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# Estimation of the PDF and the CDF of the two-parameter exponential distribution for type-II censored sample

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**Abstract:** The article addresses different estimators of the probability density function and the cumulative distribution function for the two-parameter exponential distribution for type-II censored sample. Following estimation methods are considered: maximum likelihood estimator, uniformly minimum variance unbiased estimator and plug-in uniformly minimum variance unbiased estimator. Analysis of real data sets are performed to compare the performances of the proposed methods of estimation. The maximum likelihood estimators of the PDF and the CDF are performing better in mean squared error sense. In case of unknown location and known scale parameter, plug-in uniformly minimum variance unbiased estimator is performing better.

**Keywords:** Maximum likelihood estimator; Plug-in uniformly minimum variance unbiased estimator; Uniformly minimum variance unbiased estimator.
**Mathematics Subject Classification (2010):** 62F10, 62N05

# 1 Introduction

The exponential process arises most of the times in industrial processes, and is widely studied and popular among reliability engineers and scientists. It is often used to model the reliability of electronic and electrical systems, which do not typically experience wear out type failure. A host of authors has gone through the search of structural and reliability properties, characterization results, statistical inferences of the distribution. Still, there is room for further study on this distribution.

In life testing experiments, often the data are censored. Among the different censoring schemes, Type-I and Type-II censoring schemes are the two most popular censoring schemes. In Type-I censoring scheme, the experimental time is fixed, but the number of failures is random, whereas in Type-II censoring scheme, the experimental time is

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random but the number of failures is fixed. It has been found in many problems of life testing that there are occasions when a two-parameter exponential distribution is more appropriate for fitting life test data if a minimum warranty period is offered. When the random variable $X$ follows the two-parameter exponential distribution with parameters $\mu$ and $\theta$, then the probability density function (PDF) and the cumulative distribution function (CDF) are given by

$$f(x) = \theta e^{-\theta(x-\mu)}, \ x > \mu, \tag{1}$$
$$F(x) = 1 - e^{-\theta(x-\mu)}, \ x > \mu, \tag{2}$$

where $\theta > 0$. Without loss of generality, if we take $\theta = 1$, then the PDF and the CDF of the said distribution are

$$f(x) = e^{-(x-\mu)}, \ x > \mu, \tag{3}$$
$$F(x) = 1 - e^{-(x-\mu)}, \ x > \mu.. \tag{4}$$

The two-parameter exponential distribution has been used as models in analyzing life-time data quite extensively, for example, see Lawless (1982), Epstein (1956) etc. Generally, statistical inferences are made on the parameter(s) involved in the distribution. Best linear unbiased estimator (BLUE), maximum likelihood estimator (MLE), uniformly minimum variance unbiased estimator (UMVUE) of the parameter(s) have been focused by the authors. Substitution of MLE and UMVUE of the parameter(s) in the expression of the PDF and the CDF may not preserve similar efficiency as that of the estimator of the parameter(s). Therefore, instead of studying the estimators of the parameter(s), we have emphasized on finding out unbiased estimator (UMVUE) as well as biased estimators (MLE and plug-in uniformly minimum variance unbiased estimator (PUMVUE)) of the PDF and the CDF and comparison among the estimators have been made.

Many situations demand the estimate of the PDF, the CDF or both. For instance, the PDF can be used for the estimation of differential entropy, Rényi entropy, Kullback-Leibler divergence, and Fisher information; the CDF can be used for estimation of cumulative residual entropy, the quantile function, Bonferroni curve, Lorenz curve, and both the PDF and the CDF can be used for estimation of probability weighted moments, hazard rate function, mean deviation about mean etc.

A modest number of works are available in this context. As for example Asrabadi (1990), Dixit and Jabbari (2010), Jabbari and Jabbari (2010), Dixit and Jabbari (2011), Bagheri et al. (2014), Alizadeh et al. (2015), Bagheri et al. (2016a), Bagheri et al. (2016b), Mukherjee et al. (2016), Alizadeh et al. (2017), Fathi et al. (2017), Dey et al. (2018), Maiti and Mukherjee (2018), Kumar et al. (2018), Jebely et al. (2018), Mukherjee and Maiti (2019) and the references cited therein.

The article has been arranged as follows. In Section 2, we have discussed the MLE of the PDF and the CDF. Theoretical biases and MSEs have been derived. Section 3 discusses UMVUE and compared through theoretical MSEs. Section 4 devotes to derive PUMVUE of the PDF and the CDF. Here also biases and MSEs have been derived. In Section 5, two data sets have been analyzed and the summary result has been reported. Section 6 concludes.

# 2 MLE of the PDF and the CDF

We are interested about type-II censored sample drawn from the two-parameter exponential distribution. Let $x_{(1)}, x_{(2)}, \ldots, x_{(r)}$ be failure times of $r$ items. The fact that $(n-r)$ items survived beyond $x_{(r)}$.

**(a) When both $\theta$ and $\mu$ are unknown**

The likelihood function (Sinha, 1986) of $\theta$ and $\mu$ is

$$L(\mu, \theta | x_{(1)}, x_{(2)}, \ldots, x_{(r)}) = n(n-1)\ldots(n-r+1)\theta^r e^{-\theta \sum_{i=1}^{r}(x_{(i)}-\mu)} e^{-\theta(n-r)(x_{(r)}-\mu)},$$

where $\mu \leq x_{(1)} < x_{(2)} < \ldots < x_{(r)} < \infty$. The MLE of $\mu$ and $\theta$ are $\tilde{\mu} = x_{(1)}$, the smallest order observation and $\tilde{\theta} = r/(\sum_{i=1}^{r} x_{(i)} + (n-r)x_{(r)} - nx_{(1)})$, respectively. Therefore, by using the invariance property of MLE, one can obtain the MLEs of the PDF and the CDF as $\tilde{f}(x)$ and $\tilde{F}(x)$, respectively.

Let, $U = X_{(1)}$ and $V = \sum_{i=1}^{r} X_{(i)} + (n-r)X_{(r)} - nX_{(1)}$. The PDF of U and V are given by

$$\begin{aligned} g_1(u) &= n\theta e^{-n\theta(u-\mu)};\ u > \mu, \\ g_2(v) &= \frac{\theta^{r-1}}{\Gamma(r-1)} e^{-\theta v} v^{r-2};\ v > 0, \end{aligned}$$

respectively. Note that, $U$ and $V$ are independently distributed.

**Theorem 2.1.** *The estimators, $\tilde{f}(x)$ and $\tilde{F}(x)$, are biased for $f(x)$ and $F(x)$, respectively, with*

$$\begin{aligned} E(\tilde{f}(x)) &= \frac{r\theta^{r-1}}{\Gamma(r-1)} \sum_{i=0}^{\infty} \left(\frac{r}{n\theta}\right)^i \Bigg[ 2\left\{\frac{r(x-\mu)}{\theta}\right\}^{\frac{r-i-2}{2}} \\ &\quad \times K_{r-i-2}\left(2\sqrt{r(x-\mu)\theta}\right) - e^{-n\theta(x-\mu)} \frac{\Gamma(r-i-2)}{\theta^{r-i-2}} \Bigg] \end{aligned} \tag{5}$$

*and*

$$\begin{aligned} E(\tilde{F}(x)) &= 1 - e^{-n\theta(x-\mu)} - \frac{\theta^{r-1}}{\Gamma(r-1)} \sum_{i=0}^{\infty} \left(\frac{r}{n\theta}\right)^i \Bigg[ 2\left\{\frac{r(x-\mu)}{\theta}\right\}^{\frac{r-i-1}{2}} \\ &\quad \times K_{r-i-1}\left(2\sqrt{r(x-\mu)\theta}\right) - e^{-n\theta(x-\mu)} \frac{\Gamma(r-i-1)}{\theta^{r-i-1}} \Bigg], \end{aligned} \tag{6}$$

*where $K_\nu$ denotes the modified Bessel function of the second kind of order $\nu$ defined as $K_\nu\left(2\sqrt{\beta\gamma}\right) = \frac{1}{2}\left(\frac{\gamma}{\beta}\right)^{\frac{\nu}{2}} \int_0^\infty x^{\nu-1} e^{-\frac{\beta}{x}-\gamma x} dx;\ \beta, \gamma > 0.$*

*Proof.*

$$E(\tilde{f}(x)) = \int_0^\infty \int_\mu^x \tilde{f}(x) g(u,v)\, du\, dv$$

$$\begin{aligned}
&= \int_0^\infty \int_\mu^x \frac{r}{v} e^{-\frac{r(x-u)}{v}} n\theta e^{-n\theta(u-\mu)} \frac{\theta^{r-1}}{\Gamma(r-1)} e^{-\theta v} v^{r-2} du dv \\
&= \frac{r\theta^{r-1}}{\Gamma(r-1)} \sum_{i=0}^\infty \left(\frac{r}{n\theta}\right)^i \left[2\left\{\frac{r(x-\mu)}{\theta}\right\}^{\frac{r-i-2}{2}} \right. \\
&\quad \left. \times K_{r-i-2}\left(2\sqrt{r(x-\mu)\theta}\right) - e^{-n\theta(x-\mu)} \frac{\Gamma(r-i-2)}{\theta^{r-i-2}}\right]
\end{aligned}$$

and

$$\begin{aligned}
E(\tilde{F}(x)) &= \int_0^\infty \int_\mu^x \tilde{f}(x) g(u,v) du dv \\
&= \int_0^\infty \int_\mu^x \left[1 - e^{-\frac{r(x-u)}{v}}\right] n\theta e^{-n\theta(u-\mu)} \frac{\theta^{r-1}}{\Gamma(r-1)} e^{-\theta v} v^{r-2} du dv \\
&= 1 - e^{-n\theta(x-\mu)} - \frac{\theta^{r-1}}{\Gamma(r-1)} \sum_{i=0}^\infty \left(\frac{r}{n\theta}\right)^i \left[2\left\{\frac{r(x-\mu)}{\theta}\right\}^{\frac{r-i-1}{2}} \right. \\
&\quad \left. \times K_{r-i-1}\left(2\sqrt{r(x-\mu)\theta}\right) - e^{-n\theta(x-\mu)} \frac{\Gamma(r-i-1)}{\theta^{r-i-1}}\right].
\end{aligned}$$

□

**Theorem 2.2.** *The MSEs of $\tilde{f}(x)$ and $\tilde{F}(x)$ are given by*

$$\begin{aligned}
MSE(\tilde{f}(x)) &= \frac{r^2\theta^{r-1}}{\Gamma(r-1)} \sum_{i=0}^\infty \left(\frac{2r}{n\theta}\right)^i \left[2\left\{\frac{2r(x-\mu)}{\theta}\right\}^{\frac{r-i-3}{2}} \right. \\
&\quad \left. \times K_{r-i-3}\left(2\sqrt{2r(x-\mu)\theta}\right) - e^{-n\theta(x-\mu)} \frac{\Gamma(r-i-3)}{\theta^{r-i-3}}\right] \\
&\quad -2f(x) \frac{r\theta^{r-1}}{\Gamma(r-1)} \sum_{i=0}^\infty \left(\frac{r}{n\theta}\right)^i \\
&\quad \times \left[2\left\{\frac{r(x-\mu)}{\theta}\right\}^{\frac{r-i-2}{2}} K_{r-i-2}\left(2\sqrt{r(x-\mu)\theta}\right) \right. \\
&\quad \left. -e^{-n\theta(x-\mu)} \frac{\Gamma(r-i-2)}{\theta^{r-i-2}}\right] + f^2(x)
\end{aligned}$$

*and*

$$\begin{aligned}
MSE(\tilde{F}(x)) &= 1 - e^{-n\theta(x-\mu)} - \frac{2\theta^{r-1}}{\Gamma(r-1)} \sum_{i=0}^\infty \left(\frac{r}{n\theta}\right)^i \\
&\quad \left[2\left\{\frac{r(x-\mu)}{\theta}\right\}^{\frac{r-i-1}{2}} K_{r-i-1}\left(2\sqrt{r\theta(x-\mu)}\right) \right. \\
&\quad \left. -e^{-n\theta(x-\mu)} \frac{\Gamma(r-i-1)}{\theta^{r-i-1}}\right] + \frac{\theta^{r-1}}{\Gamma(r-1)} \sum_{i=0}^\infty \left(\frac{2r}{n\theta}\right)^i
\end{aligned}$$

$$\left[2\left\{\frac{2r(x-\mu)}{\theta}\right\}^{\frac{r-i-1}{2}} K_{r-i-1}\left(2\sqrt{2r\theta(x-\mu)}\right)\right.$$
$$\left.-e^{-n\theta(x-\mu)}\frac{\Gamma(r-i-1)}{\theta^{r-i-1}}\right] - 2f(x)\left[1-e^{-n\theta(x-\mu)}\right.$$
$$-\frac{\theta^{r-1}}{\Gamma(r-1)}\sum_{i=0}^{\infty}\left(\frac{r}{n\theta}\right)^i\left\{2\left\{\frac{r(x-\mu)}{\theta}\right\}^{\frac{r-i-1}{2}}\right.$$
$$\left.\left.\times K_{r-i-1}\left(2\sqrt{r(x-\mu)\theta}\right) - e^{-n\theta(x-\mu)}\frac{\Gamma(r-i-1)}{\theta^{r-i-1}}\right\}\right] + f^2(x),$$

*respectively.*

*Proof.*

$$MSE(\tilde{f}(x)) = E(\tilde{f}(x))^2 - 2f(x)E(\tilde{f}(x)) + f^2(x). \tag{7}$$

So,

$$\begin{aligned} E(\tilde{f}(x))^2 &= \int_0^\infty\int_\mu^x \tilde{f}^2(x)g(u,v)dudv \\ &= \frac{r^2\theta^{r-1}}{\Gamma(r-1)}\sum_{i=0}^{\infty}\left(\frac{2r}{n\theta}\right)^i\left[2\left\{\frac{2r(x-\mu)}{\theta}\right\}^{\frac{r-i-3}{2}}\right. \\ &\quad \left.\times K_{r-i-3}\left(2\sqrt{2r(x-\mu)\theta}\right) - e^{-n\theta(x-\mu)}\ \frac{\Gamma(r-i-3)}{\theta^{r-i-3}}\right]. \end{aligned} \tag{8}$$

If we put the values of equation (1), (5) and (8) in (7), we get the expression for MSE of MLE of the PDF. Similarly,

$$MSE(\tilde{F}(x)) = E(\tilde{F}(x))^2 - 2F(x)E(\tilde{F}(x)) + F^2(x), \tag{9}$$

where,

$$\begin{aligned} E(\tilde{F}(x))^2 &= \int_0^\infty\int_\mu^x \tilde{F}^2(x)g(u,v)dudv \\ &= 1-e^{-n\theta(x-\mu)} - \frac{2\theta^{r-1}}{\Gamma(r-1)}\sum_{i=0}^{\infty}\left(\frac{r}{n\theta}\right)^i\left[2\left\{\frac{r(x-\mu)}{\theta}\right\}^{\frac{r-i-1}{2}}\right. \\ &\quad \left.\times K_{r-i-1}\left(2\sqrt{r\theta(x-\mu)}\right) - e^{-n\theta(x-\mu)}\frac{\Gamma(r-i-1)}{\theta^{r-i-1}}\right] \\ &\quad +\frac{\theta^{r-1}}{\Gamma(r-1)}\sum_{i=0}^{\infty}\left(\frac{2r}{n\theta}\right)^i\left[2\left\{\frac{2r(x-\mu)}{\theta}\right\}^{\frac{r-i-1}{2}}\right. \\ &\quad \left.\times K_{r-i-1}\left(2\sqrt{2r\theta(x-\mu)}\right) - e^{-n\theta(x-\mu)}\ \frac{\Gamma(r-i-1)}{\theta^{r-i-1}}\right]. \end{aligned} \tag{10}$$

If we put the values of equation (2), (6) and (10) in (9), we get the expression for MSE of MLE of the CDF. □

**(b) When $\theta$ is known but $\mu$ is unknown**

The MLE of the location parameter, $\mu$ with the PDF in (3) is $\tilde{\mu} = x_{(1)}$. Therefore, by using the invariance property of the MLE, one can obtain the MLE of the PDF and the CDF, respectively, as

$$\begin{aligned}\tilde{f}(x) &= e^{-(x-\tilde{\mu})};\ x > \tilde{\mu}\\ \tilde{F}(x) &= 1 - e^{-(x-\tilde{\mu})};\ x > \tilde{\mu},\end{aligned}$$

Define, $U = X_{(1)}$, a complete sufficient statistic. The distribution of $U$ is

$$f(u) = ne^{-n(u-\mu)};\ u > \mu.$$

The censoring scheme has no impact on the distribution of $X_(1)$. Hence, all the results are true for complete sample.

**Theorem 2.3.** *The estimators, $\tilde{f}(x)$ and $\tilde{F}(x)$, are biased for $f(x)$ and $F(x)$, respectively, with*

$$E(\tilde{f}(x)) = \frac{n}{n-1}\left[e^{-(x-\mu)} - e^{-n(x-\mu)}\right], \tag{11}$$

$$E(\tilde{F}(x)) = \frac{1}{n-1}\left[n-1+e^{-n(x-\mu)} - ne^{-(x-\mu)}\right]. \tag{12}$$

*Proof.*

$$\begin{aligned}E(\tilde{f}(x)) &= \int_{\mu}^{x} \tilde{f}(x)f(u)du\\ &= \int_{\mu}^{x} e^{-(x-u)} \cdot ne^{-n(u-\mu)}du\\ &= \frac{n}{n-1}[e^{-(x-\mu)} - e^{-n(x-\mu)}]\end{aligned}$$

and

$$\begin{aligned}E(\tilde{F}(x)) &= \int_{\mu}^{x} \tilde{F}(x)f(u)du\\ &= \int_{\mu}^{x} [1-e^{-(x-u)}] \cdot ne^{-n(u-\mu)}du\\ &= \frac{1}{n-1}[n-1+e^{-n(x-\mu)} - ne^{-(x-\mu)}].\end{aligned}$$

□

**Theorem 2.4.** *The MSE of $\tilde{f}(x)$ and $\tilde{F}(x)$, respectively, are given by*

$$\begin{aligned}MSE(\tilde{f}(x)) &= \left(\frac{1}{n-1}\right)\left[(2n-3)e^{-2(x-\mu)} - 2ne^{-(x-\mu)} - 3ne^{-n(x-\mu)}\right]\\ MSE(\tilde{F}(x)) &= \left(\frac{1}{n-1}\right)\Big[2ne^{-(x-\mu)} - (n+1)e^{-2(x-\mu)} - (2n+1)e^{-n(x-\mu)}\\ &\quad +2e^{-(n+1)(x-\mu)} + 2n\left\{e^{-(nx-\mu)} - e^{-(x+(n-2)\mu)}\right\}\Big].\end{aligned}$$

*Proof.*

$$\begin{aligned} MSE(\tilde{f}(x)) &= E(\tilde{f}(x))^2 - 2f(x)E(\tilde{f}(x)) + f^2(x), & (13)\\ E(\tilde{f}(x))^2 &= \int_\mu^x (\tilde{f}(x))^2 f(u)du \\ &= \int_\mu^x e^{-2(x-u)} \cdot ne^{-n(u-\mu)}du \\ &= \frac{n}{n-1}[e^{-2(x-\mu)} - e^{-n(x-\mu)}]. & (14) \end{aligned}$$

If we put the value of expressions (3), (11) and (14) in (13), we get the expression for the MSE of the MLE of the PDF. Similarly,

$$MSE(\tilde{F}(x)) = E(\tilde{F}(x))^2 - 2F(x)E(\tilde{F}(x)) + F^2(x), \quad (15)$$

where

$$\begin{aligned} E(\tilde{F}(x))^2 &= \int_\mu^x (\tilde{F}(x))^2 f(u)du \\ &= \int_\mu^x \left(1 - e^{-(x-u)}\right)^2 \cdot ne^{-n(u-\mu)}du \\ &= 1 - \left(\frac{2n-1}{n-1}\right) e^{-n(x-\mu)} \\ &\quad + \left(\frac{2n}{n-1}\right)\left[e^{-(nx-\mu)} - e^{-n(x+(n-2)\mu)}\right]. & (16) \end{aligned}$$

If we put the value of expressions (4), (12) and (16) in (15), we get the expression for the MSE of the MLE of the CDF. □

It is to be noted that there is no impact of censoring point ($r$) in the expressions when $\theta$ is known.

# 3 UMVUE of the PDF and the CDF

In this section, we obtain the UMVUE of the PDF and the CDF, and the MSE (in this case variance) of these estimators in the form of theorems.

### (a) When both $\theta$ and $\mu$ are unknown

**Theorem 3.1.** *The UMVUEs of the PDF and the CDF of the two-parameter exponential distribution for type-II censored sample (Laurent, 1963) are given by*

$$\hat{f}(x) = \begin{cases} \frac{1}{n}; \ x = u \\ (r-2)\left(1 - \frac{1}{n}\right)\frac{1}{v}\left\{1 - \frac{x-u}{v}\right\}^{r-3}; u < x < u + v, \end{cases}$$

*and*

$$\hat{F}(x) = \begin{cases} 0; \ x < u \\ \frac{1}{n}; \ x = u \\ 1 - \left(1 - \frac{1}{n}\right)\left\{1 - \frac{x-u}{v}\right\}^{r-2}; \ u < x < u + v \\ 1; \ x \geq u + v, \end{cases}$$

*respectively.*

**Theorem 3.2.** *The MSEs of UMVUE of the PDF and the CDF of two-parameter exponential distribution for type-II censored sample are given by*

$$\begin{aligned} MSE(\hat{f}(x)) &= \frac{\theta}{n}e^{-n\theta(x-\mu)} + \left(1 - \frac{1}{n}\right)^2 (r-2)^2 \frac{n\theta^r}{\Gamma(r-1)} \\ &\times \sum_{k=0}^{(2r-6)} \binom{2r-6}{k} \int_\mu^x \frac{1}{\theta^{k-r+3}} e^{-n\theta(u-\mu)}(u-x)^{2r-6-k} \\ &\times \Gamma(k+3-r, \theta(x-u))du - f^2(x), \end{aligned}$$

*and*

$$\begin{aligned} MSE(\hat{F}(x)) &= \frac{\theta}{n}e^{-n\theta(x-\mu)} + \frac{n\theta}{\Gamma(r-1)} \int_\mu^x e^{-n\theta(u-\mu)} \\ &\times \Gamma(r-1, \theta(x-u))du + \left(1 - \frac{1}{n}\right)^2 \frac{n}{\Gamma(r-1)} \\ &\times \sum_{k=0}^{(2r-4)} \binom{2r-4}{k} \frac{1}{\theta^{k+3-2r}} \int_\mu^x e^{-n\theta(u-\mu)}(u-x)^{2r-4-k} \\ &\times \Gamma(k-r+3, \theta(x-u))du \\ &-2\left(1 - \frac{1}{n}\right) \frac{n\theta^r}{\Gamma(r-1)} \sum_{k=0}^{(r-2)} \binom{r-2}{k} \frac{1}{\theta^{k+1}} \\ &\times \int_\mu^x e^{-n\theta(u-\mu)}(u-x)^{r-2-k}\Gamma(k+1, \theta(x-u))du - F^2(x). \end{aligned}$$

*Proof.*

$$MSE(\hat{f}(x)) = E(\hat{f}^2(x)) - f^2(x). \tag{17}$$

Now,

$$\begin{aligned} E(\hat{f}^2(x)) &= \int_\mu^x \int_{x-u}^\infty \hat{f}^2(x)g(u,v)dvdu \\ &= \int_\mu^x \int_{x-u}^\infty \left(1 - \frac{1}{n}\right)^2 (r-2)^2 \frac{1}{v^2} \left\{1 - \frac{x - x_{(1)}}{v}\right\}^{2(r-3)} \\ &\times n\theta e^{-n\theta(u-\mu)} e^{-\theta v} v^{r-2} \frac{\theta^{r-1}}{\Gamma(r-1)} dvdu + \frac{1}{n}\theta e^{-n\theta(x-\mu)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\theta}{n}e^{-n\theta(x-\mu)} + \left(1-\frac{1}{n}\right)^2 (r-2)^2 \frac{n\theta^r}{\Gamma(r-1)} \sum_{k=0}^{(2r-6)} \binom{2r-6}{k} \\
&\times \int_\mu^x \frac{1}{\theta^{k-r+3}} e^{-n\theta(u-\mu)}(u-x)^{2r-6-k}\Gamma(k+3-r,\theta(x-u))du. \quad (18)
\end{aligned}
$$

Substituting (1), (18) in (17), we will get the MSE of UMVUE of the PDF.

$$
MSE(\hat{F}(x)) = E(\hat{F}^2(x)) - F^2(x). \quad (19)
$$

$$
\begin{aligned}
E(\hat{F}^2(x)) &= \int_\mu^x \int_{x-u}^\infty \hat{F}^2(x) g(u,v)dvdu \\
&= \frac{\theta}{n}e^{-n\theta(x-\mu)} + \int_\mu^x \int_{x-u}^\infty \left[1-(1-\frac{1}{n})\left\{1-\frac{x-x_{(1)}}{v}\right\}^{r-2}\right]^2 \\
&\times n\theta e^{-n\theta(u-\mu)}e^{-\theta v}v^{r-2}\frac{\theta^{r-1}}{\Gamma(r-1)}dvdu \\
&= \frac{\theta}{n}e^{-n\theta(x-\mu)} + \frac{n\theta}{\Gamma(r-1)}\int_\mu^x e^{-n\theta(u-\mu)}\Gamma(r-1,\theta(x-u))du \\
&+ \left(1-\frac{1}{n}\right)^2 \frac{n}{\Gamma(r-1)} \\
&\times \sum_{k=0}^{(2r-4)} \binom{2r-4}{k} \frac{1}{\theta^{k+3-2r}} \int_\mu^x e^{-n\theta(u-\mu)}(u-x)^{2r-4-k} \\
&\times \Gamma(k-r+3,\theta(x-u))du \\
&-2\left(1-\frac{1}{n}\right)\frac{n\theta^r}{\Gamma(r-1)} \sum_{k=0}^{(r-2)} \binom{r-2}{k}\frac{1}{\theta^{k+1}} \\
&\times \int_\mu^x e^{-n\theta(u-\mu)}(u-x)^{r-2-k}\Gamma(k+1,\theta(x-u))du. \quad (20)
\end{aligned}
$$

Similarly, by putting (2), (20) in (19), we will get the MSE of UMVUE of the CDF. □

**(b) When $\theta$ is known but $\mu$ is unknown**

**Theorem 3.3.** *The UMVUE of the PDF and the CDF, respectively, is given by*

$$
\hat{f}(x) = \begin{cases} \frac{1}{n};\ x=u \\ \frac{n-1}{n}e^{-(x-u)};\ x>u, \end{cases}
$$

$$
\hat{F}(x) = \begin{cases} 0; x<u \\ \frac{1}{n};\ x=u \\ 1-\frac{n-1}{n}e^{-(x-u)};\ x>u. \end{cases}
$$

*Proof.* Here, using (3) and (4) we get

$$
g(u) = ne^{-n(u-\mu)}\ ; u>\mu.
$$

Now, we have to obtain $P(X_1 \le u|X_{(1)} = u)$.
**Case-I:**

$$P(X_1 \le u|X_{(1)} = u) = P(X_1 = u, X_i \ge u) = f(u)(1 - F(u))^{n-1}.$$

Therefore,

$$P(X_1 \le t|X_{(1)} = u) = \frac{f(u)(1 - F(u))^{n-1}}{nf(u)(1 - F(u))^{n-1}} = \frac{1}{n} \text{ if } t = u.$$

**Case-II:** When, $t > u$

$$\begin{aligned} P(X_1 \le t|X_{(1)} = u) &= 1 - P(X_1 > t|X_{(1)} = u) \\ &= 1 - \frac{(n-1)f(u)(1 - F(t))(1 - F(u))^{n-2}}{nf(u)(1 - F(u))^{n-1}} \\ &= 1 - \frac{n-1}{n} e^{-(t-u)}. \end{aligned}$$

Hence, the required results are obtained. □

The MSE of the UMVUE of the PDF and the CDF of the two-parameter exponential distribution (when $\theta$ is known) are given by

$$\begin{aligned} MSE(\hat{f}(x)) &= \frac{(n-1)^2}{n(n-2)}[e^{-2(x-\mu)} - e^{-n(x-\mu)}] + \frac{1}{n}[ne^{-2(x-\mu)} - e^{-n(x-\mu)}] \\ MSE(\hat{F}(x)) &= \left(\frac{n+1}{n}\right) e^{-n(x-\mu)} - e^{-2(x-\mu)} + \frac{(n-1)^2}{n(n-2)}[e^{-2(x-\mu)} - e^{-n(x-\mu)}]. \end{aligned}$$

It is also to be noticed that censoring point $(r)$ has no impact on UMVUE of the PDF and the CDF.

# 4 PUMVUE of the PDF and the CDF

In this section, we obtain the PUMVUE of the PDF and the CDF of the two-parameter exponential distribution for the type-II censored sample. Also, we obtain the MSE of these estimators.

**(a) When both $\theta$ and $\mu$ are unknown**

The UMVUE of $\theta$ and $\mu$ (Sinha, 1986) are given by

$$\hat{\hat{\theta}} = \frac{r-1}{V} \tag{21}$$

$$\hat{\hat{\mu}} = U - \frac{V}{n(r-1)}, \tag{22}$$

respectively. By plugging-in (21) and (22) in (1), (2), we get the PUMVUE of the PDF and the CDF as $\hat{\hat{f}}(x)$ and $\hat{\hat{F}}(x)$, respectively.

**Theorem 4.1.** *The estimators, $\hat{\hat{f}}(x)$ and $\hat{\hat{F}}(x)$, are biased for $f(x)$ and $F(x)$, respectively, with*

$$E(\hat{\hat{f}}(x)) = \frac{(r-1)\theta^{r-1}}{\Gamma(r-1)}\sum_{i=0}^{\infty}\left(\frac{r-1}{n\theta}\right)^{i}\left[2e^{\frac{-1}{n}}\left\{\frac{(r-1)(x-\mu)}{\theta}\right\}^{\frac{r-i-2}{2}} \times K_{r-i-2}\left(2\sqrt{(r-1)(x-\mu)\theta}\right) - e^{-n\theta(x-\mu)}\Gamma(r-i-2)\left(\frac{r-1}{r\theta}\right)^{r-i-2}\right] \tag{23}$$

$$E(\hat{\hat{F}}(x)) = 1 - e^{-n\theta(x-\mu)}\left(\frac{r-1}{r}\right)^{r-1} - \frac{\theta^{r-1}}{\Gamma(r-1)}\sum_{i=0}^{\infty}\left(\frac{r-1}{n\theta}\right)^{i} \times\left[2e^{\frac{-1}{n}}\left\{\frac{(r-1)(x-\mu)}{\theta}\right\}^{\frac{r-i-1}{2}} K_{r-i-1}\left(2\sqrt{(r-1)(x-\mu)\theta}\right) - e^{-n\theta(x-\mu)}\,\Gamma(r-i-1)\left(\frac{r-1}{r\theta}\right)^{r-i-1}\right]. \tag{24}$$

*Proof.*

$$\begin{aligned}
E(\hat{\hat{f}}(x)) &= \int_0^{\infty}\int_{\mu}^{x+\frac{v}{n(r-1)}} \hat{\hat{f}}(x) g(u,v)\,du\,dv \\
&= \int_0^{\infty}\int_{\mu}^{x+\frac{v}{n(r-1)}} \frac{(r-1)}{v} e^{-\frac{(r-1)}{v}\left(x-u+\frac{v}{n(r-1)}\right)} n\theta e^{-n\theta(u-\mu)} \\
&\quad \times \frac{\theta^{r-1}}{\Gamma(r-1)} e^{-\theta v} v^{r-2}\,du\,dv \\
&= \frac{(r-1)\theta^{r-1}}{\Gamma(r-1)}\sum_{i=0}^{\infty}\left(\frac{r-1}{n\theta}\right)^{i}\left[2e^{\frac{-1}{n}}\left\{\frac{(r-1)(x-\mu)}{\theta}\right\}^{\frac{r-i-2}{2}}\right. \\
&\quad \left.\times K_{r-i-2}\left(2\sqrt{(r-1)(x-\mu)\theta}\right) - e^{-n\theta(x-\mu)}\Gamma(r-i-2)\left(\frac{r-1}{r\theta}\right)^{r-i-2}\right]
\end{aligned}$$

and

$$\begin{aligned}
E(\hat{\hat{F}}(x)) &= \int_0^{\infty}\int_{\mu}^{x+\frac{v}{n(r-1)}} \hat{\hat{F}}(x) g(u,v)\,du\,dv \\
&= \int_0^{\infty}\int_{\mu}^{x+\frac{v}{n(r-1)}} \left[1 - e^{-\frac{(r-1)}{v}\left(x-u+\frac{v}{n(r-1)}\right)}\right] n\theta e^{-n\theta(u-\mu)} \\
&\quad \times \frac{\theta^{r-1}}{\Gamma(r-1)} e^{-\theta v} v^{r-2}\,du\,dv \\
&= 1 - e^{-n\theta(x-\mu)}\left(\frac{r-1}{r}\right)^{r-1} - \frac{\theta^{r-1}}{\Gamma(r-1)}\sum_{i=0}^{\infty}\left(\frac{r-1}{n\theta}\right)^{i} \\
&\quad \times\left[2e^{\frac{-1}{n}}\left\{\frac{(r-1)(x-\mu)}{\theta}\right\}^{\frac{r-i-1}{2}} K_{r-i-1}\left(2\sqrt{(r-1)(x-\mu)\theta}\right)\right.
\end{aligned}$$

$$-e^{-n\theta(x-\mu)}\ \Gamma(r-i-1)\left(\frac{r-1}{r\theta}\right)^{r-i-1}\Bigg].$$

□

**Theorem 4.2.** *The MSE of $\hat{\hat{f}}(x)$, and $\hat{\hat{F}}(x)$ are given by*

$$\begin{aligned}
MSE(\hat{\hat{f}}(x)) &= \frac{(r-1)^2\theta^{r-1}}{\Gamma(r-1)}\sum_{i=0}^{\infty}\left(\frac{2(r-1)}{n\theta}\right)^{i} \\
&\quad\times\left[2e^{-\frac{2}{n}}\left\{\frac{2(r-1)(x-\mu)}{n\theta}\right\}^{\frac{r-i-3}{2}}\right. \\
&\quad\times K_{r-i-3}\left(2\sqrt{2(r-1)(x-\mu)\theta}\right)-e^{-n\theta(x-\mu)} \\
&\quad\left.\times\Gamma(r-i-3)\left(\frac{r-1}{r\theta}\right)^{r-i-3}\right] \\
&\quad-2f(x)\frac{(r-1)\theta^{r-1}}{\Gamma(r-1)}\sum_{i=0}^{\infty}\left(\frac{r-1}{n\theta}\right)^{i} \\
&\quad\times\left[2e^{\frac{-1}{n}}\left\{\frac{(r-1)(x-\mu)}{\theta}\right\}^{\frac{r-i-2}{2}}\right. \\
&\quad\times K_{r-i-2}\left(2\sqrt{(r-1)(x-\mu)\theta}\right)-e^{-n\theta(x-\mu)} \\
&\quad\left.\times\Gamma(r-i-2)\left(\frac{r-1}{r\theta}\right)^{r-i-2}\right]+f^2(x)
\end{aligned}$$

*and*

$$\begin{aligned}
MSE(\hat{\hat{F}}(x)) &= 1-e^{n\theta(x-\mu)}\left(\frac{r-1}{r}\right)^{r-1}+\frac{\theta^{r-1}}{\Gamma(r-1)}\sum_{i=0}^{\infty}\left(\frac{2(r-1)}{n\theta}\right)^{i} \\
&\quad\times\left[2e^{-\frac{2}{n}}\left\{\frac{2(r-1)(x-\mu)}{\theta}\right\}^{\frac{r-i-1}{2}}\right. \\
&\quad K_{r-i-1}\left(2\sqrt{2(r-1)\theta(x-\mu)}\right)-e^{-n\theta(x-\mu)}\Gamma(r-i-1) \\
&\quad\left.\times\left(\frac{r-1}{r\theta}\right)^{r-i-1}\right]-\frac{2\theta^{r-1}}{\Gamma(r-1)} \\
&\quad\times\sum_{i=0}^{\infty}\left(\frac{r-1}{n\theta}\right)^{i}\left[2e^{-\frac{1}{n}}\left\{\frac{(r-1)(x-\mu)}{\theta}\right\}^{\frac{r-i-1}{2}}\right. \\
&\quad\times K_{r-i-1}\left(2\sqrt{(r-1)\theta(x-\mu)}\right) \\
&\quad\left.-e^{-n\theta(x-\mu)}\Gamma(r-i-1)\left(\frac{r-1}{r\theta}\right)^{r-i-1}\right]
\end{aligned}$$

$$
\begin{aligned}
&-2F(x)\Bigg[1-e^{-n\theta(x-\mu)}\left(\frac{r-1}{r}\right)^{r-1}\\
&-\frac{\theta^{r-1}}{\Gamma(r-1)}\sum_{i=0}^{\infty}\left(\frac{r-1}{n\theta}\right)^{i}\Bigg[2e^{\frac{-1}{n}}\left\{\frac{(r-1)(x-\mu)}{\theta}\right\}^{\frac{r-i-1}{2}}\\
&\times K_{r-i-1}\left(2\sqrt{(r-1)(x-\mu)\theta}\right)-e^{-n\theta(x-\mu)}\\
&\times\Gamma(r-i-1)\left(\frac{r-1}{r\theta}\right)^{r-i-1}\Bigg]\Bigg]+F^2(x),
\end{aligned}
$$

*respectively.*

*Proof.*

$$
MSE(\hat{\hat{f}}(x))=E(\hat{\hat{f}}(x))^2-2f(x)E(\hat{\hat{f}}(x))+f^2(x). \tag{25}
$$

Here,

$$
\begin{aligned}
E(\hat{\hat{f}}(x))^2 &= \int_0^{\infty}\int_{\mu}^{x+\frac{v}{n(r-1)}}\hat{\hat{f}}^2(x)g(u,v)dudv\\
&= \int_0^{\infty}\int_{\mu}^{x+\frac{v}{n(r-1)}}\left[\frac{(r-1)}{v}e^{-\frac{(r-1)}{v}\left(x-u+\frac{v}{n(r-1)}\right)}\right]^2\\
&\quad\times n\theta e^{-n\theta(u-\mu)}\frac{\theta^{r-1}}{\Gamma(r-1)}e^{-\theta v}v^{r-2}dudv\\
&= \frac{(r-1)^2\theta^{r-1}}{\Gamma(r-1)}\sum_{i=0}^{\infty}\left(\frac{2(r-1)}{n\theta}\right)^{i}\\
&\quad\Bigg[2e^{-\frac{2}{n}}\left\{\frac{2(r-1)(x-\mu)}{n\theta}\right\}^{\frac{r-i-3}{2}}K_{r-i-3}\left(2\sqrt{2(r-1)(x-\mu)\theta}\right)\\
&\quad-e^{-n\theta(x-\mu)}\,\Gamma(r-i-3)\left(\frac{r-1}{r\theta}\right)^{r-i-3}\Bigg].
\end{aligned}\tag{26}
$$

If we put the values of equation (1), (23) and (26) in (25), we get the expression for MSE of PUMVU estimator of the PDF. Similarly,

$$
MSE(\hat{\hat{F}}(x))=E(\hat{\hat{F}}(x))^2-2F(x)E(\hat{\hat{F}}(x))+F^2(x). \tag{27}
$$

where,

$$
\begin{aligned}
E(\hat{\hat{F}}(x))^2 &= \int_0^{\infty}\int_{\mu}^{x+\frac{v}{n(r-1)}}\hat{\hat{F}}^2(x)g(u,v)dudv\\
&= \int_0^{\infty}\int_{\mu}^{x+\frac{v}{n(r-1)}}\left[1-e^{-\frac{(r-1)}{v}\left(x-u+\frac{v}{n(r-1)}\right)}\right]^2 n\theta e^{-n\theta(u-\mu)}\\
&\quad\times\frac{\theta^{r-1}}{\Gamma(r-1)}e^{-\theta v}v^{r-2}dudv
\end{aligned}
$$

$$
\begin{aligned}
&= 1-e^{n\theta(x-\mu)}\left(\frac{r-1}{r}\right)^{r-1}+\frac{\theta^{r-1}}{\Gamma(r-1)}\sum_{i=0}^{\infty}\left(\frac{2(r-1)}{n\theta}\right)^{i}\\
&\quad\times\left[2e^{-\frac{2}{n}}\left\{\frac{2(r-1)(x-\mu)}{\theta}\right\}^{\frac{r-i-1}{2}}\right.\\
&\quad\times K_{r-i-1}\left(2\sqrt{2(r-1)\theta(x-\mu)}\right)\\
&\quad\left.-e^{-n\theta(x-\mu)}\Gamma(r-i-1)\left(\frac{r-1}{r\theta}\right)^{r-i-1}\right]-\frac{2\theta^{r-1}}{\Gamma(r-1)}\\
&\quad\times\sum_{i=0}^{\infty}\left(\frac{r-1}{n\theta}\right)^{i}\left[2e^{-\frac{1}{n}}\left\{\frac{(r-1)(x-\mu)}{\theta}\right\}^{\frac{r-i-1}{2}}\right.\\
&\quad\times K_{r-i-1}\left(2\sqrt{(r-1)\theta(x-\mu)}\right)\\
&\quad\left.-e^{-n\theta(x-\mu)}\Gamma(r-i-1)\left(\frac{r-1}{r\theta}\right)^{r-i-1}\right]. \qquad (28)
\end{aligned}
$$

If we put the values of equation (2), (24) and (28) in (27), we get the expression for MSE of PUMVUE of the CDF. □

**(b) When $\theta$ is known but $\mu$ is unknown**

By Lehmann-Scheffe theorem, we get the UMVUE of $\mu$ which is $\hat{\hat{\mu}}$ (Sinha, 1986),

$$\hat{\hat{\mu}} = X_{(1)} - \frac{1}{n} = U - \frac{1}{n}.$$

The PUMVUE of the PDF and the CDF of the two-parameter exponential distribution (when $\theta$ is known) are given by

$$
\begin{aligned}
\hat{\hat{f}}(x) &= e^{-(x-\hat{\hat{\mu}})};\ x > \hat{\hat{\mu}},\\
\hat{\hat{F}}(x) &= 1-e^{-(x-\hat{\hat{\mu}})};\ x > \hat{\hat{\mu}}.
\end{aligned}
$$

**Theorem 4.3.** *The estimators, $\hat{\hat{f}}(x)$ and $\hat{\hat{F}}(x)$, are biased for $f(x)$ and $F(x)$, respectively, with*

$$
\begin{aligned}
E(\hat{\hat{f}}(x)) &= \frac{n}{n-1}e^{-\frac{1}{n}}\left[e^{-(x-\mu)}-e^{-n(x-\mu)}\right]\\
E(\hat{\hat{F}}(x)) &= 1-e^{-n(x-\mu)}-\frac{n}{n-1}e^{-\frac{1}{n}}\left[e^{-(x-\mu)}-e^{-n(x-\mu)}\right].
\end{aligned}
$$

*Proof.* Proof is simple and hence omitted. □

**Theorem 4.4.** *The MSE of the PUMVUE of the PDF and the CDF of the two-parameter exponential distribution (when $\theta$ is known) are given by*

$$
MSE(\hat{\hat{f}}(x)) = \left(\frac{n}{n-2}\right)e^{-\frac{2}{n}}\left[e^{-2(x-\mu)}-e^{-n(x-\mu)}\right]
$$

$$-\frac{2n}{n-1}e^{\frac{-1}{n}} \cdot e^{-(x-\mu)}\left[e^{-(x-\mu)} - e^{-n(x-\mu)}\right]$$

*and*

$$\begin{aligned} MSE(\hat{\hat{F}}(x)) &= \left(1 - e^{-n(x-\mu)}\right) + \frac{n}{n-2} \cdot e^{-\frac{2}{n}}\left(e^{-2(x-\mu)} - e^{-n(x-\mu)}\right) \\ &\quad -\frac{2n}{n-1} \cdot e^{-\frac{1}{n}}\left(e^{-(x-\mu)} - e^{-n(x-\mu)}\right), \end{aligned}$$

*respectively.*

*Proof.* Proof is straight forward and hence omitted. □

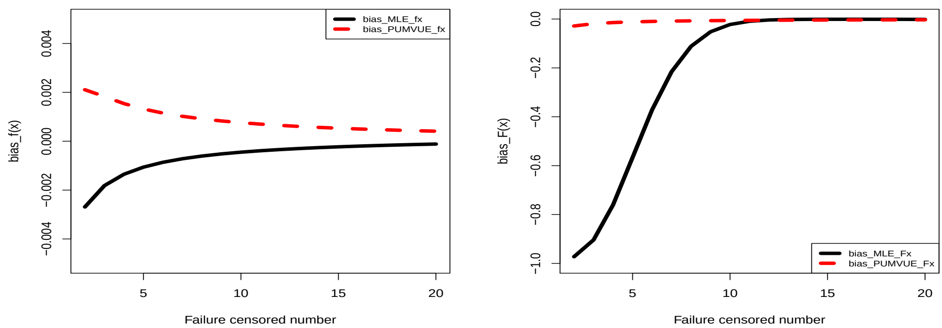


Figure 1: Graph of the theoretical bias of the MLE and PUMVUE of the PDF and the CDF of the two-parameter exponential distribution for type-II censored sample for $\mu = 5$, $\theta = 0.4$ and $x = 16$.

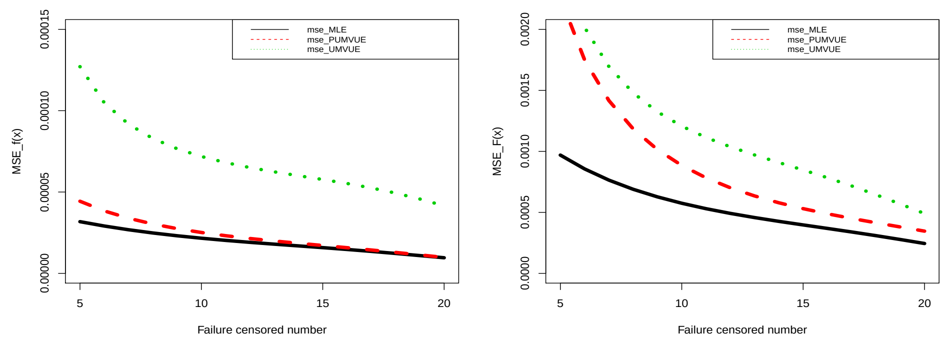


Figure 2: Graphs of the theoretical MSE of the MLE, UMVUE and PUMVUE of the PDF and the CDF of the two-parameter exponential distribution for type-II censored sample for $\mu = 5$, $\theta = 0.4$ and $x = 16$.

Here, it is also observed that censoring point $(r)$ has no impact on PUMVUE of the PDF and the CDF. Graphical representations of theoretical bias of MLE and PUMVUE of the PDF and the CDF of two-parameter exponential distribution for type-II censored sample have been shown in Figure 1. From Figure 1, it is observed that the MLE of the PDF and the CDF is negatively biased. The PUMVUE of the PDF is positively biased whereas that of the CDF is negatively biased. When censoring point $(r)$ increases (i.e nearer to complete sample), the biases of two estimators tend to zero. This indicates the accuracy of these estimators.

Theoretical MSE of the MLE, PUMVUE and UMVUE of the PDF and the CDF of type-II censored sample and same for the two-parameter exponential distribution when scale parameter is known and for the complete sample have been shown graphically in Figures 2−4. It is observed that the MSEs decrease with the increase of sample sizes. It verifies the consistency properties of all the estimators. From Figures 5 and 6, we observe that the UMVUE is more efficient than the MLE in MSE sense for the parameters, but this is not the case for the PDF and the CDF.

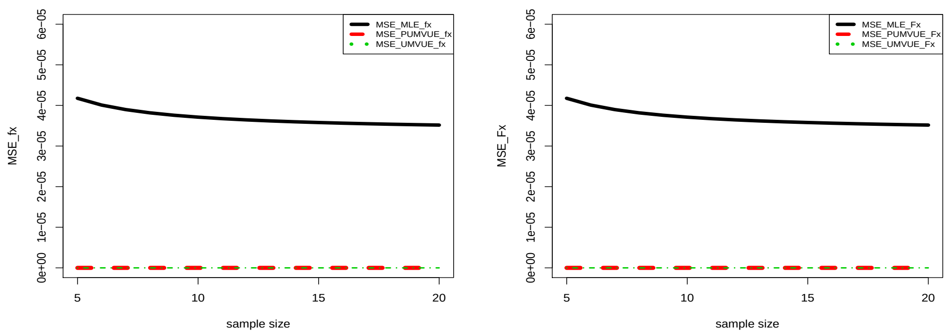


Figure 3: Graph of the theoretical MSE of the MLE and PUMVUE of the PDF and the CDF of the two-parameter exponential distribution when $\mu$ is unknown and $\theta$ is known for $\mu = 5$, $\theta = 1$ and $x = 16$.

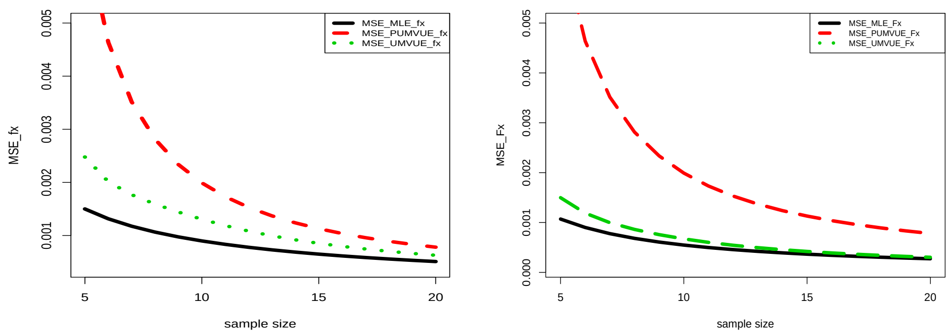


Figure 4: Graph of the theoretical MSE of the MLE, UMVUE and PUMVUE of the PDF and the CDF of the two-parameter exponential distribution for $\mu = 5$, $\theta = 0.4$ and $x = 16$.

# 5 Data analysis

In this section, we provide the analysis of a real data set for comparing the performances of the MLE, UMVUE and PUMVUE for the PDF and the CDF. For type-II censored sample, we have considered the data from Al-Mutairi et al. (2013) and is presented in Table 1. The data represents the waiting times (in minutes) before customer

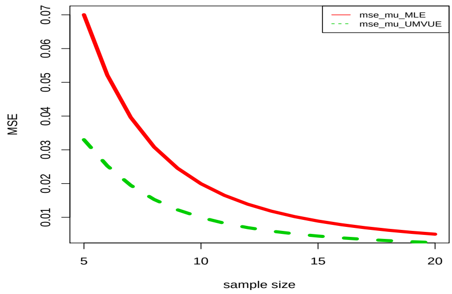


Figure 5: Graph of the theoretical MSE of the MLE and UMVUE of $\mu$ of the two-parameter exponential distribution with known scale parameter.

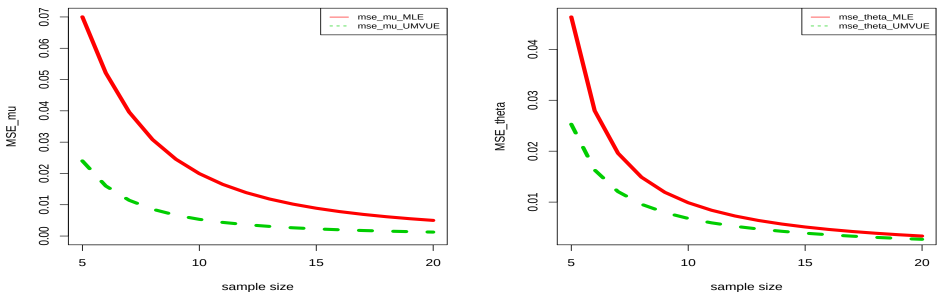


Figure 6: Graph of the theoretical MSE of the MLE and UMVUE of $\mu$ and $\theta$ of the two-parameter exponential distribution when both parameters are unknown.

service in a bank. Here, $n = 100$. The graph of the estimated the PDF and CDF of the two-parameter exponential distribution for type-II censored sample, censored at 75 $(= r)$, is given in Figure 7.

For the complete sample, the data represents the times between successive failures. This data is obtained from Rahman and Pearson (2001) and is presented in Table 2. The graph of the estimated PDF and CDF of the two-parameter exponential distribution for known $\theta$ and for both $\theta$ and $\mu$ both are unknown, are given in Figures 8 and 9, respectively. Table 3 shows the Akaike information criterion (AIC) $[AIC = -2\ln L(\hat{\mu}, \hat{\theta}) + 2k$, $k(= 2)$ is the number of parameter] values for the estimators. Lower the value of AIC indicates the better fit. In three situations, it is observed that the AIC value of the MLE is the lowest. So the MLE fits better.

# 6 Concluding remarks

Different methods of estimation of the PDF and the CDF of the two-parameter exponential distribution have been considered. The MLE, UMVUE and PUMVUE have been found out. Theoretical and simulated MSEs are observed to be in a similar tune. For the two-parameter exponential distribution with type-II censored sample $(r)$, the MLEs of the PDF and the CDF are found to be better than the other estimators in

Table 1: Waiting times (in minutes) before customer service in a Bank

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0.8 | 0.8 | 1.3 | 1.5 | 1.8 | 1.9 | 1.9 | 2.1 | 2.6 | 2.7 |
| 2.9 | 3.1 | 3.2 | 3.3 | 3.5 | 3.6 | 4 | 4.1 | 4.2 | 4.2 |
| 4.3 | 4.3 | 4.4 | 4.4 | 4.6 | 4.7 | 4.7 | 4.8 | 4.9 | 4.9 |
| 5 | 5.3 | 5.5 | 5.7 | 5.7 | 6.1 | 6.2 | 6.2 | 6.2 | 6.3 |
| 6.7 | 6.9 | 7.1 | 7.1 | 7.1 | 7.1 | 7.4 | 7.6 | 7.7 | 8 |
| 8.2 | 8.6 | 8.6 | 8.6 | 8.8 | 8.8 | 8.9 | 8.9 | 9.5 | 9.6 |
| 9.7 | 9.8 | 10.7 | 10.9 | 11 | 11 | 11.1 | 11.2 | 11.2 | 11.5 |
| 11.9 | 12.4 | 12.5 | 12.9 | 13 | 13.1 | 13.3 | 13.6 | 13.7 | 13.9 |
| 14.1 | 15.4 | 15.4 | 17.3 | 17.3 | 18.1 | 18.2 | 18.4 | 18.9 | 19 |
| 19.9 | 20.6 | 21.3 | 21.4 | 21.9 | 23 | 27 | 31.6 | 33.1 | 38.5 |

Table 2: Times between system failures data

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 5.2 | 8.4 | 0.9 | 0.1 | 5.9 | 17.9 | 3.6 | 2.5 | 1.2 | 1.8 | 1.8 | 6.1 | 5.3 |
| 1.2 | 1.2 | 3.0 | 3.5 | 7.6 | 3.4 | 0.5 | 2.4 | 5.3 | 1.9 | 2.8 | 0.1 | |

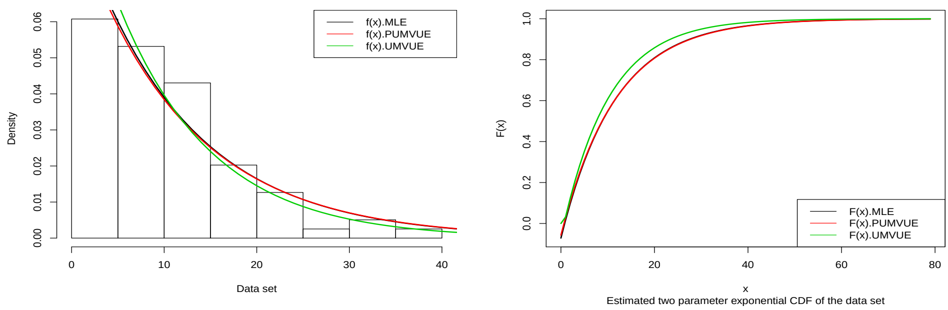


Figure 7: Graph of the estimated PDF and CDF of the two-parameter exponential distribution when the number of type-II censored sample is 75.

Table 3: Model selection criterion

| | AIC | | |
|---|---|---|---|
| | type-II censored sample ($\theta$, $\mu$ are unknown) | complete sample ($\mu$ is unknown) | complete sample ($\theta$, $\mu$ are unknown) |
| MLE | 650.676 | 116.672 | 118.670 |
| UMVUE | 656.217 | 125.734 | 127.734 |
| PUMVUE | 653.804 | 117.340 | 119.340 |

MSE sense. Data analysis result also ensures the same (i.e. MLE) in AIC sense. For the two-parameter exponential distribution with known scale parameter the PUMVUE of the PDF and the CDF is better than others in MSE sense. Since the hazard rate and mean remaining life function for the considered model is constant, we have the estimate of the functions directly from the estimates of the related parameter(s).

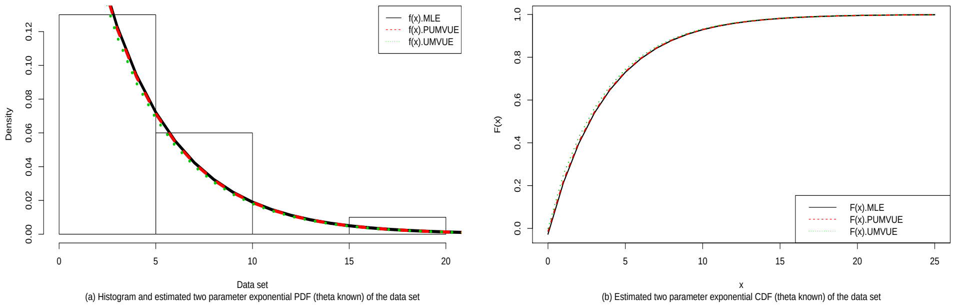


Figure 8: Graph of the estimated PDF and CDF of the two-parameter exponential distribution when $\theta$ is known but $\mu$ is unknown.

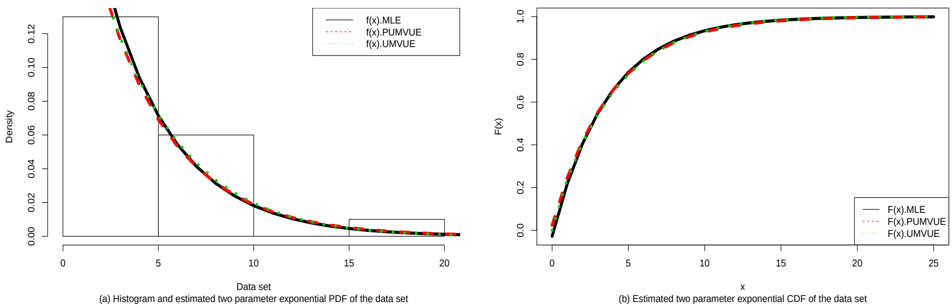


Figure 9: Graph of the estimated PDF and CDF of the two-parameter exponential distribution when $\theta$ and $\mu$ both are unknown.

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