Journal of Statistical Modelling: Theory and Applications
Vol. 2, No. 2, 2021, pp. 41-49
Yazd University Press 2021

JSMTA

Research Paper

Parametric bootstrapping in a generalized extreme value regression model for binary response: Application in health study

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Received: March 19, 2022/ Revised: April 30, 2022 / Accepted: May 07, 2022

Abstract: Generalized extreme value regression is often more adapted when we investigate a relationship between a binary response variable that represents a rare event and potential predictors. In particular, we use the quantile function of the generalized extreme value distribution as the link function. Bootstrapping assigns measures of accuracy (bias, variance, confidence intervals, prediction error, hypotheses testing) to sample estimates. This technique allows estimation of the sampling distribution of almost any statistic using random sampling methods. Bootstrapping estimates the properties of an estimator by measuring those properties when sampling from an approximating distribution. In this paper, we fit the generalized extreme value regression model and perform a parametric bootstrap method for testing hypotheses and confidence interval estimation of parameters for the generalized extreme value regression model with a real data application.

Keywords: Confidence interval; Generalized extreme value; Hypothesis testing; Parametric bootstrap; Stroke.

1 Introduction

Classical methods of statistical inference may not provide correct answers to all the concrete problems the user has. They are only valid under some specific conditions (e.g. normal distribution of populations, independence of samples). The estimation of some

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characteristics such as dispersion measures (variance, standard deviation), confidence intervals, decision tables for hypothesis tests, is also based on the mathematical expressions of the probability laws, as well as approximations of these when the calculation was not feasible.

A good estimate of the nature of the population distribution can lead to powerful results. However, the price to pay is high if the assumption of the distribution is incorrect. It is therefore important to consider other analysis methods (such as non-parametric methods, for which the conditions of application are more restrictive), which are more flexible with the choice of the distribution and based on this, bootstrap methods was introduced.

Bootstrapping is a relatively new, computer-intensive statistical methodology introduced by Efron (1979). The bootstrap method replaces complex analytical procedures with computer intensive empirical analysis. It relies heavily on Monte Carlo Method where several random resamples are drawn from a given original sample. The bootstrap method is an effective technique in situations where it is necessary to determine the sampling distribution of a complex statistic with an unknown probability distribution using these data in a single sample. The bootstrap method has been applied effectively in a variety of situations. Efron and Tibshirani (1994), Shao (1996), and Shao (2010) provide a comprehensive discussion of the bootstrap method. Andronov and Kulynska (2020) discussed the statistical properties of the approximations of the "bootstrap-generated" data sets in more detail. Many studies have shown that the bootstrap resampling technique provides a more appropriate estimate of a parameter than the analysis of any one of the samples (Carpenter and Bithell, 2000; Zoubir and Iskander, 2004; Manly, 1997; Shao and Tu, 1995).

The bootstrap method is a powerful method to assess statistical accuracy or estimate distribution from a sample's statistics (Reynolds and Templin, 2004; Davison et al., 2003; Chernick, 1998). In principle there are three different ways of obtaining and evaluating bootstrap estimates: non-parametric bootstrap which does not assume any distribution of the population; semi-parametric bootstrap, which partly has an assumption on the distribution on parameter and whose residuals have no distributional assumption; and finally parametric bootstrap which assumes a particular distribution for the sample at hand. Adjeil and Karim (2016) have considered parametric and non-parametric bootstrap in the case of a classical logistic regression model.

In this work, we aim to estimate the generalized extreme value (GEV) regression model for binary data, construct confidence intervals, and test hypotheses for unknown parameters of the model using both classical and parametric bootstrap methods.

The rest of this paper is organized as follows. In Section 2, we describe the problem of the GEV regression model and parametric bootstrap method. In Section 3, we describe a small simulation study to evaluate the usefulness of the proposed method. Section 4 presents the obtained results. A discussion and some perspectives are given in Section 5.

2 Method

2.1 Generalized extreme value regression model

GEV is used to model rare and extreme events (Coles, 2001). In the case where the dependent variable Y represents a rare event, the logistic regression model (obviously used for this category of data) shows relevant drawbacks. We suggest the quantile function of the GEV distribution as the link function to investigate the relationship between the binary response variable Y and the potential predictors $\boldsymbol{X}$ (Wang and Dey, 2010; Calabrese and Osmetti, 2013). We use a bootstrap method as a tool to estimate parameters and standard errors for a GEV regression model. For a binary response variable Y_i and the vector of explanatory variables $\boldsymbol{X}_i$, let $\pi(\boldsymbol{x}_i) = \mathbb{P}(Y_i = 1|\boldsymbol{X}_i = \boldsymbol{x}_i)$ the conditional probability of belonging to class ”1”. Since we consider the class of generalized linear models, we suggest the cumulative distribution function (cdf) of GEV proposed by Calabrese and Osmetti (2013) as the response curve given by

$$\begin{aligned}\pi(\boldsymbol{x}_i) &= 1 - \exp\{[(1 - \tau(\beta_1 + \beta_2\boldsymbol{x}_{i2} + \cdots + \beta_p\boldsymbol{x}_{ip}))_+]^{-1/\tau}\} \\ &= 1 - GEV(-\boldsymbol{x}_i'\boldsymbol{\beta};\tau),\end{aligned} \tag{1}$$

where $\boldsymbol{\beta} = (\beta_1,\ldots,\beta_p)' \in \mathbb{R}^p$ is unknown regression parameter measuring the association between potential predictors and response variable Y, $GEV(x;\tau)$ represents the cdf of GEV at x with location parameter $\mu = 0$, scale parameter $\sigma = 1$, unknown shape parameter τ, and $(a)_+ = \max(a, 0)$.

For $\tau \to 0$, the model (1) becomes the response curve of the log-log model, for $\tau > 0$ and $\tau < 0$, it becomes the Frechet and Weibull response curve, respectively, a particular case of the GEV one.

The link function of the GEV regression model is given by

$$\frac{1 - [\log(1 - \pi(\boldsymbol{x}_i))]^{-\tau}}{\tau} = \boldsymbol{x}_i'\boldsymbol{\beta} = \eta(\boldsymbol{x}_i).$$

The unknown vector parameter $\boldsymbol{\beta}$ will be estimated with $(1-2\alpha)\%$ confidence intervals ($\alpha \in [0, 1]$) and a test of hypothesis H_0: $\beta_j = 0$ by both classical approaches of GEV regression model and bootstrap methods.

2.2 Estimation procedure

Let $(Y_1, \boldsymbol{X}_1), \ldots, (Y_n, \boldsymbol{X}_n)$ be independent and identically distributed (i.i.d.) of the random vector $(Y, \boldsymbol{X})$ defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. For every individual $i = 1, \ldots, n$, Y_i is a binary response variable indicating say, the occurrence of some outcome of interest ($Y_i = 1$ if the outcome occurred and $Y_i = 0$ otherwise). Let $\boldsymbol{X}_i = (1, \boldsymbol{X}_{i2}, \ldots, \boldsymbol{X}_{ip})'$ be random vectors of predictors or covariates. The conditional probability $\pi(\boldsymbol{x}_i) = \mathbb{P}(Y_i = 1|\boldsymbol{X}_i = \boldsymbol{x}_i)$ is given in (1).
The likelihood function for the unknown p-dimensional parameter $\boldsymbol{\beta}$ and shape parameter τ from the independent sample $(y_1, \boldsymbol{x}_1), \ldots, (y_n, \boldsymbol{x}_n)$ is

$$L_n(\boldsymbol{\theta}) = \prod_{i=1}^{n} [1 - \mathrm{GEV}(-\boldsymbol{x}_i'\boldsymbol{\beta};\tau)]^{y_i} \times [\mathrm{GEV}(-\boldsymbol{x}_i'\boldsymbol{\beta};\tau)]^{1-y_i},$$

where $\boldsymbol{\theta} = (\boldsymbol{\beta}, \tau) \in \mathbb{R}^{p+1}$. We define the maximum likelihood estimator $\widehat{\beta}_n$ as the solution of the $p+1$-dimensional score equation

$$\frac{\partial \log L_n(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \mathbf{0}. \tag{2}$$

See Calabrese and Osmetti (2013) and Lo et al. (2022) for more details on the maximum likelihood estimation procedure for the parameters $\boldsymbol{\beta}$ and τ in the GEV regression for binary data.

2.3 Parametric bootstrapping

Parametric bootstraps resample a known distribution function, whose parameters are estimated from a given sample. A parametric model is fitted using parameters estimated from the distribution of the bootstrap estimates, from which confidence limits are obtained analytically. In applications, where the standard asymptotic theory does not hold, the null reference distribution can be obtained through parametric bootstrapping (Reynolds and Templin, 2004). The maximum likelihood estimators are commonly used for parametric bootstrapping although this criterion is nearly always based upon their large sample behavior.

2.3.1 Parametric bootstrap confidence interval

For an unknown distribution of parameters estimators, it will be not possible to perform confidence intervals and test hypotheses for the unknown parameter $\boldsymbol{\beta}$ in the model (1). Using an algorithm by Zoubir and Iskander (2004) or Carpenter and Bithell (2000) a parametric bootstrap confidence interval is obtained as follows:

1. *Draw B bootstrap samples $\{(Y_i^{(b)}, \boldsymbol{X}_i^{(b)}), i = 1, \ldots, n\}$ $(b = 1, \ldots, B)$ from the original data sample, and for each bootstrap sample, estimate $\boldsymbol{\beta}$ and π_i by its maximum likelihood estimators $\widehat{\beta}_n^{(b)}$ in model* (1).

2. *Calculate the bootstrap mean and standard error of $\widehat{\beta}_n$ as follows:*

$$\widehat{\beta}_n^* = \frac{1}{B}\sum_{b=1}^{B}\widehat{\beta}_n^{(b)} \quad \textit{and} \quad \widehat{s}_n = \sqrt{\frac{1}{B-1}\sum_{b=1}^{B}\left(\widehat{\beta}_n^{(b)} - \widehat{\beta}_n^*\right)^2}.$$

3. *Calculate* $(1-2\alpha)100\%$ *bootstrap confidence interval by finding quantile of bootstrap replicates*

$$\left[\widehat{\beta}_{n,L}, \widehat{\beta}_{n,U}\right] = \left[\widehat{\beta}_n^{(b),\alpha}, \widehat{\beta}_n^{(b),1-\alpha}\right].$$

Remark 2.1. *Confidence intervals given in the algorithm will allow on the one hand to define an interval belonging to the true parameter β, and on the other hand will be used to decide on comparison tests on the parameters, such as $H_0 : \beta = a$ where $a \in \mathbb{R}$. We reject the hypothesis H_0 if a does not belong to the confidence interval of β.*

2.3.2 Parametric bootstrap for test of hypotheses

We use here the algorithm for parametric hypotheses testing given by Fox (2015) defined as follows:

1. *Estimates parameters* $\widehat{\beta}_{n,j}$ *of the model (1) using the observed data and calculate observed statistic test* $t^{obs}_{\beta_j} = \widehat{\beta}_{n,j}/s_{\widehat{\beta}_{n,j}}$. *Let* $\hat{\boldsymbol{\theta}} = (\widehat{\beta}_{n,j}, t^{obs}_{\beta_j})$
2. *Draw* B *bootstrap samples* $\{(Y_i^{(b)}, \boldsymbol{X}_i^{(b)}), i = 1, \ldots, n\}$ $(b = 1, \ldots, B)$ *from the original data sample, and for each bootstrap sample, estimate* $\boldsymbol{\beta}$ *and* $t^{obs}_{\beta_j}$ *by its maximum likelihood estimators* $\widehat{\beta}_n^{(b)}$ *and* $t^{obs}_{\widehat{\beta}_n^{(b)}}$ *in the model* (1).
3. *Calculate bootstrap p-value by*

$$p - value = \frac{\#\left\{|t^{obs}_{\widehat{\beta}_n^{(b)}}| > |t^{obs}_{\widehat{\beta}_n}|\right\}}{B}.$$

Remark 2.2. *Testing hypotheses defined in the algorithm will allow us to study the impact of the explanatory variables on the response variable* Y *by studying the significance of the parameters* β, $H_0 : \beta_j = 0$. *We reject the hypothesis* H_0 *(i.e. the explanatory variable* X_j *has a significant impact on the response variable* Y*) if the p-value associated with this test is strictly lower than the fixed threshold* $\alpha \in [0, 1]$.

3 Simulation study

This simulation study aims to evaluate the usefulness of the proposed method in Sections (2.3.1) and (2.3.2). The simulation setting is as follows. We consider the following model

$$\frac{1 - [\log(1 - \pi(\boldsymbol{X}_i))]^{-\tau}}{\tau} = \beta_1 + \beta_2 X_{i2} + \beta_3 X_{i3},$$

where the covariates X_{i2} and X_{i3} are independently drawn from normal $\mathcal{N}(0, 1)$ and normal $\mathcal{N}(-1, 1)$, respectively. The true parameter $\boldsymbol{\beta}$ is set such that the proportion of 1's in the simulated data sets is around 15% (considered as Model $\mathcal{M}_1$: $\boldsymbol{\beta} = (-1, 0, 0.7)'$) and 30% (considered as Model $\mathcal{M}_2$, $\boldsymbol{\beta} = (-1.3, 1.5, 0)'$).

An i.i.d. sample of size n of the vector $(Y, \boldsymbol{X})$ is generated from the model (1), and for each individual i, we get a realization $(y_i, \boldsymbol{x}_i)$. The maximum likelihood estimator $\widehat{\beta}_n$ of $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)'$ is obtained from this dataset by solving the score equation (2), using the `optim` function of the software `R` (R Core Team, 2021). Note that τ is not the primary parameter of interest, hence we only focus on the simulation results for $\widehat{\beta}_n$. The usefulness of the proposed method was assessed for several sample sizes ($n = 100, 300, 500$).

For each configuration (sample size, proportion of 1's) of the design parameters, $B = 2000$ bootstrap samples were obtained. Based on these $B = 2000$ samples, we obtain the characteristics defined in the Sections 2.3.1 and 2.3.2. The results from the model (1) are summarized in Table 1.

From these results, it appears that the proposed method is very efficient with regard to the estimation of the model parameters. Indeed, the bootstrap maximum likelihood estimator $\widehat{\beta}_n$ provides a reasonable approximation of the true parameter value, even when the sample size is large enough ($n \geq 300$ say). Note that the length of the confidence intervals decreases as the sample size increases, and the standard errors also decrease. On the other hand, the quality of the estimates is quite poor when the sample size is less than 100 (slightly wide confidence intervals). Finally, these results indicate that a reliable statistical inference on the regression effects in the regression model for binary data with a GEV link function using bootstrap method should be based on a sample having, at least, a moderately large size ($n \geq 100$, say).

Table 1: Simulation results.

n		Model $\mathcal{M}_1$ $\hat{\beta}_{1n}$	$\hat{\beta}_{2n}$	$\hat{\beta}_{3n}$
100	BE	-0.554	- 0.006	0.609
	SE	0.181	0.184	0.185
	95% C.I.	[-0.718;-0.084]	[-0.101;0.140]	[0.330;0.936]
	P-value	0.004	0.206	< 0.0001
300	BE	- 0.580	- 0.003	0.682
	SE	0.115	0.095	0.107
	95% C.I.	[-0.662;-0.526]	[-0.080;0.062]	[0.523;0.852]
	P-value	0.003	0.265	< 0.0001
500	BE	- 0.581	- 0.001	0.689
	SE	0.087	0.071	0.079
	95% C.I.	[-0.605;-0.565]	[-0.075;0.043]	[0.655;0.756]
	P-value	< 0.0001	0.295	< 0.0001
		Model $\mathcal{M}_2$ $\hat{\beta}_{1n}$	$\hat{\beta}_{2n}$	$\hat{\beta}_{3n}$
100	BE	- 0.672	1.683	0.009
	SE	0.265	0.214	0.204
	95% C.I.	[-1.033;-0.506]	[1.240;2.273]	[-0.139;0.091]
	P-value	0.003	< 0.0001	0.153
300	BE	- 1.016	1.669	0.007
	SE	0.136	0.182	0.105
	95% C.I.	[-1.024;-0.559]	[1.359;2.026]	[-0.047;0.071]
	P-value	< 0.0001	< 0.0001	0.172
500	BE	-1.009	1.654	-0.005
	SE	0.117	0.134	0.078
	95% C.I.	[-1.017;-1.001]	[1.452;1.886]	[-0.025;0.046]
	P-value	< 0.0001	< 0.0001	0.190

BE: bootstrap estimate. SE: standard error. C.I.: confidence interval

4 Real data application

In this section, we consider an application on Stroke data in central Senegal. A stroke is a sudden neurological deficit of vascular origin caused by an infarct or hemorrhage in the brain (Bogousslavsky et al., 1993; Biousse, 1994). We consider here a database of size $n = 162$. Data were collected in the context of a prospective and analytical study, carried out over 8 months from April 5 to November 30, 2016 in central Senegal. Patients with CT confirmation of stroke were included in the study.

In Senegal, stroke is the most frequent neurological disease. Known for their high mortality and morbidity rates, they account for more than 30% of hospital admissions and nearly two-thirds of the loss of human life (Sene-Diouf et al., 2007; Touré et al., 2016). We consider the covariates in Table 2.

In this study, the dependent variable is the evolution of the health status of stroke

Table 2: Covariates definition.

Covariate	Definition	Abbreviation
Stroke type	Ischemic stroke or Hemorrhagic stroke	Stroke-Typ
Motor deficiency	affected mobility of the upper and/or lower limbs	Motor-def
Severity Cerebral commitment	displacement of parts of the nervous structure contained in the cranium through an orifice	SC-commitment
Intraventricular haemorrhages	bleeding into the ventricles of the brain	Ivh
Hospital Admission Delay	delay between the first symptoms and admission to hospital	Delay
Vital Prognosis	vital prognosis engaged	Prognosis

patients (vital prognosis). We denote Y as the binary variable defined as follows:

$$Y_i = \begin{cases} 1 & \text{if vital prognosis evolves favourably} \\ 0 & \text{if vital prognosis evolves unfavourably.} \end{cases}$$

Information about explanatory variables is given in Table 2. We aim to
1. build a bootstrap parametric confidence interval for the parameters;
2. perform a bootstrap parametric test of the hypothesis.

We ran a GEV regression analysis using the statistical software **R** of the model defined as follows:

$$\mathbb{P}(Y_i = 1|x) = 1 - GEV(-(\beta_1 + \beta_2 \times \mathit{Delay} + \beta_3 \times \mathit{Ivh} + \beta_4 \times \mathit{Stroke\text{-}typ} + \beta_5 \times \mathit{SC\text{-}commitment} + \beta_6 \times \mathit{Motor\text{-}def}; \tau).$$

Table 3: Estimation of parameters of GEV regression model.

Parameter	Estimate	SE	P-value
Intercept (β_1)	6.8133	0.5680	0.0001
Delay (β_2)	-0.6093	0.1082	0.0001
Ivh (β_3)	1.6597	0.3914	0.0001
Stroke-typ (β_4)	-0.8061	0.2546	0.0015
SC-commitment (β_5)	0.7888	0.3111	0.0112
Motor-def (β_6)	-4.0756	0.4945	0.0001

Table 3 represents the results of the maximum likelihood estimation by solving the score equation (2) using the `optim` function of the software **R**. The standard errors are calculated using the variance-covariance matrix of the estimator. The p-values are calculated using the theoretical distribution of the estimator.

Table 4 shows the obtained results from the GEV regression model in (1) by parametric bootstrap. These results lead to a similar conclusion from the classical method in the estimation of parameters. The parameter estimates are very close. The standard errors of estimates for parametric bootstrap were slightly lower compared to that of the classical approach. Also, the effect of explanatory variables is highly significant in both classical and parametric bootstrap methods. The length of the interval of the parametric method is larger than the classical method.

Table 4: Confidence intervals and P-value by parametric bootstrapping.

Parameter	Estimate	SE	95% C.I	P-value
Intercept (β_1)	6.0407	0.6008	[3.3784;7.5870]	0.0001
Delay (β_2)	-0.6687	0.1037	[-1.0485;-0.4128]	0.0001
Ivh (β_3)	1.8610	0.3631	[0.7645;3.4442]	0.0001
Stroke-typ (β_4)	-1.0418	0.2543	[-1.6633;-0.5731]	0.0002
SC-commitment (β_5)	1.0889	0.3134	[0.4147;1.8897]	0.0001
Motor-def (β_6)	-3.0105	0.4648	[-4.4482;-0.4848]	0.0001

5 Discussion and perspectives

Confidence intervals are good indicators of practical significance, unlike p-values and they also provide more information than p values. Unfortunately, confidence intervals are rarely reported in academic papers. This is because computing confidence intervals are not practical and not possible for some statistics. This is why bootstrap methods, which are resampling techniques for assessing uncertainty, have become popular.

In this study, we have performed bootstrapping parametric method on a GEV regression model. Moreover bootstrapped method was compared with the classical approach while calculating the parameters of the GEV regression model. The bootstrap technique used for estimation and testing produced flexible results. Several questions can be asked: the appropriate value of B for confidence intervals and hypotheses testing. For example, Efron and Tibshirani (1994) suggest that B should be between 1000 and 2000 for 90-95 percent confidence intervals. The nonparametric bootstrap method can also be used for this study. With the help of statistical software today, it is easy to compute confidence intervals and test hypotheses for almost any statistics of interest.

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