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Research Paper

On the weighted dynamic cumulative residual entropy and dynamic cumulative past entropy with applications: A survey

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Abstract: The main measure of the uncertainty contained in random variable X is the Shannon entropy $H(X) = -E(\log(f(X))$. The cumulative entropy is an information measure which is alternative to the Shannon entropy and is connected with reliability theory. The cumulative residual entropy (CRE) introduced by Rao et al. (2004) is a generalized measure of uncertainty which is applied in reliability. Asadi and Zohrevand (2007) defined a dynamic version of the CRE by $\varepsilon(X,t)$. In this paper, weighted residual entropy and weighted cumulative residual entropy are discussed. The properties of weighted entropy, cumulative residual entropy, weighted residual entropy, weighted cumulative residual entropy, weighted past entropy, weighted cumulative past entropy, dynamic cumulative residual entropy, dynamic cumulative past entropy, are also given.

Keywords: Cumulative residual entropy; Cumulative past entropy; Dynamic cumulative residual and past entropy; Shannon entropy; Weighted entropy.
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1 Introduction

We live in an era of extreme uncertainties. An important measure of uncertainty associated with a random variable X is the notion of entropy which is introduced by Shannon (1948). Let X be an absolutely continuous nonnegative random variable with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. A well known measure of uncertainty associated with a random variable comes from the field of information theory and is called entropy. The most widely known measure of entropy, Shannon's entropy, is given by

$$H(X) = -E(\log(f(X)) = -\int_0^\infty f(x)\log(f(x))dx. \tag{1}$$

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The integrand function on the right-hand-side of (1) depends on x only via $f(x)$. Shannon's entropy gives equal importance or weight to the occurrence of every event. Shannon (1948) is considered to be the father of information theory and was the first that incorporated the term information entropy in an information systems for measuring the uncertainty associated with a random variable.

Definition 1.1. *(Differential entropy) The differential entropy H(X) of a continuous random variable X with pdf $f(x)$ is defined as*

$$H(X) = -\int_S f(x) \log f(x) dx$$

where S is the support region of the random variable.

Example 1.2. *Consider a uniform distribution over $[a, b]$, $a < b$. Then, Shannon's entropy is*

$$H(X) = -\int_0^\infty f(x) \log(f(x)) dx = \log(b - a).$$

The differential entropy plays the central role of information theory and a large number of research work has been reported in the literature. Rest of the paper is organized as follows. In Section 2, the definition of the weighted entropy (WE) and a description of its properties are given. In Section 3, we present weighted residual entropy (WRE), cumulative residual entropy (CRE), cumulative past entropy (CPE), weighted cumulative residual entropy (WCRE), dynamic cumulative residual entropy (DCRE), dynamic cumulative past entropy (DCPE) and related examples.

2 Weighted entropy

Weighted entropy, which is a generalization of classical entropy, has been proposed by Belis and Guiasu (1968), Guiasu (1971) and Guiasu (1986). Other measures of uncertainty as suitable generalizations or modifications of the classical entropy have been proposed in the recent literature such as the WE (Di Crescenzo and Longobardi, 2006) defined as

$$H^\omega(X) = -E(X \log(f(X))) = -\int_0^\infty x f(x) \log(f(x)) dx. \tag{2}$$

The factor X in the integrand of Equation (2) represents a weight which linearly emphasizes the occurrence of the event $\{X{=}x\}$. This is a "length-biased" shift-dependent information measure assigning greater importance to larger values of X. When the weight function depends on the length of the component, the resulting distribution is called length-biased weighted function (see Misagh, 2016; Akdeniz and Çabulk, 2017).

Example 2.1. *Suppose X and Y denote random variables with pdfs*

$$f(x) = \frac{1}{2}x, \qquad 0 < x < 2,$$
$$g(y) = \frac{1}{2}(2 - y), \quad 0 < y < 2,$$

respectively. By simple calculations, we have

$$H(X) = -\int_0^2 f(x)\log(f(x))dx = -\int_0^2 \frac{x}{2}\log(\frac{x}{2})dx = \frac{1}{2},$$
$$H(Y) = -\int_0^2 g(y)\log(g(y))dy = -\int_0^2 \frac{1}{2}(2-y)\log(\frac{1}{2}(2-y))dy = \frac{1}{2}.$$

Their Shannon entropies are identical. Therefore, the expected uncertainties for $f(x)$ and $g(y)$on the predictability of the outcomes of the X and Y are identical. But, we have

$$H^{\omega}(X) = -\int_0^2 xf(x)\log(f(x))dx = -\int_0^2 x\frac{1}{2}x\log(\frac{1}{2}x)dx = \frac{4}{9},$$
$$H^{\omega}(Y) = -\int_0^2 yg(y)\log(g(y))dy = -\int_0^2 y\frac{1}{2}(2-y)\log(\frac{1}{2}(2-y))dy = \frac{5}{9}.$$

Then $H^{\omega}(X) < H^{\omega}(Y)$. Hence, even though $H(X) = H(Y)$, the expected weighted uncertainty contain in of the $f(x)$ on the predictability of the outcome of X is less than that of $g(y)$ on the predictability of the outcome of Y.

3 Weighted residual entropy

The role of Shannon entropy as a measure of uncertainty in residual lifetime distributions has been studied by many researchers (Ebrahimi and Pellerey, 1995; Ebrahimi, 1996; Belzunce et al., 2004; Kumar et al., 2015). Let X be an absolutely continuous nonnegative random variable having cdf $F(x)$ and the survival function $\bar{F}(x) = 1 - F(x)$. In reliability theory, X represents the random lifetime of an item or system with survival function $\bar{F}(x)$ and the pdf $f(x)$. Suppose X denotes the lifetime of a component/system or of a living organism and $f(t)$ denotes the lifetime pdf. (see Nanda and Paul, 2006a) If a component is known to have survived to age t then Shannon entropy is no longer useful to measure the uncertainty of remaining lifetime of the component.

Ebrahimi (1996) defined the entropy for residual lifetime $X_t = (X - t|X > t)$ as a dynamic form of uncertainty called the residual entropy at time t and defined as

$$H(X;t) = -\int_t^{\infty} \frac{f(x)}{\bar{F}(t)}\log(\frac{f(x)}{\bar{F}(t)})dx = \log(\bar{F}(t)) - \frac{1}{\bar{F}(t)}\int_t^{\infty} f(x)\log(f(x))dx, \quad (3)$$

where $F(\infty) = 1$. Ebrahimi(1996) showed that $H(X,t)$ uniquely determines the distribution function $F(t)$. Obviously $H(X,0) = H(X)$. It is well known from (3) that units which exhibitless uncertainty in life times are more reliable and hence measure (3) has much relevance in characterizing, ordering and classifying life distributions according to its behavior.

We now make use of (2) to define weighted entropy for residual lifetime that is the weighted version of entropy (3), we have

$$H^{\omega}(X,t) = -\int_t^{\infty} x\frac{f(x)}{\bar{F}(t)}\log(\frac{f(x)}{\bar{F}(t)})dx. \quad (4)$$

3.1 Cumulative residual entropy

In order to estimate the Shannon entropy for a continuous random variable, one has to obtain the estimation of pdf, which is not a trivial task. Recently, Rao et al. (2004) introduced an alternative measure of uncertainty called CRE which is based on the survival (reliability) function $\bar{F}(x) = 1 - F(x)$ instead of the pdf $f(x)$ used in the classical Shannon's entropy (1) (see Toomaj et al., 2017).

As an alternative measure of uncertainty, the CRE of X is defined by

$$\varepsilon(X) = -\int_0^\infty \bar{F}(x)\log\bar{F}(x)dx. \tag{5}$$

This measures the uncertainty contained in the survival function of X. The basic idea in their definition was to replace the pdf by the survival function in Shannon's definition. CRE is more general than the Shannon entropy and possesses more general mathematical properties than the Shannon entropy. This measure is always non-negative and its definition is valid for both continuous and discrete cases.

Example 3.1. *Let X be uniformly distributed over $[a, b]$, $a < b$. Then, the CRE is computed by (5) where $\bar{F}(x) = 1 - F(x) = \frac{b-x}{b-a}$. Thus, we have $\varepsilon(X) = (b-a)/4$.*

Example 3.2. *If X has Pareto distribution with pdf*

$$f(x) = \frac{\alpha\beta^\alpha}{(x+\beta)^\alpha}, \quad x \geq 0,\ \alpha > 1,\ \beta > 0,$$

the CRE of X is given by $\varepsilon(X) = \frac{\alpha\beta}{(1-\alpha)^2}$.

Example 3.3. *Suppose X has two parameter Weibull distribution with pdf*

$$f(x, k, b) = bkx^{k-1}e^{-bx^k}, \quad x \geq 0$$

where $k > 0$, $b > 0$. The CRE of the Weibull distribution is given by

$$\varepsilon(X) = \frac{1}{k}b^{-1/k}\Gamma(1 + \frac{1}{k}),$$

where $\Gamma(.)$ is the complete gamma function. The Weibull distribution appear in some of the entropy expressions.

CRE has many interesting applications in different branches of sciences such as reliability theory, survival analysis, computer vision and image processing. (see Zohrevand et al., 2015; Psarakos and Toomaj, 2017)

Misagh et al. (2011) and Miralia et al. (2017) defined WCRE as follows.

Definition 3.4. *(Weighted cumulative residual entropy) Let X be nonnegative continuous random variable having survival function $\bar{F}(x)$. The WCRE of X is defined by*

$$\varepsilon^\omega(X) = -\int_0^\infty x\bar{F}(x)\log(\bar{F}(x))dx. \tag{6}$$

Now we evaluate WCRE for some distributions.

Example 3.5. *Let X be uniformly distributed on $[a,b]$, $a<b$. Then*

$$\varepsilon^{\omega}(X) = \frac{b-a}{4}.\frac{5b+4a}{9} = \frac{5b+4a}{9}\varepsilon(X).$$

If $\frac{5b+4a}{9} < 1 (>1)$, *then* $\varepsilon^{\omega}(X) < (> \varepsilon(X))$ *and if* $5b+4a=9$ *then* $\varepsilon^{\omega}(X)=\varepsilon(X)$.

Example 3.6. *If X has exponential distribution with mean $\frac{1}{\lambda}$, then the CRE is computed as follows*

$$\varepsilon(X) = -\int_0^{\infty} \bar{F}(x)\log(\bar{F}(x))dx = \frac{1}{\lambda}.$$

The WCRE is given as $\varepsilon^{\omega}(X) = \frac{2}{\lambda^2}$.

3.2 The dynamic cumulative residual entropy

Asadi and Zohrevand (2007) proposed the DCRE as

$$DCRE((X;t)) = \varepsilon(X,t) = -\int_t^{\infty} \frac{\bar{F}(x)}{\bar{F}(t)}\log(\frac{\bar{F}(x)}{\bar{F}(t)})dx. \tag{7}$$

It is clear that $\varepsilon(X;0)=\varepsilon(X)$. The DCRE is a measure of the information in the residual life distribution.

Example 3.7. *Let X has exponential distribution with mean $\frac{1}{\lambda}$. Then, the DCRE is given as*

$$DCRE(X,t) = \varepsilon(X,t) = \frac{1}{\lambda} = \varepsilon(X),$$

which DCRE for exponential distribution does not depend on t.

3.3 The cumulative past entropy

In some practical situations, uncertainty is related to past life time rather than future. As an example, one can find past uncertainty of a unit that failed at time t.

Di Crescenzo and Longobardi (2002) have introduced past entropy over $(0,t)$. Since it is reasonable top resume that in many realistic situations uncertainty is not necessarily related to the future but can also refer to the past. They have also shown the necessity of past entropy and its relation with the residual entropy. If X denotes the lifetime of an item or of a living organism, then past entropy(or uncertainty of lifetime distribution) of an item is defined as

$$\bar{H}(X;t) = -\int_0^{t} \frac{f(x)}{F(t)}\log(\frac{f(x)}{F(t)})dx. \tag{8}$$

Note that (8) can be rewritten as

$$\bar{H}(X,t) = F(t) - \frac{1}{F(t)}\int_0^{t} f(x)\log(f(x)dx. \tag{9}$$

Given that at time t an item has been found to be failing, $\bar{H}(X,t)$ measure the uncertainty about its past life. Nanda and Paul (2006b) have studied some properties and applications of past entropy.

Di Crescenzo and Longobardi (2009) proposed a dual concept of CRE called CPE defined as

$$\bar{\varepsilon}(X) = -\int_0^\infty F(x)\log F(x)dx, \tag{10}$$

which measures information concerning past lifetime.

Example 3.8. *If X is uniformly distributed in $[a,b]$, then*

$$\bar{\varepsilon}(X) = \frac{b-a}{4} = \varepsilon(X).$$

We note that $H(X)$ is the differential entropy of the past lifetime of X at time t, i.e.,$[X|\, X \le t]$. We now make use of (2) to define weighted entropy for past lifetime that is the weighted version of entropy (8). Di Crescenzo and Longobardi (2006) have defined weighted past entropy. The weighted past entropy at time t of a random lifetime X is defined as

$$\bar{H}^\omega(X,t) = -\int_0^t x\frac{f(x)}{F(t)}\log(\frac{f(x)}{F(t)})dx. \tag{11}$$

Example 3.9. *Suppose X is uniformly distributed on (a,b). The weighted past entropy is*

$$\bar{H}^\omega(X,t) = \frac{t+a}{2}\log(t-a).$$

3.4 Weighted cumulative past entropy

The WCP)is defined as

$$WCPE(X) = -\int_0^\infty xF(x)\log(F(x))dx \tag{12}$$

which is proposed by Misaghet al. (2011).

Example 3.10. *If X is uniformly distributed on $(0,b)$ with $b>0$, then the WCPE of X is given by*

$$WCPE(X) = -\frac{b^2}{9}.$$

3.5 Dynamic cumulative past entropy

The dynamic cumulative past entropy (DCPE) for a nonnegative random variable X with absolutely continuous cdf $F(x)$ and pdf $f(x)$ is defined as

$$\bar{\varepsilon}(X,t) = DCPE(X) = -\int_0^t \frac{F(x)}{F(t)}\log(\frac{F(x)}{F(t)})dx \tag{13}$$

which is proposed by Di Crescenzo and Longobardi (2009).

Example 3.11. *Let X be distributed uniformly on $(0, b)$ with $b > 0$. For t >0, the DCPE of X is given as*

$$\bar{\varepsilon}(X, t) = -\frac{t}{4}.$$

It is clear that for the CPE of X we have $\bar{\varepsilon}(X) = \frac{b}{4} = \varepsilon(X)$.

4 Conclusions

In this paper, the concept of weighted entropy has been discussed. In literature of information measures, weighted entropy is a famous concept which always give a non-negative uncertainty measure. But in many survival studies for modeling statistical data information about lifetime is available. We study the properties of the resulting entropiessuch as the weighted residual entropy, cumulative residual entropy, weighted cumulative residual entropy, and weighted past entropy. The dynamic form of cumulative residual entropy measures the residual lifetime of the component has survived up to time t. Dynamic cumulative residual and past entropies are also given. Some suitable examples are presnted.

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