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The Weibull odd Burr III-G family of distributions: Properties and applications

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Abstract: A new generalized family of distributions called the Weibull Odd Burr III-G is introduced using the T-X transformation technique. Some of useful mathematical and statistical properties such as the hazard function, quantile function, moments, probability weighted moments, Rényi entropy, order statistics and stochastic orders are derived. The method of maximum likelihood estimation is used to estimate the model parameters. The usefulness of these family of distributions is demonstrated via simulated experiments and its special cases are applied to real life data sets to illustrate flexibility.

Keywords: Weibull distribution; Odd Burr-III distribution; Family of distributions; Stochastic Order; Maximum likelihood Estimation.
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1 Introduction

Recently many lifetime modeling distributions have been studied by different authors. Despite this being the case, there is always an opportunity for the discovery of new distributions which provides more flexibility in fitting various real-world problems. The approach has motivated many researchers to work on developing new flexible families of models. Consequently, new distributions have been studied across the literature. This has necessitated a growing trend of generating new families of distributions from existing models by adding one or more parameter(s) to the baseline distribution and

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assessing the behavior of pdf and hazard rate shapes together with the associated goodness-of-fits. Some well known generated families studied by various authors include the Weibull-G family of distributions by Bourguignon et al. (2014), exponentiated-generalized-G family of distributions by Cordeiro et al. (2013) and two log-gamma-G families by Amini et al. (2014) to mention just a few. Another useful generators called gamma-G were introduced by Zografos and Balakhrishnan (2009) and Ristić and Balakhrishnan (2012). Alzaatreh et al. (2013) developed a general approach, the transformed-transformer (T-X) family. Alzaghal et al. (2013) further extended T-X family and proposed exponentiated-T-X family of distributions. Aljarrah et al. (2014) introduced T-X family based on quantile function approach. Furthermore, more recent work has been developed on the generalization of the Odd Burr distributions leading to models with desirable properties and enhanced flexibility, see Alizadeh et al. (2017), Altun et al. (2017b) and Altun et al. (2017a) for more details.

Motivated by these developments, we propose a new useful family of distributions called the Weibull Odd Burr III-G (WOBIII-G) using the T-X transformation technique proposed by Alzaatreh et al. (2013). The proposed new family of distributions presents some flexible shapes compared to the baseline models. Furthermore, these models consistently give better fits than some of the generators having the same number of parameters. We are hopeful that these new family of models will find wider applications in areas such as engineering, medicine, reliability, economics and finance, just to mention a few.

The layout of this paper is as follows: Section 2, presents the proposed model and its hazard rate function, quantile function and linear representation of the pdf. Some of the special cases are presented under Section 3. We derive some of the mathematical properties for the WOBIII-G family, namely; moments, probability weighted moments, order statistics, Rényi entropy, stochastic ordering and maximum likelihood estimates under Section 4. Simulation results are presented under Section 5. Section 6 presents results on applications using real life data examples to demonstrate the applicability and flexibility of the fitted model and finally concluding remarks are given under Section 7. The elements of a score vector, other useful expansions and probability density functions (pdfs) of the non-nested models used for comparisons are given under the appendix.

2 Generating the Weibull odd Burr III-G family of distributions

In this section, we develop the new model, its sub-classes and study its mathematical properties in detail.

2.1 The model

We formulate and develop the new model using the T-X generator technique proposed by Alzaatreh et al. (2013) having the cumulative distribution function (cdf) given by

$$F(x) = \int_0^{W(G(x;\underline{\boldsymbol{\varphi}}))} r(t)dt, \qquad x \in \mathbb{R},$$

where $r(t)$ is the probability density function (pdf) of a random variable T and $W(G(x;\underline{\boldsymbol{\varphi}}))$ represent the function of the cdf for the baseline function of a random variable X. To develop the WOBIII-G family using the T-X family technique, we let $r(t)=\beta t^{\beta-1}e^{-t^{\beta}}, t>0, \beta>0$ (one parameter Weibull distribution) and $W(G(x;\underline{\boldsymbol{\varphi}})) = -\log(1-(1+(\frac{G(x;\underline{\boldsymbol{\varphi}})}{\overline{G}(x;\underline{\boldsymbol{\varphi}})})^{-a})^{-b})$, where $G(x;\underline{\boldsymbol{\varphi}})$ is the baseline cdf. From these results, we can write the cdf of the WOBIII-G family of distributions as

$$F(x;a,b,\beta,\underline{\boldsymbol{\varphi}}) = 1-\exp\left(-\left[-\log\left(1-\left(1+\left(\frac{G(x;\underline{\boldsymbol{\varphi}})}{\overline{G}(x;\underline{\boldsymbol{\varphi}})}\right)^{-a}\right)^{-b}\right)\right]^{\beta}\right), \tag{1}$$

for $a,b,\beta,\ x>0$ and parameter vector $\underline{\boldsymbol{\varphi}}$, with the corresponding pdf given by

$$\begin{aligned} f(x;a,b,\beta,\underline{\boldsymbol{\varphi}}) =& ab\beta\exp\left(-\left[-\log\left(1-\left(1+\left(\frac{G(x;\underline{\boldsymbol{\varphi}})}{\overline{G}(x;\underline{\boldsymbol{\varphi}})}\right)^{-a}\right)^{-b}\right)\right]^{\beta}\right) \\ &\times\left[-\log\left(1-\left(1+\left(\frac{G(x;\underline{\boldsymbol{\varphi}})}{\overline{G}(x;\underline{\boldsymbol{\varphi}})}\right)^{-a}\right)^{-b}\right)\right]^{\beta-1}\left(\frac{G(x;\underline{\boldsymbol{\varphi}})}{\overline{G}(x;\underline{\boldsymbol{\varphi}})}\right)^{-a-1} \\ &\times\frac{\left(1+\left(\frac{G(x;\underline{\boldsymbol{\varphi}})}{\overline{G}(x;\underline{\boldsymbol{\varphi}})}\right)^{-a}\right)^{-b-1}}{\left(1-\left(1+\left(\frac{G(x;\underline{\boldsymbol{\varphi}})}{\overline{G}(x;\underline{\boldsymbol{\varphi}})}\right)^{-a}\right)^{-b}\right)}\frac{g(x;\underline{\boldsymbol{\varphi}})}{\left(\overline{G}(x;\underline{\boldsymbol{\varphi}})\right)^{2}}. \end{aligned} \tag{2}$$

The hazard rate function (hrf) is given by

$$\begin{aligned} h(x;a,b,\beta,\underline{\boldsymbol{\varphi}}) =& ab\beta\left[-\log\left(1-\left(1+\left(\frac{G(x;\underline{\boldsymbol{\varphi}})}{\overline{G}(x;\underline{\boldsymbol{\varphi}})}\right)^{-a}\right)^{-b}\right)\right]^{\beta-1} \\ &\times\frac{\left(1+\left(\frac{G(x;\underline{\boldsymbol{\varphi}})}{\overline{G}(x;\underline{\boldsymbol{\varphi}})}\right)^{-a}\right)^{-b-1}}{\left(1-\left(1+\left(\frac{G(x;\underline{\boldsymbol{\varphi}})}{\overline{G}(x;\underline{\boldsymbol{\varphi}})}\right)^{-a}\right)^{-b}\right)}\left(\frac{G(x;\underline{\boldsymbol{\varphi}})}{\overline{G}(x;\underline{\boldsymbol{\varphi}})}\right)^{-a-1}\frac{g(x;\underline{\boldsymbol{\varphi}})}{\left(\overline{G}(x;\underline{\boldsymbol{\varphi}})\right)^{2}}. \end{aligned} \tag{3}$$

2.2 Quantile function

The quantile function for the WOBIII-G family of distributions is derived by inverting the following function

$$1-\exp\left(-\left[-\log\left(1-\left(1+\left(\frac{G(x;\underline{\boldsymbol{\varphi}})}{\overline{G}(x;\underline{\boldsymbol{\varphi}})}\right)^{-a}\right)^{-b}\right)\right]^{\beta}\right)=u,$$

for $0<u<1$, so that

$$\left(1+\left(\frac{G(x;\underline{\boldsymbol{\varphi}})}{\overline{G}(x;\underline{\boldsymbol{\varphi}})}\right)^{-a}\right)=\left(1-\exp\left(-\left(-\log\left(1-u\right)\right)^{\frac{1}{\beta}}\right)\right)^{\frac{-1}{b}}.$$

Therefore, we obtain the quantile function as

$$Q(u)=G^{-1}\left[\left(\left(\left(1-\exp\left(-\left(-\log\left(1-u\right)\right)^{\frac{1}{\beta}}\right)\right)^{\frac{-1}{b}}-1\right)^{\frac{1}{a}}+1\right)^{-1}\right], \tag{4}$$

which can be solved using iterative methods from any applicable software such as R.

2.3 Linear representation of the density function

This section presents results on linear representation of the pdf for the WOBIII-G family of distributions. The expansion of the pdf will further allow us to derive important mathematical and statistical properties. By applying several series expansions, (see appendix A for details), we can write the pdf of W-OBIII-G family of distributions given by equation (2) as

$$\begin{aligned}f(x;a,\beta,b,\underline{\boldsymbol{\varphi}})=&ab\beta\sum_{k,z,i,p,m,s=0}^{\infty}\frac{\Gamma(z+1)}{\Gamma(1)z!}b_{s,m}(-1)^{k+p}\binom{\beta(k+1)-1}{m}\\&\times\binom{-a(i+1)-1}{p}\binom{k^{*}}{i}\left(\overline{G}(x;\underline{\boldsymbol{\varphi}})\right)^{a(i+1)+p-1}g(x;\underline{\boldsymbol{\varphi}})\\&=ab\beta\sum_{k,z,i,p,w,m,s=0}^{\infty}\frac{\Gamma(z+1)}{\Gamma(1)z!}b_{s,m}(-1)^{k+p+w}\binom{\beta(k+1)-1}{m}\binom{k^{*}}{i}\\&\times\binom{a(i+1)+p-1}{w}\binom{-a(i+1)-1}{p}\left(\frac{w+1}{w+1}\right)\left(G(x;\underline{\boldsymbol{\varphi}})\right)^{w}g(x;\underline{\boldsymbol{\varphi}})\\&=\sum_{w=0}^{\infty}c_{w}g_{w}(x;\underline{\boldsymbol{\varphi}}),\end{aligned} \tag{5}$$

where $k^{*}=-b(\beta(k+1)+m+s+z)-1$, $g_{w}(x;\underline{\boldsymbol{\varphi}})=(w+1)\left(G(x;\underline{\boldsymbol{\varphi}})\right)^{w}g(x;\underline{\boldsymbol{\varphi}})$ is the exponentiated-G (Exp-G) distribution with power parameter $w+1$ and

$$\begin{aligned}c_{w}=&ab\beta\sum_{k=0}^{\infty}\sum_{z=0}^{\infty}\sum_{i,p=0}^{\infty}\frac{\Gamma(z+1)}{(w+1)\Gamma(1)z!}\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}b_{s,m}(-1)^{k+p+w}\binom{\beta(k+1)-1}{m}\\&\times\binom{-a(i+1)-1}{p}\binom{a(i+1)+p-1}{w}\binom{k^{*}}{i}.\end{aligned} \tag{6}$$

The details of these results are presented under appendix (A.1).

3 Some special cases

We consider some special cases of various sub-models obtained by changing the baseline distribution function $G(x;\underline{\varphi})$ to other flexible distributions. The parameter vector space is limited to atmost 2 component vector to avoid over parametrization and redundancy.

3.1 Weibull odd Burr III-log-logistic distribution

Consider the log-logistic distribution as the baseline distribution with parameter $\lambda > 0$ having cdf and pdf $G(x;\lambda) = 1-(1+x^{\lambda})^{-1}$ and $g(x;\lambda) = \lambda x^{\lambda-1}(1+x^{\lambda})^{-2}$, respectively. Define $t_1 = \frac{1-(1+x^{\lambda})^{-1}}{(1+x^{\lambda})^{\lambda}}$, then the cdf, pdf and hrf of the Weibull odd Burr III-log-logistic (WOBIII-LLoG) distribution, respectively, are given by

$$
\begin{aligned}
F(x;a,b,\beta,\lambda) =& 1-\exp\left(-\left[-\log\left(1-\left(1+t_1^{-a}\right)^{-b}\right)\right]^{\beta}\right),\\
f(x;a,b,\beta,\lambda) =& ab\beta\exp\left(-\left[-\log\left(1-\left(1+t_1^{-a}\right)^{-b}\right)\right]^{\beta}\right)\frac{\lambda x^{\lambda-1}(1+x^{\lambda})^{-2}}{((1+x^{\lambda})^{-1})^2}\\
&\times\left[-\log\left(1-\left(1+t_1^{-a}\right)^{-b}\right)\right]^{\beta-1}\frac{\left(1+t_1^{-a}\right)^{-b-1}}{\left(1-\left(1+t_1^{-a}\right)^{-b}\right)}t_1^{-a-1},\\
h(x;a,b,\beta,\lambda) =& ab\beta\left[-\log\left(1-\left(1+t_1^{-a}\right)^{-b}\right)\right]^{\beta-1}\frac{\left(1+t_1^{-a}\right)^{-b-1}}{\left(1-\left(1+t_1^{-a}\right)^{-b}\right)}t_1^{-a-1}\\
&\times\frac{\lambda x^{\lambda-1}(1+x^{\lambda})^{-2}}{((1+x^{\lambda})^{-1})^2},
\end{aligned}
$$

for $a, b, \beta, \lambda > 0$. Plots of the pdf and hrf for the WOBIII-LLoG distribution are given in Figure 1.

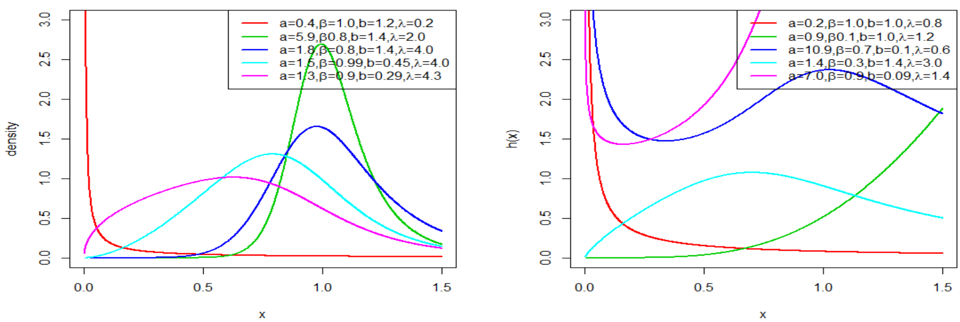

Figure 1: Plots of the pdf and hrf for the WOBIII-LLoG distribution

Figure 1 shows the flexibility of the WOBIII-LLoG distribution for selected parameter values. The pdfs of the WOBIII-LLoG distribution can take various shapes that include reverse-J, uni-modal, left or right skewed shapes. Additionally, the WOBIII-LLoG distribution hrf plots reveal decreasing, increasing, bathtub, upside down bathtub and bathtub followed by upside down bathtub shapes.

3.2 Weibull odd Burr III-logistic distribution

Consider the logistic distribution as the baseline distribution with parameter $\lambda > 0$ having cdf and pdf $G(x;\lambda) = (1 + e^{-\lambda x})^{-1}$ and $g(x;\lambda) = \lambda e^{-\lambda x}(1 + e^{-\lambda x})^{-2}$, respectively. Define $t_2 = \frac{(1+e^{-\lambda x})^{-1}}{1-(1+e^{-\lambda x})^{-1}}$, then the cdf, pdf and hrf of the Weibull odd Burr III-logistic (WOBIII-LoG) distribution, respectively, are given by

$$
\begin{aligned}
F(x;a,b,\beta,\lambda) =& 1 - \exp\left(-\left[-\log\left(1-\left(1+t_2^{-a}\right)^{-b}\right)\right]^{\beta}\right),\\
f(x;a,b,\beta,\lambda) =& ab\beta \exp\left(-\left[-\log\left(1-\left(1+t_2^{-a}\right)^{-b}\right)\right]^{\beta}\right) \frac{\lambda e^{-\lambda x}(1+e^{-\lambda x})^{-2}}{\left((1+e^{-\lambda x})^{-1}\right)^2}\\
&\times \left[-\log\left(1-\left(1+t_2^{-a}\right)^{-b}\right)\right]^{\beta-1} \frac{\left(1+t_2^{-a}\right)^{-b-1}}{\left(1-\left(1+t_2^{-a}\right)^{-b}\right)} (t_2)^{-a-1},\\
h(x;a,b,\beta,\lambda) =& ab\beta \left[-\log\left(1-\left(1+t_2^{-a}\right)^{-b}\right)\right]^{\beta-1} \frac{\lambda e^{-\lambda x}(1+e^{-\lambda x})^{-2}}{\left((1+e^{-\lambda x})^{-1}\right)^2}\\
&\times \frac{\left(1+t_2^{-a}\right)^{-b-1}}{\left(1-\left(1+t_2^{-a}\right)^{-b}\right)} (t_2)^{-a-1},
\end{aligned}
$$

for $a, b, \beta, \lambda > 0$. Plots of the pdf and hrf for the WOBIII-LoG distribution are given in Figure 2.

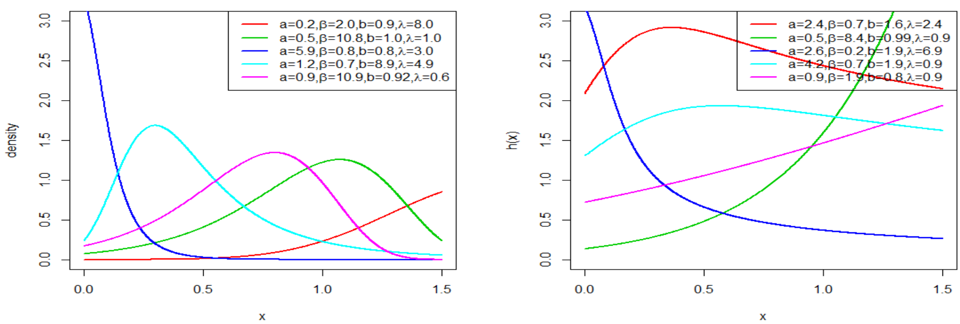

Figure 2: Plots of the pdf and hrf for the WOBIII-LoG distribution

Figure 2 demonstrates different shapes adopted by the WOBIII-LoG distribution for selected parameter values. The pdfs of the WOBIII-LoG distribution can take various shapes that include reverse-J, uni-modal, left or right skewed shapes. Also, the WOBIII-LoG distribution hrf plots give decreasing, increasing and upside down bathtub shapes.

3.3 Weibull odd Burr III-Lindley distribution

If we let Lindley distribution be the baseline distribution with pdf and cdf given by $g(x;\lambda) = \frac{\lambda^2}{(1+\lambda)}(1+x)e^{-\lambda x}$ and $G(x;\lambda) = 1 - (1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x}$, respectively, for $\lambda > 0$. Define $t_3 = \frac{1-(1+\frac{\lambda x}{1+\lambda})}{(1+\frac{\lambda x}{1+\lambda})}$ then the cdf, pdf and hrf of the Weibull odd Burr III-Lindley (WOBIII-L) distribution, respectively, are given by

$$F(x;a,b,\beta,\lambda) = 1 - \exp\left(-\left[-\log\left(1-\left(1+t_3^{-a}\right)^{-b}\right)\right]^{\beta}\right),$$

$$f(x;a,b,\beta,\lambda) = ab\beta\exp\left(-\left[-\log\left(1-\left(1+t_3^{-a}\right)^{-b}\right)\right]^{\beta}\right)\frac{\frac{\lambda^2}{(1+\lambda)}(1+x)e^{-\lambda x}}{\left((1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}\right)^2}$$
$$\times\left[-\log\left(1-\left(1+t_3^{-a}\right)^{-b}\right)\right]^{\beta-1}\frac{\left(1+t_3^{-a}\right)^{-b-1}}{\left(1-\left(1+t_3^{-a}\right)^{-b}\right)}(t_3)^{-a-1}$$

$$h(x;a,b,\beta,\lambda) = ab\beta\left[-\log\left(1-\left(1+t_3^{-a}\right)^{-b}\right)\right]^{\beta-1}\frac{\frac{\lambda^2}{(1+\lambda)}(1+x)e^{-\lambda x}}{\left((1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}\right)^2}$$
$$\times\frac{\left(1+t_3^{-a}\right)^{-b-1}}{\left(1-\left(1+t_3^{-a}\right)^{-b}\right)}(t_3)^{-a-1},$$

for $a, b, \beta, \lambda > 0$. Plots of the pdf and hrf for the WOBIII-L distribution are given in

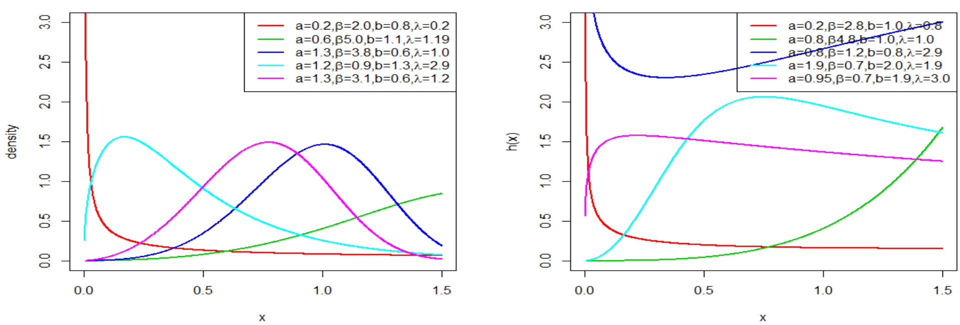

Figure 3: Plots of the pdf and hrf for the WOBIII-L distribution

Figure 3 illustrates the flexible nature of the WOBIII-L distribution for selected parameter values. The pdfs of the WOBIII-L distribution can adopt various shapes that include reverse-J, uni-modal, left or right skewed shapes. The WOBIII-L distribution hrf plots also exhibit decreasing, increasing, bathtub and upside down bathtub shapes.

3.4 Weibull odd Burr III-uniform distribution

Suppose that we take the baseline distribution to be the uniform distribution with pdf and cdf given by $g(x) = 1/\lambda$ and $G(x;\lambda) = x/\lambda$, respectively, for $0 < x < \lambda$. If let

$t_4 = \frac{x/\lambda}{1-x/\lambda}$, we can obtain the Weibull odd Burr III-uniform (WOBIII-U) distribution with cdf, pdf and hrf, respectively, as

$$
\begin{aligned}
F(x;a,b,\beta,\lambda) =& 1-\exp\left(-\left[-\log\left(1-\left(1+t_4^{-a}\right)^{-b}\right)\right]^{\beta}\right),\\
f(x;a,b,\beta,\lambda) =& ab\beta\exp\left(-\left[-\log\left(1-\left(1+t_4^{-a}\right)^{-b}\right)\right]^{\beta}\right)\frac{1/\lambda}{(1-x/\lambda)^2}\\
&\times\left[-\log\left(1-\left(1+t_4^{-a}\right)^{-b}\right)\right]^{\beta-1}\frac{\left(1+t_4^{-a}\right)^{-b-1}}{\left(1-\left(1+t_4^{-a}\right)^{-b}\right)}(t_4)^{-a-1}\\
h(x;a,b,\beta,\lambda) =& ab\beta\left[-\log\left(1-\left(1+t_4^{-a}\right)^{-b}\right)\right]^{\beta-1}\frac{1/\lambda}{(1-x/\lambda)^2}\\
&\times\frac{\left(1+t_4^{-a}\right)^{-b-1}}{\left(1-\left(1+t_4^{-a}\right)^{-b}\right)}(t_4)^{-a-1},
\end{aligned}
$$

for $a,b,\beta,\lambda > 0$. Plots of the pdf and hrf for the WOBIII-U distribution are given in Figure 4.

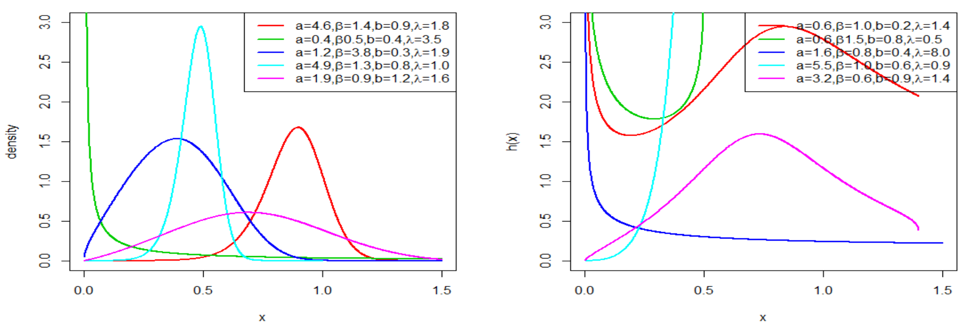

Figure 4: Plots of the pdf and hrf for the WOBIII-U distribution

Figure 4 gives different shapes from the WOBIII-U distribution for selected parameter values. The pdfs of the WOBIII-U distribution follow various shapes that include reverse-J, uni-modal, left or right skewed shapes. In addition, plots of the hrf for the WOBIII-U distribution exhibit decreasing, increasing, bathtub, upside down bathtub and bathtub followed by upside down bathtub shapes.

4 Some mathematical properties

In this section, we derive some useful mathematical and statistical properties for the WOBIII-G family of distributions such as moments, distribution of order statistics and rényi entropy.

4.1 Moments

Let $Y_{w+1} \sim Exp-G(w+1)$. Then using equation (5), the n^{th} raw moment, μ_n' of the WOBIII-G family of distributions is obtained as

$$\mu_n' = E(X^n) \quad = \quad \int_{-\infty}^{\infty} x^n f(x)dx = \sum_{w=0}^{\infty} c_w E(Y_w^n).$$

The first five moments for the WOBIII-LoG distibution together with the standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) for selected values are presented under Table 1.

Table 1: Weibull Odd Burr III-LoG Moments for selected parameter values

	(a,b,β,λ)				
	(.9,1.5,.5,.5)	(.5,1.9,1.0,.5)	(.7,2.0,.5,.3)	(.5,2.0,.9,.8)	(.7,1.6,.6,.3)
$E(X)$	0.0343	0.0339	0.0189	0.0528	0.0199
$E(X^2)$	0.0227	0.0228	0.0125	0.0355	0.0132
$E(X^3)$	0.0170	0.0171	0.0094	0.0267	0.0099
$E(X^4)$	0.0135	0.0138	0.0075	0.0214	0.0079
$E(X^5)$	0.0113	0.0115	0.0063	0.0179	0.0066
SD	0.1468	0.1470	0.1104	0.1808	0.1134
CV	1.2742	2.3352	1.8557	3.4217	2.6971
CS	3.6492	4.6897	2.4648	3.6189	3.2820
CK	4.4561	5.7754	3.9514	5.3030	4.4636

4.2 Probability weighted moments

The $(s,m)^{th}$ probability weighted moment (PWM) of X denoted by $\eta_{s,m}$ is

$$\eta_{s,m} = E(X^s(F(X))^m) = \int_{-\infty}^{\infty} x^s(F(x))^m f(x)dx.$$

By using equations (1) and (2), and if we define $t = \frac{G(x;\underline{\varphi})}{\overline{G}(x;\underline{\varphi})}$, we can write

$$\begin{aligned} f(x)(F(x))^m =& ab\beta \exp\left(-\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta}\right)\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta-1} \\ =& \frac{\left(1+t^{-a}\right)^{-b-1}}{\left(1-(1+t^{-a})^{-b}\right)} t^{-a-1} \\ & \times \left(1-\exp\left(-\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta}\right)\right)^m \frac{g(x;\underline{\varphi})}{\left(\overline{G}(x;\underline{\varphi})\right)^2}. \end{aligned}$$

Using the series expansion results (see appendix (A.2) for details), we have

$$f(x)(F(x))^m = \sum_{w=0}^{\infty} \phi_w g_w(x;\underline{\varphi}), \tag{7}$$

where $g_w(x;\underline{\varphi}) = (w+1)\left(G(x;\underline{\varphi})\right)^w g(x;\underline{\varphi})$ is the exponentiated-G (Exp-G) distribution with power parameter $w+1$ and

$$\begin{aligned}\phi_w =& ab\beta \sum_{q,k,z,i,p,m,s=0}^{\infty} \binom{m}{q} \frac{(-1)^{q+k+p+w}}{k!}(q+1)^k \frac{\Gamma(z+1)}{(w+1)\Gamma(1)z!} b_{s,m} \\ &\times \binom{-a(i+1)-1}{p}\binom{\beta(k+1)-1}{m}\binom{a(i+1)+p-1}{w} \\ &\times \binom{-b(\beta(k+1)+m+s+z)-1}{i}. \end{aligned} \tag{8}$$

Finally, the PWMs of the WOBIII-G family of distributions can be written as

$$\eta_{s,m} = \int_{-\infty}^{\infty} x^s \sum_{w=0}^{\infty} \phi_w g_w(x;\underline{\varphi})dx = \sum_{w=0}^{\infty} \phi_w \int_{-\infty}^{\infty} x^s g_w(x;\underline{\varphi})dx.$$

This shows that the $(s,m)^{th}$ PWMs of WOBIII-G family of distributions can be obtained from the moments of the E-G distribution. The details of these results are presented under appendix (A.2)

4.3 Distribution of order statistics

Let $X_1, X_2, ..., X_n$ be a random sample from the WOBIII-G family of distribution and suppose $X_{1:n} < X_{2:n}, ... < X_{n:n}$ denote the corresponding order statistics. The pdf of the k^{th} order statistic is given by

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l f(x) \left[F(x)\right]^{k+l-1}. \tag{9}$$

From the results derived under Section 4.2, (also see appendix A.3 for details), we have

$$f(x)(F(x))^{k+l-1} = \sum_{w=0}^{\infty} b_w g_w(x;\underline{\varphi}), \tag{10}$$

where $g_w(x;\underline{\varphi}) = (w+1)\left(G(x;\underline{\varphi})\right)^w g(x;\underline{\varphi})$ is the exponentiated-G (Exp-G) distribution with power parameter $w+1$ and

$$\begin{aligned}b_w =& ab\beta \sum_{q,k=0}^{\infty}\sum_{z,i=0}^{\infty}\sum_{p,m,s=0}^{\infty} \binom{k+l-1}{q}(-1)^{q+k+p+w}(q+1)^k \frac{\Gamma(z+1)}{(w+1)\Gamma(1)z!} b_{s,m} \\ &\times \binom{-a(i+1)-1}{p}\binom{\beta(k+1)-1}{m}\binom{a(i+1)+p-1}{w}\binom{k^*}{i}. \end{aligned}$$

If we substitute equation (10) into (9), we obtain

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!} \sum_{l=0}^{n-k}\sum_{w=0}^{\infty} \binom{n-k}{l} (-1)^l b_w g_w(x;\underline{\varphi}), \tag{11}$$

where $g_w(x;\underline{\varphi}) = (w+1)\left(G(x;\underline{\varphi})\right)^w g(x;\underline{\varphi})$ is the exponentiated-G (Exp-G) distribution with power parameter $w > 0$ and parameter vector $\underline{\varphi}$.

4.4 Rényi entropy

An entropy gives a measure of variation for the random variable X and the commonly known entropies are Shannon entropy (Shannon, 1951) and Rényi entropy (Rényi, 1961). Rényi entropy is defined to be

$$I_R(v) = \frac{1}{1-v}\log\left(\int_0^\infty [f(x;a,b,\beta,\underline{\varphi})]^v dx\right), v \neq 1, v > 0. \tag{12}$$

Note that $I_R(v)$ can be written as (see appendix A.4 for details):

$$I_R(v) = \frac{1}{1-v}\log\left[\sum_{w=0}^{\infty}\tau_w \exp((1-v)I_{REG})\right],$$

for $v > 0$, $v \neq 1$, where $I_{REG} = \frac{1}{1-v}\log\left[\int_0^\infty \left(\left[\frac{w}{v}+1\right](G(x;\underline{\varphi}))^{\frac{w}{v}} g(x;\underline{\varphi})\right)^v dx\right]$ is the Rényi entropy of Exp-G distribution with power parameter $\frac{w}{v}+1$, and

$$\begin{aligned}\tau_w =& \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s,z,i,p=0}^{\infty}(ab\beta)^v b_{s,m}\binom{\beta(k+v)-v}{m}(-1)^{k+p+w}v^k \\ &\times \binom{-a(v+i)-v}{p}\binom{k^{**}}{i}\binom{a(v+i)-v+p}{w}\frac{\Gamma(z+v)}{\Gamma(v)z!}\frac{1}{\left[1+\frac{w}{v}\right]^v},\end{aligned}$$

for $k^{**} = -b(\beta(k+v)+m+s+z)-v$.

4.5 Stochastic ordering

In this section, we define and derive the most commonly applied three orders for the WOBIII-G family of distributions namely; the usual stochastic order, the hazard rate order and the likelihood ratio order, see Shaked and Shanthikumar (2007) for more details.

Consider the two random variables X and Y having the cdfs $F_x(t)$ and $F_y(t)$, respectively, with $\overline{F}_x(t) = 1 - F_x(t)$ as the reliability or survival function. A random variable X is said to be stochastically smaller than the random variable Y if $\overline{F}_x(t) \leq \overline{F}_y(t)$ for all t or $F_x(t) \geq F_y(t)$ for all t. This is denoted by $X <_{st} Y$. The hazard rate order and likelihood ratio order are stronger and are given by $X <_{hr} Y$ if $h_x(t) \geq h_y(t)$ for all t, and $X <_{\ell_r} Y$ if $\frac{f_x(t)}{f_y(t)}$ is decreasing in t. It holds that $X <_{\ell_r} Y \implies X <_{hr} Y \implies X <_{st} Y$.

Now, consider X_1 and X_2 as two independent random variables following $WOBIII-G(a,b,\beta_1,\underline{\varphi})$ and $WOBIII-G(a,b,\beta_2,\underline{\varphi})$ distributions, then the pdfs of X_1 and X_2 are

$$\begin{aligned}f_1(x) =& ab\beta_1 \exp\left(-\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta_1}\right)\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta_1-1} \\ &\times \frac{\left(1+t^{-a}\right)^{-b-1}}{\left(1-(1+t^{-a})^{-b}\right)}t^{-a-1}\frac{g(x;\underline{\varphi})}{\left(\overline{G}(x;\underline{\varphi})\right)^2},\end{aligned}$$

$$f_2(x) = ab\beta_2 \exp\left(-\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta_2}\right)\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta_2-1}$$
$$\times \frac{\left(1+t^{-a}\right)^{-b-1}}{\left(1-(1+t^{-a})^{-b}\right)} t^{-a-1} \frac{g(x;\underline{\boldsymbol{\varphi}})}{\left(\overline{G}(x;\underline{\boldsymbol{\varphi}})\right)^2}.$$

The ratio, $\frac{f_1(x)}{f_2(x)}$ takes the form

$$\frac{f_1(x)}{f_2(x)} = \frac{\beta_1}{\beta_2}\frac{\exp(-z^{\beta_1})}{\exp(-z^{\beta_2})}\frac{z^{\beta_1-1}}{z^{\beta_2-1}} = \frac{\beta_1}{\beta_2}\exp(-z^{\beta_1}+z^{\beta_2})z^{\beta_1-\beta_2}, \tag{13}$$

where $z = -\log\left(1-(1+t^{-a})^{-b}\right)$. If we differentiate equation (13) with respect to x, we get

$$\frac{d}{dx}\left(\frac{f_1(x)}{f_2(x)}\right) = \frac{\beta_1}{\beta_2}\left[\exp(z^{\beta_2}-z^{\beta_1})z'(\beta_2 z^{\beta_2-1}-\beta_1 z^{\beta_1-1})z^{\beta_1-\beta_2}\right]$$
$$\times (\beta_1-\beta_2)z^{(\beta_1-\beta_2)-1}z'\exp(z^{\beta_2}-z^{\beta_1}),$$

where $z' = \frac{\left(1+t^{-a}\right)^{-b-1}}{\left(1-(1+t^{-a})^{-b}\right)}t^{-a-1}\frac{g(x;\underline{\boldsymbol{\varphi}})}{\left(\overline{G}(x;\underline{\boldsymbol{\varphi}})\right)^2}$, and finally if $\beta_2 < \beta_1$, then $\frac{d}{dx}\left(\frac{f_1(x)}{f_2(x)}\right) < 0$, and therefore, likelihood ratio order $X_1 <_{\ell_r} X_2$ exists. As a result, the random variables X_1 and X_2 are stochastically ordered.

4.6 Maximum likelihood estimation

Let $X \sim WOBIII-G(a,b,\beta,\underline{\boldsymbol{\varphi}})$ and $\boldsymbol{\Delta} = (a,b,\beta,\underline{\boldsymbol{\varphi}})^T$ be the vector of model parameters, then the log-likelihood function $\ell_n = \ell_n(\boldsymbol{\Delta})$ based on a random sample of size n from the WOBIII-G family of distributions is given by

$$\ell_n(\Delta) = n\log(a\beta b) - \sum_{i=1}^{n}\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta} + (-b-1)\sum_{i=1}^{n}\left(1+t^{-a}\right)$$
$$+ (\beta-1)\sum_{i=1}^{n}\log\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]$$
$$+ (-a-1)\sum_{i=1}^{n}\log t - \sum_{i=1}^{n}\log\left(1-\left(1+t^{-a}\right)^{-b}\right)$$
$$+ \sum_{i=1}^{n}\log\left(g(x_i;\underline{\boldsymbol{\varphi}})\right) - 2\sum_{i=1}^{n}\log\left(\overline{G}(x_i;\underline{\boldsymbol{\varphi}})\right).$$

The elements of the score vector are given in appendix B. Through the use of numerical methods such as Newton-Raphson procedure, we can obtain the maximum likelihood estimates of the parameters, denoted by $\hat{\boldsymbol{\Delta}}$ by solving the nonlinear equation $(\frac{\partial \ell_n}{\partial a}, \frac{\partial \ell_n}{\partial b}, \frac{\partial \ell_n}{\partial \beta}, \frac{\partial \ell_n}{\partial \underline{\boldsymbol{\varphi}}})^T = \mathbf{0}$. The multivariate normal distribution $N_{q+3}(\underline{0}, J(\hat{\boldsymbol{\Delta}})^{-1})$, where the mean vector $\underline{\mathbf{0}} = (0,0,0,\underline{0})^T$ and $J(\hat{\boldsymbol{\Delta}})^{-1}$ is the observed Fisher information matrix evaluated at $\hat{\boldsymbol{\Delta}}$, which is critical in the construction of confidence intervals and regions for the model parameters.

5 Monte Carlo simulation

In this section, we present the simulation results for the WOBIII-LoG distribution. Monte Carlo simulation were performed to evaluate the accuracy of the MLEs. We expect the estimates to converge towards the true values consistently as the sample size increases. We generate N=1000 random samples of size n and determine the MLEs for each sample. The bias and root mean square error (RMSE) are obtained from each sample size. The numerical results for the simulation runs are collected and presented under Tables 2 and 3. From these results, for all the considered sets of parameters, we see that the RMSE and bias decreases as the sample size increases as expected which is consistent with the theoretical properties of convergence for the MLEs.

Table 2: WOBIII-LoG distribution simulation results; mean, average bias and RMSE (Set I: a=2.3, b=1.4 , β=0.7 , λ=1.2 : Set II: a=1.4, b=1.3, β=0.6 , λ=2.3)

Parameter	n	Set I			Set II		
		Mean	Bias	RMSE	Mean	Bias	RMSE
a	35	2.848	0.548	0.764	1.849	0.449	3.465
	50	2.748	0.449	0.489	1.639	0.239	2.489
	100	2.709	0.409	0.441	1.605	0.205	2.118
	200	2.590	0.290	0.369	1.539	0.139	1.840
	400	2.509	0.209	0.331	1.470	0.070	0.679
	800	2.351	0.051	0.219	1.421	0.021	0.483
	1000	2.307	0.007	0.195	1.401	0.001	0.316
b	35	1.897	0.497	0.685	1.848	0.548	0.855
	50	1.697	0.297	0.638	1.727	0.427	0.738
	100	1.619	0.219	0.586	1.639	0.339	0.704
	200	1.577	0.177	0.537	1.490	0.190	0.658
	400	1.560	0.160	0.520	1.385	0.085	0.617
	800	1.490	0.090	0.438	1.343	0.043	0.574
	1000	1.409	0.009	0.400	1.303	0.003	0.503
β	35	0.994	0.294	0.399	0.939	0.338	0.476
	50	0.878	0.178	0.365	0.785	0.185	0.448
	100	0.855	0.155	0.311	0.760	0.160	0.426
	200	0.790	0.089	0.238	0.640	0.040	0.385
	400	0.759	0.059	0.220	0.618	0.018	0.354
	800	0.720	0.021	0.149	0.610	0.010	0.327
	1000	0.711	0.011	0.119	0.608	0.008	0.281
λ	35	1.798	0.598	0.748	2.888	0.587	0.837
	50	1.694	0.494	0.628	2.486	0.186	0.805
	100	1.484	0.284	0.604	2.388	0.088	0.784
	200	1.428	0.228	0.587	2.354	0.054	0.748
	400	1.299	0.099	0.539	2.320	0.019	0.648
	800	1.238	0.038	0.428	2.307	0.007	0.360
	1000	1.202	0.002	0.343	2.302	0.002	0.310

Table 3: WOBIII-LoG distribution simulation results; mean, average bias and RMSE (Set III: a=2.1, b=1.3 , β=0.6 , λ=1.6 : Set IV: a=1.3, b=0.9 , β=2.2 , λ=1.2)

Parameter	n	Set III			Set IV		
		Mean	Bias	RMSE	Mean	Bias	RMSE
a	35	2.537	0.437	0.684	1.849	0.549	0.394
	50	2.438	0.338	0.603	1.794	0.494	0.246
	100	2.356	0.256	0.418	1.639	0.339	0.218
	200	2.316	0.216	0.358	1.587	0.287	0.193
	400	2.300	0.200	0.288	1.395	0.095	0.152
	800	2.178	0.078	0.239	1.350	0.050	0.129
	1000	2.100	0.001	0.217	1.308	0.008	0.099
b	35	1.958	0.658	0.735	1.754	0.854	0.648
	50	1.937	0.637	0.703	1.583	0.683	0.528
	100	1.874	0.574	0.658	1.375	0.475	0.508
	200	1.814	0.514	0.527	1.037	0.137	0.483
	400	1.648	0.348	0.490	0.999	0.099	0.427
	800	1.305	0.005	0.454	0.953	0.053	0.398
	1000	1.302	0.002	0.204	0.910	0.010	0.230
β	35	0.858	0.258	0.478	2.846	0.646	0.466
	50	0.811	0.211	0.346	2.632	0.432	0.368
	100	0.758	0.158	0.318	2.438	0.238	0.343
	200	0.716	0.116	0.264	2.419	0.219	0.294
	400	0.698	0.098	0.238	2.334	0.134	0.248
	800	0.639	0.039	0.210	2.299	0.099	0.208
	1000	0.605	0.005	0.108	2.210	0.010	0.180
λ	35	1.937	0.337	0.849	1.499	0.299	0.483
	50	1.910	0.310	0.739	1.403	0.203	0.374
	100	1.827	0.227	0.626	1.339	0.139	0.320
	200	1.782	0.182	0.603	1.292	0.092	0.290
	400	1.779	0.179	0.569	1.247	0.047	0.230
	800	1.649	0.049	0.389	1.230	0.030	0.180
	1000	1.608	0.008	0.240	1.209	0.009	0.123

6 Applications

We present two data sets to demonstrate the flexibility of the WOBIII-LoG distribution compared to other models with the same number of parameters. In order to show the flexibility of WOBIII-LoG distribution, we compared its performance to other existing models namely, the Kumaraswamy odd Lindley-log logistic (KOLLLoG) by Chipepa et al. (2019), the New Modified Weibull (NMW) distribution introduced by Doostmoradi et al. (2014), the beta generalized Lindley (BGL) distribution by Oluyede and Yang (2015), the Marshall Olkin-Kappa (MO-K) distribution by Javed et al. (2019), the Kumaraswamy Weibull (KW) distribution by Cordeiro et al. (2010) and Generalized Weibull Log-logistic (GWLLoG) distribution by Cordeiro et al. (2015). The goodness-of-fit statistics that were computed and used to compare the model performances include -2 log-likelihood ($-2\ln(L)$), Kolmogorov-Smirnov (K-S) statistic and its corresponding p-values, Akaike information criterion ($AIC = 2p - 2\ln(L)$), Bayesian information criterion ($BIC = p\ln(n) - 2\ln(L)$) and consistent Akaike information criterion $\left(AICC = AIC + 2\frac{p(p+1)}{n-p-1}\right)$, where $L = L(\hat{\mathbf{\Delta}})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters. The pdfs of the non-nested models used for comparison are given under the appendix C.

6.1 Growth hormone data

The data consists of the estimated time since growth hormone medication until the children reached the target age. The dataset was analyzed by Alizadeh et al. (2018) to demonstrate the applicability of the exponentiated power Lindley power series (EPLPS) class of distributions distributions. The datasets are: 2.15, 2.20, 2.55, 2.56, 2.63, 2.74, 2.81, 2.90, 3.05, 3.41, 3.43, 3.43, 3.84, 4.16, 4.18, 4.36, 4.42, 4.51,4.60, 4.61, 4.75, 5.03, 5.10, 5.44, 5.90, 5.96, 6.77, 7.82,8.00, 8.16, 8.21, 8.72, 10.40, 13.20, 13.70.
Estimates of the parameters of WOBIII-LoG distribution and other competing models with (standard error in parentheses), AIC, AICC, BIC, and the goodness-of-fit statistics W*, A*, K-S statistic and its p-values are presented under Table 4. Plots of the fitted pdfs and the histogram, observed probability plots vs predicted probability plots are given in Figure 5.

Table 4: The models estimates for Growth hormone data (M1 = WOBIII-LoG, M2 = KOLLLoG, M3 = NMW, M4 = BGL, M5 = MO-K, M6 = KW and M7 = GWLLoG)

Model	M1	M2	M3	M4	M5	M6	M7
$\hat{a}$	0.7	$9.1e^{3}$	0.001	$1.1e^{-2}$	5.7	$9.8e^{2}$	0.3
(s.e.)	(0.1)	$(5.8e^{-6})$	(0.001)	$(8.9e^{-3})$	$(3.7e^{2})$	$(2.2e^{-7})$	(0.5)
$\hat{b}$	38.1	$8.7e^{3}$	2.9	$1.8e^{-8}$	$6.2e^{-2}$	$3.4e^{6}$	2.7
(s.e.)	(26.4)	$(4.1e^{-7})$	(0.8)	$(2.1e^{-6})$	(4.1)	$(4.5e^{-13})$	(1.3)
$\hat{\beta}$	0.6	7.2	4.1	$3.0e^{-2}$	$3.7e^{3}$	$6.1e^{3}$	2.3
(s.e.)	(0.1)	$(5.2e^{-2})$	(0.8)	$(8.5e^{-4})$	$(6.3e^{-2})$	$(4.9e^{-10})$	(1.6)
$\hat{\lambda}$	1.2	$3.2e^{-2}$	0.002	$1.0e^{2}$	$4.4e^{-2}$	$6.4e^{-2}$	1.5
(s.e.)	(0.1)	$(3.8e^{-3})$	(0.002)	$(2.2e^{-5})$	$(1.5e^{-2})$	$(5.1e^{-4})$	(0.78)
$-2\log L$	156.1	162.1	168.1	347.0	158.6	163.7	159.5
AIC	164.1	170.1	176.1	355.0	166.6	171.7	167.5
$AICC$	165.5	171.4	177.4	356.3	168.0	173.0	168.8
BIC	170.4	176.3	182.3	361.2	172.8	177.9	173.7
W^*	0.03	0.12	0.05	0.06	0.05	0.14	0.08
A^*	0.28	0.79	0.38	0.46	0.40	0.92	0.56
$K-S$	0.09	0.13	0.16	0.50	0.09	0.14	0.11
p-value	0.91	0.56	0.31	$2.8e^{-8}$	0.89	0.48	0.73

From Figure 5 above, it is clear that the proposed model provides better fits compared to other non-nested models used for comparison. We note that the fitted density of the WOBIII-LoG distribution remain closer to the sample histogram and similarly the fitted probability plot is also close to the diagonal line. This shows that the model is consistent in providing a better fit for both skewed and symmetric data than the other competing models used for comparison. From the results presented under table 4, it is also shown that the WOBIII-LoG distribution provides a better fit than the competing models since it has the smallest values for the goodness-of-fit statistics: W^*, A^*, $K-S$ and a higher p-value.

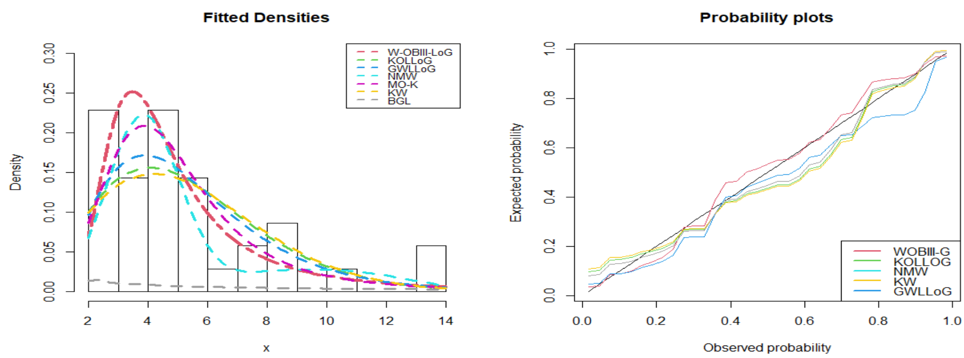

Figure 5: Fitted densities and probability plots for the growth hormone data

6.2 Tuborcharger data

The second data set (n= 40) below is from Xu et al. (2003) and it represents the time to failure (103h) of turbocharger of one type of engine. The data are: 1.6, 3.5, 4.8, 5.4, 6.0, 6.5, 7.0, 7.3, 7.7, 8.0, 8.4, 2.0, 3.9, 5.0, 5.6, 6.1, 6.5, 7.1, 7.3, 7.8, 8.1, 8.4, 2.6, 4.5, 5.1, 5.8, 6.3, 6.7, 7.3, 7.7, 7.9, 8.3, 8.5, 3.0, 4.6, 5.3, 6.0, 8.7, 8.8, 9.0. Similarly, Table 5 presents the results on the performance of WOBIII-LoG model in comparison to other existing non-nested models having equal number of parameters. The associated observed probability plots and fitted pdf curves are given under Figure 6.

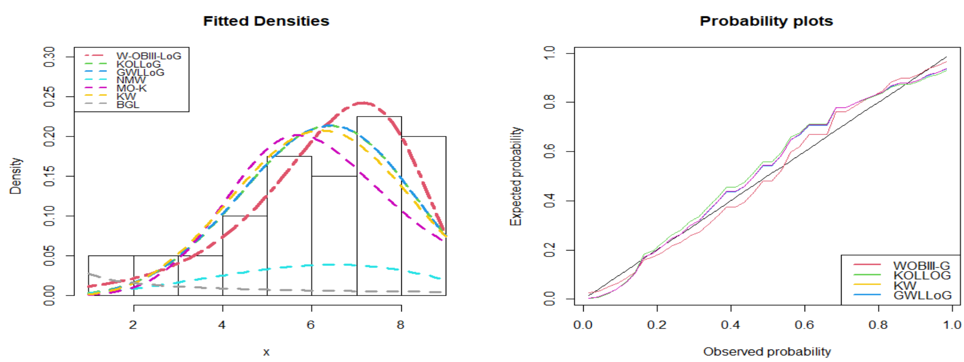

Figure 6: Fitted densities and probability plots for the tuborcharger data

Figure 6 shows that the model fits the data well compared to other competing non-nested models. The goodness-of-fit statistics W^*, A^* and K-S presented under table 5

Table 5: The models estimates for tuborcharger data (M1 = WOBIII-LoG, M2 = KOLLLoG, M3 = NMW, M4 = BGL, M5 = MO-K, M6 = KW and M7 = GWLLoG)

Model	M1	M2	M3	M4	M5	M6	M7
$\hat{a}$	1.2	$7.3e^{-2}$	$2.5e^{-3}$	$1.2e^{-2}$	1.34	$2.1e^{2}$	27.7
(s.e.)	$(2.2e^{-5})$	$(9.4e^{-2})$	$(1.0e^{-3})$	$(8.5e^{-3})$	(1.98)	$(4.8e^{-8})$	(0.5)
$\hat{b}$	$7.2e^{-2}$	$1.3e^{2}$	2.7	$9.6e^{-8}$	4.8	$5.5e^{6}$	0.4
(s.e.)	$(8.8e^{-3})$	$(1.1e^{-6})$	$(2.1e^{-2})$	$(2.1e^{-6})$	(0.7)	$(1.2e^{-13})$	(1.3)
$\hat{\beta}$	$6.7e^{2}$	$2.9e^{-4}$	$2.5e^{2}$	$3.0e^{-2}$	0.00	$1.3e^{4}$	16.3
(s.e.)	$(8.5e^{-6})$	$(1.1e^{-4})$	$(5.1e^{-19})$	$(7.6e^{-4})$	(0.00)	$(1.8e^{-10})$	(1.6)
$\hat{\lambda}$	$1.4e^{-2}$	2.7	$1.8e^{2}$	$1.0e^{2}$	23.2	$8.6e^{-2}$	8.8
(s.e.)	$(1.8e^{2})$	$(2.6e^{-2})$	$(7.3e^{-19})$	$(2.5e^{-5})$	(0.07)	$(3.3e^{-4})$	(0.78)
$-2\log L$	160.7	165.0	285.3	407.3	177.3	166.8	165.0
AIC	168.7	173.0	293.3	415.3	185.3	174.8	173.0
$AICC$	169.8	174.1	294.5	416.4	186.5	175.9	174.1
BIC	175.5	179.7	300.1	422.0	192.1	181.5	179.7
W^*	0.03	0.07	0.09	0.26	0.21	0.09	0.07
A^*	0.24	0.56	0.59	1.68	1.40	0.71	0.57
$K-S$	0.08	0.10	0.63	0.49	0.14	0.11	0.10
p-value	0.91	0.75	$1.8e^{-14}$	$4.6e^{-9}$	0.37	0.68	0.74

clearly shows the superiority of the WOBIII-LoG distribution over other comparison models. Additionally, the values of AIC and BIC also show that the WOBIII-LoG distribution is better than these competing equi-parameter models.

7 Conclusions

In this article, a new family of distributions called the Weibull Odd Burr III-G (WOBIII-G) has been developed using the T-X transformation technique and studied in details. Its various structural properties have been derived. We applied some of its special case to real life dataset to demonstrate its flexibility and goodness-of-fit. The proposed model has been found to provide best fit in modelling the real life datasets compared to other non-nested models with equal number of parameters.

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Appendix

A Some useful expansions

A.1 Series expansion of the pdf

Note that by using the series expansion

$$\exp\left(-\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta}\right)\sum_{k=0}^{\infty}(-1)^k\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta k},$$

we can write the WOBIII-G pdf as

$$\begin{aligned} f(x;a,b,\beta,\underline{\varphi}) =& ab\beta\sum_{k=0}^{\infty}(-1)^k\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta(k+1)-1} \\ &\times\frac{\left(1+t^{-a}\right)^{-b-1}}{\left(1-\left(1+t^{-a}\right)^{-b}\right)}t^{-a-1}\frac{g(x;\underline{\varphi})}{\left(\overline{G}(x;\underline{\varphi})\right)^2}. \end{aligned}$$

Furthermore, applying the expansion

$$\left[-\log(1-y)\right]^{\delta-1}=y^{\delta-1}\left[\sum_{m=0}^{\infty}\binom{\delta-1}{m}y^m\left(\sum_{s=0}^{\infty}\frac{y^s}{s+2}\right)^m\right],$$

for $0<y<1$, and the result on power series raised to a positive integer, with $a_s=(s+2)^{-1}$, that is,

$$\left(\sum_{s=0}^{\infty} a_s y^s\right)^m = \sum_{s=0}^{\infty} b_{s,m} y^s,$$

where $b_{s,m} = (sa_0)^{-1} \sum_{l=1}^{s} [m(l+1) - s] a_l b_{s-l,m}$, and $b_{0,m} = a_0^m$, see Gradshteyn and Ryzhik (2000), we have

$$\left[-\log\left(1 - \left(1 + t^{-a}\right)^{-b}\right)\right]^{\beta(k+1)-1} = \sum_{m=0}^{\infty}\sum_{s=0}^{\infty} b_{s,m} \binom{\beta(k+1)-1}{m} \times \left(1 + t^{-a}\right)^{-b(\beta(k+1)-1+m+s)}.$$

Now, the WOBIII-G pdf can be written as

$$f(x; a, b, \beta, \underline{\varphi}) = ab\beta \sum_{k=0}^{\infty} (-1)^k \sum_{m=0}^{\infty}\sum_{s=0}^{\infty} b_{s,m} \binom{\beta(k+1)-1}{m} \times \left(1 + t^{-a}\right)^{k^*} \frac{t^{-a-1}}{\left(1 - (1 + t^{-a})^{-b}\right)} \frac{g(x; \underline{\varphi})}{\left(\overline{G}(x; \underline{\varphi})\right)^2}.$$

Also, by applying the generalized binomial series expansion

$$\left(1 - \left(1 + t^{-a}\right)^{-b}\right)^{-1} = \sum_{z=0}^{\infty} \frac{\Gamma(z+1)}{\Gamma(1) z!} \left(1 + t^{-a}\right)^{-bz},$$

we get

$$f(x; a, b, \beta, \underline{\varphi}) = ab\beta \sum_{k=0}^{\infty} (-1)^k \sum_{z=0}^{\infty} \frac{\Gamma(z+1)}{\Gamma(1) z!} \sum_{m=0}^{\infty}\sum_{s=0}^{\infty} b_{s,m} \binom{\beta(k+1)-1}{m} \times \left(1 + t^{-a}\right)^{k^*} t^{-a-1} \frac{g(x; \underline{\varphi})}{\left(\overline{G}(x; \underline{\varphi})\right)^2}.$$

Now, considering $(1 + t^{-a})^{k^{**}} = \sum_{i=0}^{\infty} \binom{k^{**}}{i} t^{-ai}$, the pdf then reduces to

$$f(x; a, b, \beta, \underline{\varphi}) = ab\beta \sum_{k,z,i,m,s=0}^{\infty} \frac{\Gamma(z+1)}{\Gamma(1) z!} b_{s,m} (-1)^k \binom{\beta(k+1)-1}{m} \times \binom{k^{**}}{i} \left(G(x; \underline{\varphi})\right)^{-a(i+1)-1} \left(\overline{G}(x; \underline{\varphi})\right)^{a(i+1)+1} \frac{g(x; \underline{\varphi})}{\left(\overline{G}(x; \underline{\varphi})\right)^2}.$$

Using the generalized binomial series expansion

$$\left(G(x; \underline{\varphi})\right)^{-a(i+1)-1} = \left(1 - \overline{G}(x; \underline{\varphi})\right)^{-a(i+1)-1} = \sum_{p=0}^{\infty} \binom{-a(i+1)-1}{p} (-1)^p \overline{G}^p(x; \underline{\varphi}),$$

we have

$$f(x; a, b, \beta, \underline{\varphi}) = ab\beta \sum_{k,z,i,p,m,s=0}^{\infty} \frac{\Gamma(z+1)}{\Gamma(1) z!} b_{s,m} (-1)^{k+p} \binom{\beta(k+1)-1}{m} g(x; \underline{\varphi})$$

$$
\begin{aligned}
&\times \binom{-a(i+1)-1}{p}\binom{k^{**}}{i}\left(\overline{G}(x;\underline{\boldsymbol{\varphi}})\right)^{a(i+1)+p-1} \\
=&ab\beta \sum_{k,z,i,p,w,m,s=0}^{\infty} \frac{\Gamma(z+1)}{\Gamma(1)z!} b_{s,m}(-1)^{k+p+w}\binom{\beta(k+1)-1}{m} \\
&\times \binom{a(i+1)+p-1}{w}\binom{k^{**}}{i}\binom{-a(i+1)-1}{p} \\
&\times \frac{w+1}{w+1}\left(G(x;\underline{\boldsymbol{\varphi}})\right)^{w} g(x;\underline{\boldsymbol{\varphi}}) \\
=&\sum_{w=0}^{\infty} c_w g_w(x;\underline{\boldsymbol{\varphi}}),
\end{aligned}
\tag{14}
$$

where $g_w(x;\underline{\boldsymbol{\varphi}}) = (w+1)\left(G(x;\underline{\boldsymbol{\varphi}})\right)^{w} g(x;\underline{\boldsymbol{\varphi}})$ is the exponentiated-G (Exp-G) distribution with power parameter $w+1$ and

$$
\begin{aligned}
c_w =&ab\beta \sum_{k=0}^{\infty}\sum_{z=0}^{\infty}\sum_{i,p=0}^{\infty} \frac{\Gamma(z+1)}{(w+1)\Gamma(1)z!}\sum_{m=0}^{\infty}\sum_{s=0}^{\infty} b_{s,m}(-1)^{k+p+w}\binom{\beta(k+1)-1}{m} \\
&\times \binom{-a(i+1)-1}{p}\binom{a(i+1)+p-1}{w}\binom{-b(\beta(k+1)+m+s+z)-1}{i}.
\end{aligned}
\tag{15}
$$

A.2 Series expansion of the PWMs

Using the binomial expansion

$$
\left(1-\exp\left(-\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta}\right)\right)^{m} = \sum_{q=0}^{\infty}\binom{m}{q}(-1)^{q}H,
$$

where $H = \exp\left(-q\left[-\log\left(1-(1+t^{-a})^{-b}\right)\right]^{\beta}\right)$, we then write

$$
\begin{aligned}
f(x)(F(x))^{m} =&ab\beta\sum_{q=0}^{\infty}\binom{m}{q}(-1)^{q}\exp\left(-(q+1)\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta}\right) \\
&\times \left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta-1} \\
&\times \frac{\left(1+t^{-a}\right)^{-b-1}}{\left(1-(1+t^{-a})^{-b}\right)} t^{-a-1}\frac{g(x;\underline{\boldsymbol{\varphi}})}{\left(\overline{G}(x;\underline{\boldsymbol{\varphi}})\right)^{2}}.
\end{aligned}
\tag{16}
$$

Using the series expansion

$$
\exp\left(-(q+1)\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta}\right) = \sum_{k=0}^{\infty}\frac{(-1)^{k}(q+1)^{k}T}{k!},
$$

where $T = \left[-\log\left(1-(1+t^{-a})^{-b}\right)\right]^{\beta k}$, we can write

$$f(x)(F(x))^m = ab\beta \sum_{q=0}^{\infty}\sum_{k=0}^{\infty}\binom{m}{q}\frac{(-1)^k(q+1)^k}{k!}\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta(k+1)-1}$$
$$\times \frac{(1+t^{-a})^{-b-1}}{\left(1-(1+t^{-a})^{-b}\right)}t^{-a-1}\frac{g(x;\underline{\boldsymbol{\varphi}})}{\left(\overline{G}(x;\underline{\boldsymbol{\varphi}})\right)^2}.$$

Furthermore, applying the series expansion

$$\left[-\log(1-y)\right]^{\delta-1} = y^{\delta-1}\left[\sum_{m=0}^{\infty}\binom{\delta-1}{m}y^m\left(\sum_{s=0}^{\infty}\frac{y^s}{s+2}\right)^m\right],$$

and the result on power series raised to a positive integer, with $a_s = (s+2)^{-1}$, that is,

$$\left(\sum_{s=0}^{\infty}a_s y^s\right)^m = \sum_{s=0}^{\infty}b_{s,m}y^s,$$

where $b_{s,m} = (sa_0)^{-1}\sum_{l=1}^{s}[m(l+1)-s]a_l b_{s-l,m}$, and $b_{0,m} = a_0^m$, see Gradshteyn and Ryzhik (2000),

$$\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta(k+1)-1} = \sum_{m=0}^{\infty}\sum_{s=0}^{\infty}b_{s,m}\binom{\beta(k+1)-1}{m}\left(1+t^{-a}\right)^{q^*},$$

where $q^* = -b(\beta(k+1)-1+m+s)$, we get

$$f(x)(F(x))^m = ab\beta\sum_{q=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m,s=0}^{\infty}\binom{m}{q}\frac{(-1)^{q+k}}{k!}(q+1)^k b_{s,m}\binom{\beta(k+1)-1}{m}$$
$$\times\left(1+t^{-a}\right)^{q^*}t^{-a-1}\left(1-\left(1+t^{-a}\right)^{-b}\right)\frac{g(x;\underline{\boldsymbol{\varphi}})}{\left(\overline{G}(x;\underline{\boldsymbol{\varphi}})\right)^2}.$$

Furthermore, by applying the series expansion

$$\left(1-\left(1+t^{-a}\right)^{-b}\right)^{-1} = \sum_{z=0}^{\infty}\frac{\Gamma(z+1)}{\Gamma(1)z!}\left(1+t^{-a}\right)^{-bz},$$

we get

$$f(x)(F(x))^m = ab\beta\sum_{q=0}^{\infty}\sum_{k=0}^{\infty}\sum_{z,m,s=0}^{\infty}\binom{m}{q}\frac{(-1)^{q+k}(q+1)^k}{k!}\frac{\Gamma(z+1)}{\Gamma(1)z!}b_{s,m}\binom{\beta(k+1)-1}{m}$$
$$\times\left(1+t^{-a}\right)^{q^{**}}t^{-a-1}\frac{g(x;\underline{\boldsymbol{\varphi}})}{\left(\overline{G}(x;\underline{\boldsymbol{\varphi}})\right)^2},$$

where $q^{**} = -b(\beta(k+1)+m+s+z)-1$. Now, considering

$$\left(1+t^{-a}\right)^{q^{**}} = \sum_{i=0}^{\infty} \binom{q^{**}}{i} \left(\frac{G(x;\underline{\varphi})}{\overline{G}(x;\underline{\varphi})}\right)^{-ai},$$

we have

$$\begin{aligned} f(x)(F(x))^m =& ab\beta \sum_{q,k=0}^{\infty} \sum_{z,i=0}^{\infty} \sum_{m,s=0}^{\infty} \binom{m}{q} \frac{(-1)^{q+k}}{k!} (q+1)^k \frac{\Gamma(z+1)}{\Gamma(1) z!} b_{s,m} \binom{\beta(k+1)-1}{m} \\ &\times \binom{q^{**}}{i} \left(G(x;\underline{\varphi})\right)^{-a(i+1)-1} \left(\overline{G}(x;\underline{\varphi})\right)^{a(i+1)+1} \frac{g(x;\underline{\varphi})}{\left(\overline{G}(x;\underline{\varphi})\right)^2}. \end{aligned}$$

Using the expansion

$$\left(G(x;\underline{\varphi})\right)^{-a(i+1)-1} = \left(1-\overline{G}(x;\underline{\varphi})\right)^{-a(i+1)-1} = \sum_{p=0}^{\infty} \binom{-a(i+1)-1}{p} (-1)^p \overline{G}^p(x;\underline{\varphi}),$$

we have

$$\begin{aligned} f(x)(F(x))^m =& ab\beta \sum_{q,k,z,i,p,m,s=0}^{\infty} \binom{m}{q} \frac{(-1)^{q+k+p}}{k!} (q+1)^k \frac{\Gamma(z+1)}{\Gamma(1) z!} b_{s,m} \\ &\times \binom{\beta(k+1)-1}{m} \binom{q^{**}}{i} \binom{-a(i+1)-1}{p} \left(\overline{G}(x;\underline{\varphi})\right)^{a(i+1)+p-1} g(x;\underline{\varphi}) \\ =& ab\beta \sum_{q,k,z,i,p,w,m,s=0}^{\infty} \binom{m}{q} \frac{(-1)^{q+k+p+w}}{k!} (q+1)^k \frac{\Gamma(z+1)}{\Gamma(1) z!} b_{s,m} \\ &\times \binom{\beta(k+1)-1}{m} \binom{a(i+1)+p-1}{w} \binom{q^{**}}{i} \\ &\times \binom{-a(i+1)-1}{p} \left(\frac{w+1}{w+1}\right) \left(G(x;\underline{\varphi})\right)^w g(x;\underline{\varphi}) \\ =& \sum_{w=0}^{\infty} \phi_w g_w(x;\underline{\varphi}), \end{aligned} \tag{17}$$

where $g_w(x;\underline{\varphi}) = (w+1)\left(G(x;\underline{\varphi})\right)^w g(x;\underline{\varphi})$ is the exponentiated-G (Exp-G) distribution with power parameter $w+1$ and

$$\begin{aligned} \phi_w =& ab\beta \sum_{q,k,z,i,p,m,s=0}^{\infty} \binom{m}{q} \frac{(-1)^{q+k+p+w}}{k!} (q+1)^k \frac{\Gamma(z+1)}{(w+1)\Gamma(1) z!} b_{s,m} \\ &\times \binom{-a(i+1)-1}{p} \binom{\beta(k+1)-1}{m} \binom{a(i+1)+p-1}{w} \binom{q^{**}}{i}. \end{aligned} \tag{18}$$

A.3 Distribution of order statistics expansions

Using equations (1) and (2) we write

$$f(x)\left[F(x)\right]^{k+l-1} = ab\beta \exp\left(-\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta}\right)$$

$$\times\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta-1}$$
$$\times\frac{\left(1+t^{-a}\right)^{-b-1}}{\left(1-\left(1+t^{-a}\right)^{-b}\right)}t^{-a-1}\frac{g(x;\underline{\varphi})}{\left(\overline{G}(x;\underline{\varphi})\right)^{2}}$$
$$\times\left(1-exp\left(-\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta}\right)\right)^{k+l-1}.$$

Note that by using the binomial expansion

$$\left(1-\exp\left(-\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta}\right)\right)^{m}=\sum_{q=0}^{\infty}\binom{k+l-1}{q}(-1)^{q}H,$$

where $H=\exp\left(-q\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta}\right)$, we can write

$$\begin{aligned}f(x)(F(x))^{k+l-1}=&ab\beta\sum_{q=0}^{\infty}\binom{k+l-1}{q}\times\exp\left(-(q+1)\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta}\right)\\&\times(-1)^{q}\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta-1}\frac{\left(1+t^{-a}\right)^{-b-1}}{\left(1-\left(1+t^{-a}\right)^{-b}\right)}t^{-a-1}\\&\times\frac{g(x;\underline{\varphi})}{\left(\overline{G}(x;\underline{\varphi})\right)^{2}}.\end{aligned}\tag{19}$$

Now using the expansion

$$\exp\left(-(q+1)\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta}\right)=\sum_{k=0}^{\infty}\frac{(-1)^{k}(q+1)^{k}T}{k!},$$

where $T=\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta k}$, we get

$$\begin{aligned}f(x)(F(x))^{k+l-1}=&ab\beta\sum_{q=0}^{\infty}\sum_{k=0}^{\infty}\binom{k+l-1}{q}\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta(k+1)-1}\\&\times(-1)^{q+k}(q+1)^{k}\frac{\left(1+t^{-a}\right)^{-b-1}}{\left(1-\left(1+t^{-a}\right)^{-b}\right)}t^{-a-1}\frac{g(x;\underline{\varphi})}{\left(\overline{G}(x;\underline{\varphi})\right)^{2}}.\end{aligned}$$

Furthermore, applying the expansion

$$\left[-\log(1-y)\right]^{\delta-1}=y^{\delta-1}\left[\sum_{m=0}^{\infty}\binom{\delta-1}{m}y^{m}\left(\sum_{s=0}^{\infty}\frac{y^{s}}{s+2}\right)^{m}\right],$$

and the result on power series raised to a positive integer, with $a_s=(s+2)^{-1}$, that is,

$$\left(\sum_{s=0}^{\infty}a_{s}y^{s}\right)^{m}=\sum_{s=0}^{\infty}b_{s,m}y^{s},$$

where $b_{s,m} = (sa_0)^{-1}\sum_{l=1}^{s}[m(l+1)-s]a_l b_{s-l,m}$, and $b_{0,m} = a_0^m$, Gradshteyn and Ryzhik (2000),

$$\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta(k+1)-1} = \sum_{m=0}^{\infty}\sum_{s=0}^{\infty} b_{s,m}\binom{\beta(k+1)-1}{m}\left(1+t^{-a}\right)^{q^{**}}$$

we get

$$\begin{aligned} f(x)(F(x))^{k+l-1} =& ab\beta\sum_{q=0}^{\infty}\sum_{k=0}^{\infty}\binom{k+l-1}{q}(-1)^{q+k}(q+1)^k\sum_{m=0}^{\infty}\sum_{s=0}^{\infty} b_{s,m}\binom{\beta(k+1)-1}{m} \\ &\times\left(1+t^{-a}\right)^{q^{**}} t^{-a-1}\left(1-\left(1+t^{-a}\right)^{-b}\right)\frac{g(x;\underline{\boldsymbol{\varphi}})}{\left(\overline{G}(x;\underline{\boldsymbol{\varphi}})\right)^2}. \end{aligned}$$

Also, by applying the series expansion

$$\left(1-\left(1+t^{-a}\right)^{-b}\right)^{-1} = \sum_{z=0}^{\infty}\frac{\Gamma(z+1)}{\Gamma(1)z!}\left(1+t^{-a}\right)^{-bz},$$

we get

$$\begin{aligned} f(x)(F(x))^{k+l-1} = ab\beta&\sum_{q,k=0}^{\infty}\sum_{z,m=0}^{\infty}\sum_{s=0}^{\infty}\binom{k+l-1}{q}(-1)^{q+k}(q+1)^k\frac{\Gamma(z+1)}{\Gamma(1)z!}b_{s,m} \\ &\times\binom{\beta(k+1)-1}{m}\left(1+t^{-a}\right)^{q^{**}} t^{-a-1}\frac{g(x;\underline{\boldsymbol{\varphi}})}{\left(\overline{G}(x;\underline{\boldsymbol{\varphi}})\right)^2}. \end{aligned}$$

Now, considering

$$\left(1+t^{-a}\right)^{q^{**}} = \sum_{i=0}^{\infty}\binom{q^{**}}{i}t^{-ai} \tag{20}$$

we have

$$\begin{aligned} f(x)(F(x))^{k+l-1} =& ab\beta\sum_{q,k=0}^{\infty}\sum_{z,i=0}^{\infty}\sum_{m,s=0}^{\infty}\binom{k+l-1}{q}(-1)^{q+k}(q+1)^k\frac{\Gamma(z+1)}{\Gamma(1)z!}b_{s,m} \\ &\times\binom{\beta(k+1)-1}{m}\binom{q^{**}}{i}\left(G(x;\underline{\boldsymbol{\varphi}})\right)^{-a(i+1)-1} \\ &\times\left(\overline{G}(x;\underline{\boldsymbol{\varphi}})\right)^{a(i+1)+1}\frac{g(x;\underline{\boldsymbol{\varphi}})}{\left(\overline{G}(x;\underline{\boldsymbol{\varphi}})\right)^2}. \\ =& ab\beta\sum_{q,k=0}^{\infty}\sum_{z,i=0}^{\infty}\sum_{p,m,s=0}^{\infty}\binom{k+l-1}{q}(-1)^{q+k+p}(q+1)^k\frac{\Gamma(z+1)}{\Gamma(1)z!}b_{s,m} \\ &\times\binom{\beta(k+1)-1}{m}\binom{k^{***}-1}{i}\binom{-a(i+1)-1}{p} \\ &\times\left(\overline{G}(x;\underline{\boldsymbol{\varphi}})\right)^{a(i+1)+p-1} g(x;\underline{\boldsymbol{\varphi}}) \end{aligned}$$

$$
\begin{aligned}
=&ab\beta\sum_{q,k=0}^{\infty}\sum_{z,i=0}^{\infty}\sum_{p,w=0}^{\infty}\sum_{m,s=0}^{\infty}\binom{k+l-1}{q}(-1)^{q+k+p+w}(q+1)^{k}\\
&\times\frac{\Gamma(z+1)}{\Gamma(1)z!}b_{s,m}\binom{\beta(k+1)-1}{m}\binom{a(i+1)+p-1}{w}\binom{k^{***}-1}{i}\\
&\times\binom{-a(i+1)-1}{p}\left(\frac{w+1}{w+1}\right)\left(G(x;\underline{\boldsymbol{\varphi}})\right)^{w}g(x;\underline{\boldsymbol{\varphi}})\\
=&\sum_{w=0}^{\infty}b_{w}g_{w}(x;\underline{\boldsymbol{\varphi}}),
\end{aligned}
\tag{21}
$$

where b_w and $g_w(x;\underline{\boldsymbol{\varphi}})$ are as defined in equation (10), and $k^{***}=-b(\beta(k+1)+m+s+z)$.

A.4 Entropy

Here,

$$
\begin{aligned}
I_R(v)=&\frac{1}{1-v}\log\left(\int_0^{\infty}\left[(ab\beta)^{v}\exp\left(-v\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta}\right)\right.\right.\\
&\times\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{v(\beta-1)}\frac{\left(1+t^{-a}\right)^{-v(b+1)}}{\left(1-\left(1+t^{-a}\right)^{-b}\right)^{v}}t^{-v(a+1)}\left.\left.\frac{g^{v}(x;\underline{\boldsymbol{\varphi}})}{\left(\overline{G}(x;\underline{\boldsymbol{\varphi}})\right)^{2v}}\right]dx\right).
\end{aligned}
$$

Using the expansion

$$
\exp\left(-v\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta}\right)=\sum_{k=0}^{\infty}\frac{(-1)^{k}v^{k}}{k!}\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta k},
$$

we can write

$$
\begin{aligned}
I_R(v)=&\frac{1}{1-v}\log\left(\int_0^{\infty}\left[(ab\beta)^{v}\sum_{k=0}^{\infty}\frac{(-1)^{k}v^{k}}{k!}\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta(k+v)-v}\right.\right.\\
&\times\frac{\left(1+t^{-a}\right)^{-v(b+1)}}{\left(1-\left(1+t^{-a}\right)^{-b}\right)^{v}}t^{-v(a+1)}\left.\left.\frac{g^{v}(x;\underline{\boldsymbol{\varphi}})}{\left(\overline{G}(x;\underline{\boldsymbol{\varphi}})\right)^{2v}}\right]dx\right).
\end{aligned}
$$

We apply the expansion

$$
\Big[-\log(1-y)\Big]^{\delta-1}=y^{\delta-1}\left[\sum_{m=0}^{\infty}\binom{\delta-1}{m}y^{m}\left(\sum_{s=0}^{\infty}\frac{y^{s}}{s+2}\right)^{m}\right],
$$

and the result on power series raised to a positive integer, with $a_s=(s+2)^{-1}$, that is,

$$
\left(\sum_{s=0}^{\infty}a_{s}y^{s}\right)^{m}=\sum_{s=0}^{\infty}b_{s,m}y^{s},
$$

where $b_{s,m} = (sa_0)^{-1}\sum_{l=1}^{s}[m(l+1)-s]a_l b_{s-l,m}$, and $b_{0,m} = a_0^m$, (Gradshteyn and Ryzhik, 2000), so that

$$\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta(k+v)-v} = \sum_{m=0}^{\infty}\sum_{s=0}^{\infty} b_{s,m}\binom{\beta(k+v)-v}{m}\left(1+t^{-a}\right)^{q^{**}},$$

to obtain

$$\begin{aligned} I_R(v) =& \frac{1}{1-v}\log\left(\int_0^{\infty}\left[(ab\beta)^v\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s=0}^{\infty} b_{s,m}\binom{\beta(k+v)-v}{m}(-1)^k v^k\right.\right.\\ &\left.\left.\times\left(1+t^{-a}\right)^{-b(\beta(k+v)+m+s)-v}\frac{t^{-v(a+1)}}{\left(1-\left(1+t^{-a}\right)^{-b}\right)^v}\frac{g^v(x;\underline{\varphi})}{\left(\overline{G}(x;\underline{\varphi})\right)^{2v}}\right]dx\right). \end{aligned}$$

Also, by applying the series expansion

$$\left(1-\left(1+t^{-a}\right)^{-b}\right)^{-v} = \sum_{z=0}^{\infty}\frac{\Gamma(z+v)}{\Gamma(v)z!}\left(1+t^{-a}\right)^{-bz},$$

we get

$$\begin{aligned} I_R(v) =& \frac{1}{1-v}\log\left(\int_0^{\infty}\left[(ab\beta)^v\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s,z=0}^{\infty} b_{s,m}\binom{\beta(k+v)-v}{m}(-1)^k v^k\right.\right.\\ &\left.\left.\times\frac{\Gamma(z+v)}{\Gamma(v)z!}\left(1+t^{-a}\right)^{q^{***}} t^{-v(a+1)}\frac{g^v(x;\underline{\varphi})}{\left(\overline{G}(x;\underline{\varphi})\right)^{2v}}\right]dx\right). \end{aligned}$$

Now, considering $(1+t^{-a})^{q^{***}} = \sum_{i=0}^{\infty}\binom{q^{***}}{i}t^{-ai}$, then the Rényi entropy can be written as

$$\begin{aligned} I_R(v) =& \frac{1}{1-v}\log\left(\int_0^{\infty}\left[(ab\beta)^v\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s,z,i=0}^{\infty} b_{s,m}\binom{\beta(k+v)-v}{m}(-1)^k v^k\right.\right.\\ &\left.\left.\times\binom{q^{***}}{i}\frac{\Gamma(z+v)}{\Gamma(v)z!}\left(G(x;\underline{\varphi})\right)^{-a(v+i)-v}\left(\overline{G}(x;\underline{\varphi})\right)^{a(v+i)+v}\frac{g^v(x;\underline{\varphi})}{\left(\overline{G}(x;\underline{\varphi})\right)^{2v}}\right]dx\right). \end{aligned}$$

Now, using the expansion

$$\left(G(x;\underline{\varphi})\right)^{-a(v+i)-v} = \left(1-\overline{G}(x;\underline{\varphi})\right)^{-a(v+i)-v} = \sum_{p=0}^{\infty}(-1)^p\binom{-a(v+i)-v}{p}\overline{G}^p(x;\underline{\varphi}),$$

we have

$$I_R(v) = \frac{1}{1-v}\log\left(\int_0^{\infty}\left[(ab\beta)^v\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s,z,i,p,w=0}^{\infty} b_{s,m}\binom{\beta(k+v)-v}{m}(-1)^{k+p+w}v^k\right.\right.$$

$$
\begin{aligned}
&\times \binom{-a(v+i)-v}{p}\binom{q^{***}}{i}\binom{a(v+i)-v+p}{w}\frac{\Gamma(z+v)}{\Gamma(v)z!} \\
&\times \left(G(x;\underline{\varphi})\right)^{w} g^{v}(x;\underline{\varphi})\Bigg]dx\Bigg) \\
=&\frac{1}{1-v}\log\Bigg((ab\beta)^{v}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s,z,i,p,w=0}^{\infty} b_{s,m}\binom{\beta(k+v)-v}{m}(-1)^{k+p+w}v^{k} \\
&\times \binom{-a(v+i)-v}{p}\binom{q^{***}}{i}\binom{a(v+i)-v+p}{w}\frac{\Gamma(z+v)}{\Gamma(v)z!} \\
&\times \int_{0}^{\infty} g^{v}(x;\underline{\varphi})G^{w}(x;\underline{\varphi})dx\Bigg).
\end{aligned}
$$

Consequently, the Rényi entropy is given by

$$
\begin{aligned}
I_R(v) =&\frac{1}{1-v}\log\Bigg((ab\beta)^{v}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s,z,i,p,w=0}^{\infty} b_{s,m}\binom{\beta(k+v)-v}{m}(-1)^{k+p+w}v^{k} \\
&\times \binom{-a(v+i)-v}{p}\binom{q^{***}}{i}\binom{a(v+i)-v+p}{w}\frac{\Gamma(z+v)}{\Gamma(v)z!} \\
&\times \frac{1}{\left[1+\frac{w}{v}\right]^{v}}\int_{0}^{\infty}\left(\left[\frac{w}{v}+1\right]\left(G(x;\underline{\varphi})\right)^{\frac{w}{v}} g(x;\underline{\varphi})\right)^{v} dx\Bigg) \\
=&\frac{1}{1-v}\log\left[\sum_{w=0}^{\infty}\tau_{w}\exp((1-v)I_{REG})\right],
\end{aligned}
$$

for $v > 0$, $v \neq 1$, where $I_{REG} = \frac{1}{1-v}\log\left[\int_0^{\infty}\left(\left[\frac{w}{v}+1\right]\left(G(x;\underline{\varphi})\right)^{\frac{w}{v}} g(x;\underline{\varphi})\right)^{v} dx\right]$ is the Rényi entropy of Exp-G distribution with power parameter $\frac{w}{v}+1$, and

$$
\begin{aligned}
\tau_{w} =&\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s,z,i,p=0}^{\infty}(ab\beta)^{v} b_{s,m}\binom{\beta(k+v)-v}{m}(-1)^{k+p+w}v^{k} \\
&\times \binom{-a(v+i)-v}{p}\binom{q^{***}}{i}\binom{a(v+i)-v+p}{w}\frac{\Gamma(z+v)}{\Gamma(v)z!}\frac{1}{\left[1+\frac{w}{v}\right]^{v}}.
\end{aligned}
$$

B Elements of the score vector

The elemets of a score vector, $U(\boldsymbol{\Delta})$ are given by

$$
\frac{\partial \ell}{\partial a} =\frac{n}{a}+\beta b\sum_{i=1}^{n}\frac{\left(1+t^{-a}\right)^{-b-1}t^{-a}\ln t}{\left(1-\left(1+t^{-a}\right)^{-b}\right)}\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta-1}
$$

$$
\begin{aligned}
&- (\beta - 1) \sum_{i=1}^{n} \frac{b\left(1+t^{-a}\right)^{-b-1} t^{-a} \ln t}{\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]\left(1-\left(1+t^{-a}\right)^{-b}\right)} \\
&+ (-b-1) \sum_{i=1}^{n} t^{-a} \ln t - \sum_{i=1}^{n} \ln t - \sum_{i=1}^{n} \frac{b\left(1+t^{-a}\right)^{-b-1} t^{-a} \ln t}{\left(1-\left(1+t^{-a}\right)^{-b}\right)},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \ell}{\partial b} =& \frac{n}{b} - \beta b \sum_{i=1}^{n} \frac{\left(1+t^{-a}\right)^{-b} \ln\left(1+t^{-a}\right)}{\left(1-\left(1+t^{-a}\right)^{-b}\right)} \left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta-1} \\
&+ (\beta - 1) \sum_{i=1}^{n} \frac{\left(1+t^{-a}\right)^{-b} \ln\left(1+t^{-a}\right)}{\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]\left(1-\left(1+t^{-a}\right)^{-b}\right)} \\
&- \sum_{i=1}^{n} \left(1+t^{-a}\right) + \sum_{i=1}^{n} \frac{\left(1+t^{-a}\right)^{-b} \ln\left(1+t^{-a}\right)}{\left(1-\left(1+t^{-a}\right)^{-b}\right)},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \ell}{\partial \beta} =& \frac{n}{\beta} - \sum_{i=1}^{n} \left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta} \ln\left(-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right) \\
&+ \sum_{i=1}^{n} \ln\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \ell}{\partial \underline{\boldsymbol{\varphi}}_k} =& - \beta b a \sum_{i=1}^{n} \frac{\left(1+t^{-a}\right)^{-b-1} t^{-a-1} \frac{\partial G(x_i;\underline{\boldsymbol{\varphi}})}{\partial \underline{\boldsymbol{\varphi}}_k}}{\left(1-\left(1+t^{-a}\right)^{-b}\right) \overline{G}^2(x_i;\underline{\boldsymbol{\varphi}})} \left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]^{\beta-1} \\
&+ (\beta - 1) \sum_{i=1}^{n} \frac{ba\left(1+t^{-a}\right)^{-b-1}}{\left[-\log\left(1-\left(1+t^{-a}\right)^{-b}\right)\right]\left(1-\left(1+t^{-a}\right)^{-b}\right)} \\
&\times \frac{t^{-a-1} \frac{\partial G(x_i;\underline{\boldsymbol{\varphi}})}{\partial \underline{\boldsymbol{\varphi}}_k}}{\overline{G}^2(x_i;\underline{\boldsymbol{\varphi}})} - a(-b-1) \sum_{i=1}^{n} t^{-a-1} \frac{\frac{\partial G(x_i;\underline{\boldsymbol{\varphi}})}{\partial \underline{\boldsymbol{\varphi}}_k}}{\overline{G}^2(x_i;\underline{\boldsymbol{\varphi}})} \\
&+ (-a-1) \sum_{i=1}^{n} \frac{\overline{G}(x_i;\underline{\boldsymbol{\varphi}})}{G(x_i;\underline{\boldsymbol{\varphi}})} \frac{\frac{\partial G(x_i;\underline{\boldsymbol{\varphi}})}{\partial \underline{\boldsymbol{\varphi}}_k}}{\overline{G}^2(x_i;\underline{\boldsymbol{\varphi}})} + \sum_{i=1}^{n} \frac{ab\left(1+t^{-a}\right)^{-b-1} t^{-a-1}}{\left(1-\left(1+t^{-a}\right)^{-b}\right)} \\
&\times \frac{\frac{\partial G(x_i;\underline{\boldsymbol{\varphi}})}{\partial \underline{\boldsymbol{\varphi}}_k}}{\overline{G}^2(x_i;\underline{\boldsymbol{\varphi}})} + \sum_{i=1}^{n} \frac{\frac{\partial g(x_i;\underline{\boldsymbol{\varphi}})}{\partial \underline{\boldsymbol{\varphi}}_k}}{(g(x_i;\underline{\boldsymbol{\varphi}}))} - 2 \sum_{i=1}^{n} \frac{\frac{\partial \overline{G}(x_i;\underline{\boldsymbol{\varphi}})}{\partial \underline{\boldsymbol{\varphi}}_k}}{(\overline{G}(x_i;\underline{\boldsymbol{\varphi}}))}.
\end{aligned}
$$

C Pdfs of the non-nested models

C.1 KW distribution

$$
\begin{aligned}
f(x;a,b,\beta,\lambda) =& ab\beta\lambda^{\beta} x^{\beta-1} \exp\{-(\lambda x)^{\beta}\}[1-\exp\{-(\lambda x)^{\beta}\}]^{a-1} \\
&\times \{1-[1-\exp\{-(\lambda x)^{\beta}\}]^{a}\}^{b-1},
\end{aligned}
$$

for $a, b, \beta, \lambda > 0$ and $x > 0$.

C.2 GWLLoG distribution

$$f(x;a,b,\beta,\lambda) = \frac{ab\beta x^{\beta-1}}{\lambda^{\beta}}\left(1+\left(\frac{x}{\lambda}\right)^{\beta}\right)^{-1}\left[\log\left(1+\left(\frac{x}{\lambda}\right)^{\beta}\right)\right]^{b-1} \times \exp\left\{-a\left[\log\left(1+\left(\frac{x}{\lambda}\right)^{\beta}\right)\right]^{b}\right\},$$

for $a, b, \beta, \lambda > 0$ and $x > 0$.

C.3 NMW distribution

$$f(x;a,b,\beta,\lambda) = \left(abx^{b-1}e^{ax^{b}} + \beta\lambda x^{\beta-1}e^{-\lambda x^{\beta}}\right)e^{-e^{ax^{b}}+e^{-\lambda x^{\beta}}}, \quad x > 0,$$

for $a, b, \beta, \lambda > 0$ and $x > 0$.

C.4 BGL distribution

$$f(x;a,b,\beta,\lambda) = \frac{\beta b^{2}}{B(a,b)(1+b)}(1+x)e^{-bx}\left[1-\frac{1+b+bx}{1+b}e^{-bx}\right]^{\beta\beta-1} \times \left[1-\left(1-\frac{1+b+bx}{1+b}e^{-bx}\right)^{\beta}\right]^{\lambda-1},$$

for $a, b, \beta, \lambda > 0$ and $x > 0$.

C.5 KOLLLoG distribution

$$f(x;a,b,\beta,\lambda) = ab\left[\frac{\beta^{2}}{(1+\beta)}\frac{\lambda x^{\lambda-1}}{(1+x^{\lambda})^{-1}}\exp\left\{-\beta\frac{1-(1+x^{\lambda})^{-1}}{(1+x^{\lambda})^{-1}}\right\}\right] \times \left[1-\frac{\beta+(1+x^{\lambda})^{-1}}{(1+\beta)(1+x^{\lambda})^{-1}}\exp\left\{-\beta\frac{(1-(1+x^{\lambda})^{-1})}{(1+x^{\lambda})^{-1}}\right\}\right]^{a-1} \times \left(1-\left[1-\frac{\beta+(1+x^{\lambda})^{-1}}{(1+\beta)(1+x^{\lambda})^{-1}}\exp\left\{-\beta\frac{(1-(1+x^{\lambda})^{-1})}{(1+x^{\lambda})^{-1}}\right\}\right]^{a}\right)^{b-1},$$

for $a, b, \beta, \lambda > 0$ and $x > 0$.

C.6 MO-K distribution

$$f(x;a,b,\beta,\lambda) = \frac{\frac{\lambda a\beta}{b}\left(\frac{x}{b}\right)^{\beta-1}\left(a+\left(\frac{x}{b}\right)^{a\beta}\right)^{\frac{-(a+1)}{a}}}{\left(\lambda+(1-\lambda)\left(\frac{\left(\frac{x}{b}\right)^{a\beta}}{a+\left(\frac{x}{b}\right)^{a\beta}}\right)^{\frac{1}{a}}\right)^{2}},$$

for $a, b, \beta, \lambda > 0$ and $x > 0$.