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Improving the parameter estimation under interval-censored exponential lifetimes

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Abstract: The present study proposes several methods for improving the parameter estimation of interval-censored exponential lifetime data. These methods can be regarded as the versions of substitution method which are based on the most probable time of occurring failures or the meantime of failures in each inspection interval. Based on simulation studies, the methods proposed in this present study can improve the existing estimators which are based on the midpoint of inspection intervals "or" the exponential probability plot.

Keywords: Exponential distribution; Inspection times; Interval censoring; Pseudo-likelihood function.
Mathematics Subject Classification (2010): 62N02, 62N05.

1 Introduction

The exponential distribution is one of the most common lifetimes widely used in industry and medical survival analysis. Assume that the lifetime distribution of a random sample of n experimental units is exponential with hazard rate θ:

$$F(x;\theta) = \begin{cases} 1 - \exp(-\theta x) & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases} \tag{1}$$

To estimate the parameter θ, due to time constraints and cost reduction, different censoring data schemes might be used by the experimenter. One of these schemes is

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the type I censoring in which the test is terminated at a prefixed time t. In this case, the maximum likelihood estimator (MLE) of θ is obtained by

$$\hat{\theta} = \frac{y}{\sum_{i=1}^{y} x_i + (n-y)t}, \tag{2}$$

where n is the sample size, y denotes the number of failures that are observed until t, and $x_1, \ldots, x_y$ denote those failure times, (Rausand and Høyland, 2004).

On the other hand, experimental units might not be continuously monitored, and if so, the collected data are related to the interval-censored data. In this scheme, n individuals are inspected at k predetermined time points, $t_1, t_2, \ldots, t_k$, and the numbers of failures occurring in the time intervals between inspection times are observed. The exact failure time of an individual is not known. The only information we have is that the individual has survived at the beginning of the interval, and it has failed before the end of the interval. Let $t_0 = 0$, $[t_{j-1}, t_j), j = 1, 2, \ldots, k$ be the j-th inspection time interval and $y_j; j = 1, 2, \ldots, k$ denote the number of failures occurring during the j-th time interval. Since $(Y_1, Y_2, ..., Y_k)$ follows a multinomial distribution, hence for the collected data $(t_j, y_j); j = 1, 2, \ldots, k$, the likelihood function is proportional to

$$\begin{aligned} L(\theta|y_1, y_2, \ldots, y_k) \quad \propto \quad & [F(t_1;\theta) - F(t_0;\theta)]^{y_1} \, [F(t_2;\theta) - F(t_1;\theta)]^{y_2} \\ & \times \ldots [F(t_k;\theta) - F(t_{k-1};\theta)]^{y_k} \, [1 - F(t_k;\theta)]^{n - \sum_{j=1}^{k} y_j}. \end{aligned} \tag{3}$$

The MLE of the model parameter θ is obtained by maximizing the likelihood function, with respect to this parameter. There is no closed-form analytical solution for the following equation

$$\frac{\partial L(\theta|y_1, y_2, \ldots, y_k)}{\partial \theta} = 0.$$

Instead, a numerical method can be used to find the MLE of θ. On the other hand, the determination of the best inspection times $t_1, t_2, \ldots, t_k$ is a fundamental topic in the reliability analysis of interval-censored data. To run a test for inspecting the individuals, given a fixed overall budget, the determination of optimal inspection times will significantly save time. The simplest scheme is the one in which the spaced inspection time is equal for all inspection intervals, (Nanasi, 2014). El-Shaarawi and Naderi (1991), gave guidelines for choosing statistically efficient inspection times and the approximate sample size that achieve a specified degree of precision for estimating a particular quantile of a Weibull time-to-failure distribution. Also, Nelson (1997) gave the optimum inspection times for a demonstration test using an exponential distribution. Sundberg (2001), compared several asymptotic confidence intervals for the type I censored exponential data, with a common censoring time, that have been proposed in the literature.

In this work, we consider some of the existing parameter estimators of the exponential lifetimes under interval censoring, namely, the substitution method, the probability plot method (regression line), and a method that is a version of type-I censored data. We will first give a brief description of these existing estimators and then describe how to improve them by introducing our three methods.

In interval-censored data, the exact failure times of individuals are not observed, while in the substitution method, the approximate failure times are taken into consideration. Using the mean value theorem, for a continuous and differentiable distribution function F on (t_i, t_j), we have

$$F(t_j) - F(t_i) \approx f(c)(t_j - t_i),$$

where $c \in (t_i, t_j)$, and $f(x) = \frac{d}{dx}F(x)$ is the density function. In (Meeker, 1986), by setting c to be the midpoint of the interval (t_{i-1}, t_i), the authors obtained a closed form of the approximate maximum likelihood estimators for the mean and variance of the normal distribution.

To estimate parameter θ in the exponential distribution case, one can apply this theorem to the pseudo-likelihood function in Equation (3) with $t_0 = 0$, to get

$$\begin{aligned} L^*(\theta|y_1, y_2, \ldots, y_k) \quad \propto \quad & [t_1 f(t_1^*; \theta)]^{y_1} \; [(t_2 - t_1) f(t_2^*; \theta)]^{y_2} \\ & \times \ldots [(t_k - t_{k-1}) f(t_k^*; \theta)]^{y_k} \; [1 - F(t_k; \theta)]^{n - \sum_{j=1}^{k} y_j}, \end{aligned}$$

where $t_i^* \in \{t_{i-1}, t_i, \frac{t_{i-1}+t_i}{2}\}, i = 1, \ldots, k$.

Therefore, by substituting t_i^* instead of failure times and using Equation (2), the estimator of θ in this substitution method is obtained by

$$\hat{\theta}_a = \frac{\sum_{j=1}^{k} y_j}{\sum_{j=1}^{k} y_j t_{j-1}^* + (n - \sum_{j=1}^{k} y_j) t_k},$$

where $t_{j-1}^* = t_{j-1}, t_j, \frac{t_{j-1}+t_j}{2}, j = 1, 2, \ldots, k$.

Based on exponential probability plot, Chen and Lio (2010) under interval censoring scheme, we have another estimator for θ and it is given by

$$\frac{1}{k} \sum_{j=1}^{k} \frac{-1}{t_j} \log(1 - \frac{\sum_{r=1}^{j} y_r}{n}).$$

Finally, based on the type I censored exponential reliability data without exact failure times, which is defined by Zhang et al. (2013), the estimator is given as follows

$$\frac{-1}{t_k} \log(1 - \frac{1}{n} \sum_{r=1}^{k} y_r).$$

Remark 1.1. *Suppose all the failures have occurred before the time t_k, that is, $\sum_{j=1}^{k} y_j = n$, then two recent estimators are infinite. In this case, to overcome this problem, we use the following estimators $\hat{\theta}_b = \frac{1}{k}\sum_{j=1}^{k} \hat{\theta}_j$, where, $\hat{\theta}_j = \frac{-1}{t_j} \log(1 - \frac{\sum_{r=1}^{j} y_r}{n+1})$, $j = 1, \ldots, k$, and $\hat{\theta}_c = \frac{-1}{t_k} \log(1 - \frac{\sum_{r=1}^{k} y_r}{n+1})$.*

It must be noted that the estimator $\hat{\theta_a}$, is based on the mathematical concept of the mean value theorem. But in the other methods, that are presented in this work, we used the statistical concepts like mode of the distribution.

The rest of this paper is structured as follows. Section 2 presents new methods for improving $\hat{\theta_a}$, $\hat{\theta_b}$ and $\hat{\theta_c}$. For applying the new methods, first, we use the existing estimators for interval-censored data, and then we adjust them. The determination of the best inspection times is studied in Section 3. In Section 4, we evaluate our proposed methods by using simulation studies, and finally, the conclusion is given in Section 5.

2 Main results

In this section, we introduce three methods for improving $\hat{\theta_a}$, $\hat{\theta_b}$ and $\hat{\theta_c}$. These methods use approximate values instead of unknown failure times. They are based on the number of failures in the equally likely sub-intervals (Method I), the meantime of failures (Method II), and the number and the most probable time of occurring failures in each inspection interval (Method III). Note that Method I is used only to improve $\hat{\theta_a}$, whereas both Methods II and III can be used for the improvement of the $\hat{\theta_a}$, $\hat{\theta_b}$ and $\hat{\theta_c}$.

2.1 Method I

Since the imputation method tends to be strongly biased and has a better performance when the length of the inspection time intervals is short, we use a finer partition for each interval. More precisely, we divide each inspection time interval $I_j = [t_{j-1}, t_j)$ into y_j equally likely sub-intervals:

$$[t_{j-1}, t_{j-1,1}), [t_{j-1,1}, t_{j-1,2}), \ldots, [t_{j-1,y_j-1}, t_{j-1,y_j}),$$

such that

$$\frac{F(t_{j-1,i}) - F(t_{j-1})}{F(t_j) - F(t_{j-1})} \equiv \frac{i}{y_j}, i = 1, 2, \ldots, y_j,$$

or equivalently,

$$\frac{e^{-\theta t_{j-1}} - e^{-\theta t_{j-1,i}}}{e^{-\theta t_{j-1}} - e^{-\theta t_j}} \equiv \frac{i}{y_j}, i = 1, 2, \ldots, y_j,$$

which implies that

$$t_{j-1,i} = \frac{1}{\theta} \log(\frac{y_j}{(y_j - i)e^{-\theta t_{j-1}} + ie^{-\theta t_j}}).$$

Since θ is unknown, it can be estimated by $\hat{\theta_a}$, $\hat{\theta_b}$ or $\hat{\theta_c}$. Suppose that θ is estimated by $\hat{\theta_a}$, then we have

$$\hat{t}_{j-1,i} = \frac{1}{\hat{\theta_a}} \log(\frac{y_j}{(y_j - i)e^{-\hat{\theta_a} t_{j-1}} + ie^{-\hat{\theta_a} t_j}}), i = 1, 2, \ldots, y_j.$$

Substituting either the midpoint or the endpoints of sub-intervals instead of failure times, in Equation (2), another estimation for θ is obtained by

$$\hat{\hat{\theta}}_{a,I} = \frac{\sum_{j=1}^{k} y_j}{\sum_{j=1}^{k} \sum_{i=1}^{y_j} t^*_{j-1,i} + (n - \sum_{j=1}^{k} y_j)t_k},$$

where, $t^*_{j-1,i} = \hat{t}_{j-1,i}, \hat{t}_{j,i}, \frac{\hat{t}_{j-1,i}+\hat{t}_{j,i}}{2}, i = 1, \ldots, y_j, j = 1, 2, \ldots, k.$

2.2 Method II

Let $X_1, X_2 \ldots, X_n$ be a random sample from $F(x) = 1 - \exp(-\theta x); x > 0$. Since the event on which $Y_1 = y_1, Y_2 = y_2, \ldots, Y_k = y_k$ is equivalent to the event on which

$0 < X_{1:n} < \ldots < X_{y_1:n} < t_1 < \ldots < t_{k-1} < X_{\sum_{i=1}^k y_i:n} < t_k$, we have for $i = 1, 2, \ldots, \sum_{j=1}^k y_j$

$$\begin{aligned}&\mathbb{E}(X_{i:n}|Y_1 = y_1, Y_2 = y_2, \ldots, Y_k = y_k)\\ &\quad = \mathbb{E}(X_{i:n}|0 < X_{1:n} < \ldots < X_{y_1:n} < t_1 < \ldots < t_{k-1} < X_{\sum_{i=1}^k y_i:n} < t_k).\end{aligned}$$

On the other hand, for $i = 1, 2, \ldots, y_j$ and $j = 1, 2, \ldots, k$, we have

$$\mathbb{E}(X_{(i+\sum_{i=1}^{j-1} y_i):n}| \ldots < t_{j-1} < X_{(\sum_{i=1}^{j-1} y_i+1):n} < \ldots < X_{(\sum_{i=1}^{j} y_i):n} < t_j < \ldots) = \mathbb{E}(Z_{i:y_j}),$$

where $Z_{i:y_j}$ is i-th order statistics of a random sample of size y_j from the following distribution

$$\frac{F(x) - F(t_{j-1})}{F(t_j) - F(t_{j-1})}. \tag{4}$$

In fact Z is equivalent in distribution to the random variable $X|t_{j-1} < X < t_j$. After estimating all failure times by

$$\mathbb{E}(Z_{1:y_1}), \ldots, \mathbb{E}(Z_{y_1:y_1}), \mathbb{E}(Z_{1:y_2}), \ldots, \mathbb{E}(Z_{y_2:y_2}), \ldots, \mathbb{E}(Z_{1:y_k}), \ldots, \mathbb{E}(Z_{y_k:y_k}),$$

and using the pseudo likelihood function, we obtain the following estimator

$$\frac{\sum_{j=1}^k y_j}{\sum_{j=1}^k \sum_{i=1}^{y_j} E(Z_{i:y_j}) + (n - \sum_{j=1}^k y_j) t_k}. \tag{5}$$

Note that

$$\begin{aligned}\sum_{i=1}^{y_j} \mathbb{E}(Z_{i:y_j}) &= \sum_{i=1}^{y_j} \mathbb{E}(X|t_{j-1} < X < t_j)\\ &= y_j\Big(\frac{1}{\theta} + \frac{t_{j-1}e^{-\theta t_{j-1}} - t_j e^{-\theta t_j}}{e^{-\theta t_{j-1}} - e^{-\theta t_j}}\Big).\end{aligned} \tag{6}$$

By substituting (6) in (5), we have the following estimator for θ:

$$\hat{\hat{\theta}}_{II} = \frac{\sum_{j=1}^k y_j}{\frac{\sum_{j=1}^k y_j}{\theta} + \sum_{j=1}^k y_j \frac{t_{j-1}e^{-\theta t_{j-1}} - t_j e^{-\theta t_j}}{e^{-\theta t_{j-1}} - e^{-\theta t_j}} + (n - \sum_{j=1}^k y_j) t_k}.$$

In this case, θ can also be estimated by $\hat{\theta}_a$, $\hat{\theta}_b$ or $\hat{\theta}_c$. Therefore, re-estimation of the parameter can be obtained by

$$\hat{\hat{\theta}}_{r,II} = \frac{\sum_{i=1}^k y_i}{\frac{\sum_{j=1}^k y_j}{\hat{\theta}_r} + \sum_{j=1}^k y_j \frac{t_{j-1}e^{-t_{j-1}\hat{\theta}_r} - t_j e^{-t_j\hat{\theta}_r}}{e^{-t_{j-1}\hat{\theta}_r} - e^{-t_j\hat{\theta}_r}} + (n - \sum_{i=1}^k y_i) t_k},$$

where $r = a, b, c$.

2.3 Method III

In this subsection, an improved estimator is introduced based on the most probable time of occurring failures in the j-th interval of inspection, $j = 1, 2, \ldots, k$. It is worth noting that, in this method, we did not use the usual MLE method (in which, the unknown parameter θ is estimated by finding the values that maximize the likelihood function of failure times). In Method III, the unknown failure times were first estimated by time values whose probability of observing failures was maximized. In other words, we found the most probable time at which the failure occurred. Then, we obtained the MLE of unknown parameter based on these estimated failure times.

Let y_j denote the number of failures that occur in the interval $[t_{j-1}, t_j)$. We assume that failures occur at the most probable time in the j-th interval.

Let $X^{I_j}_{i:y_j}$ denote the i-th order statistics obtained from a sample size of y_j from a population of failure times in the interval $I_j := [t_{j-1}, t_j)$. Then, we have the density function of $X^{I_j}_{i:y_j}$ for $x \in I_j$ and $i = 1, 2, \ldots, y_j$, as follows

$$f_{X^{I_j}_{i:y_j}}(x) = \frac{y_j!}{(i-1)!(y_j-i)!}[F_{X^{I_j}}(x)]^{i-1} f_{X^{I_j}}(x)[1 - F_{X^{I_j}}(x)]^{y_j-i},$$

where the distribution of X^{I_j} has been defined in (4). Hence,

$$\begin{aligned} f_{X^{I_j}_{i:y_j}}(x) &= \frac{y_j!}{(i-1)!(y_j-i)!}[\frac{F(x)-F(t_{j-1})}{F(t_j)-F(t_{j-1})}]^{i-1}\frac{f(x)}{F(t_j)-F(t_{j-1})} \\ &\quad \times[1 - \frac{F(x)-F(t_{j-1})}{F(t_j)-F(t_{j-1})}]^{y_j-i}. \end{aligned} \tag{7}$$

Since $\bar{F}(x) = \mathrm{e}^{-\theta x}$ hence $f(x) = \theta\bar{F}(x)$. Put, $u_j := \frac{\bar{F}(t_{j-1})-\bar{F}(x)}{\bar{F}(t_{j-1})-\bar{F}(t_j)}$ and $m_j := \frac{\bar{F}(t_{j-1})}{\bar{F}(t_{j-1})-\bar{F}(t_j)}$ (> 1) in (7) to get

$$\begin{aligned} f_{X^{I_j}_{i:y_j}}(u_j) &\propto u_j^{i-1}(m_j - u_j)(1-u_j)^{y_j-i} \\ &= m_j u_j^{i-1}(1-u_j)^{y_j-i} - u_j^i(1-u_j)^{y_j-i}. \end{aligned}$$

The mode/modes are obtained from the following equation

$$\begin{aligned} \frac{d}{dx} f_{X^{I_j}_{i:y_j}}(x) &= \frac{d}{du_j} f_{X^{I_j}_{i:y_j}}(u_j) \times \frac{du_j}{dx} \\ &= u_j^{i-2}(1-u_j)^{y_j-i-1}\{y_j u_j^2 - (m_j(y_j-1)+i)u_j + m_j(i-1)\} \equiv 0. \end{aligned}$$

Therefore, the possible modes of $f_{X^{I_j}_{i:y_j}}(x)$ occur at $x = t_{j-1}$when $u_j = 0$, and at $x = t_j$ when $u_j = 1$. Two other modes at two different roots of the equation are obtained as follows

$$y_j u_j^2 - (m_j(y_j-1)+i)u_j + m_j(i-1) = 0. \tag{8}$$

Let $u_{j_1} = \frac{m_j(y_j-1)+i-\sqrt{\Delta}}{2y_j}$ and $u_{j_2} = \frac{m_j(y_j-1)+i+\sqrt{\Delta}}{2y_j}$, where

$$\Delta = (m_j(y_j-1)+i)^2 - 4m_j y_j(i-1).$$

Note that we must have $0 \le u_{j_1}, u_{j_2} \le 1$ in order to find the corresponding value of x. Observe that $\Delta \ge 0$ and the only acceptable root of (8) is u_{j_1}. Since

$$u_{j_1} = \frac{\bar{F}(t_{j-1}) - \bar{F}(x)}{\bar{F}(t_{j-1}) - \bar{F}(t_j)} = m_j(1 - \frac{\bar{F}(x)}{\bar{F}(t_{j-1})}),$$

we have

$$\frac{\bar{F}(x)}{\bar{F}(t_{j-1})} = \frac{m_j(y_j+1) - i + \sqrt{\Delta}}{2m_j y_j},$$

and therefore we obtain

$$x = t_{j-1} - \frac{\log(\frac{m_j(y_j+1)-i+\sqrt{\Delta}}{2m_j y_j})}{\theta}. \tag{9}$$

As m_j is a function of θ, we can estimate x in (9) by $\hat{x} = t_{j-1} - \frac{1}{\hat{\theta}} \log g(i, y_j, \hat{m}_j)$, where $\hat{m}_j = \frac{e^{-\hat{\theta} t_{j-1}}}{e^{-\hat{\theta} t_{j-1}} - e^{-\hat{\theta} t_j}}$, and

$$g(i, y_j, \hat{m}_j) = \frac{\hat{m}_j(y_j+1) - i + \sqrt{(\hat{m}_j(y_j-1)+i)^2 - 4\hat{m}_j y_j(i-1)}}{2\hat{m}_j y_j},$$

where $\hat{\theta}$ is one of the estimators of θ such as $\hat{\theta_a}$, $\hat{\theta_b}$ or $\hat{\theta_c}$.

By applying the pseudo-likelihood function, and using θ_a, θ_b or θ_c, θ can be re-estimated as follows

$$\hat{\hat{\theta}}_{r,III} = \frac{\sum_{j=1}^k y_j}{\sum_{j=1}^k \sum_{i=1}^{y_j} x^*_{i:y_j,r} + (n - \sum_{j=1}^k y_j) t_k}, \quad r = a, b, c,$$

where $\hat{m}_{j,r} = \frac{e^{-\hat{\theta}_r t_{j-1}}}{e^{-\hat{\theta}_r t_{j-1}} - e^{-\hat{\theta}_r t_j}}$, and $x^*_{i:y_i,r} = t_{j-1} - \frac{\log g(i, y_j, \hat{m}_{j,r})}{\hat{\theta}_r}$.

3 Further consideration

As it was pointed out above, the determination of the best inspection times, $t_1, t_2, \ldots, t_k$, is a fundamental topic in the study of interval-censored data. This section proposes an approach to choosing the optimum last inspection time when $\hat{\theta_c}$ is used for estimation. Since $(Y_1, Y_2, ..., Y_k)$ follows Multinomial $(n, F(t_1), F(t_2) - F(t_1), ..., F(t_k) - F(t_{k-1}))$ and $\sum_{i=1}^k Y_i \sim Bin(n, F(t_k))$, therefore using delta method, the asymptotic distribution of $\hat{\theta_c}$ is $N(\theta, \frac{F(t_k)}{nt_k^2(1-F(t_k))})$. Given this observation, we find the value of t_k such that t_k minimizes the asymptotic variance of $\hat{\theta_c}$. Let $v(t) := \frac{F(t)}{nt^2(1-F(t))}$, where $F(t) = 1 - e^{-\theta t}, t > 0$. Then,

$$\frac{dv(t)}{dt} = te^{-\theta t}[\theta t + 2e^{-\theta t} - 2].$$

The numerical computation gives us the approximate solution $t \cong \frac{\pi}{2\theta}$ for the equation $\frac{dv(t)}{dt} = 0$. Therefore, if we choose the last inspection time, t_k as $\frac{\pi}{2\theta}$, then we will have the minimum value for the variance of the asymptotic distribution of $\hat{\theta}_c$. However, since the value of θ is unknown, we use the information of failures up to the time t_{k-1} in order to estimate t_k as $t_k = \frac{\pi}{2\hat{\theta}^*}$ with $\hat{\theta}^* = \frac{-1}{t_{k-1}} \log(1 - \frac{\sum_{j=1}^{k-1} Y_j}{n+1})$.

Remark 3.1. *The first inspection time is known, and it has been determined before the inspection. When $k = 2$, then the estimator of the last optimal inspection time is t_2, and the related estimator is*

$$\frac{-\log(1 - \frac{\sum_{i=1}^{2} Y_i}{n+1})}{t_2} = \frac{2\log(1 - \frac{Y_1}{n+1})\log(1 - \frac{\sum_{i=1}^{2} Y_i}{n+1})}{\pi t_1}.$$

4 Simulation study

In this section, we compare the performance of our three methods for the improvement of $\hat{\theta_a}$, $\hat{\theta_b}$ or $\hat{\theta_c}$, through simulation studies. For a given sample size n and k inspection times, $t_1, t_2, \ldots, t_k$, samples from the distribution F in (1) were generated using MATLAB.

For exponential density $f(x) = \theta \exp(-\theta x)$, it is worth noting that, the slope of the tangent line at $x = t$ is equal to $-\theta^2 e^{-\theta t}$, and the tangent line at $(0, \theta)$ intercepts the x-axis at $x = \frac{1}{\theta}$. Also the graph $f(x)$ is approximately flat when θt is large. Hence, for a fixed θ, the behaviour of the exponential distribution function becomes completely different when x becomes very small or very large, especially when θ is large. Then, the lengths of inspection times should be determined in such a way that the exponential distribution function behaves almost the same in each interval. Therefore, the largeness of θ and the choosing of a large value for t_1 will reduce the accuracy of $\hat{\theta_a}$.

Preliminary information about the unknown parameter can be useful in the determination of appropriate inspection time t_1. However, we may not have any preliminary information about the unknown parameter in practical works, or else, it might not be possible to consider a small value for t_1, and this can lead to decrease precision of estimators. In this case, as you will see, applying the proposed methods can be useful.

Let y_j be the number of failures that occur in the interval $[t_{j-1}, t_j)$ and in the substitution method, only midpoint selection will be considered, since it works better than the selection of endpoints.

Therefore, we will consider three estimators and seven re-estimators which are defined as follows

$$\hat{\theta}_a = \frac{\sum_{j=1}^{k} y_j}{\sum_{j=1}^{k} y_j \frac{t_{j-1}+t_j}{2} + (n-y)t_k}, \qquad \hat{\theta}_b = \frac{1}{k}\sum_{j=1}^{k} \hat{\theta}_j, \qquad \hat{\theta}_c = \hat{\theta}_k,$$

$$\hat{\hat{\theta}}_{a,I} = \frac{\sum_{j=1}^{k} y_j}{\frac{\sum_{j=1}^{k}\sum_{i=1}^{y_j}(A_{ij}+B_{ij})}{2\hat{\theta_a}} + (n - \sum_{j=1}^{k} y_j)t_k}.$$

$$\hat{\hat{\theta}}_{r,II} = \frac{\sum_{j=1}^{k} y_j}{\frac{\sum_{j=1}^{k} y_j}{\hat{\theta_r}} + \sum_{j=1}^{k} y_j \frac{t_{j-1}e^{-t_{j-1}\hat{\theta_r}} - t_j e^{-t_j\hat{\theta_r}}}{e^{-t_{j-1}\hat{\theta_r}} - e^{-t_j\hat{\theta_r}}} + (n - \sum_{j=1}^{k} y_j)t_k}, \qquad r = a, b, c,$$

$$\hat{\hat{\theta}}_{r,III} = \frac{\sum_{j=1}^{k} y_j}{\sum_{j=1}^{k}\sum_{i=1}^{y_j} x^*_{i:y_j,r} + (n - \sum_{j=1}^{k} y_j)t_k}, \qquad r = a, b, c,$$

where

$$\begin{aligned}
\hat{\theta}_j &= \frac{-\log(1-\frac{W_j}{n+1})}{t_j}, \quad \text{with} \quad W_j = \sum_{i=1}^{j} Y_i,\\
A_{ij} &= \log\Big(\frac{y_j}{(y_j+1-i)e^{-\hat{\theta}_a t_{j-1}} + (i-1)e^{-\hat{\theta}_a t_j}}\Big),\\
B_{ij} &= \log\Big(\frac{y_j}{(y_j-i)e^{-\hat{\theta}_a t_{j-1}} + ie^{-\hat{\theta}_a t_j}}\Big),\\
x^*_{i:y_j,r} &= t_{j-1} - \frac{\log g(i, y_j, \hat{m_{j,r}})}{\hat{\theta}_r},
\end{aligned}$$

with $\hat{m}_{i,r} = \frac{e^{-\hat{\theta}_r t_{i-1}}}{e^{-\hat{\theta}_r t_{i-1}} - e^{-\hat{\theta}_r t_i}}$,

$$g(i, y_j, \hat{m}_{j,r}) = \frac{\hat{m_{j,r}}(y_j+1) - i + \sqrt{(i + (y_j-1)\hat{m_{j,r}})^2 - 4y_j\hat{m_{j,r}}(i-1)}}{2\hat{m_{j,r}}y_j}.$$

Since the number of inspection times and the size of censoring intervals are the crucial factors influencing the efficiency of the estimators, it seems $\hat{\theta_a}$, $\hat{\theta_b}$ and $\hat{\theta_c}$ may need improvement when the size of censoring intervals is large and the number of inspection times is small. For a preliminary assessment of the proposed methods, we first consider different values of $\theta = 0.02, 0.04, 0.08, 0.1$, and $k = 2$ as the number of inspection times. Besides, $t_1 = [F^{-1}(0.6; 0.02) + \ldots + F^{-1}(0.6; 0.1)]/5$ and $t_2 = [F^{-1}(0.9; 0.02) + \ldots + F^{-1}(0.9; 0.1)]/5$ have been considered as two common inspection times for all of θ's. Then, by considering $k = 3$ and $\theta = 0.02, 0.04, 0.08, 0.1$, $t_1 = [F^{-1}(0.4; 0.02) + \ldots + F^{-1}(0.4; 0.1)]/5$, $t_2 = [F^{-1}(0.64; 0.02) + \ldots + F^{-1}(0.64; 0.1)]/5$ and $t_3 = [F^{-1}(0.84; 0.02) + \ldots + F^{-1}(0.84; 0.1)]/5$, the influence of decreasing the size of censoring intervals and increasing the values of k has been studied.

The results summarized in Tables 1-6 are the averages over 10000 repetitions, in which $y_j > 0 : j = 1, 2, \ldots, k$. For each scenario, bias, the sum of square error (SSE), and combined error (C.E.) were calculated. In addition, for the comparison of estimators and their corresponding proposed re-estimators, the relative combined error (R.C.E.) was taken into consideration. These measures of accuracy are defined as follows

$$\begin{aligned}
Bias(\hat{\theta}) &= \hat{\theta} - \theta,\\
C.E.(\hat{\theta}) &= |Bias(\hat{\theta})| + \sqrt{\frac{SSE(\hat{\theta})}{10000}},\\
R.C.E.(\hat{\theta_1}, \hat{\theta_2}) &= \frac{C.E.(\hat{\theta_2})}{C.E.(\hat{\theta_1})}.
\end{aligned}$$

Table 1: Comparison of $\hat{\theta}_a$ with its re-estimators, when $t_1 = 22, t_2 = 53$.

θ	$\hat{\theta}$	n	$Bias(\hat{\theta})$	$SSE(\hat{\theta})$	$R.C.E(\hat{\theta}, \hat{\theta}_a)$
0.02	$\hat{\theta}_a$	20	0.0001	0.2963	1
	$\hat{\theta}_{a,I}$		0.0007	0.3546	0.837
	$\hat{\theta}_{a,II}$		0.0007	0.3562	0.834
	$\hat{\theta}_{a,III}$		0.0008	0.3627	0.820
	$\hat{\theta}_a$	30	-0.00007	0.1971	1
	$\hat{\theta}_{a,I}$		0.0004	0.2319	0.856
	$\hat{\theta}_{a,II}$		0.0005	0.2323	0.854
	$\hat{\theta}_{a,III}$		0.0005	0.2353	0.842
	$\hat{\theta}_a$	50	-0.0003	0.1138	1
	$\hat{\theta}_{a,I}$		0.0002	0.1307	0.956
	$\hat{\theta}_{a,II}$		0.0002	0.1308	0.955
	$\hat{\theta}_{a,III}$		0.0002	0.1318	0.945
0.04	$\hat{\theta}_a$	20	-0.0019	0.7364	1
	$\hat{\theta}_{a,I}$		0.0015	1.1597	0.853
	$\hat{\theta}_{a,II}$		0.0017	1.1852	0.838
	$\hat{\theta}_{a,III}$		0.0021	1.2979	0.781
	$\hat{\theta}_a$	30	-0.0023	0.5059	1
	$\hat{\theta}_{a,I}$		0.0010	0.7324	0.979
	$\hat{\theta}_{a,II}$		0.0011	0.7382	0.972
	$\hat{\theta}_{a,III}$		0.0013	0.7673	0.932
	$\hat{\theta}_a$	50	-0.0027	0.3259	1
	$\hat{\theta}_{a,I}$		0.0004	0.3917	1.272
	$\hat{\theta}_{a,II}$		0.0004	0.3925	1.268
	$\hat{\theta}_{a,III}$		0.0005	0.4014	1.227
0.1	$\hat{\theta}_a$	20	-0.0296	9.4903	1
	$\hat{\theta}_{a,I}$		-0.0115	3.2463	2.049
	$\hat{\theta}_{a,II}$		-0.0098	3.2065	2.182
	$\hat{\theta}_{a,III}$		-0.004	3.1429	2.729
	$\hat{\theta}_a$	30	-0.0287	8.8479	1
	$\hat{\theta}_{a,I}$		-0.0093	2.6296	2.294
	$\hat{\theta}_{a,II}$		-0.0083	2.6502	2.376
	$\hat{\theta}_{a,III}$		-0.0048	2.6498	2.770
	$\hat{\theta}_a$	50	-0. 0283	8.44300	1
	$\hat{\theta}_{a,I}$		-0.0082	1.9919	2.567
	$\hat{\theta}_{a,II}$		-0.0079	1.9968	2.609
	$\hat{\theta}_{a,III}$		-0.0060	1.9699	2.858

Table 2: Comparison of $\hat{\theta}_a$ with its re-estimators, when $t_1 = 12, t_2 = 24, t_3 = 44$.

θ	$\hat{\theta}$	n	$Bias(\hat{\theta})$	$SSE(\hat{\theta})$	$R.C.E(\hat{\theta}, \hat{\theta}_a)$
0.02	$\hat{\theta}_a$	20	0.0007	0.3580	1
	$\hat{\theta}_{a,I}$		0.0009	0.3814	0.945
	$\hat{\theta}_{a,II}$		0.0009	0.3829	0.941
	$\hat{\theta}_{a,III}$		0.0009	0.3847	0.933
	$\hat{\theta}_a$	30	0.0004	0.2407	1
	$\hat{\theta}_{a,I}$		0.0006	0.2556	0.94
	$\hat{\theta}_{a,II}$		0.0006	0.2560	0.939
	$\hat{\theta}_{a,III}$		0.0006	0.2574	0.933
	$\hat{\theta}_a$	50	0.0001	0.1390	1
	$\hat{\theta}_{a,I}$		0.0003	0.1466	0.936
	$\hat{\theta}_{a,II}$		0.0003	0.1467	0.933
	$\hat{\theta}_{a,III}$		0.0003	0.1473	0.928
0.04	$\hat{\theta}_a$	20	0.0003	0.9439	1
	$\hat{\theta}_{a,I}$		0.0015	1.1437	0.817
	$\hat{\theta}_{a,II}$		0.0016	1.1599	0.807
	$\hat{\theta}_{a,III}$		0.0018	1.2110	0.78
	$\hat{\theta}_a$	30	-0.00007	0.6365	1
	$\hat{\theta}_{a,I}$		0.0011	0.7656	0.814
	$\hat{\theta}_{a,II}$		0.0012	0.7698	0.809
	$\hat{\theta}_{a,III}$		0.0013	0.7848	0.793
	$\hat{\theta}_a$	50	-0.0005	0.3603	1
	$\hat{\theta}_{a,I}$		0.0007	0.4249	0.899
	$\hat{\theta}_{a,II}$		0.0007	0.4256	0.897
	$\hat{\theta}_{a,III}$		0.0008	0.4309	0.882
0.1	$\hat{\theta}_a$	20	-0.0132	3.8959	1
	$\hat{\theta}_{a,I}$		-0.0038	3.6984	1.387
	$\hat{\theta}_{a,II}$		-0.0024	4.0125	1.422
	$\hat{\theta}_{a,III}$		0.0018	5.1582	1.299
	$\hat{\theta}_a$	30	-0.0114	3.1843	1
	$\hat{\theta}_{a,I}$		-0.0010	2.9406	1.574
	$\hat{\theta}_{a,II}$		-0.0003	3.1320	1.593
	$\hat{\theta}_{a,III}$		0.0023	3.7414	1.322
	$\hat{\theta}_a$	50	-0. 01769	2.3179	1
	$\hat{\theta}_{a,I}$		-0.00003	2.0753	1.801
	$\hat{\theta}_{a,II}$		0.0003	2.1420	1.743
	$\hat{\theta}_{a,III}$		0.0016	2.4009	1.522

Table 3: Comparison of $\hat{\theta}_b$ with its re-estimators, when $t_1 = 22, t_2 = 53$.

θ	$\hat{\theta}$	n	$Bias(\hat{\theta})$	$SSE(\hat{\theta})$	$R.C.E(\hat{\theta}, \hat{\theta}_b)$
0.02	$\hat{\theta}_b$	20	-0.0008	0.3172	1
	$\hat{\theta}_{b,II}$		0.0007	0.3540	0.958
	$\hat{\theta}_{b,III}$		0.0008	0.3604	0.943
	$\hat{\theta}_b$	30	-0.0005	0.2222	1
	$\hat{\theta}_{b,II}$		0.0004	0.2319	0.993
	$\hat{\theta}_{b,III}$		0.0005	0.2349	0.979
	$\hat{\theta}_b$	50	-0.0004	0.1348	1
	$\hat{\theta}_{b,II}$		0.0002	0.1310	1.054
	$\hat{\theta}_{b,III}$		0.0002	0.1321	1.042
0.04	$\hat{\theta}_b$	20	-0.0026	0.8333	1
	$\hat{\theta}_{b,II}$		0.0016	1.1861	0.942
	$\hat{\theta}_{b,III}$		0.0020	1.2978	0.878
	$\hat{\theta}_b$	30	-0.0016	0.6052	1
	$\hat{\theta}_{b,II}$		0.0011	0.7612	0.956
	$\hat{\theta}_{b,III}$		0.0014	0.7939	0.915
	$\hat{\theta}_b$	50	-0.0011	0.3649	1
	$\hat{\theta}_{b,II}$		0.0005	0.4107	1.029
	$\hat{\theta}_{b,III}$		0.0007	0.4211	0.994
0.08	$\hat{\theta}_b$	20	-0.0169	3.7875	1
	$\hat{\theta}_{b,II}$		-0.0020	2.5925	2.002
	$\hat{\theta}_{b,III}$		0.0015	3.350464	1.831
	$\hat{\theta}_b$	30	-0.0122	2.4194	1
	$\hat{\theta}_{b,II}$		-0.0007	2.1369	1.808
	$\hat{\theta}_{b,III}$		0.0016	2.6168	1.564
	$\hat{\theta}_b$	50	-0.0082	1.4616	1
	$\hat{\theta}_{b,II}$		-0.0005	1.4568	1.609
	$\hat{\theta}_{b,III}$		0.0006	1.6451	1.503
0.1	$\hat{\theta}_b$	20	-0.0295	9.4922	1
	$\hat{\theta}_{b,II}$		-0.0097	3.2403	2.176
	$\hat{\theta}_{b,III}$		-0.0043	3.1955	2.714
	$\hat{\theta}_b$	30	-0.0228	6.1137	1
	$\hat{\theta}_{b,II}$		-0.00621	2.6843	2.104
	$\hat{\theta}_{b,III}$		-0.0024	2.8653	2.461
	$\hat{\theta}_b$	50	-0. 0162	3.5046	1
	$\hat{\theta}_{b,II}$		-0.0035	2.0207	1.965
	$\hat{\theta}_{b,III}$		-0.0012	2.2020	2.178

Table 4: Comparison of $\hat{\theta}_b$ with its re-estimators, when $t_1 = 12, t_2 = 24, t_3 = 44$

θ	$\hat{\theta}$	n	$Bias(\hat{\theta})$	$SSE(\hat{\theta})$	$R.C.E(\hat{\theta}, \hat{\theta}_b)$
0.02	$\hat{\theta}_b$	20	-0.0006	0.3888	1
	$\hat{\theta}_{b,II}$		0.0009	0.3813	0.964
	$\hat{\theta}_{b,III}$		0.0009	0.3830215	0.955
	$\hat{\theta}_b$	30	-0.0003	0.2797	1
	$\hat{\theta}_{b,II}$		0.0006	0.2554	1
	$\hat{\theta}_{b,III}$		0.0006	0.2567	0.994
	$\hat{\theta}_b$	50	-0.0002	0.1720	1
	$\hat{\theta}_{b,II}$		0.0003	0.1466	1.066
	$\hat{\theta}_{b,III}$		0.0003	0.1472	1.061
0.04	$\hat{\theta}_b$	20	-0.0020	0.9395	1
	$\hat{\theta}_{b,II}$		0.0015	1.1452	0.956
	$\hat{\theta}_{b,III}$		0.0017	1.1912	0.926
	$\hat{\theta}_b$	30	-0.0012	0.6850	1
	$\hat{\theta}_{b,II}$		0.0011	0.7667	0.955
	$\hat{\theta}_{b,III}$		0.0012	0.7813	0. 937
	$\hat{\theta}_b$	50	-0.0007	0.4096	1
	$\hat{\theta}_{b,II}$		0.0007	0.4261	0.985
	$\hat{\theta}_{b,III}$		0.0008	0.4315	0.969
0.08	$\hat{\theta}_b$	20	-0.0112	2.6915	1
	$\hat{\theta}_{b,II}$		0.0008	3.1345	1.487
	$\hat{\theta}_{b,III}$		0.0031	3.9060	1.208
	$\hat{\theta}_b$	30	-0.0071	1.8251	1
	$\hat{\theta}_{b,II}$		0.0015	2.3793	1.221
	$\hat{\theta}_{b,III}$		0.0027	2.7398	1.071
	$\hat{\theta}_b$	50	-0.0042	1.1925	1
	$\hat{\theta}_{b,II}$		0.0011	1.4711	1.149
	$\hat{\theta}_{b,III}$		0.0016	1.5740	1.071
0.1	$\hat{\theta}_b$	20	-0.0210	5.7930	1
	$\hat{\theta}_{b,II}$		-0.0035	3.7724	1.965
	$\hat{\theta}_{b,III}$		0.0004	4.7081	2.036
	$\hat{\theta}_b$	30	-0.0145	3.4146	1
	$\hat{\theta}_{b,II}$		-0.0007	3.0091	1.823
	$\hat{\theta}_{b,III}$		0.0018	3.5660	1.595
	$\hat{\theta}_b$	50	-0. 0089	1.9702	1
	$\hat{\theta}_{b,II}$		0.0005	2.1604	1.504
	$\hat{\theta}_{b,III}$		0.0019	2.4293	1.312

Table 5: Comparison of $\hat{\theta}_c$ with its re-estimators, when $t_1 = 22, t_2 = 53$.

θ	$\hat{\theta}$	n	$Bias(\hat{\theta})$	$SSE(\hat{\theta})$	$R.C.E(\hat{\theta}, \hat{\theta}_c)$
0.02	$\hat{\theta}_c$	20	-0.0009	0.3038	1
	$\hat{\theta}_{c,II}$		0.0007	0.3530	0.957
	$\hat{\theta}_{c,III}$		0.0008	0.3589	0.942
	$\hat{\theta}_c$	30	-0.0006	0.2152	1
	$\hat{\theta}_{c,II}$		0.0004	0.2317	0.995
	$\hat{\theta}_{c,III}$		0.0005	0.2348	0.981
	$\hat{\theta}_c$	50	-0.0004	0.1296	1
	$\hat{\theta}_{c,II}$		0.0002	0.1310	1.051
	$\hat{\theta}_{c,III}$		0.00024	0.1320	1.040
0.04	$\hat{\theta}_c$	20	-0.0037	0.9542	1
	$\hat{\theta}_{c,II}$		0.0015	1.1328	1.114
	$\hat{\theta}_{c,III}$		0.0018	1.2123	1.053
	$\hat{\theta}_c$	30	-0.0023	0.7422	1
	$\hat{\theta}_{c,II}$		0.0011	0.7445	1.128
	$\hat{\theta}_{c,III}$		0.0013	0.7720	1.084
	$\hat{\theta}_c$	50	-0.0015	0.4746	1
	$\hat{\theta}_{c,II}$		0.0005	0.4074	1.218
	$\hat{\theta}_{c,III}$		0.0007	0.4177	1.177
0.08	$\hat{\theta}_c$	20	-0.0261	7.1873	1
	$\hat{\theta}_{c,II}$		-0.0048	2.0296	2.774
	$\hat{\theta}_{c,III}$		-0.0019	2.3909	3.052
	$\hat{\theta}_c$	30	-0.02033	4.6648	1
	$\hat{\theta}_{c,II}$		-0.0003	1.6165	2.640
	$\hat{\theta}_{c,III}$		-0.0014	1.8438	2.805
	$\hat{\theta}_c$	50	-0.0140	2.7067	1
	$\hat{\theta}_{c,II}$		-0.0022	1.1673	2.344
	$\hat{\theta}_{c,III}$		-0.0013	1.2521	2.444
0.1	$\hat{\theta}_c$	20	-0.0439	19.4018	1
	$\hat{\theta}_{c,II}$		-0.0147	3.5774	2.615
	$\hat{\theta}_{c,III}$		-0.0102	2.9577	3.209
	$\hat{\theta}_c$	30	-0.03711	13.9994	1
	$\hat{\theta}_{c,II}$		-0.0114	2.6294	2.697
	$\hat{\theta}_{c,III}$		-0.0083	2.3522	3.144
	$\hat{\theta}_c$	50	-0. 02876	8.6107	1
	$\hat{\theta}_{c,II}$		-0.0082	1.7563	2.705
	$\hat{\theta}_{c,III}$		-0.0064	1.6647	3.008

Table 6: Comparison of $\hat{\theta}_c$ with its re-estimators, when $t_1 = 12, t_2 = 24, t_3 = 44$.

θ	$\hat{\theta}$	n	$Bias(\hat{\theta})$	$SSE(\hat{\theta})$	$R.C.E(\hat{\theta}, \hat{\theta}_c)$
0.02	$\hat{\theta}_c$	20	-0.0006	0.3275	1
	$\hat{\theta}_{c,II}$		0.0009	0.3812	0.896
	$\hat{\theta}_{c,III}$		0.0009	0.3827	0.888
	$\hat{\theta}_c$	30	-0.0004	0.2363	1
	$\hat{\theta}_{c,II}$		0.0006	0.2554	0.938
	$\hat{\theta}_{c,III}$		0.0006	0.2567	0.933
	$\hat{\theta}_c$	50	-0.0003	0.1720	1
	$\hat{\theta}_{c,II}$		0.0003	0.1466	1.002
	$\hat{\theta}_{c,III}$		0.0003	0.1472	0.997
0.04	$\hat{\theta}_c$	20	-0.0028	1.0180	1
	$\hat{\theta}_{c,II}$		0.0015	1.1347	1.059
	$\hat{\theta}_{c,III}$		0.0017	1.1710	1.031
	$\hat{\theta}_c$	30	-0.0018	0.7620	1
	$\hat{\theta}_{c,II}$		0.0011	0.7624	1.072
	$\hat{\theta}_{c,III}$		0.0012	0.7754	1.052
	$\hat{\theta}_c$	50	-0.0010	0.4779	1
	$\hat{\theta}_{c,II}$		0.0007	0.4254	1.102
	$\hat{\theta}_{c,III}$		0.0008	0.4308	1.084
0.08	$\hat{\theta}_c$	20	-0.0191	4.6202	1
	$\hat{\theta}_{c,II}$		-0.00006	2.7751	2.431
	$\hat{\theta}_{c,III}$		0.0019	3.3745	2.006
	$\hat{\theta}_c$	30	-0.0138	3.0749	1
	$\hat{\theta}_{c,II}$		0.0008	2.1357	2.038
	$\hat{\theta}_{c,III}$		0.0018	2.4127	1.808
	$\hat{\theta}_c$	50	-0.0086	2.0517	1
	$\hat{\theta}_{c,II}$		0.0006	1.3682	1.863
	$\hat{\theta}_{c,III}$		0.0011	1.4408	1.754
0.1	$\hat{\theta}_c$	20	-0.0344	12.3515	1
	$\hat{\theta}_{c,II}$		-0.0055	3.3061	2.938
	$\hat{\theta}_{c,III}$		-0.0020	3.8676	3.204
	$\hat{\theta}_c$	30	-0.02739	8.202775	1
	$\hat{\theta}_{c,II}$		-0.0026	2.5404	3.019
	$\hat{\theta}_{c,III}$		-0.0005	2.8913	3.205
	$\hat{\theta}_c$	50	-0. 0192	4.6671	1
	$\hat{\theta}_{c,II}$		-0.0009	1.8276	2.819
	$\hat{\theta}_{c,III}$		0.0001	2.0020	2.857

5 Conclusions

Tables 1-2 show that the negative bias of $\hat{\theta_a}$ can be reduced by using the proposed methods. Methods I and II perform better than Method III when $\frac{1}{\theta} < t_1$, but Method III is preferred for large value of n. However, $\hat{\theta_a}$ can be improved in a better way, by using Method III, when $\frac{2}{\theta} < t_1$. For example, as you see in Table 1, for $t_1 = 22, t_2 = 53$, $\theta = 0.1$ and $n = 50$, the combined error of $\hat{\theta_a}$ is about one-thirds, when Method III is used. Tables 3-6 show that the negative bias of $\hat{\theta_b}$ and $\hat{\theta_c}$ are reduced by using Methods II and III when $\frac{1}{\theta} < t_1$. However, Method III is preferred when $\frac{2}{\theta} < t_1$. It must be noted that $\hat{\theta_a}$, $\hat{\theta_b}$, $\hat{\theta_c}$ (and even other estimators of θ) might be improved by the proposed methods when the size of the first censoring interval is small and when the next censoring intervals are large. According to the results given in the tables, it can be concluded that Methods I, II and III can
a) reduce the negative biases of $\hat{\theta_a}, \hat{\theta_b}$ and $\hat{\theta_c}$ when their values are bigger than $\frac{1}{t_1}$,
b) improve $\hat{\theta_a}$, $\hat{\theta_b}$ and $\hat{\theta_c}$ when their values are bigger than $\frac{1}{t_1}$, and when the sample size increases,
c) improve $\hat{\theta_a}$, $\hat{\theta_b}$ and $\hat{\theta_c}$ when their values are bigger than $\frac{2}{t_1}$ even for small values of n.

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