Journal of Statistical Modelling: Theory and Applications
Vol. 1, No. 1, 2018, pp. 95-104
Yazd University Press 2018

JSMTA

Mathematics of evidences in dynamic systems with exponential component lifetimes and optimal sample size determination

Majid Hashempour, Mahdi Doostparast*
Department of Statistics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran

Received: April 18, 2018/ Revised: May 28, 2018/ Accepted: June 2, 2018

Abstract: In this paper, statistical evidences in lifetimes of sequential r-out-of-n systems, which are modelled by the concept of sequential order statistics (SOS), coming from homogeneous exponential populations are considered. Weak and misleading evidences in SOS for hypotheses about the population parameter are derived in explicit expressions and their behaviours with respect to the model parameters are studied in details. Optimal sample sizes given a minimum desired level for the *decisive* and the *correct* probabilities are provided. It is shown that the optimal sample size does not depend on some model parameters.

Keywords: Exponential model; Hypotheses testing; Likelihood ratio; Sequential order statistics; Strong and weak evidences.
Mathematics Subject Classification (2010): 62F03, 62G30, 62N05.

1 Introduction

The concept of *sequential order statistics* (SOSs), introduced by Kamps (1995), is an extension of the usual order statistics (OSs) and used for modelling lifetimes of sequential r-out-of-n systems. Specifically, consider a given r-out-of-n system consisting of n components and $X_1, \cdots, X_n$ denote the corresponding component lifetimes. Then, the system lifetime (T) coincide to the r-th order statistics among $X_1, \cdots, X_n$, denoted by $X_{r:n}$. In the usual r-out-of-n systems, it is assumed that the lifetimes $X_1, \cdots, X_n$ are independent and identically distributed (i.i.d.) with a common cumulative distribution function (CDF), say F. Notice that in these systems failing a component does not change distributions of lifetimes of surviving components. Motivated by Cramer and Kamps (1996, 2001a), in practice, the failure of a component may result in a higher load on the remaining components and hence causes the distribution of the surviving

*Corresponding author: doustparast@um.ac.ir

components change. Examples of such phenomena include automobile industries, gas and oil transmission pipelines, etc. In these cases, the system lifetimes are usually modelled by SOSs. To see this, suppose that F_j, for $j = 1, \cdots n$, denotes the common CDF of the i.i.d lifetime components when $n - j + 1$ components are working. The components begin to work at time $t = 0$ independently with the common CDF F_1. When at time x_1, the first failure occurs, the remaining $n - 1$ components work independently with the common CDF F_2. This process continues to $n - r + 1$ components independently with the common CDF F_r work until the r-th failure occurs at time x_r and hence the whole system fails. The mentioned system is called dynamic system or *sequential r-out-of-n system* and the system lifetime coincides to r-th component failure time, denoted by $X^{\star}_{(r)}$. In the literature, $(X^{\star}_{(1)}, \cdots, X^{\star}_{(n)})$ is called SOSs from $F_1, F_2, \cdots, F_n$ and denoted by $(X^{\star}_{(1)}, \cdots, X^{\star}_{(n)}) \sim SOS(F_1, F_2, \cdots, F_n)$; For other formal definitions and properties of SOS, see, e.g., Cramer and Kamps (1996, 2001a,b, 2003).

The problem of estimating parameters on the basis of SOS has been considered in the literature. For example, Cramer and Kamps (1996) considered the problem of estimating the parameters on the basis of s independent multiple SOSs samples under a conditional proportional hazard rates (CPHR) model, defined by $\bar{F}_j(t) = \bar{F}_0^{\alpha_j}(t)$ for $j = 1, \cdots, r$, where the underlying CDF $F_0(t)$ is the exponential distribution, i.e.

$$F_0(x; \sigma) = 1 - \exp\left\{-\left(\frac{x}{\sigma}\right)\right\}, \quad x > 0, \quad \sigma > 0. \tag{1}$$

In this case, the hazard rate function of the CDF F_j, defined by $h_j(t) = f_j(t)/\bar{F}_j(t)$ for positive real numbers $\alpha_1, \cdots, \alpha_r$, $t > 0$ and $j = 1, \cdots, n$, is proportional to the hazard rate function of the baseline CDF F_0, i.e. $h_j(t) = \alpha_j h_0(t)$. Following Cramer and Kamps (1996), the available data may be represented as

$$\mathbf{x} = \begin{bmatrix} x_{11} & \cdots & x_{1r} \\ \vdots & \ddots & \vdots \\ x_{s1} & \cdots & x_{sr} \end{bmatrix}, \tag{2}$$

where the i-th row of the matrix $\mathbf{x}$ in (2) denotes the SOS sample coming from the i-th population. Statistical inference on the basis of SOS has been considered in literature; see, e.g, Balakrishnan et al. (2012), Bedbur (2010), Beutner and Kamps (2009), Burkschat and Navarro (2011), Cramer and Kamps (2001a), Esmailian and Doostparast (2014), Hashempour and Doostparast (2016, 2017), Schenk et al. (2011), Shafay et al. (2014) and references therein. Notice that for the special case $r = n$ and $\alpha_1 = \cdots = \alpha_n$, the SOS reduce to ordinary order statistics based on a random sample from the CDF $F1$. See also Table 1 of Cramer and Kamps (2001a).

In this paper, we consider evidences in independent multiple SOS samples coming from homogeneous exponential populations given by (2) under the above-mentioned CPHR model. Therefore, the rest of this paper is organized as follow: In Section 2 , a review on statistical evidence is given. In Section 3, statistical evidences in SOS from exponential populations are derived in explicit expressions and their behaviours with

respect to the model parameters are studied in details. In Section 4, optimal sample sizes given a minimum desired level for the *decisive* and the *correct* probabilities are provided. Section 5 concludes.

2 A review on statistical evidence

In this section, we provide a brief review on the topic of statistical evidence due to Royall (1997). Some non-statistical scientists misuse statistical methods which lead to the misinterpretation of observations. For example, the decision-making paradigms since the work of Neyman and Pearson in the 1930s, have been formulated not in terms of interpreting data as evidence, but in terms of choosing between alternative course of actions. This lead to the current situation in which the Neyman-Pearson theory view common statistical procedures as decision-making tools, while much of statistical practice consists of using the same procedures for a different purpose, namely, interpreting data as evidence. In the Neyman-Pearson theory, a test of two hypotheses H_1 and H_2 is represented as a procedure for choosing between two actions. But in applications, when an optimal test chooses H_2, it is often taken to mean that data are evidence favoring H_2 over H_1. This interpretation can be quite wrong. For more details, see Blume (2002, 2011) and Royall (1997, 2000).

As mentioned above, the errors are usually quantitative, as when statistical evidence is judged to be weaker or stronger than it really is. So evidence is judged to support one hypothesis over another when the opposite is true. A key question is "when a certain hypothesis is preferred to others". In other words, when is it right to say that the observations are evidence in favour of one hypothesis vis-a-vis another? The answer to this fundamental question can be answered by Bayesian methods. But, the Bayesian methods need prior knowledge on the hypotheses. To avoid this problem, one may use non-informative priors or references analysis which are solely based on the observed data. In other words, one may consider the objective priors and then derive the posterior distributions of the hypotheses. Then the mentioned question can be answered by the posteriors; see, e.g., Berger (1985) and references therein. This paper considers an alternative approach called evidential statistics which is also solely based on data. Following Royall (1997), let $\lambda(>0)$ be any data-based measure of support of H_1 against H_2. Large (Small) values of λ are interpreted as evidence given by data in favor of $H_1(H_2)$. The probabilities of observing strong misleading evidence under H_i $(i = 1, 2)$ is

$$M_1^\star = P\left(\lambda < \frac{1}{k}\middle| H_1 \text{ is correct}\right), \tag{3}$$

and

$$M_2^\star = P\left(\lambda > k\middle| H_2 \text{ is correct}\right), \tag{4}$$

respectively, where k is a known constant greater than unity. The probability of weak evidence under H_i $(i = 1, 2)$ is

$$W_i^\star = P\left(\frac{1}{k} \leq \lambda \leq k \middle| H_i \text{ is correct}\right). \tag{5}$$

3 SOS-based evidences

Let $X^\star_{(1)}, \cdots, X^\star_{(r)}$ be the first r SOS. The joint probability density function of $(X^\star_{(1)}, \cdots, X^\star_{(r)})$ is (Cramer and Kamps (2001a))

$$f(y_1, \cdots, y_r) = A \prod_{j=1}^{r-1} \left[f_j(y_j) \left(\frac{\bar{F}_j(y_j)}{\bar{F}_{j+1}(y_j)} \right)^{n-j} \right] f_r(y_r) \bar{F}_r(y_r)^{n-r}, \tag{6}$$

for $y_1 < y_2 < \cdots < y_r$, $r = 1, \cdots, n$, where $A := n(n-1)\cdots(n-r+1)$ and $\bar{F}_j(.) = 1 - F_j(.)$, $j = 1, \cdots, n$. From (6), the LF of the data given by (2) reads

$$L(F_1, \cdots, F_r; \mathbf{x}) = A^s \prod_{i=1}^{s} \left(\prod_{j=1}^{r-1} \left[f_j(x_{ij}) \left(\frac{\bar{F}_j(x_{ij})}{\bar{F}_{j+1}(x_{ij})} \right)^{n-j} \right] f_r(x_{ir}) \bar{F}_r(x_{ir})^{n-r} \right). \tag{7}$$

Under the CPHR modelin section 1 and assuming that the baseline CDF in the i-th parent population, for $i = 1, \cdots, s$, follows the exponential distribution with mean σ_i, the likelihood function (LF) of the available data is (Cramer and Kamps (1996))

$$L(\sigma_1, \cdots, \sigma_s, \boldsymbol{\alpha}; \mathbf{x}) = A^s \left(\prod_{j=1}^{r} \alpha_j \right)^s \left(\prod_{i=1}^{s} \frac{1}{\sigma_i} \right)^r \exp\Big\{ - \sum_{i=1}^{s} \sum_{j=1}^{r} \Big(\frac{x_{ij} m_j}{\sigma_i} \Big) \Big\}, \tag{8}$$

where $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_r)$ and $m_j = (n-j+1)\alpha_j - (n-j)\alpha_{j+1}$, for $j = 1, \cdots, r$, with convention $\alpha_{r+1} \equiv 0$. When the baseline exponential populations are homogeneous, the LF (8) reduces to

$$L(\sigma, \boldsymbol{\alpha}; \mathbf{x}) = A^s \left(\prod_{j=1}^{r} \alpha_j \right)^s \left(\frac{1}{\sigma} \right)^{sr} \exp\left\{ - \left(\frac{\sum_{i=1}^{s} \sum_{j=1}^{r} x_{ij} m_j}{\sigma} \right) \right\}, \tag{9}$$

where σ is the common mean of the baseline CDFs.

Remark 3.1. Cramer and Kamps (1996) show that $2\sigma^{-1} \sum_{i=1}^{s} \sum_{j=1}^{r} x_{ij} m_j \sim \chi_{2rs}$, where χ_ν stands for the chi-square distribution ν on degrees of freedom.

In sequel, we consider evidences in the available data given by (2) for the problem of hypotheses testing

$$H_1 : \sigma = \sigma_1 \quad v.s \quad H_2 : \sigma = \sigma_2 \tag{10}$$

where σ_1 and σ_2 are known positive constants and $0 < \sigma_1 < \sigma_2$. To do this, Equation (9) gives the likelihood ratio (LR) for the null hypothesis H_1 against the alternative H_2, i.e.

$$\lambda = \frac{L_1}{L_2} = \left(\frac{\sigma_2}{\sigma_1}\right)^{sr} \exp\left\{-\left(\frac{1}{\sigma_1}-\frac{1}{\sigma_2}\right)\sum_{i=1}^{s}\sum_{j=1}^{r} x_{ij}m_j\right\}. \tag{11}$$

From Remark 3.1 and Equations (3) and (11), the misleading probability is then derived as

$$\begin{aligned} M_1^\star &= P\left(\left(\frac{\sigma_2}{\sigma_1}\right)^{sr} \exp\left\{-\left(\frac{1}{\sigma_1}-\frac{1}{\sigma_2}\right)\sum_{i=1}^{s}\sum_{j=1}^{r} x_{ij}m_j\right\} < \frac{1}{k}\middle|\, \sigma=\sigma_1\right) \\ &= 1 - P\left(\sum_{i=1}^{s}\sum_{j=1}^{r} x_{ij}m_j < \left(\frac{\ln\left(k\left(\frac{\sigma_2}{\sigma_1}\right)^{sr}\right)}{\frac{1}{\sigma_1}-\frac{1}{\sigma_2}}\right)\middle|\, \sigma=\sigma_1\right) \\ &= 1 - P\left(\chi_{2rs} < \left(\frac{2\ln\left(k\left(\frac{\sigma_2}{\sigma_1}\right)^{sr}\right)}{\sigma_1\left(\frac{1}{\sigma_1}-\frac{1}{\sigma_2}\right)}\right)\middle|\, \sigma=\sigma_1\right) \\ &= 1 - F_{\chi_{2rs}}\left(\frac{2\ln\left(k\left(\frac{\sigma_2}{\sigma_1}\right)^{sr}\right)}{1-\frac{\sigma_1}{\sigma_2}}\right), \end{aligned}$$

where F_{χ_ν} is the CDF of the χ_ν-distribution and "ln" calls for the natural logarithm. Similar procedures yield the following proposition. The details are given in the appendix.

Proposition 3.2. *Let $\tau = \sigma_2/\sigma_1 \geq 1$. The misleading and weak evidences based on independent s SOS samples from homogeneous exponential population under the CPHR model are*

$$M_1^\star = 1 - F_{\chi_{2rs}}\left(\frac{2\ln\left(k\tau^{sr}\right)}{1-\tau^{-1}}\right), \tag{12}$$

$$M_2^\star = F_{\chi_{2rs}}\left(\frac{2\ln\left(\tau^{sr}/k\right)}{\tau-1}\right), \tag{13}$$

$$W_1^\star = F_{\chi_{2rs}}\left(\frac{2\ln\left(k\tau^{sr}\right)}{1-\tau^{-1}}\right) - F_{\chi_{2rs}}\left(\frac{2\ln\left(\tau^{sr}/k\right)}{1-\tau^{-1}}\right), \tag{14}$$

and

$$W_2^\star = F_{\chi_{2rs}}\left(\frac{2\ln\left(k\tau^{sr}\right)}{\tau-1}\right) - F_{\chi_{2rs}}\left(\frac{2\ln\left(\tau^{sr}/k\right)}{\tau-1}\right). \tag{15}$$

In particular, the probabilities in Equations (12)-(15) *are free of the size n of the system and the parameter vector* $\boldsymbol{\alpha}=(\alpha_1,\cdots,\alpha_r)$ *of the CPHR model.*

Remark 3.3. Applying the well-known L' Hopital rule, one can prove that

- $\lim_{\tau\to+\infty} M_i^\star = \lim_{\tau\to+\infty} W_i^\star = 0$. It may be noticed that when σ_2 tends to infinity, the distance between the means of two populations will be increasing as much as possible. Thus, the probabilities of misleading and weak evidences tend to zero. So, even with few data we can make the decision about true hypothesis.
- $\lim_{\tau\to1^+} M_i^\star = \lim_{\tau\to1^+} (1 - W_i^\star) = 0$. Note that when σ_2 tends to σ_1, the distance between the means of two populations will be decreasing as much as possible. So, $M_1^\star$ and $M_2^\star$ vanish and $W_1^\star$ and $W_2^\star$ tend to one. Hence, one can not make decision based on the available data and needs more SOS samples.

An interesting topic in statistical evidence is determination of the global maximum of the misleading evidences. Here, the maximization of $M_1^\star$ in Equation (12) is equivalent to minimization of $h(\tau) := \ln(k\tau^{sr})/(1-\tau^{-1})$ with respect to $\tau \geq 1$. After some algebraic manipulations, one can see that the global minimum of $h(\tau)$ is derived by solving the non-linear equation $\partial h(\tau)/\partial\tau = 0$, or equivalently $\tau - \ln(\tau) = 1 + \ln(k)/sr$. Note that the function $h(\tau)$ is convex and therefore the solution of the mentioned equation is unique. Similar arguments for the misleading $M_2^\star$ in Equation (13) imply the next proposition.

Proposition 3.4. *Let $u(t) := t - \ln(t) - \ln(k)/(sr) - 1$, for $t \geq 1$. The points of global maximum of $M_1^\star$ and $M_2^\star$, as a function of τ, are derived as the unique solutions of the non linear equations $u(\tau) = 0$ and $u(1/\tau) = 0$, respectively.*

4 Optimal sample size

Here, we seek an optimal value for s by minimizing $P^\star = \max\{M_1^\star, M_2^\star\}$ with a constraint on the $P^D = \min\{D_1, D_2\}$ where D_1 and D_2 are called *decisive* and *correct* evidences, defined by $D_1 = P(\lambda > k|H_1 \text{ is correct})$ and $D_2 = P(\lambda < 1/k|H_2 \text{ is correct})$, respectively.
Notice that, $D_i + M_i + W_i = 1$, for $i = 1, 2$. Based on the available data (2) and under the CPHR model, the *decisive* and *correct* evidences are given by, respectively

$$D_1 = F_{\chi_{2rs}}\left(\frac{2\ln(\tau^{sr}/k)}{1-\tau^{-1}}\right), \tag{16}$$

and

$$D_2 = 1 - F_{\chi_{2rs}}\left(\frac{2\ln(k\tau^{sr})}{\tau-1}\right). \tag{17}$$

Notice that P^D is free of the sample size (n) and the parameter vector $\boldsymbol{\alpha}= (\alpha_1, \cdots, \alpha_r)$ in the CPHR model.
As mentioned by De Santis (2004), a sample size that guarantees P^D reaches a desired level ξ, is often enough to also bound the probabilities of weak and misleading evidences. Hence, for chosen $\xi(0, 1)$ and k, we then need to solve the following optimization problem:

$$s^\star = \min\{s \geq 1 : P^D \geq \xi\}. \tag{18}$$

Table 1: Optimal sample size for some selected values of r, k, τ and ξ.

			ξ				
r	k	τ	0.7	0.8	0.9	0.95	0.99
3	3	2	4	5	8	11	19
		5	1	1	2	3	4
		8	1	1	1	2	3
	8	2	6	7	10	13	21
		5	2	2	2	3	4
		8	1	1	2	2	3
5	3	2	2	3	5	7	11
		5	1	1	1	2	3
		8	1	1	1	1	2
	8	2	4	5	6	8	13
		5	1	1	2	2	3
		8	1	1	1	1	2
15	3	2	1	1	2	3	4
		5	1	1	1	1	1
		8	1	1	1	1	1
	8	2	2	2	2	3	5
		5	1	1	1	1	1
		8	1	1	1	1	1

Table 1 presents the values of the optimal sample size $s^\star$ given by (18) for some selected values of n, r, k, τ and ξ. From Table 1, one can empirically see that the optimal value $s^\star$ is non-decreasing (non-increasing) in ξ and k (in τ and r), as we expected.

5 Conclusions and further remarks

Here, we considered statistical evidences in independent SOS arising from exponential populations. Weak and misleading evidences for simple hypotheses about the population parameter were derived in explicit expressions under the CPHR model. We assumed that the parameter vector $\boldsymbol{\alpha}= (\alpha_1, \cdots, \alpha_r)$ of the CPHR model is known. One can see that the measure λ given by (11) does not depend on the vector $\boldsymbol{\alpha}$. Hence, our findings in the preceding sections hold for the cases when the vector $\boldsymbol{\alpha}$ in the CPHR model is unknown. Also, we show that the optimal sample size $s^\star$ given by (18) is free of the vector $\boldsymbol{\alpha}$ in the CPHR model and n. The results of this paper may be extended in some directions. For example, one may consider statistical evidence for composite hypotheses. To do this, new measure of supports needs to be developed. Also, one may consider other lifetime distributions such as Pareto and Log-normal distributions.

Works in theses directions are currently studied by the authors and we hope to report findings in a future paper.

References

Balakrishnan, N. Kamps, U. and Kateri, M. (2012). A Sequential Order Statistics Approach to Step-Stress Testing. *Annals of the Institute of Statistical Mathematics,* **64**(2), 302–318.

Berger, J. (1985). *Statistical Decision Theory and Bayesian Analysis*. Springer-Verlag, New York.

Bedbur, S. (2010). UMPU Tests based on Sequential order statistics. *Journal of Statistical Planning and Inference,* **140**, 2520–-2530.

Beutner, E. and Kamps, U. (2009). Order restricted statistical inference for scale parameters based on sequential order statistics. *Journal of Statistical Planning and Inference*, **139**, 2963–2969.

Blume, J.D. (2002). Likelihood methods for measuring statistical evidence. *Statistics in Medicine*, **21**(17), 2563–2599.

Blume, J.D. (2011). Likelihood and its evidential framework. *In: Gabbay DM, Woods J (eds) Handbook of the philosophy of science: philosophy of statistics. North Holland, San Diego*, 493–511.

Burkschat, M. and Navarro, J. (2011). Aging properties of sequential order statistics. *Probability in the Engineering and Informational Science*, **25**, 449–467.

Cramer, E. and Kamps, E. (1996). Sequential order statistics and k-out-of-n systems with sequentially adjusted failure rates. *Annals of the Institute of Statistical Mathematics*, **48**(3), 535–549.

Cramer, E. and Kamps, U. (2001a). Estimation with sequential order Statistics from exponential distributions. *Annals of the Institute of Statistical Mathematics*, **53**(2), 307–324.

Cramer, E. and Kamps, U. (2001b). Sequential k-out-of-n systems. In Balakrishnan, N. and Rao, E. editors, *Handbook of Statistics, Advances in Reliability*, **20**(12), 301–372.

Cramer, E. and Kamps, U. (2003). Marginal distributions of sequential and generalized order statistics. *Metrika*, **58**, 293–310.

De Santis, F. (2004). Statistical evidence and sample size determination for Bayesian hypothesis testing. *Journal of Statistical Planning and Inference*, **34**(124), 121–144.

Doostparast, M. and Emadi, M. (2006). Statistical evidence methodology for model acceptance based on record values. *Journal of the Korean Statistical Society*, **35**(2), 167–177.

Doostparast, M. and Emadi, M. (2014). Evidential inference and optimal sample size determination on the basis of record values and record times under random sampling scheme. *Statistical Methods and Applications*, **23**, 41–50.

Esmailian, M. and Doostparast, M. (2014). Estimation based on sequential order statistics with random removals. *Probability and Mathematical Statistics*, **34**(1), 81–95.

Hashempour, M. and Doostparast, M. (2016). Statistical evidences in sequential order statistics arising from a general family of lifetime distributions. *ISTATISTIK, Journal of the Turkish Statistical Association*, **9**(1), 29–41.

Hashempour, M. and Doostparast, M. (2017). Bayesian inference on multiply sequential order statistics from heterogeneous exponential populations with GLR test for homogeneity. *Communications in Statistics-Theory and Methods*, **46**(16), 8086-8100.

Kamps, U. (1995). A concept of generalized order statistics. *Journal of Statistical Planning and Inference*, **48**, 1–23.

Royall, R. (1997). *Statistical Evidence: A Likelihood Paradigm*. Chapman and Hall, New York.

Royall, R. (2000). On the probability of observing misleading statistical evidence, *Journal of the American Statistical Association*, **95**, 760–780.

Schenk, N., Burkschat, M., Cramer, E. and Kamps, U. (2011). Bayesian Estimation and Prediction with Multiply Type-II Censored Samples of Sequential Order statistics from one- and two-Parameter Exponential Distributions. *Journal of Statistical Planning and Inference*, **141**, 1575–1587.

Shafay, A.R., Balakrishnan, N. and Sultan, K.S. (2014). Two-sample Bayesian prediction for sequential order statistics from exponential distribution based on multiply Type-II censored samples. *Journal of Statistical Computation and Simulation*, **84**(3), 526–544.

Appendix

Proof of Proposition 3.2: By Remark 3.1 and Equations (4), (5) and (11), we have

$$\begin{aligned}
M_2^\star &= P\left(\left(\frac{\sigma_2}{\sigma_1}\right)^{sr}\exp\left\{-\left(\frac{1}{\sigma_1}-\frac{1}{\sigma_2}\right)\sum_{i=1}^{s}\sum_{j=1}^{r}x_{ij}m_j\right\} > k \,\middle|\, \sigma=\sigma_2\right)\\
&= P\left(\sum_{i=1}^{s}\sum_{j=1}^{r}x_{ij}m_j < \left(\frac{\ln\left(\left(\frac{\sigma_2}{\sigma_1}\right)^{sr}/k\right)}{\frac{1}{\sigma_1}-\frac{1}{\sigma_2}}\right)\,\middle|\, \sigma=\sigma_2\right)\\
&= P\left(\chi_{2rs} < \left(\frac{2\ln\left(\left(\frac{\sigma_2}{\sigma_1}\right)^{sr}/k\right)}{\sigma_2\left(\frac{1}{\sigma_1}-\frac{1}{\sigma_2}\right)}\right)\,\middle|\, \sigma=\sigma_2\right)\\
&= F_{\chi_{2rs}}\left(\frac{2\ln\left(\left(\frac{\sigma_2}{\sigma_1}\right)^{sr}/k\right)}{\frac{\sigma_2}{\sigma_1}-1}\right)\\
&= F_{\chi_{2rs}}\left(\frac{2\ln(\,\tau^{sr}/k)}{\tau-1}\right),
\end{aligned}$$

Similarly

$$W_1^\star = P\left(\frac{1}{k}<\lambda<k\mid H_1\right) = F_{\chi_{2rs}}\left(\frac{2\ln\left(k\left(\frac{\sigma_2}{\sigma_1}\right)^{sr}\right)}{1-\frac{\sigma_1}{\sigma_2}}\right) - F_{\chi_{2rs}}\left(\frac{2\ln\left(\frac{\left(\frac{\sigma_2}{\sigma_1}\right)^{sr}}{k}\right)}{1-\frac{\sigma_1}{\sigma_2}}\right),$$

and

$$W_2^\star = P\left(\frac{1}{k}<\lambda<k\mid H_2\right) = F_{\chi_{2rs}}\left(\frac{2\ln\left(k\left(\frac{\sigma_2}{\sigma_1}\right)^{sr}\right)}{\frac{\sigma_2}{\sigma_1}-1}\right) - F_{\chi_{2rs}}\left(\frac{2\ln\left(\frac{\left(\frac{\sigma_2}{\sigma_1}\right)^{sr}}{k}\right)}{\frac{\sigma_2}{\sigma_1}-1}\right).$$