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Research Paper

On some properties of transmuted Weibull distribution

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Abstract: A new generalized version of the Weibull distribution, which is called the perturbed Weibull distribution, is introduced in this paper. The present distribution provide enough flexibility for analyzing different types of data with increasing, decreasing, constant, bathtub shaped, unimodal, increasing-decreasing-increasing and decreasing-increasing-decreasing hazard functions in comparing with former extensions of the Weibull distribution. We study its properties including servival and hazard functions, moments, moment generating and characteristic functions, quantiles and Renyi entropy. Estimation of parameters using the methods of moment and maximum likelihood is obtained. We show the consistency of the moments and maximum likelihood estimators using some simulation study. Finally, the flexibility of the new distribution is illustrated in an application to two real data sets.

Keywords: Hazard rate function; Moment generating function; Weibull distribution; Likelihood function.
Mathematics Subject Classification (2010): 62E15, 62H10.

1 Introduction

The Weibull distribution, introduced by Weibull (1951), is a very popular model and it has been extensively used over the past decades for modeling data in reliability, engineering and biological studies. For see many different fields with many applications of this distribution one can see Murthy et al. (2003).

The hazard function of the Weibull distribution can only be increasing, decreasing or constant. Unfortunately the bathtub shaped and unimodal hazard functions which are prevalent in some phenomenon such as human mortality and machine life cycles cannot yield from the Weibull distribution. Many researchers have been introduced various extensions and modified forms of the Weibull distribution to resolving this captivity. See for example, Bebbington et al. (2007), Zhang and Xie (2011), Mudholkar and

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Srivastava (1993), Marshall and Olkin (1997), Xie et al. (2002), Lai and Xie (2003), Soliman (2012), Xie and Lai (1995), Famoye et al. (2005), Cordeiroet al. (2010), Silva (2010), Shahbaz (2010), Nadarajah et al. (2011), Aryal et al. (2011) and Mahmoudi and Sepahdar (2013).

In this work we present a new distribution called the transmuted Weibull (TW) distribution with three parameters. We use the method which proposed by Mirhosseini et al. (2011) to construct the new distribution. In our opinion, this new distribution offers several advantages. (i) The present distribution provide enough flexibility for analyzing different types of data with increasing, decreasing, constant, bathtub shaped, unimodal, increasing-decreasing-increasing and decreasing-increasing-decreasing hazard functions in comparing with former extensions of the Weibull distribution. (ii) The TW distribution has a compact form and all the moments can be computed explicitly. Therefore mean, variance, skewness, kurtosis, moment generating function (mgf), characteristic function and hazard rate function, all can be computed explicitly. (iii) It has a natural interpretation in terms of the lifetime of a system with two series or parallel components.

This study examines various properties of this model. The paper is organized as follows:
In Section 2, we introduce the cumulative distribution function (cdf) of the TW distribution. The survival and hazard rate functions are obtained in section 3. Section 4 provides some properties of TW distribution such as moments, moment generating function, characteristic function, quantile and Renyi entropy. In Section 5, we discuss moments and maximum likelihood estimation and calculate the elements of the observed information matrix. Simulation study is given in Section 6. Application of the TW distribution to two real data sets is given in Section 7. Finally, Section 8 concludes the paper.

2 The transmuted Weibull distribution

Let X_1 and X_2 be two independent and identically distributed random variables of Weibull distribution with cdf $F(x) = 1 - e^{-\lambda x^{\beta}}$. For $-1 \leq \alpha \leq 1$, consider the random variable Y defined by

$$Y = \begin{cases} X_{(1)} & \text{w.p. } \frac{1+\alpha}{2}, \\ X_{(2)} & \text{w.p. } \frac{1-\alpha}{2}, \end{cases}$$

where $X_{(1)} = \min(X_1, X_2)$ and $X_{(2)} = \max(X_1, X_2)$ are the corresponding order statistics of X_1 and X_2. The cdf of Y, denoted by G, is given by

$$G(x) = 1 + (\alpha - 1)e^{-\lambda x^{\beta}} - \alpha e^{-2\lambda x^{\beta}}, \tag{1}$$

where $-1 \leq \alpha \leq 1$, $\beta > 0$ and $\lambda > 0$. The probability density function (pdf) corresponding to (1) is

$$g(x) = \lambda \beta x^{\beta-1} e^{-\lambda x^{\beta}} \left[(1-\alpha) + 2\alpha e^{-\lambda x^{\beta}}\right], \tag{2}$$

If X is a random variable with pdf (2), we write $X \sim TW(\alpha, \beta, \lambda)$. The Weibull (W) and generalize exponential (GE) distributions are clearly sub-models of (2). The sub-models can be immediately defined from Table 1.

Table 1: Some sub-models of the TW distribution.

distribution	α	λ	β
EW$(2, \lambda, \beta)$	-1	-	-
W(λ, β)	0	-	-
W$(2\lambda, \beta)$	1	-	-
E(2λ)	1	-	1
PR(α, λ)	-	-	2
ER$(2, \lambda)$	-1	-	2
R(2λ)	1	-	2

The density function of the TW distribution in (2) can take various forms depending on the values of the shape parameters α and β. In particular, for $-1 < \alpha < 1,\ 0 < \beta < 1$ the density function is a decreasing function, whereas for $-1 < \alpha < 1,\ \beta > 1$, the density function becomes a skewed unimodal density. Also, for α and λ fixed, as the shape parameter β increases, the density function becomes more and more symmetric. It can be shown that

$$\lim_{x \to 0} g(x) = \begin{cases} \infty & \beta < 1, \\ 1 + \alpha & \beta = 1, \\ 0 & \beta > 1, \end{cases}$$

and $\lim_{x \to \infty} g(x) = 0$. Figure 1 illustrates some possible shapes of the probability density function (2). We have the following theorem.

Theorem 2.1. *The TW density function is log-concave (unimodal) if* $-1 < \alpha < 1$ *and* $\beta > 1$, *and it is log-convex if* $-1 < \alpha < 1$ *and* $0 < \beta < 1$.

Proof. See Appendix. □

Figure 1 shows the graphs of density function of the TW distribution for different values α, β and $\lambda = 1$.

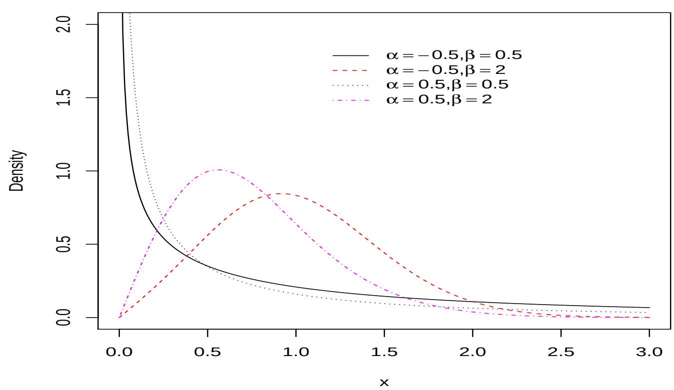


Figure 1: Graphs of density function of the TW distribution for different values α, β and $\lambda = 1$.

3 Hazard rate and survival functions

The survival and hazard rate functions defined by $\overline{G}(x) = 1 - G(x)$ and $h(x) = \frac{g(x)}{s(x)}$, respectively are the two important quantity characterizing life phenomena for the TW distribution, $h(x)$ and $s(x)$ take the forms

$$\overline{G}(x) = (1-\alpha)e^{-\lambda x^{\beta}} + \alpha e^{-2\lambda x^{\beta}}, \tag{3}$$

and

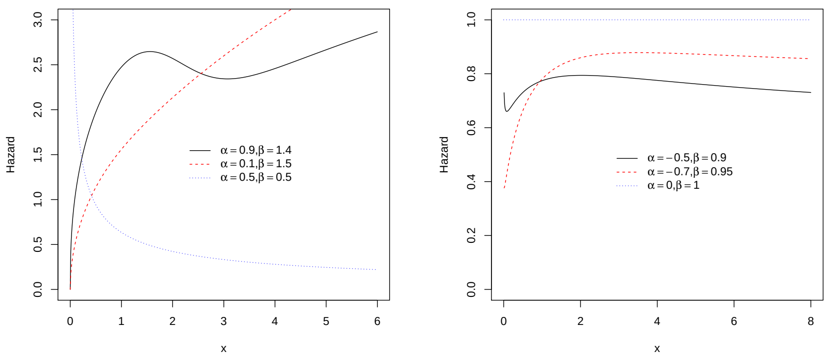
$$h_G(x) = \frac{\lambda\beta x^{\beta-1}(1-\alpha+2\alpha e^{-\lambda x^{\beta}})}{1-\alpha+\alpha e^{-\lambda x^{\beta}}}. \tag{4}$$

It is obvious

$$\lambda\beta x^{\beta-1} \leq h_G(x) \leq (1+\alpha)\lambda\beta x^{\beta-1}, \qquad (0 \leq \alpha \leq 1),$$

$$(1+\alpha)\lambda\beta x^{\beta-1} \leq h_G(x) \leq \lambda\beta x^{\beta-1}, \qquad (-1 \leq \alpha \leq 0).$$

The graphs of hazard rate function of the TW distribution for different values α, β and $\lambda = 1$ are shown in Figure 2. We see that the TW distribution allows for all possible hazard shapes :constant, increasing, decreasing, (upside-down) unimodal, bathtub, increasing-decreasing-increasing and decreasing-increasing-decreasing failure rates.



(a) (b)

Figure 2: Graphs of hazard function of the TW distribution for different values α, β and $\lambda = 1$.

4 Properties

In this section, we provide some properties for transmuted Weibull distribution.

4.1 Moments

Moments play a basic role in finding the important properties of distributions such as skewness, kurtosis, asymmetric and etc. In this part, we derive the rth moment and the rth central moment of the TW distribution.

Proposition 4.1. *If the random variable X has the TW distribution (2) then*

$$\mu'_r = E(X^r) = \Gamma(\frac{r}{\beta}+1)\left(\frac{1-\alpha}{\lambda^{r/\beta}} + \frac{\alpha}{(2\lambda)^{r/\beta}}\right). \tag{5}$$

Proof. According to Equation (2), the rth moment about zero of the random variable X with TW distribution is given by

$$\begin{aligned} E(X^r) &= (1-\alpha)\int_0^\infty \lambda\beta x^{\beta+r-1}e^{-\lambda x^\beta}dx + \alpha\int_0^\infty 2\lambda\beta x^{\beta+r-1}e^{-2\lambda x^\beta}dx \\ &= (1-\alpha)I + \alpha II, \end{aligned} \tag{6}$$

change the variable $y = \lambda x^\beta$ makes I and II as follow

$$I = \int_0^\infty \left(\frac{y}{\lambda}\right)^{r/\beta} e^{-y}dy = \frac{1}{\lambda^{r/\beta}}\Gamma(\frac{r}{\beta}+1), \tag{7}$$

$$II = \int_0^\infty \left(\frac{y}{\lambda}\right)^{r/\beta} e^{-2y}dy = \frac{1}{(2\lambda)^{r/\beta}}\Gamma(\frac{r}{\beta}+1). \tag{8}$$

The result in (5) is obtained by substituting from (7) and (8) in (6). □

Corollary 4.2. *If the random variable X has the TW distribution then $E(X^n)$ is decreasing in α. In particulare for $\alpha = 0$, -1 and 1 we have*

$$E_{\alpha=1}(X^n) \le E_{\alpha=0}(X^n) \le E_{\alpha=-1}(X^n).$$

Corollary 4.3. *If the random variable X has the TW distribution (2) then*

$$\begin{aligned} Var(X) &= \frac{1-\alpha}{\lambda^{2/\beta}}\left\{\Gamma(\frac{2}{\beta}+1) - \left[1-\alpha-\frac{2\alpha}{2^{1/\beta}}\right]\Gamma^2(\frac{1}{\beta}+1)\right\} \\ &\quad + \frac{\alpha}{(2\lambda)^{2/\beta}}\left\{\Gamma(\frac{2}{\beta}+1) - \alpha\Gamma^2(\frac{1}{\beta}+1)\right\}. \end{aligned}$$

By using the expression $\mu_r = E(X-\mu'_1)^r = \sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}\mu'_j(\mu'_1)^{r-j}$, the rth central moment of X can be calculated as

$$\mu_r = \sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}\mu'_j(\mu'_1)^{r-j}$$

$$= \sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}\Gamma(\frac{j}{\beta}+1)\left(\frac{1-\alpha}{\lambda^{j/\beta}}+\frac{\alpha}{(2\lambda)^{j/\beta}}\right) \times\left\{\Gamma(\frac{1}{\beta}+1)\left(\frac{1-\alpha}{\lambda^{1/\beta}}+\frac{\alpha}{(2\lambda)^{1/\beta}}\right)\right\}^{r-j},$$

where $\mu_1' = E(X)$ is the expectation of X.

Other important indices of the shape of the distribution are the skewness, $\gamma_1 = \frac{\mu_3}{\sigma^3}$ and kurtosis, $\gamma_2 = \frac{\mu_4}{\sigma^4}$, where μ_3, μ_4 and σ are the third and fourth central moment about the mean and standard deviation, respectively. Table 2 gives the values of $E(X)$, $Var(X)$, γ_1 and γ_2 of the TW distribution for $\beta = 3$, $\lambda = 2$ and different values of α.

Table 2: Expectation, variance, skewness and kurtosis of the TW distribution for various values of α.

α	-1	-0.75	-0.5	-0.25	0	0.25	0.5	0.75	1
E(X)	0.8546	0.8184	0.7815	0.7450	0.7086	0.6722	0.6357	0.5991	0.5628
Var(X)	0.0490	0.0560	0.0632	0.0660	0.0660	0.0642	0.0592	0.0518	0.0416
γ_1	0.1843	0.2265	0	0.1180	0.2359	0.2458	0.3541	0.3732	0.2120
γ_2	3.040	2.296	2.954	2.916	2.732	2.838	2.967	3.131	2.542

4.2 Moment generating and characteristic functions

Here, we derive the moment generating and the characteristic functions of the TW distribution. The moment generating function (mgf) of the random variable X with TW distribution is given by

$$M_X(t) = \sum_{j=0}^{\infty}\frac{t^j}{j!}E(X^j) = \sum_{j=0}^{\infty}\frac{t^j}{j!}\Gamma(\frac{j}{\beta}+1)\left(\frac{1-\alpha}{\lambda^{j/\beta}}+\frac{\alpha}{(2\lambda)^{j/\beta}}\right).$$

The characteristic function of X defined by $\phi(t) = E(e^{itX})$ takes the form

$$\phi(t) = \sum_{j=0}^{\infty}\frac{(it)^j}{j!}\Gamma(\frac{j}{\beta}+1)\left(\frac{1-\alpha}{\lambda^{j/\beta}}+\frac{\alpha}{(2\lambda)^{j/\beta}}\right), \tag{9}$$

where $i = \sqrt{-1}$ is the complex imaginary unit.

4.3 Quantile

Let X be a random variable of TW distributions, by straightforward calculating, we have

$$G^{-1}(q) = \left(-\frac{1}{\lambda}\ln\frac{\alpha-1+\sqrt{(1+\alpha)^2-4\alpha q}}{2\alpha}\right)^{\frac{1}{\beta}}.$$

In particular, the median of G is given by

$$G^{-1}(0.5) = \left(-\frac{1}{\lambda} \ln(\frac{\alpha - 1 + \sqrt{1+\alpha^2}}{2\alpha}) \right)^{\frac{1}{\beta}}.$$

4.4 Renyi entropy

An entropy of a random variable X is a measure of variation of the uncertainty. One of the popular entropy measure is the Renyi entropy Renyi (1961) defined by

$$I_R(\gamma) = \frac{1}{1-\gamma} \log \left\{ \int_0^\infty g^\gamma(x) dx \right\}, \tag{10}$$

where $g(x)$ is the pdf of X, $\gamma > 0$ and $\gamma \neq 0$. If a random variable X has the TW distribution, we can write from (2)

$$\begin{aligned} H(\gamma) &= \int_0^\infty g^\gamma(x) dx \\ &= \int_0^\infty (\lambda\beta)^\gamma x^{\gamma(\beta-1)} \exp(-\lambda\gamma) x^\beta \left[(1-\alpha) + 2\alpha \exp(-\lambda x^\beta)\right]^\gamma dx \\ &= \frac{(1-\alpha)^{\gamma+1}}{2\alpha} \int_0^\infty (\lambda\beta)^{\gamma-1} x^{(\gamma-1)(\beta-1)} \exp\{-\lambda(\gamma-1)x^\beta\}(1+u)^\gamma du, \end{aligned}$$

where $u = \frac{2\alpha}{1-\alpha}\exp(-\lambda x^\beta)$. Expanding the binomial terms, we can write

$$\begin{aligned} H(\gamma) &= B \int_0^\infty x^{(\gamma-1)(\beta-1)} \exp\{-\lambda(\gamma-1)x^\beta\} \sum_{j=0}^\infty \binom{\gamma}{j} u^j du \\ &= B \sum_{j=0}^\infty \binom{\gamma}{j} (\frac{2\alpha}{1-\alpha})^j \int_0^\infty x^{(\gamma-1)(\beta-1)} \exp\{-\lambda(\gamma+j-1)x^\beta\} dx, \end{aligned}$$

where $B = \frac{(1-\alpha)^{\gamma+1}}{2\alpha}(\lambda\beta)^{\gamma-1}$. But $\int_0^\infty x^{a-1}\exp(-\delta x^c)dx = c^{-1}\delta^{-a/c}\Gamma(\frac{a}{c})$, and then

$$H(\gamma) = \lambda^{\gamma-1}\beta^{\gamma-2}\Gamma(\rho) \sum_{j=0}^\infty \binom{\gamma}{j} (1-\alpha)^{\gamma-j+1}(2\alpha)^{j-1}\left[\lambda(\gamma+j-1)\right]^{-\rho}, \tag{11}$$

where $\rho = \frac{(\gamma-1)(\beta-1)+1}{\beta}$. So, the Renyi entropy follows from Equation (11) as $I_R(\gamma) = \frac{1}{1-\gamma}\log\{H(\gamma)\}$.

5 Estimation

Here, we consider estimation of the parameters of the TW distribution by the method of moments and maximum likelihood and provide expressions for the associated Fisher information matrix.

5.1 Method of moments

Suppose that $x_1, \cdots, x_n$ is a random sample from the pdf (2). For the moments estimation, let $m_1 = (1/n)\sum_{j=1}^n x_j$, $m_2 = (1/n)\sum_{j=1}^n x_j^2$ and $m_3 = (1/n)\sum_{j=1}^n x_j^3$. By equating three first moments of (2) with the sample moments, we have the equations

$$\Gamma(\frac{1}{\beta}+1)\left(\frac{1-\alpha}{\lambda^{1/\beta}}+\frac{\alpha}{(2\lambda)^{1/\beta}}\right)=m_1,$$
$$\Gamma(\frac{2}{\beta}+1)\left(\frac{1-\alpha}{\lambda^{2/\beta}}+\frac{\alpha}{(2\lambda)^{2/\beta}}\right)=m_2,$$
$$\Gamma(\frac{3}{\beta}+1)\left(\frac{1-\alpha}{\lambda^{3/\beta}}+\frac{\alpha}{(2\lambda)^{3/\beta}}\right)=m_3.$$

These equations cannot be solved analytically and statistical software can be used to obtain the MMEs numerically. A simulation method based on `R` software being used in obtaining the MMEs of the parameters α, β and λ in Section 6.

5.2 Method of maximum likelihood

In this part we consider estimation of the parameters of the TW distribution by the method of maximum likelihood. Consider that X follows the TW distribution and let $\theta = (\alpha, \beta, \lambda)^T$ be the parameter vector. The log-likelihood $\ell = \ell(\alpha, \beta, \lambda)$ for a single observation x of X is

$$\ell(\theta) = \ln(\lambda\beta) + (\beta-1)\ln(x) - \lambda x^\beta + \ln\left(1-\alpha+2\alpha \mathrm{e}^{-\lambda x^\beta}\right).$$

The components of the unit score vector $\mathbf{U} = (\frac{\partial\ell}{\partial\alpha}, \frac{\partial\ell}{\partial\beta}, \frac{\partial\ell}{\partial\lambda})^T$ are obtained by the following differentiations

$$\begin{aligned}
\frac{\partial\ell}{\partial\alpha} &= \frac{-1+2\mathrm{e}^{-\lambda x^\beta}}{1-\alpha+2\alpha\mathrm{e}^{-\lambda x^\beta}},\\
\frac{\partial\ell}{\partial\beta} &= \frac{1}{\beta}+\ln(x)-\lambda x^\beta\ln(x)-2\frac{\alpha\lambda x^\beta\ln(x)\,\mathrm{e}^{-\lambda x^\beta}}{1-\alpha+2\,\alpha\mathrm{e}^{-\lambda x^\beta}},\\
\frac{\partial\ell}{\partial\lambda} &= \frac{1}{\lambda}-x^\beta-2\frac{\alpha x^\beta\mathrm{e}^{-\lambda x^\beta}}{1-\alpha+2\alpha\mathrm{e}^{-\lambda x^\beta}}.
\end{aligned}$$

For a random sample $x = (x_1, \ldots, x_n)^T$ of size n from X, the total log-likelihood is $\ell_n = \ell_n(\theta) = \sum_{i=1}^n \ell^{(i)}(\theta)$, where $\ell^{(i)}(\theta)$ is the log-likelihood for the ith observation ($i = 1, \ldots, n$). The total score function is $\mathbf{U}_n = \sum_{i=1}^n \mathbf{U}^{(i)}$, where $\mathbf{U}^{(i)}$ has the form given before for $i = 1, \ldots, n$. The MLE $\widehat{\theta}$ of θ is the solution of the system of nonlinear equations $\mathbf{U}_n = 0$. Thus we have

$$\begin{aligned}
\frac{\partial\ell_n}{\partial\alpha} &= \sum_{i=1}^n \frac{-1+2\,\mathrm{e}^{-\lambda\,x_i^\beta}}{1-\alpha+2\,\alpha\,\mathrm{e}^{-\lambda\,x_i^\beta}},\\
\frac{\partial\ell_n}{\partial\beta} &= \frac{n}{\beta}+\sum_{i=1}^n \ln(x_i)-\lambda\sum_{i=1}^n x_i^\beta\ln(x_i)+\sum_{i=1}^n -2\,\frac{\alpha\,\lambda\,x_i^\beta\ln(x_i)\,\mathrm{e}^{-\lambda\,x_i^\beta}}{1-\alpha+2\,\alpha\,\mathrm{e}^{-\lambda\,x_i^\beta}},
\end{aligned}$$

$$\frac{\partial \ell_n}{\partial \lambda} \;=\; \frac{n}{\lambda} - \sum_{i=1}^{n} x_i^{\beta} + \sum_{i=1}^{n} -2\, \frac{x_i^{\beta} \alpha\, \mathrm{e}^{-\lambda\, x_i^{\beta}}}{1-\alpha+2\,\alpha\, \mathrm{e}^{-\lambda\, x_i^{\beta}}}.$$

The second-order derivatives of this log-likelihood with respect to α, β and λ are

$$\begin{aligned}
\frac{\partial^2 \ell_n}{\partial \alpha^2} &= -\sum_{i=1}^{n} \frac{\left(-1+2\,\mathrm{e}^{-\lambda\, x_i^{\beta}}\right)^2}{\left(1-\alpha+2\,\alpha\,\mathrm{e}^{-\lambda\, x_i^{\beta}}\right)^2},\\
\frac{\partial^2 \ell_n}{\partial \beta^2} &= -\frac{n}{\beta^2} - \lambda \sum_{i=1}^{n} x_i^{\beta} \left(\ln x_i\right)^2\\
&\quad - 2\alpha\lambda \sum_{i=1}^{n} \frac{x_i^{\beta} \left(\ln x_i\right)^2 \mathrm{e}^{-\lambda x_i^{\beta}} \left(1-\alpha+2\alpha \mathrm{e}^{-\lambda x_i^{\beta}} - \lambda x_i^{\beta}(1-\alpha)\right)}{\left(1-\alpha+2\alpha \mathrm{e}^{-\lambda x_i^{\beta}}\right)^2},\\
\frac{\partial^2 \ell_n}{\partial \lambda^2} &= -\frac{n}{\lambda^2} + 2\alpha(1-\alpha) \sum_{i=1}^{n} \frac{x_i^{2\beta} \mathrm{e}^{-\lambda x_i^{\beta}}}{\left(1-\alpha+2\alpha \mathrm{e}^{-\lambda x_i^{\beta}}\right)^2},\\
\frac{\partial^2 \ell_n}{\partial \alpha \partial \beta} &= -2\lambda \sum_{i=1}^{n} \frac{x_i^{\beta} \ln x_i \mathrm{e}^{-\lambda x_i^{\beta}}}{\left(1-\alpha+2\alpha \mathrm{e}^{-\lambda x_i^{\beta}}\right)^2},\\
\frac{\partial^2 \ell_n}{\partial \alpha \partial \lambda} &= -2 \sum_{i=1}^{n} \frac{x_i^{\beta} \mathrm{e}^{-\lambda x_i^{\beta}}}{\left(1-\alpha+2\alpha \mathrm{e}^{-\lambda x_i^{\beta}}\right)^2},\\
\frac{\partial^2 \ell_n}{\partial \beta \partial \lambda} &= -\sum_{i=1}^{n} x_i^{\beta} \left(\ln x_i\right)^2 - 2\alpha \sum_{i=1}^{n} \frac{x_i^{\beta} \ln x_i \mathrm{e}^{-\lambda x_i^{\beta}} \left(1-\alpha+2\alpha \mathrm{e}^{-\lambda x_i^{\beta}} - \lambda x_i^{\beta}(1-\alpha)\right)}{\left(1-\alpha+2\alpha \mathrm{e}^{-\lambda x_i^{\beta}}\right)^2}.
\end{aligned}$$

Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\widehat{\theta}-\theta)$ is $N_3(0, I_n^{-1})$ where

$$I_n = -\frac{1}{n} \left[\begin{array}{ccc} E(\frac{\partial^2 \ell_n}{\partial \alpha^2}) & E(\frac{\partial^2 \ell_n}{\partial \alpha \partial \beta}) & E(\frac{\partial^2 \ell_n}{\partial \alpha \partial \lambda}) \\ E(\frac{\partial^2 \ell_n}{\partial \beta \partial \alpha}) & E(\frac{\partial^2 \ell_n}{\partial \beta^2}) & E(\frac{\partial^2 \ell_n}{\partial \beta \partial \lambda}) \\ E(\frac{\partial^2 \ell_n}{\partial \lambda \partial \alpha}) & E(\frac{\partial^2 \ell_n}{\partial \lambda \partial \beta}) & E(\frac{\partial^2 \ell_n}{\partial \lambda^2}) \end{array} \right].$$

This asymptotic behavior remains valid if I_n is replaced by the average sample information matrix, say $\widehat{I}_n$. The estimated asymptotic multivariate normal $N_3((\alpha,\beta,\lambda)^T$, $\widehat{I}_n^{-1})$ distribution can be used to construct approximate confidence intervals and hyphothesis test for the parameters. An asymptotic confidence interval with significance level γ for each parameter θ_r is

$$(\widehat{\theta}_r - z_{\gamma/2}\sqrt{\widehat{I}_n^{rr}}, \widehat{\theta}_r + z_{\gamma/2}\sqrt{\widehat{I}_n^{rr}}),$$

where $\widehat{I}_n^{rr}$ is the rth diagonal element of I_n^{-1} for $r = 1, \cdots, 3$ and $z_{\gamma/2}$ is the quantile $1-\gamma/2$ of the standard normal distribution.

6 Simulation

We carried out Monte Carlo simulations in order to compare the performance of all the estimators proposed in the preceding section. We used several different sample sizes and parameter values. The programs were written in `R`.

The sample sizes considered were n =20, 50, 70, 100 and the shape parameter was taken as α = -0.7, -0.3, 0.3 and 0.7 . In all cases, we set $\beta = 3$ and $\lambda = 2$. For each model parameters and for each sample size, we compute the MMEs and MLEs of α, β and λ. We repeat this process 1000 times and compute the average estimators (AE), the mean squared errors (MSE). The results are reported in Table 3.

Table 3: Simulated average MMEs and MLEs, mean squared errors of the parameters of the TW distribution.

		MME							
		$\alpha = -0.7$		$\alpha = -0.3$		$\alpha = 0.3$		$\alpha = 0.7$	
n		AE	MSE	AE	MSE	AE	MSE	AE	MSE
20	α	-0.655	0.151	-0.170	0.309	0.241	0.256	0.527	0.227
	β	3.188	0.725	3.133	0.587	3.049	0.394	3.016	0.399
	λ	2.122	0.268	2.059	0.309	2.177	0.437	2.144	0.251
50	α	-0.669	0.095	-0.22	0.258	0.204	0.218	0.506	0.208
	β	3.102	0.361	3.044	0.271	2.942	0.157	2.964	0.133
	λ	2.016	0.095	1.952	0.185	2.133	0.342	2.201	0.256
70	α	-0.672	0.088	-0.222	0.233	0.195	0.21	0.554	0.205
	β	3.073	0.316	3.038	0.212	2.951	0.116	2.968	0.112
	λ	1.993	0.077	1.945	0.153	2.122	0.310	2.213	0.259
100	α	-0.698	0.078	-0.256	0.202	0.206	0.198	0.617	0.155
	β	3.062	0.252	3.030	0.163	2.966	0.099	2.973	0.072
	λ	1.995	0.059	1.964	0.132	2.100	0.277	2.159	0.199
		MME							
		$\alpha = -0.7$		$\alpha = -0.3$		$\alpha = 0.3$		$\alpha = 0.7$	
n		AE	MSE	AE	MSE	AE	MSE	AE	MSE
20	α	-0.572	0.19	-0.174	0.231	0.164	0.193	0.458	0.294
	β	3.373	0.725	3.303	0.587	3.119	0.394	3.149	0.399
	λ	2.058	0.322	2.037	0.369	2.363	0.709	2.440	1.143
50	α	-0.581	0.178	-0.179	0.217	0.185	0.19	0.462	0.194
	β	3.218	0.361	3.132	0.271	2.932	0.157	2.957	0.133
	λ	1.947	0.137	1.953	0.189	2.185	0.349	2.350	0.367
70	α	-0.583	0.166	-0.187	0.214	0.190	0.182	0.474	0.176
	β	3.181	0.316	3.112	0.212	2.949	0.116	2.978	0.112
	λ	1.949	0.117	1.957	0.175	2.148	0.274	2.323	0.342
100	α	-0.599	0.141	-0.194	0.185	0.201	0.178	0.559	0.136
	β	3.144	0.252	3.09	0.163	2.984	0.099	2.983	0.072
	λ	1.953	0.093	1.962	0.139	2.118	0.252	2.276	0.285

7 Application

To show the superiority of the TW distribution, we compare the results of fitting the TW distribution to some models such as Weighted Wiebul (WW)Shahbaz (2010), Kumaraswamy Weibull (KW)Cordeiroet al. (2010), Weibull (W) and Generalized Exponential (GE)Gupta and Kundu (2007) distributions, using two real data sets. The required numerical evaluations are implemented using the `R` software.

In many applications, there is a qualitative information about the failure rate function shape, which can help in selecting a particular model. In this context, a device called the total time on test (TTT) plot is useful Aarset (1987). The TTT-plot is obtained by plotting $G(r/n) = [(\sum_{i=1}^{n} T_{i:n}) + (n-r)T_{r:n}]/(\sum_{i=1}^{n} T_{i:n})$, where $r = 1, \dots, n$ and T_i, $i = 1, \dots, n$, are the order statistics of the sample, against r/n Mudholkar et al. (1996).

The scaled TTT transform is convex (concave) if the hazard rate is decreasing (increasing), and for bathtub (unimodal) hazard rates, the scaled TTT transform is first convex (concave) and then concave (convex).

The first data we consider an uncensored data set from Hinkley (1977). This data consists of thirty successive values of March precipitation (in inches) in Minneapolis/St Paul. The TTT plot for this data in Figure 3 shows an increasing hazard rate function and indicates that appropriateness of the PHN distribution to fit this data.

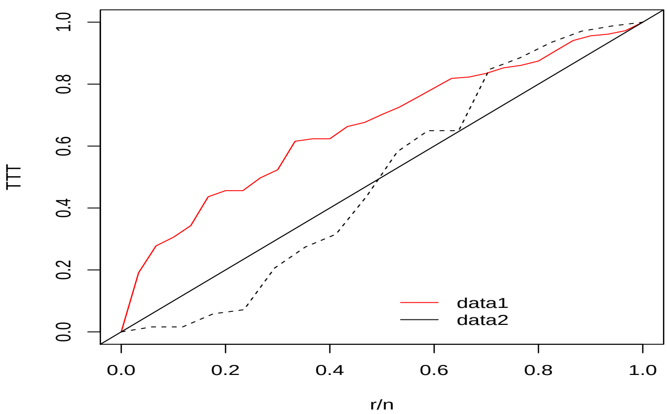


Figure 3: TTT-plots on Hinkley data (Data 1) and Feigl and Zelen (Data 2).

Table 4 lists the MLEs of the parameters and their standard errors, the values of K-S (Kolmogorov-Smirnov) statistic with its respective p-value, -2log-likelihood (-2L), the Cramer-von Mises (CM) test and Watson (WA) test. The CM and WA test statistics are described in details in Chen and Balakrishnan (1995) and Watson (1961), respectively. In general, the smaller the values of CM and WA, the better the fit to the data. From the values of these statistics, we conclude that the TW distribution provides a better fit to this data than the WW, KW, W and GE models.

Figure 4 plots the empirical and estimated survival functions of the transmuted Weibull distribution. Further, Figure 4 plots the histogram of the data and the fitted TW distribution. We conclude that the TW distribution provides a good fit for these data. As a second application, we consider observed survival times and white blood

Table 4: MLEs, SE, -2L, K-S statistics, CM, WA and p-values for Hinkley's data.

Model	MLEs (SE)	-2L	K-S	CM	WA	p-value
TW	$\widehat{\alpha}=0.617, \widehat{\beta}=1.946, \widehat{\lambda}=0.195$ (0.653) (0.270) (0.104)	76.798	0.056	0.096	0.095	1
WW	$\widehat{\alpha}=11.828, \widehat{\beta}=1.605, \widehat{\lambda}=0.406$ (32.311) (0.444) (0.232)	76.064	0.061	0.094	0.094	0.9999
KW	$\widehat{a}=11.321, \widehat{b}=9.430,$ (91.127) (79.895) $\widehat{c}=0.392, \widehat{\lambda}=2.138$ (1.926) (28.208)	76.180	0.061	0.095	0.095	0.9999
W	$\widehat{\beta}=1.809, \widehat{\lambda}=0.315$ (0.249) (0.091)	77.287	0.069	0.101	0.099	0.9988
GE	$\widehat{\alpha}=3.461, \widehat{\lambda}=1.167$ (1.064) (0.215)	76.189	0.065	0.096	0.096	0.9995

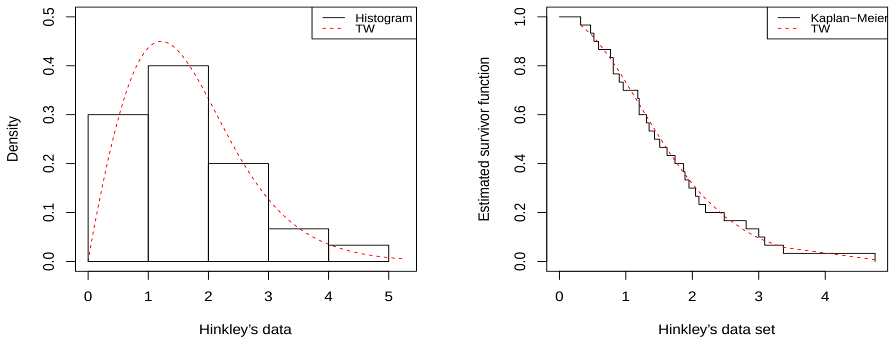
Table 5: MLEs, SE, -2L, K-S statistics, CM, WA and p-values for Feigl and Zelen's data.

Model	MLEs(SE)	-2L	K-S	CM	WA	p-value
TW	$\widehat{\alpha}=-0.215, \widehat{\beta}=0.848, \widehat{\lambda}=0.035$ (0.555) (0.209) (0.038)	174.088	0.145	0.153	0.150	0.8653
WW	$\widehat{\alpha}=0.010, \widehat{\beta}=0.550, \widehat{\lambda}=0.233$ (1.114) (0.107) (0.172)	175.546	0.161	0.174	0.165	0.7684
KW	$\widehat{a}=15.358, \widehat{b}=257.228,$ (101.127) (703.656) $\widehat{c}=0.102, \widehat{\lambda}=0.100$ (8.134) (36.452)	175.352	0.159	0.171	0.163	0.7838
W	$\widehat{\beta}=0.885, \widehat{\lambda}=0.0270$ (0.181) (0.023)	174.219	0.148	0.158	0.153	0.8506
GE	$\widehat{\alpha}=0.757, \widehat{\lambda}=0.013$ (0.225) (0.004)	173.642	0.148	0.155	0.145	0.8531

counts (AG Positive) from Feigl and Zelen (1965).
Figure 3 shows that the TTT-plot for these data has first a convex shape and then a concave shape. It indicates a bathtub shaped hazard rate function. the MLEs of the parameters and their standard errors, the values of K-S statistic with its respective p-value, -2L, CM and WA tests are listed in Table 5. From these values, we note that the TW model is better than the other distributions. Plots of the empirical and estimated survival functions of the TW distribution, and the histogram of the data and the fitted TW distribution are given in Figures 5.

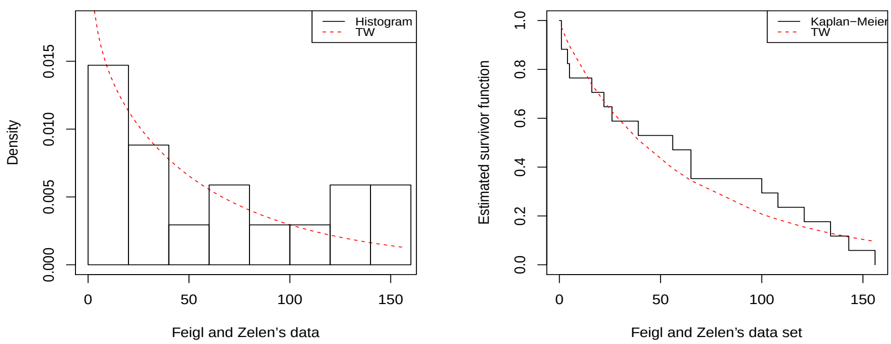
8 Conclusion

We introduce a three parameter lifetime distribution called "transmuted Weibull (TW) distribution". The new model extends several distributions widely used in the lifetime



(a) (b)

Figure 4: a) Estimated densities of the TW models for Hinkley's data. b) Estimated survival function from the fitted TW distribution and the empirical survival for Hinkley's data.



(a) (b)

Figure 5: a) Estimated densities of the TW models for Feigl and Zelen's data. b) Estimated survival function from the fitted TW distribution and the empirical survival for Feigl and Zelen's data.

literature and is more flexible than the submodels. The proposed distribution could have constant, increasing, decreasing, bathtub and unimodal hazard rate functions. We derived important properties of the new distribution and obtained closed-form expressions for its moments. Also, we discuss method of moments and maximum likelihood estimations. Applications of the TW distribution to two real data sets are given to show that the new distribution provides consistently better fits than the Weighted Wiebul (WW), Kumaraswamy Weibull (KW), Weibull (W) and Generalized Exponential (GE) distributions. We hope that this generalization may attract wider applications in the lifetime analysis.

Appendix

Proof of Theorem 2.1. Instead of working with $g(x)$, it is easy to cosider $\log(g(x))$. λ is the scale parameter and does not change the shape of the density and failure rate functions, thus without of generality we take $\lambda = 1$.
We have

$$\frac{d}{dx}\log(g(x)) = \frac{[(\beta - 1 - \beta x^{\beta})(1 - \alpha + 2\alpha e^{-x^{\beta}}) - 2\alpha\beta x^{\beta} e^{-x^{\beta}}]}{x(1 - \alpha + 2\alpha e^{-x^{\beta}})}.$$

The denominator of the above expreesion is positive, so its sign is the same as the sign of $s(x) = (\beta - 1 - \beta x^{\beta})(1 - \alpha + 2\alpha e^{-x^{\beta}}) - 2\alpha\beta x^{\beta} e^{-x^{\beta}}$. Consider the following cases:

a) Let $0 < \beta < 1$ and $0 < \alpha < 1$, since $(1 - \alpha + 2\alpha e^{-x^{\beta}}) > 0$ thus $\frac{d}{dx}\log(g(x)) < 0$ and hence $g(x)$ is strictly decreasing in this case.

b) Let $\beta > 1$ and $-1 < \alpha < 1$. Consider $g_1(x) = (\beta - 1 - \beta x^{\beta})(1 - \alpha + 2\alpha e^{-x^{\beta}})$ and $g_2(x) = 2\alpha\beta x^{\beta} e^{-x^{\beta}}$. Note that $0 = g_2(0) < g_1(0) = (\beta - 1)(\alpha + 1)$ and $-\infty = \lim_{x\to\infty} g_1(x) < \lim_{x\to\infty} g_2(x) = 0$, thus there exist x_0, such that for each $0 < x < x_0$, $s(x) = g_1(x) - g_2(x) > 0$ and for each $x > x_0$, $s(x) = g_1(x) - g_2(x) < 0$, thus $g(x)$ is firstly increasing then decreasing and therefore it is unimodal.

c) Let $0 < \beta < 1$ and $-1 < \alpha < 0$. In this case according to the previous case $(\beta-1)(\alpha+1) = g_1(0) < g_2(0) = 0$ and $-\infty = \lim_{x\to\infty} g_1(x) < \lim_{x\to\infty} g_2(x) = 0$, thus for each $0 < \beta < 1$ and $-1 < \alpha < 0$ and each $x > 0$, $\frac{d}{dx}\log(g(x)) < 0$ and hence $g(x)$ is a decreasing function in this case.

On the other hand, consider the change in variable $z = e^{-x^{\beta}}$, which implies $x = (-\log(z))^{1/\beta} = \phi(z)$, where $0 < z < 1,\ x > 0$. Now, rewriting the TW density as function of z, $\eta(z)$ say, we obtain

$$\eta(z) = f(\phi(z)) = \beta z(-\log(z))^{\frac{\beta-1}{\beta}}(1 - \alpha - 2\alpha z).$$

The result follows by noting that the first and second derivative of $\log(\eta(z))$, i.e.;

$$\frac{\partial}{\partial z}\log(\eta(z)) = \frac{\beta - 1}{\beta z\log(z)} + \frac{1}{z} + \frac{2\alpha}{1 - \alpha + 2\alpha z},$$
$$\frac{\partial^2}{\partial z^2}\log(\eta(z)) = \frac{(1 - \beta)(1 + \log(z))}{\beta(z\log(z))^2} - \frac{1}{z^2} - \frac{4\alpha^2}{(1 - \alpha + 2\alpha z)^2}.$$

□

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