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Research Paper

Bayesian D-optimal design for a quadratic beta regression model with a known nuisance parameter considering prior uniform and normal distributions for parameters

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Abstract: One of the practical and important issue in statistics is the fitness of regression models. Optimal design is a way to obtain suitable fitness of this type of models. In addition, we need to use some criteria for attaining optimal design in regression models. The D-optimality criterion is one of the most famous criteria which is used here. An appropriate method to obtain the optimal designs is the Bayesian method that need to the prior distribution for the parameters of the model (coefficient regression). In this paper, by using Bayesian methods, D-optimal designs are obtained for quadratic beta regression model. Also, uniform and normal distributions are considered as the prior distributions and obtained results are analyzed.

Keywords: Bayesian D-optimal design; Beta regression model; D-optimality criterion; Fisher information matrix.
Mathematics Subject Classification (2010): 62K05.

1 Introduction

In nonlinear models, Fisher information matrix is dependent on unknown parameters, which causes the dependence of optimality criteria to the parameters of the model because the considered criterion in this paper is a function of the information matrix. However, the purpose of presenting an appropriate design is to find optimal points, collect data, and estimate unknown parameters of the model. So dependency of optimality criteria to unknown parameters causes a kind of conflict. The first and the easiest solution is to replace unknown parameters with the initial guesses. The designs

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obtained by this method are called locally optimal designs (Chernoff, 1953). Selection of a suitable initial guess greatly affects the efficiency of local optimal design. If the initial guess is far from its actual value, the optimal design which is obtained will not be suitable. Furthermore, the parameter space often consists of an infinite number of points, which cannot be considered all and only a limited number is investigated. Thus, other methods were recommended to solve the problem such as Bayesian optimality criteria. Instead of an initial guess, a prior distribution is used for unknown parameters of the model in Bayesian method. The Bayesian optimal design will be obtained by maximizing an appropriate optimality criterion based on a prior distribution. Lindley (1972) proposed the decision theory and provided a mathematical foundation for the selection of optimal Bayesian designs. Chaloner and Larntz (1989) presented a unified theory for the Bayesian optimal design in nonlinear models. Chaloner and Verdinelli (1995) conducted a general overview of Bayesian optimal designs.

This paper calculates the Bayesian D-optimal design of the quadratic beta regression model, when the nuisance parameter is known. In addition, uniform and normal prior distributions for unknown parameters are considered. Also, locally D-optimal design for this model was proposed by Latif and ZafarYab (2014).

Given the statements, this paper is organized as follows: The beta regression model is introduced in section 2 and the information matrix is calculated in section 3. The Bayesian D-optimal design and the equivalence theorem for quadratic beta regression model are introduced in section 4 and 5, respectively. Section 6 is dedicated to the calculation of the Bayesian D-optimal design in the beta regression model and discussion and conclusion obtained from the paper is provided in section 7.

2 Beta Regression Model

The beta random variable has a probability distribution with two parameters. This variable is defined on the interval [0,1]. If $Y_i \sim Beta(p,q); i = 1,2,...,n$ and n is the number of observations, then the its density function is as follows;

$$f_{p,q}(y_i) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} y_i^{p-1}(1-y_i)^{q-1} \quad ; y_i \in (0,1), p,q > 0. \tag{1}$$

Consider $\phi = p+q$, then the density function will be written as follows;

$$f_{\mu_i,\phi}(y_i) = \frac{\Gamma(\phi)}{\Gamma(\mu_i\phi)\Gamma((1-\mu_i)\phi)} y_i^{\mu_i\phi-1}(1-y_i)^{(1-\mu_i)\phi-1}; y_i \in (0,1), 0 < \mu_i < 1, \phi > 0, \tag{2}$$

where ϕ is the nuisance parameter and μ_i is the expected value of random variable Y_i, such that:

$$\mu_i = E(Y_i) = \frac{p}{p+q}.$$

Now, a link function is introduced as follows:

$$g(\mu_i) = \mathbf{f}^T(x_i)\boldsymbol{\beta},$$

where

$$\mathbf{f}^T(x_i) = (f_0(x_i), f_1(x_i), \cdots, f_{r-1}(x_i))^T, \boldsymbol{\beta} = (\beta_0, \beta_1, \cdots, \beta_{r-1})^T,$$

and $g(.), \boldsymbol{\beta}, \mathbf{f}(x_i)$ and x_i is an appropriate link function, vector of unknown parameters, given vector of functions of predictors and predictors, respectively. We assume $x_i \in [0,1]$ and since $\mu_i \in [0,1]$, the appropriate link function that can be considered in this case is the *Logit* link function, that is given by;

$$Logit(\mu_i) = \mathbf{f}^T(x_i)\boldsymbol{\beta}.$$

The considered model in this paper is a quadratic beta regression model. Thus;

$$\mathbf{f}(x_i) = (1 \quad x_i \quad x_i^2)^T; r = 3.$$

3 Fisher information matrix

Given that most of the optimality criteria are functions of information matrix (Atkinson et al., 2007), at first, it is necessary to present Fisher information matrix related to model (2). The Fisher information matrix is generally defined as follows:

$$\boldsymbol{M}(\boldsymbol{\beta}, x_i) = -E\left(\frac{\partial \ln(f_{\boldsymbol{\beta}}(y \mid x_i))}{\partial \boldsymbol{\beta} \partial \boldsymbol{\beta}^T}\right), \tag{3}$$

where $\boldsymbol{\beta}$ and $\ln(.)$ are the vector of unknown parameters and natural logarithm, respectively. Fisher information matrix is a symmetric matrix. Now, by using (3), the Fisher information matrix for quadratic beta regression model with a known nuisance parameter is as follows:

$$\boldsymbol{M}(\boldsymbol{\beta}, x_i) = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ & M_{22} & M_{23} \\ & & M_{33} \end{pmatrix}, \tag{4}$$

where

$$\begin{aligned}
M_{11} &= \phi^2 \mu_i^2 (1-\mu_i)^2 \left[\Psi^{'}(\mu_i \phi) + \Psi^{'}((1-\mu_i)\phi)\right] \\
M_{12} &= \phi^2 \mu_i^2 (1-\mu_i)^2 \left[\Psi^{'}(\mu_i \phi) + \Psi^{'}((1-\mu_i)\phi)\right] x_i \\
M_{13} &= \phi^2 \mu_i^2 (1-\mu_i)^2 \left[\Psi^{'}(\mu_i \phi) + \Psi^{'}((1-\mu_i)\phi)\right] x_i^2 \\
M_{22} &= \phi^2 \mu_i^2 (1-\mu_i)^2 \left[\Psi^{'}(\mu_i \phi) + \Psi^{'}((1-\mu_i)\phi)\right] x_i^2 \\
M_{23} &= \phi^2 \mu_i^2 (1-\mu_i)^2 \left[\Psi^{'}(\mu_i \phi) + \Psi^{'}((1-\mu_i)\phi)\right] x_i^3 \\
M_{33} &= \phi^2 \mu_i^2 (1-\mu_i)^2 \left[\Psi^{'}(\mu_i \phi) + \Psi^{'}((1-\mu_i)\phi)\right] x_i^4,
\end{aligned}$$

and $\Psi^{'}(.)$ represents trigamma function (Ferrari and Cribari-Neto, 2004).

4 Bayesian D-optimal design

ξ is an arbitrary design as follows:

$$\xi = \left\{ \begin{array}{cccc} x_1 & x_2 & \dots & x_k \\ w_1 & w_2 & \dots & w_k \end{array} \right\} \in \Xi;\ k \geq r, \tag{5}$$

where x_i, w_i, k and Ξ are the ith design point, weight of ith design point, the number of design points and a set of all possible designs, respectively. Ξ is defined as follows:

$$\Xi = \left\{ \xi \mid \sum_{i=1}^{k} w_i = 1, \quad 0 \leq w_i \leq 1 \right\}.$$

The information matrix related to design (5) is given by Atkinson et al. (2007);

$$\boldsymbol{M}(\boldsymbol{\beta}, \xi) = \sum_{i=1}^{k} w_i \boldsymbol{M}(\boldsymbol{\beta}, x_i),$$

The Bayesian D-optimality as an optimality criterion is defined as follows (Atkinson et al., 2007);

$$\Phi_\pi(\xi) = E_{\pi(\boldsymbol{\beta})}(-\log\det(\boldsymbol{M}(\boldsymbol{\beta}, \xi))) = -\int \log\det(\boldsymbol{M}(\boldsymbol{\beta}, \xi))\pi(\boldsymbol{\beta})d\boldsymbol{\beta}, \tag{6}$$

where $\pi(\boldsymbol{\beta})$ and $\det(.)$ are the prior joint probability function of unknown parameters and determinant of a matrix, respectively. By using (6), ξ^* is called a Bayesian D-optimal design if and only if

$$\xi^* = \arg\min_{\xi\in\Xi} E_{\pi(\boldsymbol{\beta})}(-\log\det(\boldsymbol{M}(\boldsymbol{\beta}, \xi))). \tag{7}$$

Then,

$$\xi^* = \left\{ \begin{array}{cccc} x_1^* & x_2^* & \cdots & x_m^* \\ w_1^* & w_2^* & \cdots & w_m^* \end{array} \right\}; m \leq k, \tag{8}$$

where m is the number of points of optimal design. The number of points of the optimal design satisfies in following inequality (Silvey, 1980);

$$r \leq m \leq \frac{r(r+1)}{2}.$$

5 Equivalence theorem

The equivalence theorem is an important tool for optimal designs. The theorem is used to prove the optimality of the obtained designs and identify support points. Before expressing the equivalence theorem, it is necessary to define the Frechet derivative. The Frechet derivative of the criterion function Φ at ξ and in direction η is as follows:

$$F_\Phi(\xi, \eta) = \lim_{\alpha\to 0^+} \frac{1}{\alpha}(\Phi((1-\alpha)\xi + \alpha\eta) - \Phi(\xi)).$$

If the criterion function Φ is convex, $F_\Phi(\xi, \eta) \geq 0$ and if it is concave, $F_\Phi(\xi, \eta) \leq 0$ (Silvey, 1980).

Theorem 5.1. *If Φ is convex, then a Φ-optimal design ξ^* can be equivalently characterized by any of the three conditions (Whittle, 1973):*
i) ξ^* minimizes $\Phi(\xi)$
ii) ξ^* maximizes $\sup_{x\in X} F_\Phi(\xi, \eta_x)$
iii) $\sup_{x\in X} F_\Phi(\xi^*, \eta_x) = 0$.
The values of x that condition (iii) is valid for them, are called support points.

The Frechet derivative for (6) is calculated as follows:

$$F_{\Phi}(\xi, \eta_x) = -E_{\pi(\boldsymbol{\beta})} \left\{ tr \left(M^{-1}(\xi) M(\eta_x) \right) \right\} + p, \tag{9}$$

where $tr(.)$ indicate the trace of a matrix and p is the number of unknown parameters in the model. In equation (9), $E_{\pi(\boldsymbol{\beta})} \left\{ tr \left(M^{-1}(\xi) M(\eta_x) \right) \right\}$ is called sensitivity function and displayed by the symbol $d(x, \xi)$. Thus,

$$d(x, \xi) = E_{\pi(\boldsymbol{\beta})} \left\{ tr \left(M^{-1}(\xi) M(\eta_x) \right) \right\} = \int tr \left(M^{-1}(\xi) M(\eta_x) \right) \pi(\boldsymbol{\beta}) d(\boldsymbol{\beta}). \tag{10}$$

According to condition (i) in equivalence theorem, the following equation should be valid to prove the optimality of the designed obtained, $d(x, \xi^*) \leq p$. The equality holds for the support points.

6 The Bayesian D-optimal design in quadratic beta regression model

According to the model (2), Jafari and Pirmohammadi (2016) calculated the Bayesian D-optimal design for a simple beta regression model.

In this section Bayesian D-optimal designs are calculated for the quadratic beta regression model based on equation (7) and matrix (4). Two values of 50 and 100 are considered for ϕ. Also, various uniform and normal distributions are considered as prior distributions of other parameters of the model (coefficients regression). Unknown parameters of the model are independent. Therefore, prior joint probability functions are products of their marginal prior probability functions. Since there is no closed form for the optimality criterion, numerical methods are used to solve the integral of the expected value in (6), and this paper uses the Simpson method by R package.

The Bayesian D-optimal designs in the quadratic beta regression model are three point designs as follows:

$$\begin{pmatrix} 0 & x_2^* & 1 \\ 0.333 & 0.333 & 0.333 \end{pmatrix}.$$

x_2^* is presented in Tables 1-7 and 8-14 for $\phi = 50$ and $\phi = 100$ by considering different prior distributions, respectively.

Before analyzing the obtained designs, it is necessary to investigate the optimality of the designs according to equivalence theorem. There are three parameters in the quadratic beta regression model. Each of the computed designs based on part (i) of the equivalence theorem is Bayesian D-optimal if $d(x, \xi^*) \leq 3$, where

$$d(x, \xi^*) = \int tr \left(M^{-1}(\xi) M(\eta_x) \right) \pi(\boldsymbol{\beta}) d(\boldsymbol{\beta}). \tag{11}$$

Since the integral in (11) can not be solved by usual methods, there is no closed form for $d(x, \xi^*)$ and calculations were done by using numerical methods of the software R i386 3.2.0. It can be concluded that these designs are Bayesian D-optimal designs.

A) $\phi = 50$: According to Table 1, $\beta_0, \beta_1, \beta_2 \sim U(-a, a); a = 1, 5, 9$. By fixing the distribution of β_0 and β_1 and increasing domain of β_2 distribution, x_2^* is decreased.

In addition, by fixing the distributions of β_0 and β_2 and increasing domain of β_1 distribution, x_2^* is decreased and gets closer to zero. Similarly, various trends of x_2^* can be seen in Tables 2-8 by changing distributions of parameters. For example, in Table 2, by increasing the location parameter in the distribution of β_1 as $\beta_0, \beta_2 \sim N(5,1)$, x_2^* increases and gets closer to one. In addition, when $\beta_0 \sim N(10,1)$ and the location parameter in the distribution of β_2 increases, $x_2^* = 0.500$ is repeated without any changes. Also, we can fix distribution of β_1 or β_2 and see trend of x_2^*, when β_0 is changing.

B) $\phi = 100$: It is shown, in Table 8, that by fixing the distribution of β_0 and β_1 and increasing domain of β_2 distribution x_2^* is decreased and vice versa. Similar to tables for $\phi = 50$, there are different trends in this case. In addition, we can compare tables when ϕ changes.

Table 1: x_2^*: $\beta_0, \beta_1, \beta_2 \sim U(-a, a); a = 1, 5, 9$

		β_1		
		$U(-1,1)$	$U(-5,5)$	$U(-9,9)$
		$\beta_0 \sim U(-1,1)$		
	$U(-1,1)$	0.493	0.421	0.348
β_2	$U(-5,5)$	0.455	0.407	0.344
	$U(-9,9)$	0.409	0.381	0.336
		$\beta_0 \sim U(-5,5)$		
	$U(-1,1)$	0.499	0.487	0.456
β_2	$U(-5,5)$	0.494	0.480	0.452
	$U(-9,9)$	0.481	0.468	0.443
		$\beta_0 \sim U(-9,9)$		
	$U(-1,1)$	0.500	0.500	0.498
β_2	$U(-5,5)$	0.500	0.500	0.497
	$U(-9,9)$	0.500	0.499	0.495

Table 2: x_2^*: $\beta_0, \beta_1, \beta_2 \sim N(a, 1); a = 0, 5, 10$

		β_1		
		$N(0,1)$	$N(5,1)$	$N(10,1)$
		$\beta_0 \sim N(0,1)$		
	$N(0,1)$	0.484	0.348	0.233
β_2	$N(5,1)$	0.415	0.283	0.206
	$N(10,1)$	0.342	0.247	0.188
		$\beta_0 \sim N(5,1)$		
	$N(0,1)$	0.508	0.497	0.500
β_2	$N(5,1)$	0.480	0.499	0.500
	$N(10,1)$	0.492	0.500	0.500
		$\beta_0 \sim N(10,1)$		
	$N(0,1)$	0.500	0.500	0.500
β_2	$N(5,1)$	0.500	0.500	0.500
	$N(10,1)$	0.500	0.500	0.500

Table 3: x_2^*: $\beta_0, \beta_1, \beta_2 \sim N(0,a); a = 1,4,9$

		β_1		
		$N(0,1)$	$N(0,4)$	$N(0,9)$
		$\beta_0 \sim N(0,1)$		
β_2	$N(0,1)$	0.484	0.460	0.432
	$N(0,4)$	0.472	0.451	0.426
	$N(0,9)$	0.455	0.438	0.418
		$\beta_0 \sim N(0,4)$		
β_2	$N(0,1)$	0.491	0.475	0.456
	$N(0,4)$	0.483	0.469	0.452
	$N(0,9)$	0.471	0.460	0.445
		$\beta_0 \sim N(0,9)$		
β_2	$N(0,1)$	0.495	0.486	0.475
	$N(0,4)$	0.490	0.482	0.472
	$N(0,9)$	0.483	0.476	0.467

Table 4: x_2^*: $\beta_0, \beta_1 \sim U(-a,a); a = 1,5,9,\ \beta_2 \sim N(a,1); a = 0,5,10$

		β_1		
		$U(-1,1)$	$U(-5,5)$	$U(-9,9)$
		$\beta_0 \sim U(-1,1)$		
β_2	$N(0,1)$	0.489	0.420	0.347
	$N(5,1)$	0.410	0.382	0.337
	$N(10,1)$	0.334	0.327	0.310
		$\beta_0 \sim U(-5,5)$		
β_2	$N(0,1)$	0.499	0.486	0.456
	$N(5,1)$	0.483	0.468	0.444
	$N(10,1)$	0.443	0.432	0.417
		$\beta_0 \sim U(-9,9)$		
β_2	$N(0,1)$	0.500	0.500	0.498
	$N(5,1)$	0.500	0.499	0.495
	$N(10,1)$	0.499	0.496	0.485

Table 5: x_2^*: $\beta_0, \beta_2 \sim N(a,1); a = 0,5,10,\ \beta_1 \sim U(-a,a); a = 1,5,9$

		β_1		
		$U(-1,1)$	$U(-5,5)$	$U(-9,9)$
		$\beta_0 \sim N(0,1)$		
β_2	$N(0,1)$	0.491	0.428	0.361
	$N(5,1)$	0.417	0.392	0.350
	$N(10,1)$	0.342	0.336	0.323
		$\beta_0 \sim N(5,1)$		
β_2	$N(0,1)$	0.505	0.541	0.551
	$N(5,1)$	0.479	0.485	0.507
	$N(10,1)$	0.493	0.481	0.477
		$\beta_0 \sim N(10,1)$		
β_2	$N(0,1)$	0.500	0.500	0.508
	$N(5,1)$	0.500	0.500	0.501
	$N(10,1)$	0.500	0.500	0.500

Table 6: x_2^*: $\beta_0, \beta_1 \sim N(0,a); a = 1,4,9,\ \beta_2 \sim U(-a,a); a = 1,5,9$

		β_1		
		$N(0,1)$	$N(0,4)$	$N(0,9)$
		$\beta_0 \sim N(0,1)$		
	$U(-1,1)$	0.487	0.462	0.434
β_2	$U(-5,5)$	0.456	0.439	0.418
	$U(-9,9)$	0.414	0.405	0.392
		$\beta_0 \sim N(0,4)$		
	$U(-1,1)$	0.493	0.477	0.457
β_2	$U(-5,5)$	0.472	0.460	0.445
	$U(-9,9)$	0.440	0.433	0.424
		$\beta_0 \sim N(0,9)$		
	$U(-1,1)$	0.496	0.487	0.475
β_2	$U(-5,5)$	0.484	0.477	0.467
	$U(-9,9)$	0.462	0.458	0.452

Table 7: x_2^*: $\beta_0, \beta_2 \sim U(-a,a); a = 1,5,9,\ \beta_1 \sim N(0,a); a = 1,4,9$

		β_1		
		$N(0,1)$	$N(0,4)$	$N(0,9)$
		$\beta_0 \sim U(-1,1)$		
	$U(-1,1)$	0.485	0.457	0.426
β_2	$U(-5,5)$	0.451	0.432	0.410
	$U(-9,9)$	0.406	0.396	0.383
		$\beta_0 \sim U(-5,5)$		
	$U(-1,1)$	0.498	0.494	0.486
β_2	$U(-5,5)$	0.493	0.488	0.480
	$U(-9,9)$	0.480	0.475	0.467
		$\beta_0 \sim U(-9,9)$		
	$U(-1,1)$	0.500	0.500	0.500
β_2	$U(-5,5)$	0.500	0.500	0.500
	$U(-9,9)$	0.500	0.500	0.499

Table 8: x_2^*: $\beta_0, \beta_1, \beta_2 \sim U(-a,a); a = 1,5,9$

		β_1		
		$U(-1,1)$	$U(-5,5)$	$U(-9,9)$
		$\beta_0 \sim U(-1,1)$		
	$U(-1,1)$	0.493	0.418	0.336
β_2	$U(-5,5)$	0.454	0.402	0.332
	$U(-9,9)$	0.406	0.376	0.323
		$\beta_0 \sim U(-5,5)$		
	$U(-1,1)$	0.499	0.479	0.441
β_2	$U(-5,5)$	0.489	0.471	0.437
	$U(-9,9)$	0.471	0.456	0.427
		$\beta_0 \sim U(-9,9)$		
	$U(-1,1)$	0.500	0.500	0.496
β_2	$U(-5,5)$	0.500	0.499	0.495
	$U(-9,9)$	0.500	0.498	0.491

Table 9: x_2^*: $\beta_0, \beta_1, \beta_2 \sim N(a, 1); a = 0, 5, 10$

		β_1		
		$N(0,1)$	$N(5,1)$	$N(10,1)$
		$\beta_0 \sim N(0,1)$		
	$N(0,1)$	0.483	0.339	0.219
β_2	$N(5,1)$	0.411	0.275	0.195
	$N(10,1)$	0.338	0.239	0.179
		$\beta_0 \sim N(5,1)$		
	$N(0,1)$	0.410	0.490	0.500
β_2	$N(5,1)$	0.453	0.497	0.500
	$N(10,1)$	0.471	0.499	0.500
		$\beta_0 \sim N(10,1)$		
	$N(0,1)$	0.500	0.500	0.500
β_2	$N(5,1)$	0.500	0.500	0.500
	$N(10,1)$	0.500	0.500	0.500

Table 10: x_2^*: $\beta_0, \beta_1, \beta_2 \sim N(0, a); a = 1, 4, 9$

		β_1		
		$N(0,1)$	$N(0,4)$	$N(0,9)$
		$\beta_0 \sim N(0,1)$		
	$N(0,1)$	0.483	0.457	0.426
β_2	$N(0,4)$	0.470	0.448	0.420
	$N(0,9)$	0.453	0.434	0.410
		$\beta_0 \sim N(0,4)$		
	$N(0,1)$	0.489	0.471	0.448
β_2	$N(0,4)$	0.480	0.464	0.443
	$N(0,9)$	0.467	0.454	0.453
		$\beta_0 \sim N(0,9)$		
	$N(0,1)$	0.495	0.483	0.468
β_2	$N(0,4)$	0.489	0.479	0.464
	$N(0,9)$	0.481	0.471	0.459

Table 11: x_2^*: $\beta_0, \beta_1 \sim U(-a, a); a = 1, 5, 9,\ \beta_2 \sim N(a, 1); a = 0, 5, 10$

		β_1		
		$U(-1,1)$	$U(-5,5)$	$U(-9,9)$
		$\beta_0 \sim U(-1,1)$		
	$N(0,1)$	0.489	0.416	0.336
β_2	$N(5,1)$	0.408	0.377	0.323
	$N(10,1)$	0.331	0.321	0.296
		$\beta_0 \sim U(-5,5)$		
	$N(0,1)$	0.498	0.479	0.441
β_2	$N(5,1)$	0.472	0.457	0.427
	$N(10,1)$	0.424	0.415	0.397
		$\beta_0 \sim U(-9,9)$		
	$N(0,1)$	0.500	0.500	0.496
β_2	$N(5,1)$	0.500	0.498	0.491
	$N(10,1)$	0.498	0.491	0.479

Table 12: x_2^*: $\beta_0, \beta_2 \sim N(a,1); a = 0,5,10,\ \beta_1 \sim U(-a,a); a = 1,5,9$

		β_1		
		$U(-1,1)$	$U(-5,5)$	$U(-9,9)$
		$\beta_0 \sim N(0,1)$		
	$N(0,1)$	0.490	0.423	0.346
β_2	$N(5,1)$	0.414	0.385	0.334
	$N(10,1)$	0.339	0.329	0.306
		$\beta_0 \sim N(5,1)$		
	$N(0,1)$	0.506	0.546	0.555
β_2	$N(5,1)$	0.452	0.470	0.501
	$N(10,1)$	0.473	0.457	0.460
		$\beta_0 \sim N(10,1)$		
	$N(0,1)$	0.500	0.501	0.518
β_2	$N(5,1)$	0.500	0.500	0.502
	$N(10,1)$	0.500	0.500	0.500

Table 13: x_2^*: $\beta_0, \beta_1 \sim N(0,a); a = 1,4,9,\ \beta_2 \sim U(-a,a); a = 1,5,9$

		β_1		
		$N(0,1)$	$N(0,4)$	$N(0,9)$
		$\beta_0 \sim N(0,1)$		
	$U(-1,1)$	0.487	0.459	0.427
β_2	$U(-5,5)$	0.454	0.435	0.411
	$U(-9,9)$	0.410	0.399	0.384
		$\beta_0 \sim N(0,4)$		
	$U(-1,1)$	0.491	0.473	0.450
β_2	$U(-5,5)$	0.468	0.454	0.436
	$U(-9,9)$	0.432	0.424	0.412
		$\beta_0 \sim N(0,9)$		
	$U(-1,1)$	0.496	0.485	0.469
β_2	$U(-5,5)$	0.482	0.472	0.459
	$U(-9,9)$	0.456	0.449	0.441

Table 14: x_2^*: $\beta_0, \beta_2 \sim U(-a,a); a = 1,5,9,\ \beta_1 \sim N(0,a); a = 1,4,9$

		β_1		
		$N(0,1)$	$N(0,4)$	$N(0,9)$
		$\beta_0 \sim U(-1,1)$		
	$U(-1,1)$	0.485	0.455	0.421
β_2	$U(-5,5)$	0.449	0.430	0.404
	$U(-9,9)$	0.404	0.393	0.377
		$\beta_0 \sim U(-5,5)$		
	$U(-1,1)$	0.497	0.490	0.479
β_2	$U(-5,5)$	0.488	0.481	0.471
	$U(-9,9)$	0.469	0.464	0.455
		$\beta_0 \sim U(-9,9)$		
	$U(-1,1)$	0.500	0.500	0.500
β_2	$U(-5,5)$	0.500	0.500	0.499
	$U(-9,9)$	0.500	0.499	0.498

7 Discussion and conclusions

In this paper, Bayesian D-optimal designs in a quadratic beta regression model are obtained. By using these designs, a researcher finds that what proportions of which regressors should be used to get proper estimates of coefficient regression. We use Bayesian methods in the quadratic beta regression model for achieving optimal designs. Since, there are three parameters in the quadratic beta regression model, the optimal designs can be three points to six point designs. By considering uniform and normal prior distributions, all obtained Bayesian D-optimal designs are three points. In this study, interval $[0, 1]$ is assumed for regressors. First and third points in Bayesian D-optimal designs are zero and one, respectively. These points are fixed in all designs. However, by changing prior distributions of unknown parameters, second point of optimal designs changes. Trend of these changes are shown in presented tables. It should be noticed that every optimal point has same optimal weight. Therefore, these points should be used with the same proportions. As a future work, statisticians can use other prior distributions in this model.

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