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Research Paper

Interval estimation of stress-strength reliability parameter for exponential-inverted exponential model: Frequentist and Bayesian approaches

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Abstract: This paper introduces the problem of interval estimation for stress strength reliability parameter $P(X < Y)$, where random variables X and Y stand for stress and strength, respectively. In most of the research papers, the authors assumed that X and Y come from the same family of distribution. By taking into account some situations arise, in this paper we assume that X and Y follow exponential and inverted exponential distributions, respectively. Our goal is to construct a confidence interval for reliability parameter in this model by using some (approximately) exact and strong methods such as bootstrap, generalized and highest posterior distribution approaches. Also, we compare these methods by means of the expected length and coverage probability criteria. Finally, a real data set is given and we apply the above methods of estimation on it to inference about the parameter of interest.

Keywords: Bootstrap method; Highest posterior distribution approach; Maximum likelihood estimation; Modified Bessel function.
Mathematics Subject Classification (2010): 60E05, 62F12.

1 Introduction

In this paper we consider the problem of interval estimation of the stress-strength reliability parameter $P(X < Y)$. In the theory of reliability analyze the random variable X stands for stress factor and the random variable Y stands for strength of a system. The reliability of a component is defined as the probability that strength Y be greater than the stress X imposed to the component. So, the system is still working whenever the strength exceeds the stress during the entire interval. In other words, the reliability parameter R is defined as $R := P(X < Y)$ which means that if the stress exceeds the strength then the component would fail.

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In recent years, the estimation of stress-strength parameter of the discrete and continuous distributions and also a mixture of them has attracted the attention of many researchers. The term stress-strength was first introduced by Birnbaum (1956). There is an enormous amount of research papers for inference about the parameter R. Here, we only refer to some of them for the following distributions: Owen et al. (1964), Govidarajulu (1967), Downtown (1973) and Woodward and Kelley (1977) for normal distribution, Tong (1977) considered this problem for exponential families, the gamma case has been studied by Constantine and Karson (1986), Ismail et al. (1986) and Constantine et al. (1990), McCool (1991) for Weibull distribution, Baklizi and Quader El-Masri (2004) for two-parameter exponential distribution with common location parameter, Genć (2013) for Topp-Leone distribution, Nadarajah (2004) for Laplace distribution, Rezaei et al. (2010) consider this problem for generalized Pareto distribution, Kundu and Gupta (2005, 2006) considered generalized exponential and Weibull Distributions cases, respectively, Krishnamoorthy et al. (2007) for the two-parameter exponential distribution, Kakade et al. (2008) for Gumbel case, Babayi et al. (2014) for generalized logistic distribution, Bai et al. (2019) for truncated proportional hazard rate distribution under progressively type-II censored samples, Mahmoudi et al. (2019) used minimum risk sequential point estimation for exponential distribution (ED) and Khalifeh et al. (2020) used sequential fixed-accuracy confidence intervals (CIs) for the ED.

For stress and stress follow some bivariate distribution, Pak et al. (2014), Nadarajah (2005a,b), Nadarajah and Kotz (2006) considered bivariate Rayleigh, bivariate beta, some bivariate gamma and some bivariate exponential distributions, respectively. For both stress and strength follow discrete distributions one can see the papers of Maiti (1995) and Ahmad et al. (1995) for the geometric distribution and Sathe and Dixit (2001) for the negative binomial distribution and also for mixture of discrete and continuous distributions for inference about the reliability parameter we can refer to Jovanović (2017) for geometric-exponential model.

In the stress-strength analysis, usually, it is assumed that X and Y come from the same family of distribution. But there are situations that we can not consider the same family of distribution for both stress and strength, such as conditions that arise in Jovanović (2017) and Obradović et al. (2015). As an example of such models, consider for example two groups of patients with a same type of disease and observe that both groups would be subject to two different treatments, for example treatments A and B. We want to determine that which group of patients, receiving one of the treatments A and B, has higher survival probability than another group. A real data set considered here is head and neck cancer (HNC) data which is used by some authors for example, Efron (1988) and Makkar et al. (2014). Recently Sharma (2017) considered the Bayesian analysis of reliability parameter for this data set. They assumed the generalized inverse Lindley distribution for both stress and strength of model. In this paper we take two different distributions for stress and strength. We assume that the random variable X (stress) follows an ED and independent of X the random variable Y (strength) follows an inverted exponential distribution (IED). Clearly we consider the case that random variable X follows an ED with parameter θ denoted by $X \sim ED(\theta)$ and random variable Y follows an IED with parameter τ denoted by $Y \sim IED(\tau)$.

The main aim of this paper is the investigation of constructing some CIs for the parameter R. So, the plan of the paper is as follows: In Section 2, we derive the maximum likelihood estimator (MLE) of the parameter θ and τ and their sampling distributions. Then, we derive the MLE of reliability parameter R and its asymptotic distribution. In Section 3, we find some CIs such as asymptotic confidence interval (ACI), bootstrap confidence interval (BCI) and generalized confidence interval (GCI). In Bayesian viewpoint, we also discuss to construct a CI for parameter R by using the highest posterior distribution (HPD) approach. We compare these CIs by using some simulation studies in Section 4. In Section 5, for the HNC data as pointed out above, we apply four different CIs (ACI, BCI, GCI and HPD) for this data to construct CIs for parameter R. We finish the paper with our conclusions in Section 6.

2 Preliminary results

A random variable $X \sim ED(\theta)$ has the probability density function (PDF)

$$f_\theta(x) = \theta e^{-\theta x}; x > 0, \theta > 0, \tag{1}$$

and the cumulative distribution function (CDF) $F_\theta(x) = 1 - e^{-\theta x}$.

The ED has many applications in statistical inference for life-time data analysis and is defined as the time until the first success occurs. As we know that ED has a constant failure rate. But there are some cases that in which the hazard rate initially increases and reaches a peak after some finite period of time and then declines slowly. For these cases we may fit the IED.

A random variable $Y \sim IED(\tau)$ has the PDF

$$f_\tau(y) = \frac{\tau}{y^2} e^{-\frac{\tau}{y}}; y > 0, \tau > 0. \tag{2}$$

and the CDF $F_\tau(y) = e^{-\frac{\tau}{y}}$.

For more information and properties of IED, one an see Singh et al. (2013) and references there in. Using (1) and (2), one can find that

$$P(X < Y) = 1 - \theta \int_0^\infty e^{-(\theta x + \frac{\tau}{x})} dx := R_{\tau,\theta}. \tag{3}$$

The following lemma shows that $R_{\tau,\theta}$ can be presented as the modified Bessel function of the second type ($K_\upsilon(z)$).

Lemma 2.1. *Let $X \sim ED(\theta)$ and $Y \sim IED(\tau)$. Then, we have*

$$R = 1 - 2\sqrt{\theta\tau} K_1(2\sqrt{\theta\tau}) =: R_{\tau,\theta},$$

where $K_\upsilon(z)$ denotes the modified Bessel function of the second type.

Proof. It can be easily seen that $R_{\tau,\theta}$ in (3) could be rewritten as

$$R_{\tau,\theta} = 1 - \tau \int_0^\infty \frac{1}{y^2} e^{-(\theta y + \frac{\tau}{y})} dy. \tag{4}$$

The following integral representation of $K_\upsilon(z)$ can be helpful (See, for example Watson, 1944, page 183),

$$(\frac{1}{2})(\frac{1}{2}z)^{\upsilon}\int_0^\infty \frac{1}{\xi^{(\upsilon+1)}}e^{-(\xi+\frac{z^2}{4\xi})}d\xi = K_\upsilon(z).$$

So, by taking $z = 2\sqrt{\theta\tau}$ for $R_{\tau,\theta}$ in (4) we have

$$\tau\int_0^\infty \frac{1}{y^2}e^{-(\theta y+\frac{\tau}{y})}dy = \frac{z^2}{4\theta}\int_0^\infty \frac{1}{y^2}e^{-(\theta y+\frac{z^2}{4\theta y})}dy = \frac{z^2}{4}\int_0^\infty \frac{1}{(\theta y)^2}e^{-(\theta y+\frac{z^2}{4\theta y})}d(\theta y)$$
$$= \frac{\frac{z^2}{4}(\frac{1}{2})(\frac{1}{2}z)\int_0^\infty \frac{1}{\xi^2}e^{-(\xi+\frac{z^2}{4\xi})}d\xi}{[(\frac{1}{2})(\frac{1}{2}z)]} = zK_1(z).$$

□

2.1 Maximum likelihood estimator of $R_{\tau,\theta}$

Let $X_1, X_2, \ldots, X_{n_1}$ and $Y_1, Y_2, \ldots, Y_{n_2}$ be two independent random samples from $ED(\theta)$ and $IED(\tau)$, respectively. As we know that the MLE of parameters θ and τ are as

$$\widehat{\theta} = \frac{n_1}{\sum_{i=1}^{n_1} X_i} = \frac{1}{\overline{X}} \quad and \quad \widehat{\tau} = \frac{n_2}{\sum_{i=1}^{n_2}(\frac{1}{Y_i})}. \tag{5}$$

So, by using the invariance property of MLEs, the MLE of $R_{\tau,\theta}$ is

$$\widehat{R}_{\tau,\theta} = 1 - 2\sqrt{\widehat{\theta}\widehat{\tau}}K_1(2\sqrt{\widehat{\theta}\widehat{\tau}}) \tag{6}$$

It is easily verified that, if $X \sim ED(\theta)$ then $2\theta X \sim \chi^2_{(2)}$, so, for $\widehat{\theta}$ we have

$$\frac{2n_1\theta}{\widehat{\theta}} \sim \chi^2_{(2n_1)}. \tag{7}$$

In a similar manner as above, for $\widehat{\tau}$ we find that

$$\frac{2n_2\tau}{\widehat{\tau}} \sim \chi^2_{(2n_2)}. \tag{8}$$

The following equations which are related to $K_\upsilon(z)$ can be used to obtain promising results. These equations are indeed the derivative of function $K_\upsilon(z)$ on z. By referring to relation 202 of Andrews (1985), the partial derivative of $R_{\tau,\theta}$ in both τ and θ can be obtained as

$$i)\quad \frac{\partial R_{\tau,\theta}}{\partial\theta} = 2\tau K_0(2\sqrt{\theta\tau}), \tag{9}$$

$$ii)\quad \frac{\partial R_{\tau,\theta}}{\partial\tau} = 2\theta K_0(2\sqrt{\theta\tau}). \tag{10}$$

The above equations may be useful to simplifying the asymptotic variance of the $\widehat{R}_{\tau,\theta}$ in the next part of the paper.

2.2 Asymptotic distribution of $\widehat{R}_{\tau,\theta}$

It is well known that under the regularity condition, the asymptotic distribution of the MLE of model parameters is multivariate normal with mean vector and variance-covariance matrix equal to vector of corresponding parameters and inverse of Fisher information matrix, respectively. So, for the asymptotic distribution of MLE of parameters in this paper, let $I_1(\theta) = \frac{1}{\theta^2}$ and $I_2(\tau) = \frac{1}{\tau^2}$ be the expected Fisher information with respect to density functions $f_\theta(x)$ and $f_\tau(y)$, respectively. Then, it is verified that,

$$\sqrt{n_1}(\hat{\theta} - \theta) \Longrightarrow N(0, I_1^{-1}(\theta)), \tag{11}$$

$$\sqrt{n_2}(\hat{\tau} - \tau) \Longrightarrow N(0, I_2^{-1}(\tau)), \tag{12}$$

where $\Longrightarrow$ stands for convergence in law. So, we can state the following theorem for obtaining the asymptotic distribution of $\widehat{R}_{\tau,\theta}$.

Theorem 2.2. *Let $X_1, X_2, \ldots, X_{n_1}$ and $Y_1, Y_2, \ldots, Y_{n_2}$ be two independent samples from $ED(\theta)$ and $IED(\tau)$, respectively. Then the asymptotic distribution of $\widehat{R}_{\tau,\theta}$ is*

$$\sqrt{n_1 + n_2}\left(\widehat{R}_{\tau,\theta} - R_{\tau,\theta}\right) \Longrightarrow N\left(0, \sigma_R^2\right), \tag{13}$$

as $n_1 \to \infty, n_2 \to \infty$, $\frac{n_1}{n_1+n_2} \to \lambda$, where $\sigma_R^2 = \frac{1}{\lambda(1-\lambda)}\left(2\theta\tau K_0(2\sqrt{\theta\tau})\right)^2$.

Proof. According to Kotz et al. (2003) page 122, since $f_\theta(x)$ $(f_\tau(y))$ is the function of θ (τ) only, so the asymptotic distribution of $\widehat{R}_{\tau,\theta}$ is normal with mean $R_{\tau,\theta}$ and variance σ_R^2, where

$$\sigma_R^2 = \frac{1}{(\lambda I_1(\theta))}\left(\frac{\partial R_{\tau,\theta}}{\partial \theta}\right)^2 + \frac{1}{((1-\lambda)I_2(\tau))}\left(\frac{\partial R_{\tau,\theta}}{\partial \tau}\right)^2,$$

and $I_1(\theta)$ $(I_2(\tau))$ is the expected Fisher information with respect to $f_\theta(x)$ $(f_\tau(y))$. So, the proof is completed using (9) and (10). □

Remark 2.3. *Another asymptotic distribution for $\widehat{R}_{\tau,\theta}$ can be obtained via the following manner. Using* (12) *by considering $n_1 \longrightarrow \infty$, $n_2 \longrightarrow \infty$ and $\frac{n_1}{n_2} \longrightarrow p$, it is observe that*

$$\sqrt{n_1}(\hat{\tau} - \tau) \Longrightarrow N(0, p\tau^2), \tag{14}$$

and then because of independence of $\hat{\theta}$ and $\hat{\tau}$,

$$\sqrt{n_1}\begin{pmatrix} \hat{\theta} - \theta \\ \hat{\tau} - \tau \end{pmatrix} \Longrightarrow \boldsymbol{N}\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \theta^2 & 0 \\ 0 & p\tau^2 \end{pmatrix}\right]. \tag{15}$$

So, using Section 7 *of Ferguson (1996), we can conclude that*

$$\sqrt{n_1}\left(\widehat{R}_{\tau,\theta} - R_{\tau,\theta}\right) \Longrightarrow N\left(0, \sigma_R^2\right), \tag{16}$$

where $\sigma_R^2 = (1 + p)\left(2\theta\tau K_0(2\sqrt{\theta\tau})\right)^2$.

3 Interval estimation for $R_{\tau,\theta}$

In this section we construct a $100(1-\alpha)\%$ for parameter $R_{\tau,\theta}$ based on the asymptotic distribution of $\widehat{R}_{\tau,\theta}$ which stated in Theorem 2.2. As we will see in the next section simulation study shows that the CI for $R_{\tau,\theta}$ based on the asymptotic distribution of $\widehat{R}_{\tau,\theta}$ is suitable only for larger sample size. So, we introduce some well known computational methods to improve the CI for the involved parameter. These methods are bootstrap, generalized and Bayesian CIs and we consider them in this section.

3.1 Asymptotic confidence interval

In Theorem 2.2, we showed that

$$\sqrt{n_1+n_2}\left(\widehat{R}_{\tau,\theta}-R_{\tau,\theta}\right)\Longrightarrow N\left(0,\sigma_R^2\right).$$

For constructing an asymptotic CI, we need to estimate the σ_R^2. A natural consistent estimator for σ_R^2 is

$$\tilde{\sigma}_R^2=\frac{1}{\tilde{\lambda}(1-\tilde{\lambda})}\left(2\hat{\theta}\hat{\tau}K_0(2\sqrt{\hat{\theta}\hat{\tau}})\right)^2, \tag{17}$$

where $\tilde{\lambda}=\frac{n_1}{n_1+n_2}$. So, a two sided $100(1-\alpha)\%$ CI for $R_{\tau,\theta}$ is

$$R_{\tau,\theta}\in\left(\widehat{R}_{\tau,\theta}\pm\sqrt{\chi^2_{(1,1-\alpha)}}\frac{\tilde{\sigma}_R}{\sqrt{(n_1+n_2)}}\right), \tag{18}$$

where $\chi^2_{(\nu,\gamma)}$ is the γ-th quantile of a chi-square distribution with ν degrees of freedom.

Remark 3.1. *Also a* $100(1-\alpha)\%$ *one sided lower CI for* $R_{\tau,\theta}$ *is*

$$R_{\tau,\theta}\in\left(\widehat{R}_{\tau,\theta}-\sqrt{\chi^2_{(1,1-2\alpha)}}\frac{\tilde{\sigma}_R}{\sqrt{(n_1+n_2)}},1\right). \tag{19}$$

3.2 Bootstrap confidence interval for $R_{\tau,\theta}$

The bootstrap approach is a computer-based method which was introduced by Efron and Tibshirani (1994) that is applied to the observed data by Monte Carlo simulation. As we will see in the simulation section, the asymptotic CI does not act well in both coverage probability and expected length for small sample sizes. Bootstrap procedure provides a better approximation to exactness in most situations. In this section we utilize a bootstrap-t methods introduced by Hall (1988). Let $X_1,X_2,\ldots,X_{n_1}$ and $Y_1,Y_2,\ldots,Y_{n_2}$ be two independent samples from $ED(\theta)$ and $IED(\tau)$, respectively. By the following Algorithm one can use bootstrap-t method to construct a bootstrap-t CI for parameter $R_{\tau,\theta}$.

Algorithm 3.2. *The algorithm is carried out in five steps:*
Step 1. Obtain $\widehat{R}_{\tau,\theta}$ *using* (6).
Step 2. Generate bootstrap samples $x_1^*,x_2^*,\ldots,x_{n_1}^*$ *and* $y_1^*,y_2^*,\ldots,y_{n_2}^*$ *from* $ED(\theta)$ *and*

$IED(\tau)$, respectively.
Step 3. Obtain $\hat{\tau}^*$ and $\hat{\theta}^*$ based on the bootstrap samples by (5) *and Compute $\widehat{R}^*_{\tau,\theta}$ by replacing $\hat{\tau}^*$ and $\hat{\theta}^*$ instead of $\hat{\tau}$ and $\hat{\theta}$ in* (6)*, respectively.*
Step 4. Repeat Steps 2 and 3 for M times (M = 100; 000) and obtain the value of $\widehat{R}^*_{\tau,\theta}$ for each repetition, say $\widehat{R}^{*(1)}_{\tau,\theta}, ..., \widehat{R}^{*(M)}_{\tau,\theta}$.
Step 5. A $100(1-\alpha)\%$ *BCI for $R_{\tau,\theta}$ is given by*

$$\left(\widehat{R}_{\tau,\theta} - T^*_{(1-\alpha/2)}\sqrt{Var(\widehat{R}^*_{\tau,\theta})}, \widehat{R}_{\tau,\theta} - T^*_{(\alpha/2)}\sqrt{Var(\widehat{R}^*_{\tau,\theta})}\right), \tag{20}$$

where $Var(\widehat{R}^*_{\tau,\theta})$ is the variance of the values $\widehat{R}^{*(1)}_{\tau,\theta}, ..., \widehat{R}^{*(M)}_{\tau,\theta}$ and $T^*_{(\gamma)}$ denotes the γ-th quantile of $T^*_i = \frac{\widehat{R}^*_{\tau,\theta} - \widehat{R}_{\tau,\theta}}{\sqrt{Var(\widehat{R}^*_{\tau,\theta})}}, i = 1, 2, ..., M$.

Remark 3.3. *Also a* $100(1-\alpha)\%$ *one-sided lower bootstrap-t CI for $R_{\tau,\theta}$ is*

$$\left(\widehat{R}_{\tau,\theta} - T^*_{(1-\alpha)}\sqrt{Var(\widehat{R}^*_{\tau,\theta})}, 1\right). \tag{21}$$

3.3 Generalized confidence interval for $R_{\tau,\theta}$

This section utilizes the generalized variable (GV) method for parameters τ, θ and then of the $R_{\tau,\theta}$ to construct the GCI for $R_{\tau,\theta}$. The concepts of generalized pivotal variable defined by Weerahandi (1993) and this approach is very applicable in statistical inference. To understand how GV methods can be applied in statistical problems one can see the book of Weerahandi (1995).

For constructing generalized pivot quantity for parameter $R_{\tau,\theta}$ we should first construct the generalized pivot quantities for parameters τ and θ. Let $\widehat{\theta}_0$ and $\widehat{\tau}_0$ be the observed values of $\widehat{\theta}$ and $\widehat{\tau}$, respectively. From (7) and (8), the GV quantities for τ and θ are $G_\theta = \frac{2n_1\theta}{\widehat{\theta}}\frac{\widehat{\theta}_0}{2n_1} = \frac{\widehat{\theta}_0 V_1}{2n_1}$ and $G_\tau = \frac{\widehat{\tau}_0 V_2}{2n_2}$,

where $V_1 \sim \chi^2_{(2n_1)}$, $V_2 \sim \chi^2_{(2n_2)}$. So, a generalized pivotal variable for $R_{\tau,\theta}$ can be obtained by replacing the parameters in the form of the $R_{\tau,\theta}$ by their generalized pivotal variables as below

$$G_{R_{\tau,\theta}} = 1 - 2\sqrt{G_\tau G_\theta}K_1(2\sqrt{G_\tau G_\theta}). \tag{22}$$

Using the following algorithm we can construct a $100(1-\alpha)\%$ GCI for parameter of interest $R_{\tau,\theta}$.

Algorithm 3.4. *For given random samples $(x_1, x_2, \ldots, x_{n_1})$ and $(y_1, y_2, \ldots, y_{n_2})$, compute the MLEs $\widehat{\theta}_0$ and $\widehat{\tau}_0$.*
Step 1. Generate $V_1 \sim \chi^2_{(2n_1)}$ and $V_2 \sim \chi^2_{(2n_2)}$.
Step 2. Compute $G_{R_{\tau,\theta}}$ in (22)*.*
Step 3. Repeat the steps 1 and 2 a large number of times, say, $M = 100,000$ times. Then from these M values, the $100(\alpha/2)th$ and $100(1-\alpha/2)th$ percentile of $G_{R_{\tau,\theta}}$ is a $100(1-\alpha)\%$ *CI for $R_{\tau,\theta}$.*

Remark 3.5. *The $100\alpha th$ percentiles of these M generated $G_{R_{\tau,\theta}}$ is a* $100(1-\alpha)\%$ *lower confidence limit for $R_{\tau,\theta}$.*

3.4 Highest posterior distribution approach

In this section we construct a CI for parameter $R_{\tau,\theta}$ using the method of highest posterior distribution (HPD). Indeed in this section we try to use the Bayesian method for constructing a $100(1-\alpha)\%$ CI for the model parameter. Two types of useful Bayesian CIs are Bayesian credible and HPD CIs. The Bayesian credible intervals are easy to obtained. Let $\pi(\theta)$ be a prior of parameter θ. So, by using the density $f(data;\theta)$ the posterior density of θ is $\pi(\theta|data)$. Let $\{\theta_i, i = 1, 2, \ldots, M\}$ be a Markov chain Monte Carlo (MCMC) from $\pi(\theta|data)$. Then a $100(1-\alpha)\%$ credible interval is

$$\left(\theta_{[(\frac{\alpha}{2})M]}, \theta_{[(1-\frac{\alpha}{2})M]}\right), \tag{23}$$

where $\theta_{[\alpha M]}$ is the $[\alpha M]$th smallest of $\{\theta_i\}$. These intervals are easy to obtain. One can use another more complicated Bayesian intervals such as HPD intervals.

Definition 3.6. *A region C is called a HPD region of content $1-\alpha$ if*
a) $\int_C \pi(\theta|data)d\theta = 1-\alpha$,
b) for any $\theta \in C$ and $\theta^* \notin C$, we have $\pi(\theta|data) \geq \pi(\theta^*|data)$.

For more details see Berger (1985) and Knight (2000) books. If the posterior $\pi(\theta|data)$ is unimodal then C will be an HPD interval. In this part we consider two independent gamma priors for parameters θ and τ. We assume that $\theta \sim gamma(a_1, a_2)$ and $\tau \sim gamma(b_1, b_2)$, where a_1,a_2,b_1 and b_2 are positive known constants. So, it can be shown that the posterior distribution of θ and τ are

$$\pi(\theta|data) = \frac{(n_1+a_1)^{(a_2+t_x)}}{\Gamma(n_1+a_1)}\theta^{n_1+a_1-1}exp\{-(a_2+t_x)\theta\}, \tag{24}$$

$$\pi(\tau|data) = \frac{(n_2+b_1)^{(b_2+t_y)}}{\Gamma(n_2+b_1)}\tau^{n_2+b_1-1}exp\{-(b_2+t_y)\tau\}, \tag{25}$$

where $t_x = \sum x_i$ and $t_y = \sum \frac{1}{y_i}$.

Chen and Shao (1999) introduce an approximate HPD CI. This kind of interval is based on all Bayesian credible intervals of content $1-\alpha$. Then the HPD CI is one which has the shortest length. This method also can be used to construct a HPD CI for a function of the parameters. We briefly explain this approach for our problem. Let $\{(\theta_i, \tau_i), i = 1, 2, \ldots, M\}$ be an ergodic MCMC samples from $\pi(\theta, \tau|data) = \pi(\theta|data)\pi(\tau|data)$ (because of independence of θ and τ) and generate $R^i_{\tau,\theta} = 1 - 2\sqrt{(\theta_i\tau_i)}K_1(2\sqrt{(\theta_i\tau_i)})$ for $i = 1, 2, \ldots, M$. Then compute all Bayesian credible CI of content $1-\alpha$ for $R_{\tau,\theta}$ as $(R^{(j)}_{\tau,\theta}, R^{(j+[(1-\alpha)M])}_{\tau,\theta})$, for $j = 1, 2, \ldots, M-[(1-\alpha)M]$ where $R^{(i)}_{\tau,\theta}$ are the ordered values of $R^i_{\tau,\theta}$. Then the approximate HPD CI for $R_{\tau,\theta}$ is shortest length interval between all above Bayesian credible intervals of content $1-\alpha$.

4 Simulation study

In this section, some simulation studies carried out to compare the sufficiency of the asymptotic confidence interval (ACI), bootstrap confidence interval (BCI), generalized confidence interval (GCI) and highest posterior distribution (HPD) approaches for producing some CIs for reliability parameter $R_{\tau,\theta}$. Comparing of these above approaches are based on their expected lengths (EL) and coverage probabilities (CP). We run the simulations by producing two random samples n_1 and n_2 from $ED(\theta)$ and $IED(\tau)$, respectively. Also, and let (n_1, n_2) and (θ, τ) vary in the sets $\{(10, 10), (10, 15), (15, 10), (15, 15), (20, 20)\}$ and $\{(3.5, 0.5), (1, 0.5), (0.25, 0.25)\}$, respectively. Also, for using the method of HPD, the parameters of the prior distributions are taken as $a_1 = 0.001$, $a_2 = 0.1$, $b_1 = 0.001$ and $b_2 = 0.1$. Note that the values of $R_{\tau,\theta}$ for different quantities of (θ, τ) in $\{(3.5, 0.5), (1, 0.5), (0.25, 0.25)\}$ are 0.83, 0.55 and 0.17, respectively.

The results are given in Table 1. In this table, for the above four approaches the values of CPs as well as the values of ELs in parenthesis were obtained. The following results were conducted from Table 1:
i) It is observed that in each cases, if $R_{\tau,\theta}$ decreases then the CPs of ACI and BCI are close to confidence coefficient (COC) 95%. For example from Table 1, for $(n_1, n_2) = (10, 10)$ and $(\tau, \theta) = (3.5, 0.5), (1, 0.5), (0.25, 0.25)$ the CPs are 0.873, 0.928 and 0.947 and the values of $R_{\tau,\theta}$ are 0.83, 0.55 and 0.17, respectively.
ii) The CPs of GCI are close to COC for all cases. Indeed the GCI is an exact method to constructing the CI for parameter $R_{\tau,\theta}$.
iii) Although the act of HPD approach is approximately acceptable, the CPs of HPD method are less than the COC, in all cases.
iv) It is observed that the ELs of all approaches decrease whenever the sample size increases.
v) For cases with $R_{\tau,\theta} > 0.5$, the ELs of all approaches are greater than that of with $R_{\tau,\theta} < 0.5$. For example, the ELs of the case $(\tau, \theta) = (3.5, 0.5)$ with $R_{\tau,\theta} = 0.83$ are greater than that of case $(\tau, \theta) = (0.25, 0.25)$ with $R_{\tau,\theta} = 0.17$, for all approaches.

5 Real data

The following real data set represents the survival times of two groups of patients suffering from HNC disease. The first group of patients, denoted by X, was treated with radiotherapy and the second group of patients, denoted by Y, was treated with combined radiotherapy and chemotherapy. As noted before, these data set was considered by Sharma (2017) and it was assumed that both stress (X) and strength (Y) of model follow the generalized inverse Lindley distribution with utilizing the Bayesian analysis for reliability parameter for this data set. In this paper we assume X and Y follow ED and IED, respectively. It is important to detect that the combined radiotherapy and chemotherapy treatment has more effect on survival times of the HNC patients rather than the radiotherapy treatment. The data sets are as follows:
Data (X):
6.53, 7, 10.42, 14.48, 16.10, 22.70, 34, 41.55, 42, 45.28, 49.40, 53.62, 63, 64, 83, 84, 91, 108, 112, 129, 133, 133, 139, 140, 140, 146, 149, 154, 157, 160, 160, 165, 146, 149, 154, 157, 160, 160, 165, 173, 176, 218, 225, 241, 248, 273, 277, 297, 405, 417, 420, 440, 523, 583, 594, 1101, 1146, 1417
Data (Y):
12.20, 23.56, 23.74, 25.87, 31.98, 37, 41.35, 47.38, 55.46, 58.36, 63.47, 68.46, 78.26, 74.47, 81, 43, 84, 92, 94, 110, 112, 119, 127, 130, 133, 140, 146, 155, 159, 173, 179, 194, 195, 209, 249, 281, 319, 339, 432, 469, 519, 633, 725, 817, 1776

Table 1: Coverage probabilities as well as expected lengths in parenthesis of ACI, BCI, GCI and HPD approaches for various sample sizes and parameters values.

		(θ, τ)		
(n_1, n_2)	Approach	(3.5,0.5)	(1,0.5)	(0.25,0.25)
(10,10)	ACI	0.873 (0.2937)	0.928 (0.4055)	0.947 (0.2174)
	BCI	0.993 (0.4970)	0.974 (0.4866)	0.950 (0.2223)
	GCI	0.950 (0.3129)	0.951 (0.3948)	0.948 (0.2097)
	HPD	0.941 (0.3028)	0.939 (0.3884)	0.942 (0.2028)
(10,15)	ACI	0.897 (0.2760)	0.937 (0.3729)	0.955 (0.1960)
	BCI	0.982 (0.4195)	0.968 (0.4309)	0.958 (0.1991)
	GCI	0.947 (0.2908)	0.958 (0.3646)	0.957 (0.1903)
	HPD	0.944 (0.2790)	0.935 (0.3587)	0.941 (0.1853)
(15,10)	ACI	0.875 (0.2757)	0.904 (0.3717)	0.959 (0.1970)
	BCI	0.986 (0.4183)	0.950 (0.4321)	0.964 (0.2001)
	GCI	0.942 (0.2903)	0.939 (0.3640)	0.955 (0.1911)
	HPD	0.936 (0.2772)	0.935 (0.3588)	0.941 (0.1847)
(15,15)	ACI	0.914 (0.2479)	0.943 (0.3355)	0.949 (0.1718)
	BCI	0.980 (0.3436)	0.966 (0.3743)	0.952 (0.1734)
	GCI	0.963 (0.2586)	0.954 (0.3293)	0.953 (0.1678)
	HPD	0.950 (0.2510)	0.939 (0.3242)	0.943 (0.1648)
(20,20)	ACI	0.915 (0.2186)	0.927 (0.2919)	0.937 (0.1476)
	BCI	0.972 (0.2754)	0.946 (0.3158)	0.940 (0.1486)
	GCI	0.952 (0.2254)	0.945 (0.2879)	0.937 (0.1448)
	HPD	0.945 (0.2186)	0.937 (0.2835)	0.943 (0.1427)

To check the correctness of fitting ED and IED to X and Y, we report one sample Kolmogorov-Smirnov (K-S) statistic with its p-value. The K-S statistics as well as their p-values in parenthesis of fitting the ED and IED to X and Y are D = 0.1257 (p-value = 0.3957) and D = 0.0947 (p-value = 0.7783), respectively. So, we can not reject the ED and IED for distributions of X and Y, respectively. Then, we can compute the various CIs utilized in this paper for parameter $R_{\tau,\theta}$. For HPD approach, the parameters a_1, a_2, b_1, b_2 in the prior density functions $\pi(\theta)$ and $\pi(\tau)$ are considered as $10, 4, 10, 0.1$, respectively. The MLE of $R_{\tau,\theta}$ is 0.6028 and the 95% CIs for reliability parameter $R_{\tau,\theta}$ based on ACI, BCI, GCI and HPD are as (0.5062, 0.6993), (0.5027, 0.7028), (0.4983, 0.6927) and (0.5183, 0.6906), respectively.

6 Conclusion

In this paper, we considered the problem of constructing the CI for reliability parameter in stress-strength models and we assumed that the stress and strength of model follow ED and IED distributions, respectively. We utilized four methods ACI, BCI, GCI and HPD for constructing the CI for reliability parameter $R_{\tau,\theta}$ and the performance of these methods were examined via some simulation studies and we see that the performance of GCI and HPD methods are better than ACI and BCI for all cases, but the CP of GCI approach is approximately close to confidence coefficient. So, we recommend to use GCI and also in the Bayesian viewpoint, HPD approach is suitable. At the end, all four approaches were used to constructing the CI for reliability parameter $R_{\tau,\theta}$ in

HNC real data example. From obtained results for this data set, we can conclude that combined radiotherapy and chemotherapy treatment has more effect on survival times of the HNC patients rather than the radiotherapy treatment alone.

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