X_{1:3}}(1, \boldsymbol{\beta}, -\log(\mathbf{1} - \mathbf{p})) &\simeq 0.0024787 \\ &> \bar{F}_{Y_{1:3}}(1, \boldsymbol{\beta}^*, -\log(\mathbf{1} - \mathbf{p}^*)) \simeq 0.001863 \\ \bar{F}_{X_{1:3}}(3, \boldsymbol{\beta}, -\log(\mathbf{1} - \mathbf{p})) &\simeq 5.03 \times 10^{-45} \\ &< \bar{F}_{Y_{1:3}}(3, \boldsymbol{\beta}^*, -\log(\mathbf{1} - \mathbf{p}^*)) \simeq 5.23 \times 10^{-44} \end{aligned}$$

Thus, these distribution functions cross, which means that  $X_{1:3} \not\geq_{st} Y_{1:3}$ .



**Theorem 3.2.** Suppose  $X_1, \dots, X_n$  and  $Y_1, \dots, Y_n$  are two sets of independent non-negative random variables with  $X_i \sim DW(\beta_i, p_i)$  and  $Y_i \sim DW(\beta_i^*, p_i^*)$ ,  $i = 1, \dots, n$ . If  $(\beta_1, \dots, \beta_n) \succ^m \log(\mathbf{1} - \mathbf{p})$   $(\beta_1^*, \dots, \beta_n^*)$  on the  $\mathcal{D}_n^\pi$  and  $-\log(\mathbf{1} - \mathbf{p}) \succ^{uo} -\log(\mathbf{1} - \mathbf{p}^*)$  then we have  $X_{1:n} \leq_{st} Y_{1:n}$ .

*Proof.* Suppose  $Z_1, \dots, Z_n$  is another set of independent non-negative random variables with  $Z_i \sim DW(\beta_i^*, p_i)$ ,  $i = 1, \dots, n$ . Setting  $s_i = -\log(1 - p_i)$ , for  $i = 1, \dots, n$ . Then, the survival function of  $X_{1:n}$  is as follows:

$$\bar{F}_{X_{1:n}}(x; \mathbf{s}, \boldsymbol{\beta}) = P(X_{1:n} > x) = \prod_{i=1}^n e^{-s_i x^{\beta_i}}, \quad x = 1, 2, \dots$$

It is evident to observe that for all  $\mathbf{s} \in \mathbb{R}^{+n}$ ,  $\boldsymbol{\beta} \in \mathbb{R}^{+n}$  and  $\boldsymbol{\pi} \in P$ ,

$$\bar{F}_{X_{1:n}}(x; \mathbf{s}, \boldsymbol{\beta}) = \bar{F}_{X_{1:n}}(x; \mathbf{s}^\pi; \boldsymbol{\beta}^\pi).$$

Now, we must be check the (5). The partial derivative of  $\bar{F}_{X_{1:n}}(x; \mathbf{s}, \boldsymbol{\beta})$  with respect to  $\beta_i$  is as follows:

$$\frac{\partial \bar{F}_{X_{1:n}}(x; \mathbf{s}, \boldsymbol{\beta})}{\partial \beta_i} = -s_i \log(x) x^{\beta_i} \bar{F}_{X_{1:n}}(x; \mathbf{s}, \boldsymbol{\beta}),$$

then we have

$$\begin{aligned} I &= (\beta_i - \beta_j) \left( \frac{1}{s_i} \frac{\partial \bar{F}_{X_{1:n}}(x; \mathbf{s}, \boldsymbol{\beta})}{\partial \beta_i} - \frac{1}{s_j} \frac{\partial \bar{F}_{X_{1:n}}(x; \mathbf{s}, \boldsymbol{\beta})}{\partial \beta_j} \right) \\ &= (\beta_i - \beta_j) \log(x) \bar{F}_{X_{1:n}}(x; \mathbf{s}, \boldsymbol{\beta}) \{x^{\beta_j} - x^{\beta_i}\} \end{aligned}$$

It is easy to observe that  $I \leq 0$ , and then we can conclude that  $X_{1:n} \leq_{st} Z_{1:n}$ .

Now, suppose that  $Z_i \sim DW(\beta_i^*, p_i)$  and  $Y_i \sim DW(\beta_i^*, p_i^*)$ ,  $i = 1, \dots, n$ . Assume that  $-\log(\mathbf{1} - \mathbf{p}) \succ^{uo} -\log(\mathbf{1} - \mathbf{p}^*)$  or equivalently  $\mathbf{s} \succ^{uo} \mathbf{s}^*$ . The survival function of  $Z_{1:n}$  is as follows:

$$\bar{F}_{Z_{1:n}}(x; \mathbf{s}, \boldsymbol{\beta}^*) = \prod_{i=1}^n e^{-s_i x^{\beta_i^*}}, \quad x = 1, 2, \dots$$

The partial derivative of  $\bar{F}_{Z_{1:n}}(x; \mathbf{s}, \boldsymbol{\beta}^*)$  with respect to  $s_i$  is as follows:

$$\frac{\partial \bar{F}_{Z_{1:n}}(x; \mathbf{s}, \boldsymbol{\beta}^*)}{\partial s_i} = -x^{\beta_i^*} \bar{F}_{Z_{1:n}}(x; \mathbf{s}, \boldsymbol{\beta}^*).$$

Since  $\boldsymbol{\beta}^* \in \mathcal{D}_n$ , then  $\beta_i^* \geq -\beta_{i+1}^*$  and we do get

$$\frac{\partial \bar{F}_{Z_{1:n}}(x; \mathbf{s}, \boldsymbol{\beta}^*)}{\partial s_i} = -x^{\beta_i^*} \bar{F}_{Z_{1:n}}(x; \mathbf{s}, \boldsymbol{\beta}^*) \leq -x^{\beta_{i+1}^*} \bar{F}_{Z_{1:n}}(x; \mathbf{s}, \boldsymbol{\beta}^*) = \frac{\partial \bar{F}_{Z_{1:n}}(x; \mathbf{s}, \boldsymbol{\beta}^*)}{\partial s_{i+1}},$$

which according to Lemma 4, it shows that  $Z_{1:n} \leq_{st} Y_{1:n}$ . Now, by combining  $X_{1:n} \leq_{st} Z_{1:n}$  and  $Z_{1:n} \leq_{st} Y_{1:n}$  hence the theorem.  $\square$

The following example highlights the difference between Theorem 3.1 and Theorem 3.2.

**Example 3.4.** Suppose  $X_1, \dots, X_4$  and  $Y_1, \dots, Y_4$  are two sets of independent non-negative random variables with  $X_i \sim DW(\beta_i, p_i)$  and  $Y_i \sim DW(\beta_i^*, p_i^*)$ ,  $i = 1, \dots, 4$ . Assume that  $(\beta_1, \beta_2, \beta_3, \beta_4) = (1, 4, 5, 7)$ ,  $(\beta_1^*, \beta_2^*, \beta_3^*, \beta_4^*) = (1.1, 3.9, 4.9, 6.3)$ ,  $(p_1, p_2, p_3, p_4) = (1 - e^{-5.7}, 1 - e^{-6}, 1 - e^{-8.5}, 1 - e^{-0.5})$  and  $(p_1^*, p_2^*, p_3^*, p_4^*) = (1 - e^{-4.5}, 1 - e^{-6.5}, 1 - e^{-7.5}, 1 - e^{-2})$ . It is obvious to observe that  $\beta \succ^m \beta^*$ ,  $-\log(\mathbf{1} - \mathbf{p}) \succ^w -\log(\mathbf{1} - \mathbf{p}^*)$ ,  $(\beta, -\log(\mathbf{1} - \mathbf{p})) \in \mathcal{S}_n$ , and  $(\beta^*, -\log(\mathbf{1} - \mathbf{p}^*)) \in \mathcal{S}_n$ . Thus, according to Theorem 3.1, we conclude that  $X_{1:4} \geq_{st} Y_{1:4}$ . On the other hand, since  $(\beta_1, \dots, \beta_n) \not\prec^m (\beta_1^*, \dots, \beta_n^*)$  on the  $\mathcal{D}_n^+$ , therefore Theorem 3.2 is not applicable in this case.

## 4 Stochastic comparisons for parallel systems

In this section, the usual stochastic order between parallel systems with discrete Weibull components is discussed. For this purpose, we need the following lemma.

**Lemma 4.1.** (Balakrishnan et al. (2015)) Let the function  $\psi : (0, \infty) \times (0, 1) \rightarrow (0, \infty)$  be defined as

$$\psi(\alpha, t) = \frac{\alpha(1-t)t^{\alpha-1}}{1-t^\alpha}.$$

Then,

- (i) for each  $0 < t < 1$ ,  $\psi(\alpha, t)$  is decreasing with respect to  $\alpha$ ;
- (ii) for each  $0 < \alpha \leq 1$ ,  $\psi(\alpha, t)$  is decreasing with respect to  $t$ ; and
- (iii) for each  $\alpha \geq 1$ ,  $\psi(\alpha, t)$  is increasing with respect to  $t$ .

**Theorem 4.1.** Suppose  $X_1, \dots, X_n$  and  $Y_1, \dots, Y_n$  are two sets of independent non-negative random variables with  $X_i \sim DW(\beta_i, p_i)$  and  $Y_i \sim DW(\beta_i, p_i^*)$ ,  $i = 1, \dots, n$ . If  $-\log \mathbf{p} \succ^w -\log \mathbf{p}^*$  on  $\mathcal{D}_n^+$ , then for  $\beta$  on  $\mathcal{D}_n^+$ , we have  $X_{n:n} \leq_{st} Y_{n:n}$ .

*Proof.* The distribution function of  $X_{n:n}$  is as follows:

$$F_{X_{n:n}}(x) = P(X_{n:n} \leq x) = \prod_{i=1}^n \left(1 - (1 - p_i)^{x^{\beta_i}}\right), \quad x = 1, 2, \dots$$

Let us define the function  $\phi : \mathbb{R}^{+n} \times \mathbb{R}^{+n} \rightarrow \mathbb{R}^+$  as

$$\phi(\beta, \mathbf{s}) = F_{X_{n:n}}(x) = \prod_{i=1}^n (1 - (1 - e^{-s_i})^{x^{\beta_i}}),$$

where  $s_i = -\log p_i$ ,  $i = 1, \dots, n$ . From the assumption  $-\log \mathbf{p} \in \mathcal{D}_n^+$ , we conclude that  $s_1 \geq \dots \geq s_n > 0$ . For obtaining the desired result, according to Part (ii) of Lemma

2, we must show that  $(\partial/\partial s_k) \phi(\boldsymbol{\beta}, \mathbf{s})$  is non-positive and decreasing with respect to  $k$ , when  $\mathbf{s} \in \mathcal{E}_n^+$  and  $\boldsymbol{\beta} \in \mathcal{E}_n^+$ . Therefore, we have

$$\frac{\partial \phi(\boldsymbol{\beta}, \mathbf{s})}{\partial s_k} = -x^{\beta_k} \frac{e^{-s_k} (1 - e^{-s_k})^{x^{\beta_k} - 1}}{1 - (1 - e^{-s_k})^{x^{\beta_k}}} \phi(\boldsymbol{\beta}, \mathbf{s}) = -\psi(x^{\beta_k}, 1 - e^{-s_k}),$$

which  $\psi$  is defined in Lemma 4.1 and shows that  $(\partial/\partial s_k) \phi(\boldsymbol{\beta}, \mathbf{s})$  is non-positive with respect to  $k$ . On the other hand,

$$\begin{aligned} \frac{\partial \phi(\boldsymbol{\beta}, \mathbf{s})}{\partial s_k} &= -x^{\beta_k} \frac{e^{-s_k} (1 - e^{-s_k})^{x^{\beta_k} - 1}}{1 - (1 - e^{-s_k})^{x^{\beta_k}}} \phi(\boldsymbol{\beta}, \mathbf{s}) \\ &= -\psi(x^{\beta_k}, 1 - e^{-s_k}) \\ &= -\psi(x^{\beta_{k+1}}, 1 - e^{-s_k}) \quad (\text{Part(i) of Lemma 4.1 and } \boldsymbol{\beta} \in \mathcal{D}_n^+) \\ &= -\psi(x^{\beta_{k+1}}, 1 - e^{-s_{k+1}}) \quad (\text{Part(iii) of Lemma 4.1}) \\ &= \frac{\partial \phi(\boldsymbol{\beta}, \mathbf{s})}{\partial s_{k+1}} \end{aligned}$$

which shows that  $(\partial/\partial s_k) \phi(\boldsymbol{\beta}, \mathbf{s})$  is decreasing with respect to  $k$ . Now, by combining these observations, we conclude  $X_{n:n} \leq_{st} Y_{n:n}$ .  $\square$

## Conclusion remarks

In this paper, we obtain the usual stochastic order between series systems comprising heterogeneous discrete Weibull components when the vector  $\boldsymbol{\beta}$  is switched to the vector  $\boldsymbol{\beta}^*$  with respect to the majorization order, and when the vector  $-\mathbf{log}(\mathbf{1} - \mathbf{p})$  is switched to the vector  $-\mathbf{log}(\mathbf{1} - \mathbf{p}^*)$  in the sense of the weak supermajorization order. We also discuss the usual stochastic order between series systems by using the unordered majorization between the vectors  $-\mathbf{log}(\mathbf{1} - \mathbf{p})$  and  $-\mathbf{log}(\mathbf{1} - \mathbf{p}^*)$ , and the  $p$ -majorization order between the parameters  $\boldsymbol{\beta}$  and  $\boldsymbol{\beta}^*$ . It is also shown that the usual stochastic order between parallel systems comprising heterogeneous discrete Weibull components when the vector  $-\mathbf{log} \mathbf{p}$  is switched to the vector  $-\mathbf{log} \mathbf{p}^*$  in the sense of the weak supermajorization order.

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