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Preliminary test estimation in Rayleigh distribution under a squared-log error loss

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Abstract: The problem of pretest estimation in Rayleigh type-II censored data under the squared-log error loss (SLEL) is considered. The risk-unbiased estimator is derived and its risk is computed under the SLEL. The pretest estimator based on a point guess about the parameter of interest is constructed and the bias and risk is computed. A comparison study is performed between the pretest estimator and the risk-unbiased estimator. The optimal level of significance and critical values of pretest is obtained using regret minimax method. A real data set is used for illustrative purposes.

Keywords: Censored data; Pretest estimators; Rayleigh distribution; Squared log error loss.
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1 Introduction

In some situations, the experimenter has a prior point information θ_0 about θ, the parameter of interest, which is available from past investigation or any other sources. For example, a producer considers the failure times (in minutes) for a sample of electronic components in an accelerated life test. He/she can estimate the mean life in classical methods by natural estimator such as maximum likelihood estimator (MLE) or unbiased estimator or in Bayesian perspective by employing a flexible prior distribution for the parameter of interest. Suppose that he/she knows that the mean life of electronic components products is close to 50 minutes. This information can be used for a pretest estimation in the hope that it will perform better than the natural estimator. The problem of pretest estimation has received significant attention in recent years, see Baklizi (2005), Baklizi (2008), Mirfarah and Ahmadi (2014), Balaghi et al. (2015) and Baklizi et al. (2016) among others.

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Let X has a one parameter Rayleigh distribution with probability density function (p.d.f.)

$$f(x|\theta) = \frac{x}{\theta} \exp\left\{-\frac{x^2}{2\theta}\right\}, \quad x > 0. \tag{1}$$

In this paper, we deal with pretest estimation in Rayleigh distribution based on type-II censored data under the SLEL function. We provide some preliminaries about loss function, censored data and the risk-unbiased estimator in Section 2. Section 3 is concerned to construct the pretest estimators on the basis of the risk-unbiased estimator of θ. The bias of the pretest estimator is computed numerically and plotted for various cases. After calculating the risk of the pretest estimator, we compare it with respect to the risk-unbiased estimator via relative efficiency. The optimal levels of significance are obtained using regret minimax criterion. In Section 4, a numerical example is presented for illustrative purposes. We end the paper with some concluding remarks.

2 Some preliminaries

2.1 Loss function

Consider the SLEL function for estimating θ as

$$L(\theta, \delta) = (\log \delta - \log \theta)^2 = \log^2\left(\frac{\delta}{\theta}\right) = \log^2 \Delta. \tag{2}$$

This loss is proposed by Brown (1968); see also Pal and Ling (1996). The SLEL is not symmetric and convex; it is convex for $\Delta = \frac{\delta}{\theta} \leq e$ and concave otherwise, but has a unique minimum at $\Delta = 1$. Also when $\Delta > 1$, this loss increases sublinearly, while when $0 < \Delta < 1$, it rises rapidly to infinity at zero. The SLEL function is useful in situations where underestimation is more serious than overestimation. Zellner (1986) stated that in dam construction, an underestimation of the peak water level is usually much more serious than an overestimation. Sanjari and Zakerzadeh (2005), Kiapour and Nematollahi (2011), Naghizadeh Qomi and Barmoodeh (2015) and Naghizadeh Qomi (2017) used this loss for some estimation problems.

2.2 Type-II censored data

Suppose that n units are placed on test simultaneously and the test terminates immediately after r units have failed. r is fix, is chosen before the data are collected and the length of experiment is a random variable. The following lemma is a key for future derivations of the paper.

Lemma 2.1. (Arnold et al. (1998)) Let $X_{1:n}, \cdots, X_{r:n}$ be the observed failure times for the first r units under the Rayleigh model (1). Then, we have:

(i) The likelihood function of $\mathbf{X} = (X_{1:n}, \cdots, X_{r:n})$ at $\mathbf{x} = (x_{1:n}, \cdots, x_{r:n})$ is given by

$$L(\theta|\mathbf{x}) = \frac{n! \prod_{i=1}^{r} x_{i:n}}{(n-r)! \theta^r} \exp\left\{-\frac{\sum_{i=1}^{r} x_{i:n}^2 + (n-r)x_{r:n}^2}{2\theta}\right\}. \tag{3}$$

(ii) The MLE of θ, denoted by $\hat{\theta}$ is given by

$$\hat{\theta} = \frac{\sum_{i=1}^{r} X_{i:n}^2 + (n-r)X_{r:n}^2}{2r}. \tag{4}$$

(iii) The spacings $Z_i = (n-i+1)(X_{i:n}^2 - X_{i-1:n}^2)/\theta$ for $i = 1, ..., r$ $(X_{0:n} \equiv 0)$ constitute a random sample of χ_2^2 random variable and then $2r\hat{\theta}/\theta \sim \chi_{2r}^2$.

2.3   A risk-unbiased estimator of θ

Following Lehmann (1951) an estimator $\tilde{\theta}$ of θ is said to be risk-unbiased under the loss function $L(\theta, \tilde{\theta})$ if it satisfies

$$E[L(\theta, \tilde{\theta})] \leq E[L(\theta', \tilde{\theta})], \quad \forall \theta' \neq \theta. \tag{5}$$

With respect to SLEL, the estimator $\tilde{\theta}$ of θ is risk-unbiased if $E[\log \tilde{\theta}] = \log \theta$, $\forall \theta$ or equivalently $E[\log(\tilde{\theta}/\theta)] = 0$, $\forall \theta$, see Naghizadeh Qomi and Barmoodeh (2015). It is easy to check that if $X_{1:n}, \cdots, X_{r:n}$ denote the type-II censored data from the Rayleigh model with p.d.f. given in (1), then the estimator $\hat{\theta}_{RU} = re^{-\Psi(r)}\hat{\theta}$ is a risk-unbiased for θ under the SLEL, where $\hat{\theta}$ is the MLE of θ, $\Psi(r) = \frac{d}{dr}\ln\Gamma(r) = \frac{\Gamma'(r)}{\Gamma(r)}$ is the digamma function and $\Gamma(r)$ denotes the complete gamma function given by $\Gamma(r) = \int_0^\infty t^{r-1}e^{-t}dt$. Moreover, the risk of the risk-unbiased estimator under the SLEL is $\Psi'(r)$, where $\Psi'(r) = \frac{d}{dr}\Psi(r)$ is the trigamma function, for detail, see Naghizadeh Qomi (2017).

3   Pretest estimation

Assume that we have a priori θ_0 about θ. A pretest for

$$\begin{cases} H_0 : \theta = \theta_0 \\ H_1 : \theta \neq \theta_0, \end{cases} \tag{6}$$

may be performed for checking that θ is near to θ_0. For testing (6), the likelihood ratio test statistic is $U = 2r\hat{\theta}/\theta \sim \chi_{2r}^2$, then, H_0 is rejected at the level of significance α, if $U < q_1$ or $U > q_2$, where $q_1 = \chi_{\frac{\alpha}{2},2r}^2$ and $q_2 = \chi_{1-\frac{\alpha}{2},2r}^2$ are the values of the lower and upper $100\alpha/2\%$ points of a chi-square distribution with $2r$ degrees of freedom.

The pretest estimator is

$$\hat{\theta}_{PT} = \begin{cases} \theta_0 & q_1\delta \leq U \leq q_2\delta \\ \hat{\theta}_{RU} & U < q_1\delta \text{ or } U > q_2\delta, \end{cases} = \hat{\theta}_{RU} + (\theta_0 - \hat{\theta}_{RU})I(A), \tag{7}$$

where $\delta = \theta_0/\theta$ and $A = \{U : q_1\delta \leq U \leq q_2\delta\}$.

3.1   Bias of $\hat{\theta}_{PT}$

Considering $d = e^{-\Psi(r)}$ we get $\hat{\theta}_{RU}/\theta = dU/2$ and then the Bias of $\hat{\theta}_{PT}$ under the SLEL is given by

$$B_{\hat{\theta}_{PT}}(\delta) = E\left[\log\left(\frac{\hat{\theta}_{PT}}{\theta}\right)\right]$$

$$= E\left[\log\left(\frac{\hat{\theta}_{RU} + (\theta_0 - \hat{\theta}_{RU})I(A)}{\theta}\right)\right]$$

$$= E\left[\log\left(\frac{dU}{2} + (\delta - \frac{dU}{2})I(A)\right)\right]$$

$$= \int_0^\infty \left[\log\left(\frac{du}{2} + (\delta - \frac{du}{2})I(A)\right)\right]g(u)du$$

$$= \int_A \log\delta \; g(u)du + \int_{A^c} \log\left(\frac{du}{2}\right)g(u)du$$

$$= \int_0^\infty \log\left(\frac{du}{2}\right)g(u)du + \int_{q_1\delta}^{q_2\delta}\left[\log\delta - \log\left(\frac{du}{2}\right)\right]g(u)du$$

$$= 0 + \int_{q_1\delta}^{q_2\delta}\left[\log\delta - \log\left(\frac{du}{2}\right)\right]g(u)du$$

$$= \int_{q_1\delta}^{q_2\delta}\left[\log\delta - \log\left(\frac{du}{2}\right)\right]g(u)du. \tag{8}$$

Figure 1 shows the values of bias $B_{\hat{\theta}_{PT}}(\delta)$ for selected values of $r = 4(2)10$ and $\alpha = 0.01, 0.05, 0.1, 0.2$ with respect to δ. We observe that the bias may be negative, zero or positive, then we can state that the pretest estimator $\hat{\theta}_{PT}$ may be negatively risk-biased, risk-unbiased or positively risk-biased. Also note that $B_{\hat{\theta}_{PT}}(\delta)$ goes to zero when $\delta \to 0$ or 1 or ∞ or equivalently $\theta_0 \to 0$ or θ or ∞. The values δ_{min}, δ_{max} and $\delta_0 \in (\delta_{min}, \delta_{max})$ where the pretest estimator has the minimum, maximum and zero bias, respectively, are summarized in Table 1 for selected values of r and α.

Table 1: The values δ_{min}, δ_0 and δ_{max} of δ where the pretest estimator has minimum, zero and maximum bias.

| r	$\alpha = 0.01$			$\alpha = 0.05$			$\alpha = 0.1$			$\alpha = 0.2$		
	δ_{min}	δ_0	δ_{max}	δ_{min}	δ_0	δ_{max}	δ_{min}	δ_0	δ_{max}	δ_{min}	δ_0	δ_{max}
4	0.4757	1.0038	3.4721	0.5371	1.0146	2.0109	0.5683	1.0277	2.2165	0.6058	1.0587	2.0109
6	0.5346	1.0024	2.5876	0.5939	1.0094	2.0416	0.6228	1.0179	1.8657	0.6554	1.0380	1.7360
8	0.5751	1.0018	2.2097	0.6322	1.0069	1.8239	0.6592	1.0132	1.6951	0.6888	1.0281	1.5978
10	0.6054	1.0014	1.9972	0.6605	1.0055	1.6951	0.6862	1.0105	1.5918	0.7136	1.0222	1.5125
15	0.6576	1.00091	1.7232	0.7084	1.0037	1.5208	0.7314	1.0069	1.4492	0.7552	1.0147	1.3928
20	0.6919	1.0007	1.5862	0.7395	1.0027	1.4296	0.7606	1.0051	1.3731	0.7819	1.0109	1.3279
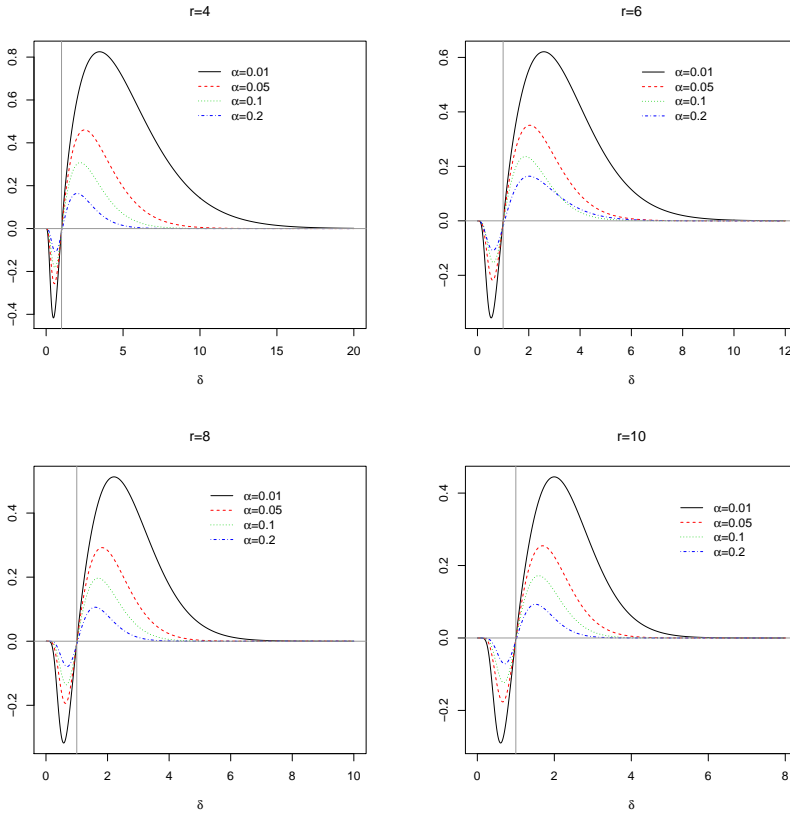
Figure 1: Bias of the pretest estimators for selected values of r and α in δ.

3.2 Risk of $\hat{\theta}_{PT}$

Going along the similar lines for deriving the bias in (8), the risk of $\hat{\theta}_{PT}$ as a function of δ and α under the SLEL is given by

$$
\begin{aligned}
R(\theta, \hat{\theta}_{PT}) &= E\left[\log^2\left(\frac{\hat{\theta}_{PT}}{\theta}\right)\right] = E\left[\log^2\left(\frac{\hat{\theta}_{RU} + (\theta_0 - \hat{\theta}_{RU})I(A)}{\theta}\right)\right] \\
&= \int_0^\infty \left[\log^2\left(\frac{du}{2} + (\delta - \frac{du}{2})I(A)\right)\right]g(u)du \\
&= \int_0^\infty \log^2\left(\frac{du}{2}\right)g(u)du + \int_{q_1\delta}^{q_2\delta}\left[\log^2\delta - \log^2\left(\frac{du}{2}\right)\right]g(u)du \\
&= \int_{q_1\delta}^{q_2\delta}\left[\log^2\delta - \log^2\left(\frac{du}{2}\right)\right]g(u)du + \Psi'(r). \qquad (9)
\end{aligned}
$$

For comparison of $\hat{\theta}_{PT}$ and $\hat{\theta}_{RU}$, the relative efficiency (RE) is calculated as

$$
RE(\hat{\theta}_{PT}, \hat{\theta}_{RU}) = \frac{R(\theta, \hat{\theta}_{RU})}{R(\theta, \hat{\theta}_{PT})}, \qquad (10)
$$

Table 2: The values of δ_{max} and the range of δ that $\hat{\theta}_{PT}$ dominates $\hat{\theta}_{RU}$.

	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.1$		$\alpha = 0.2$	
r	δ_{max}	$[\delta_1, \delta_2]$	δ_{max}	$[\delta_1, \delta_2]$	δ_{max}	$[\delta_1, \delta_2]$	δ_{max}	$[\delta_1, \delta_2]$
4	1.0026	[0.6179,1.6214]	1.0085	[0.6364,1.5428]	1.0146	[0.6484,1.5077]	1.0277	[0.6674,1.4845]
6	1.0018	[0.6805,1.4719]	1.0056	[0.6974,1.4173]	1.0096	[0.7077,1.3926]	1.0181	[0.7228,1.3751]
8	1.0013	[0.7189,1.3928]	1.0042	[0.7346,1.3497]	1.0071	[0.7439,1.3300]	1.0134	[0.7568,1.3155]
10	1.001	[0.7457,1.3425]	1.0034	[0.7605,1.3062]	1.0056	[0.7689,1.2895]	1.0106	[0.7804,1.2768]
15	1.0007	[0.7885,1.2693]	1.0022	[0.8014,1.2423]	1.0037	[0.8087,1.2297]	1.0071	[0.8179,1.2197]
20	1.0005	[0.8146,1.2283]	1.0017	[0.8263,1.2062]	1.0028	[0.8328,1.1958]	1.0053	[0.8408,1.1872]
30	1.0003	[0.8464,1.1819]	1.0011	[0.8565,1.165]	1.0019	[0.8619,1.1569]	1.0035	[0.8685,1.1501]

and plotted in Figure 2 for selected values of $r = 4(2)10$ and $\alpha = 0.01, 0.05, 0.1, 0.2$ with respect to δ. It is observed that when the guess value is near to the true value of the parameter, i.e., $\delta \approx 1$, the pretest estimator $\hat{\theta}_{PT}$ with small α is better than the risk-unbiased estimator $\hat{\theta}_{RU}$. Moreover, for δ closer to one and fix values of r, the pretest estimators with smaller α perform better than other pretest estimators. Also, the pretest estimator with larger r is preferable when α is fixed and $\delta = 1$, see Figure 3. The value of δ_{max} (the value of δ with maximum RE) and the range of δ that $\hat{\theta}_{PT}$ dominates $\hat{\theta}_{RU}$ are given in Table 2.

3.3   Optimal level of significance

As we mentioned in the previous subsection, the risk of $\hat{\theta}_{PT}$ in (9), is a function of α (through q_1 and q_2) and δ, then we consider the risk by $R_{PT}(\delta, \alpha)$, hereafter. From (9) we get $R_{PT}(\delta, \alpha) \to R_{PT}(\delta, 1)$ when δ goes to 0 or ∞. This can be observed from Figure 2. We obtain the roots of the equation $R_{PT}(\delta, 0) = R_{PT}(\delta, 1)$, or equivalently $\log^2 \delta = \Psi'(r)$, which are $\delta_1 = e^{-\sqrt{\Psi'(r)}}$ and $\delta_2 = e^{\sqrt{\Psi'(r)}}$. Then, an optimal value of α is $\alpha = 0$ if $\delta_1 \leq \delta \leq \delta_2$ and $\alpha = 1$ otherwise. For finding an optimal value of α with reasonable risk $R_{PT}(\delta, \alpha)$, we use the regret function $REG(\delta, \alpha) = R_{PT}(\delta, \alpha) - \inf_\alpha R_{PT}(\delta, \alpha)$ with

$$\inf_\alpha R_{PT}(\delta, \alpha) = \begin{cases} R_{PT}(\delta, 0) & e^{-\sqrt{\Psi'(r)}} < \delta < e^{\sqrt{\Psi'(r)}} \\ R_{PT}(\delta, 1) & \delta \leq e^{-\sqrt{\Psi'(r)}} \text{ or } \delta \geq e^{\sqrt{\Psi'(r)}}. \end{cases}$$

We explore the value of α, say α_{opt}, so that $REG(\delta_L, \alpha_{opt}) = REG(\delta_U, \alpha_{opt})$, where δ_L and δ_U are the values which $REG(\delta, \alpha)$ takes a maximum for $\delta < e^{\sqrt{\Psi'(r)}}$ and $\delta > e^{\sqrt{\Psi'(r)}}$, respectively. Figure 4, show the shapes of $REG(\delta, \alpha)$ for $r = 2, 4, 6, 8$.

The values of α_{opt}, δ_L, δ_U, q_1 and q_2 are summarized in Table 3 for $r = 2(1)20$, where q_1 and q_2 are $100\frac{\alpha_{opt}}{2}$ left quantiles of chi-square distribution with $2r$ degrees of freedom. For example, if $r = 6$, then $\delta_1 = 0.6532$ and $\delta_2 = 1.5308$. For $\delta \leq 1.5308$, $REG(\delta, \alpha)$ takes a maximum value at $\delta_L = 0.9567$ and For $\delta > 1.5308$, $REG(\delta, \alpha)$ takes a maximum value at $\delta_U = 2.1631$. Therefore, the optimal value of α is $\alpha_{opt} = 0.1959$.
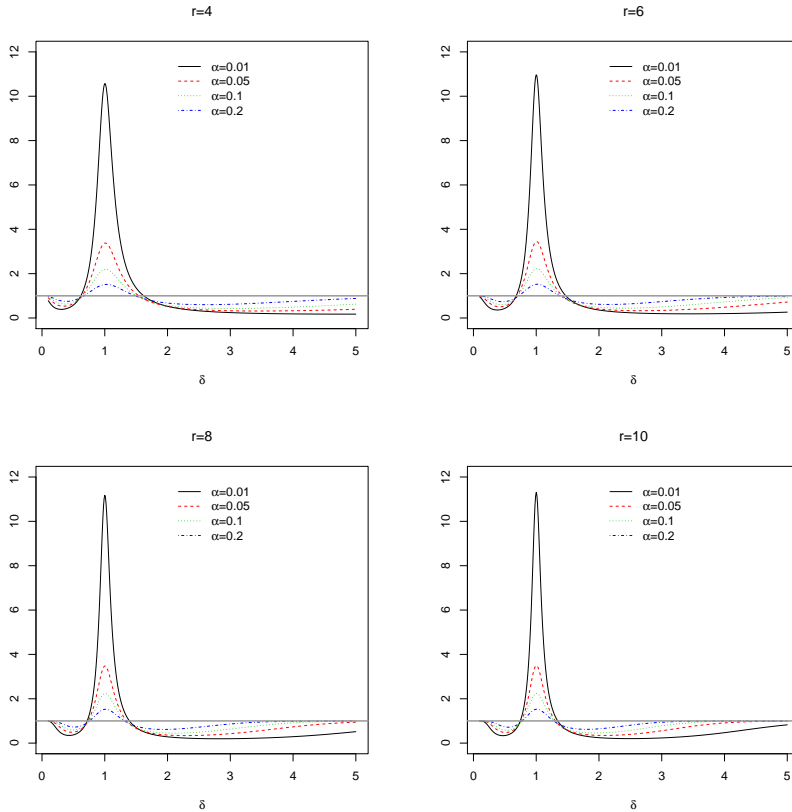
Figure 2: RE between the pretest and the risk-unbiased estimator for selected values of $r = 4(2)10$ and $\alpha = 0.01, 0.05, 0.1, 0.2$ with respect to δ.

4 An illustrative example

The following data set due to Lawless (2003)

```
1.4, 5.1, 6.3, 10.8, 12.1, 18.5, 19.7, 22.2,
23, 30.6, 37.3, 46.3, 53.9, 59.8, 66.2.
```

are the failure times (in minutes) for a sample of 15 electronic components in an accelerated life test. A Kolmogorov-Smirnov (K-S) test with the test statistic $D = 0.2341$ and a corresponding $p - value = 0.3837$ shows adequacy of the fitness of the Rayleigh distribution with $\theta = 580.5973$, see MirMostafaee et al. (2016).

Assume that all 15 components were placed on test and we have failed to observe the last ten ordered data so that $r = 10$ and $n = 15$. The MLE of θ is $\hat{\theta} = 385.053$. Also, the risk-unbiased estimator of θ is $\hat{\theta}_{RU} = (10)e^{-\Psi(10)}(385.053) = 405.1322$ with corresponding risk $R(\theta, \hat{\theta}_{RU}) = \Psi'(10) = 0.1052$. From Table 3, the optimal value of α is $\alpha_{opt} = 0.1914$ with quantiles $q_1 = \chi^2_{0.0957,20} = 12.3286$ and $q_2 = \chi^2_{0.9043,20} = 28.6112$. Consider the point guess values $\theta_0 \in \{100, 400, 700\}$ for true value θ. The test
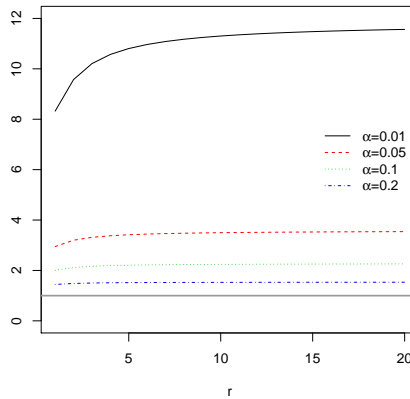
Figure 3: RE between the pretest and the risk-unbiased estimator for selected values of $\alpha = 0.01, 0.05, 0.1, 0.2$ with respect to r when $\delta = 1$.

Table 3: The values of optimal α, δ_L, δ_U and quantiles q_1 and q_2 for selected values of $r = 2(1)19$.

	r								
	2	3	4	5	6	7	8	9	10
α_{opt}	0.2061	0.2024	0.1997	0.1975	0.1959	0.1945	0.1933	0.1923	0.1914
δ_L	0.8873	0.9207	0.9382	0.9492	0.9567	0.9622	0.9664	0.9697	0.9724
δ_U	4.0902	3.0611	2.6023	2.3376	2.1630	2.0381	1.9434	1.8689	1.8084
q_1	1.0830	2.2159	3.4875	4.8456	6.2666	7.7333	9.2366	10.7702	12.3286
q_2	7.7038	10.6101	13.3663	16.0309	18.6264	21.1742	23.6834	26.1602	28.6112

	r								
	11	12	13	14	15	16	17	18	19
α_{opt}	0.1906	0.1899	0.1893	0.1887	0.1882	0.1878	0.1873	0.1869	0.1866
δ_L	0.9747	0.9766	0.9782	0.9796	0.9808	0.9819	0.9829	0.9837	0.9845
δ_U	1.7582	1.7157	1.6791	1.6472	1.6192	1.5943	1.5720	1.5519	1.5336
q_1	1.033334	2.1535	3.4194	4.7756	6.1956	7.6637	9.16762	10.7025	12.2637
q_2	7.9002	10.7941	13.5373	16.1889	18.7751	21.3120	23.8138	26.2835	28.7258

statistic under $H_0 : \theta = \theta_0$ is $U = 2r\hat{\theta}/\theta_0$. We obtain $U = 77.0106, 19.2526, 11.0015$ corresponding to the values θ_0. For deriving the risk of pretest estimator, we need to estimate δ. We have two estimate of δ at hand. The first is the MLE of δ of the form $\hat{\delta}_{ML} = \theta_0/\hat{\theta}$. The second is the risk-unbiased estimate of δ as $\hat{\delta}_{RU} = \theta_0/\hat{\theta}_{RU}$, because

$$
\begin{aligned}
E[\log \hat{\delta}_{RU}] = E\left[\log \left(\frac{\theta_0}{\hat{\theta}_{RU}} \right) \right] &= E[\log \theta_0] - E[\log \hat{\theta}_{RU}] \\
&= \log \theta_0 - \log \theta = \log \delta.
\end{aligned}
$$

Table 4 shows the risk of $\hat{\theta}_{RU}$ and the bias and the risk of $\hat{\theta}_{PT}$. It is observed from
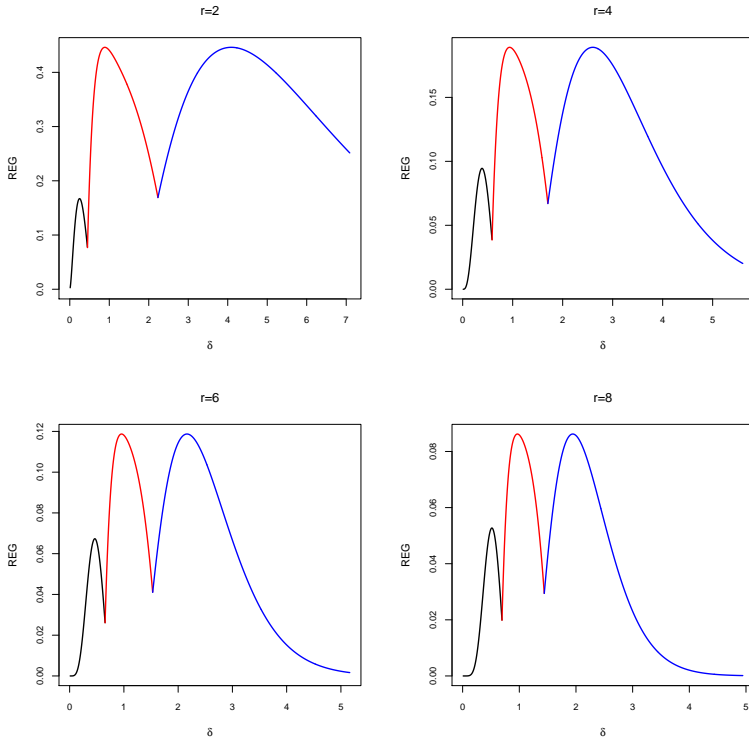
Figure 4: The shapes of $REG(\delta, \alpha)$ for selected values $r = 2(2)8$.

this table that the pretest estimator corresponding to $\theta_0 = 400$ has smaller risk than the risk-unbiased estimator.

5   Concluding remarks

In this paper, we consider the problem of constructing the pretest estimators for the scale parameter of a Rayleigh distribution based on censored data under the SLEL function. The risk-unbiased estimator is derived under the SLEL and the bias and risk of them plotted for different cases. Comparisons between the pretest estimator and the risk-unbiased estimator show that the pretest estimator is better than the risk-unbiased estimator when the point guess is near to the true parameter ($\delta \to 1$).

The key question in pretest estimation is selection of guess value θ_0. A choice is to select the θ_0 using a form of inference, called fiducial inference, see Casella and Berger (2001). In this inference, $M(\mathbf{x})L(\theta|\mathbf{x})$ interprets as a p.d.f. for θ, where $M(\mathbf{x}) = (\int_{-\infty}^{\infty} L(\theta|\mathbf{x})d\theta)^{-1}$. After finding the fiducial distribution of θ, the mean of distribution can be considered as θ_0, which is an empirical estimation of θ.

Table 4: The risk of $\hat{\theta}_{RU}$ and the bias and risk of $\hat{\theta}_{PT}$ for selected values of $\theta_0 = 100, 400, 700$.

θ_0			$R(\theta, \hat{\theta}_{RU})$	$B(\hat{\delta})$	$R(\hat{\delta}, \alpha_{opt})$
100	$\hat{\delta}_{RU}$	0.24	0.1052	-0.0008	0.1075
	$\hat{\delta}_{ML}$	0.26	0.1052	-0.0013	0.1084
400	$\hat{\delta}_{RU}$	0.98	0.1052	-0.0148	0.0677
	$\hat{\delta}_{ML}$	1.04	0.1052	0.0068	0.0679
700	$\hat{\delta}_{RU}$	1.73	0.1052	0.0867	0.1714
	$\hat{\delta}_{ML}$	1.82	0.1052	0.0779	0.1728

Acknowledgement

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