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The gamma odd power generalized Weibull-G family of distributions with applications

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Abstract: A new generalized distribution called the gamma odd power generalized Weibull-G family of distributions is developed and studied. Some special models of the new family of distribution are explored. Statistical properties of the new family of distributions including the quantile function, ordinary and incomplete moments, probability weighted moments, stochastic ordering, distribution of the order statistics, and Rényi entropy are presented. The maximum likelihood method is used for estimating model parameters, and Monte Carlo simulation is conducted to examine the performance of the model. The flexibility of the new family of distributions is demonstrated by means of two applications to real data sets.

Keywords: Generalized distribution; Maximum likelihood estimation; Power generalized Weibull distribution.
Mathematics Subject Classification (2010): 62E99, 60E05, 62E15.

1 Introduction

The development of generalized families of distributions for modelling data in many areas such as economics, biology, insurance, finance, and engineering to mention a few is of tremendous practical importance. This has motivated researchers to study and develop many methods of generating new families of distributions from well-known distributions in order to get greater flexibility when modelling data in practice.

Several studies in the literature include the work by Oluyede et al. (2018), where they developed the gamma-Weibull-G (GWG) family of distributions via the gamma-G

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transformation by Zografos and Balakrishnan (2009). For a continuous baseline cumulative distribution function (cdf) F(x), Zografos and Balakrishnan (2009) developed a gamma-generator with the cdf and probability density function (pdf) given by

$$\begin{aligned} G(x;\delta,\beta,\psi) &= \frac{1}{\Gamma(\delta)\beta^{\delta}}\int_0^{-\log(\overline{F}(x;\psi))} t^{\delta-1}e^{\frac{-t}{\beta}}dt = \frac{\gamma(\delta,-\beta^{-1}\log(\overline{F}(x;\psi)))}{\Gamma(\delta)}, \\ g(x;\delta,\beta,\psi) &= \frac{1}{\Gamma(\delta)\beta^{\delta}}[-\log(\overline{F}(x;\psi))]^{\delta-1}(1-F(x;\psi))^{\frac{1}{\beta}-1}f(x;\psi), \end{aligned}$$

respectively, for $\delta > 0$, $(\beta = 1)$ and parameter vector ψ, where $\overline{F}(x;\psi) = 1 - F(x;\psi)$, $\Gamma(\delta) = \int_0^\infty t^{\delta-1}e^{-t}dt$ is the gamma function, and $\gamma(z,\delta) = \int_0^z t^{\delta-1}e^{-t}dt$ is the incomplete gamma function. Oluyede et al. (2015) proposed the generalized modified Weibull distributions via the same generator. Similarly Ristić and Balakrishnan (2011) presented an alternative gamma-generator with the cdf and pdf as follows:

$$\begin{aligned} G(x;\delta,\beta,\psi) &= 1-\frac{1}{\Gamma(\delta)\beta^{\delta}}\int_0^{-\log(F(x;\psi))} t^{\delta-1}e^{\frac{-t}{\beta}}dt, \\ g(x;\delta,\beta,\psi) &= \frac{1}{\Gamma(\delta)\beta^{\delta}}[-\log(F(x;\psi))]^{\delta-1}(F(x;\psi))^{\frac{1}{\beta}-1}f(x;\psi). \end{aligned}$$

There are more generalizations of the Weibull distribution in the literature, which include the two, three, four, and five parameter distributions. The two-parameter Weibull-extensions are Bebbington and Zitikis (2007) and Zhang and Xie (2011). The three-parameter Weibull extensions include Marshall and Olkin (1997), Xie et al. (2002), and Nadarajah and Kotz (2005). Some of the four-parameter generalizations include the additive Weibull distribution by Xie and Lai (1996), modified Weibull by Sarhan and Zaindin (2009), and the beta-Weibull which was proposed by Famoye and Olumolade (2005). The five-parameter modified Weibull distribution includes those introduced by Phani (1987) beta modified Weibull by Silva et al. (2010) and Nadarajah et al. (2011) among others. Bagdonavicius and Nikulin (2001) on the other hand developed an extension of the Weibull distribution called the power generalized Weibull (PGW) distribution which was later studied by Nikulin and Haghighi (2009). Cordeiro et al. (2015) introduced a new generalized Weibull family of distributions.

Weibull distribution has many limitations when it comes to applications in real data since it only has monotonically increasing, monotonically decreasing, and constant hazard rates. This has drawn our attention to developing the more generalized Weibull distribution that can give the desired results in applications. We introduce a new extended generator, called the gamma odd power generalized Weibull-G (GOPGW-G) family of distribution based on the previous works by Bourguignon et al. (2014).

The main aim and motivations for developing this new family of distributions are that it gives a better fit than the Odd Power Generalized Weibull-G (OPGW-G) family of distributions and that it exhibits interesting shapes of the hazard rate function. The new family of distributions can model data with monotonic and non-monotonic hazard rate functions. This family is also of interest to us because we are looking for a model that can provide a better fit than other generated distributions through the same transformation. The model can be used in applications for heavy-tailed and skewed data.

The results of this paper are organized as follows: In Section 2, we define our new model, the gamma odd power generalized Weibull-G (GOPGW-G) family of distributions and its sub-models. In Section 3, statistical properties of the GOPGW-G distribution are explored including expansion of the probability density function, hazard rate and quantile functions, moments, conditional moments, moment generating, order statistics and Rényi entropy are presented. Maximum likelihood estimates of the model parameters and observed information matrix are given in Section 4. The special cases of gamma odd power generalized Weibull-Weibull (GOPGW-W) and gamma odd power generalized Weibull-Burr XII (GOPGW-BXII) distributions are presented in details in Section 5. A Monte Carlo simulation study to examine the bias and mean square error of the maximum likelihood estimates is presented in Section 6. Section 7 contains applications of the new model to real data sets. Finally, the concluding remarks are given in Section 8.

2 The model

In a recent note, Oluyede et al. (2021) developed the odd power generalized Weibull-G family of distributions with the cdf and pdf given by

$$
\begin{aligned}
F_{OPGW-G}(x;\alpha,\beta,\xi) &= 1-\exp\left\{1-\left[1+\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\alpha}\right]^{\beta}\right\},\\
f_{OPGW-G}(x;\alpha,\beta,\xi) &= \alpha\beta\exp\left\{1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right\}\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\\
&\quad\times\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta-1}\frac{g(x;\xi)}{(1-G(x;\xi))^2},
\end{aligned}
$$

respectively, for $\alpha,\beta>0$, where $\overline{G}(x;\xi)=1-G(x;\xi)$ is the survival function with parameter vector ξ, and $g(x;\xi)=\frac{dG(x;\xi)}{dx}$. The cdf and pdf of the proposed GOPGW-G family of distributions are given by

$$
\begin{aligned}
F_{GOPGW-G}(x;\delta,\alpha,\beta,\xi) &= 1-\frac{1}{\Gamma(\delta)}\int_0^{-\log\left(F_{OPGW-G}(x;\alpha,\beta,\xi)\right)} t^{\delta-1}e^{-t}dt\\
&= 1-\frac{1}{\Gamma(\delta)}.\gamma(-\log\left(F_{OPGW-G}(x;\alpha,\beta,\xi)\right),\delta), \qquad (1)
\end{aligned}
$$

$$
\begin{aligned}
f_{GOPGW-G}(x;\delta,\alpha,\beta,\xi) &= \frac{\beta\alpha}{\Gamma(\delta)}\left\{-\log\left(1-\exp\left\{1-\left[1+\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\alpha}\right]^{\beta}\right\}\right)\right\}^{\delta-1}\\
&\quad\times\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\alpha-1}\left[1+\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\alpha}\right]^{\beta-1}\\
&\quad\times\exp\left\{1-\left[1+\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\alpha}\right]^{\beta}\right\}\frac{g(x;\xi)}{(\overline{G}(x;\xi))^2}, \qquad (2)
\end{aligned}
$$

respectively. The hazard rate function (hrf) is given by

$$h_{GOPGW-G}(x;\delta,\alpha,\beta,\xi) = \beta\alpha\left\{-\log\left(1-\exp\left\{1-\left[1+\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\alpha}\right]^{\beta}\right\}\right)\right\}^{\delta-1}$$
$$\times\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\alpha-1}\left[1+\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\alpha}\right]^{\beta-1}$$
$$\times\exp\left\{1-\left[1+\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\alpha}\right]^{\beta}\right\}\frac{g(x;\xi)}{(\overline{G}(x;\xi))^{2}}$$
$$\times\left(\gamma\left(-\log\left(1-\exp\left\{1-\left[1+\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\alpha}\right]^{\beta}\right\}\right),\delta\right)\right)^{-1},$$

for $x,\delta,\alpha,\beta>0$, and parameter vector ξ, where $\gamma(x,\delta)$ is the lower incomplete gamma function and $\Gamma(.)$ is the gamma function. If a random variable X has the GOPGW-G distribution, we write $X\sim GOPGW-G(\delta,\alpha,\beta,\xi)$.

2.1 Sub-models of the GOPGW-G distribution

Some of the sub-models of the GOPGW-G family of distributions are given in this section.

• When $\delta=1$, we obtain the OPGW-G family of distributions with the cdf given in equation (1), Oluyede et al. (2021).

• When $\delta=\beta=1$, we obtain the Weibull-G (W-G) family of distributions Bourguignon et al. (2014) with the cdf

$$F(x;\alpha,\xi)=1-\exp\left\{-\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\alpha}\right\},$$

for $\alpha>0$, and parameter vector ξ.

• If $\alpha=\delta=1$, the GOPGW-G reduces to the Odd Nadarajah-Haghighi family of distributions, with the cdf

$$F(x;\beta,\xi)=1-\exp\left\{1-\left[1+\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)\right]^{\beta}\right\},$$

for $\beta>0$, and parameter vector ξ.

• When $\beta=1$, the GOPGW-G reduces to the gamma odd Exponential-G family of distributions with the cdf

$$F(x;\delta,\alpha,\xi)=1-\frac{1}{\Gamma(\delta)}.\gamma\left(-\log\left(1-\exp\left\{-\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\alpha}\right\}\right),\delta\right),$$

for $\delta,\alpha>0$, and parameter vector ξ.

• When $\alpha=1$, we obtain gamma odd Nadarajah Haghighi-G (GONH-G) distributions with cdf given as

$$F(x;\delta,\beta,\xi)=1-\frac{1}{\Gamma(\delta)}.\gamma\left(-\log\left(1-\exp\left\{1-\left[1+\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)\right]^{\beta}\right\}\right),\delta\right).$$

• When $\beta = 1, \alpha = 2$, we have the gamma odd Rayleigh-G (GOR-G) distributions with the cdf given by

$$F(x;\delta,\xi) = 1 - \frac{1}{\Gamma(\delta)}.\gamma\left(-\log\left(1-\exp\left\{-\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^2\right\}\right),\delta\right).$$

• If $\delta = \beta = 1, \alpha = 2$, the GOPGW-G reduces to odd Rayleigh-G distributions with the cdf

$$F(x) = 1 - \exp\left\{-\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^2\right\}.$$

3 Statistical properties

In this section, expansion of the density function, quantile function, moments, incomplete moments, generating function, mean deviations, probability weighted moments, stochastic ordering, order statistics and Rényi entropy are presented.

3.1 Expansion of the GOPGW-G density function

In this section, we obtain the series expansion of the GOPGW-G family of distributions. the GOPGW-G family of distributions pdf is given by

$$\begin{aligned} f(x) &= \frac{1}{\Gamma(\delta)}\alpha\beta\sum_{i,j,k,l,p,r=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\binom{\delta-1}{m}b_{s,m}\binom{\beta-1}{j}\binom{\alpha(i+j+1)}{r} \\ &\quad\times\frac{(m+s+\delta)^k(-1)^{l+p+r}}{k!}\binom{k}{l}\binom{\beta l}{i}\binom{p-\alpha(i+j+1)+1}{r}[G(x;\xi)]^r g(x;\xi) \\ &= \sum_{r=0}^{\infty} w_{r+1}h_{r+1}(x;\xi), \end{aligned} \tag{3}$$

where

$$\begin{aligned} w_{r+1} &= \frac{1}{\Gamma(\delta)}\alpha\beta\sum_{i,j,k,l,p=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\binom{\delta-1}{m}b_{s,m}\binom{\beta-1}{j}\binom{\alpha(i+j+1)}{r} \\ &\quad\times\frac{(m+s+\delta)^k(-1)^{l+p+r}}{k!(r+1)}\binom{k}{l}\binom{\beta l}{i}\binom{p-\alpha(i+j+1)+1}{r}, \qquad (4) \\ h_{r+1}(x;\xi) &= (r+1)[G(x;\xi)]^r g(x;\xi). \end{aligned}$$

Equation (3) express the GOPGW-G density function as an infinite linear mixture of exponentiated-G (Exp-G) (Cordeiro et al. (2013)) densities, therefore, mathematical properties of the GOPGW-G family of distributions can be directly obtained from those of the Exp-G distribution. See Appendix for more details of the derivations.

3.2 Quantile function

The quantile function can be obtained by converting $F_{GOPGW-G}(x;\delta,\alpha,\beta,\xi)=u$, where $F_{GOPGW-G}(x;\delta,\alpha,\beta,\xi)$ is given in equation (1), and $0\leqslant u\leqslant 1$. Note that

$$F_{GOPGW-G}(x;\delta,\alpha,\beta,\xi)=1-\frac{\gamma(-\log\left(F_{OPGW-G}(x;\alpha,\beta,\xi)\right),\delta)}{\Gamma(\delta)}=u,$$

so that

$$\begin{aligned}\gamma(-\log\left(F_{OPGW-G}(x;\alpha,\beta,\xi)\right),\delta) &= (1-u)\Gamma(\delta),\\ 1-\exp\left\{1-\left[1+\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\alpha}\right]^{\beta}\right\} &= \exp\{-\gamma^{-1}((1-u)\Gamma(\delta),\delta)\}.\end{aligned}$$

That is,

$$\left[1+\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\alpha}\right]^{\beta}=1-\log\left(1-\exp\{-\gamma^{-1}((1-u)\Gamma(\delta),\delta)\}\right).$$

Furthermore,

$$\frac{G(x;\xi)}{\overline{G}(x;\xi)}=\left([1-\log(1-\exp\{-\gamma^{-1}((1-u)\Gamma(\delta),\delta)\})]^{\frac{1}{\beta}}-1\right)^{\frac{1}{\alpha}},$$

thus the quantile function of the GOPGW-G family of distributions reduces to

$$Q_{G(u;\delta,\alpha,\beta,\xi)}=G^{-1}\left(\frac{\left([1-\log(1-\exp\{-\gamma^{-1}((1-u)\Gamma(\delta),\delta)\})]^{\frac{1}{\beta}}-1\right)^{\frac{1}{\alpha}}}{1+\left([1-\log(1-\exp\{-\gamma^{-1}((1-u)\Gamma(\delta),\delta)\})]^{\frac{1}{\beta}}-1\right)^{\frac{1}{\alpha}}}\right). \quad (5)$$

3.3 Moments, incomplete moments and generating function

The q^{th} moment of the GOPGW-G family of distributions is obtained from equation (3) as

$$E(X^q)=\int_{-\infty}^{\infty}x^q f_{GOPGW-G}(x;\alpha,\beta,\delta,\xi)dx=\sum_{r=0}^{\infty}w_{r+1}E(Y_{r+1}^q), \quad (6)$$

where $E(Y_{r+1}^q)$ is the q^{th} moment of Y_{r+1} which follows the Exp-G distribution with power parameter $r+1$ and w_{r+1} is defined in (4). The incomplete moment and the moment generating function (mgf) are given by

$$\begin{aligned}I_X(t) &= \int_{-\infty}^{t}x^q f_{GOPGW-G}(x;\alpha,\beta,\delta,\xi)dx=\sum_{r=0}^{\infty}w_{r+1}I_{r+1}(t),\\ M_X(t) &= E\left(e^{tX}\right)=\sum_{r=0}^{\infty}w_{r+1}E\left(e^{tY_{r+1}}\right),\end{aligned}$$

where $I_{r+1}(t)=\int_{-\infty}^{t}x^q(r+1)[G(x;\xi)]^r g(x;\xi)dx$ and $E\left(e^{tY_{r+1}}\right)$ is the mgf of Exp-G family of distributions with power parameter $r+1$ and w_{r+1} is defined in equation (4).

3.4 Probability weighted moments

The probability weighted moments (PWMs) of the GOPGW-G family of distributions are presented in this subsection. The PWMs were introduced by Greenwood et al. (1979) which can be an alternative method for parameter estimation of probability distributions. From equations (1) and (2), we have

$$\begin{aligned} f(x)(F(x))^n &= \frac{1}{\Gamma(\delta)}\beta\alpha\left\{-\log\left(1-\exp\left\{1-\left[1+\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\alpha}\right]^{\beta}\right\}\right)\right\}^{\delta-1} \\ &\times\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\alpha-1}\left[1+\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\alpha}\right]^{\beta-1} \\ &\times\exp\left\{1-\left[1+\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\alpha}\right]^{\beta}\right\}\frac{g(x;\xi)}{(\overline{G}(x;\xi))^2} \\ &\times\left[1-\frac{1}{\Gamma(\delta)}\gamma\left(-\log\left(1-\exp\left\{1-\left[1+\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\alpha}\right]^{\beta}\right\}\right),\delta\right)\right]^{n}, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} f(x)(F(x))^n &= \beta\alpha\sum_{z,s,m,p=0}^{\infty}\binom{\delta+m-1}{z}b_{s,z}d_{m,p}\frac{(-1)^p}{(\Gamma(\delta))^{p+1}}\binom{n}{p} \\ &\times\sum_{k,l,i,j,r,q=0}^{\infty}\frac{(z+s+\delta+m)^k(-1)^{l+j+q+r}}{k!}\binom{k}{l}\binom{\beta l}{i}\binom{\beta-1}{j} \\ &\times\binom{\alpha(j+i+1)-1}{q}\binom{q-\alpha(j+i+1)-1}{r}[G(x;\xi)]^r g(x;\xi) \\ &= \sum_{r=0}^{\infty}v_{r+1}h_{r+1}(x;\xi), \end{aligned}$$

where

$$\begin{aligned} v_{r+1} &= \beta\alpha\sum_{z,s,m,p=0}^{\infty}\binom{\delta+m-1}{z}b_{s,z}d_{m,p}\frac{(-1)^p}{(\Gamma(\delta))^{p+1}}\binom{n}{p} \\ &\times\sum_{k,l,i,j,q=0}^{\infty}\frac{(z+s+\delta+m)^k(-1)^{l+j+q+r}}{k!}\binom{k}{l}\binom{\beta l}{i}\binom{\beta-1}{j} \\ &\times\binom{\alpha(j+i+1)-1}{q}\binom{q-\alpha(j+i+1)-1}{r}, \end{aligned}$$

and $h_{r+1}(x;\xi)=(r+1)\left(G(x;\xi)\right)^r g(x;\xi)$ is the Exp-G density with the power parameter $r+1$ and parameter vector ξ. Consequently the PWM's of the GOPGW-G family of distributions is given by

$$\eta_{a,n}=\int_0^{\infty}x^a\sum_{r=0}^{\infty}v_{r+1}h_{r+1}(x;\xi)dx=\sum_{r=0}^{\infty}v_{r+1}\int_0^{\infty}x^a h_{r+1}(x;\xi)dx.$$

See Appendix for more details of the derivations.

3.5 Stochastic ordering

In this subsection, the stochastic orders for the GOPGW-G family of distribution are presented. Suppose we have two random variables X and Y with distribution functions $F_X(r)$ and $F_Y(r)$, respectively, and $\overline{F}_X(r) = 1 - F_X(r)$ the survival function, X is stochastically smaller than Y if $\overline{F}_X(r) \leq \overline{F}_Y(r)$ for all r or $F_X(r) \geq F_Y(r)$ for all r. This is denoted by $X <_s Y$. Hazard rate order and likelihood ratio order are stronger and are given by $X <_{hr} Y$ if $h_X(r) \geq h_Y(r)$ for all r, and $X <_{lr} Y$ if $\frac{f_X(r)}{f_Y(r)}$ is decreasing in r, (Shaked and Shanthikumar (2007)). We know that $X <_{lr} Y \Rightarrow X <_{hr} Y \Rightarrow X <_s Y$.

If we let X_1 and X_2 be the two independent random variables following $GOPGW-G(\alpha,\beta,\delta_1,\xi)$ and $GOPGW-G(\alpha,\beta,\delta_2,\xi)$ distributions, then

$$\begin{aligned} f_i(x) &= \frac{1}{\Gamma(\delta_i)}\beta\alpha\left\{-\log\left(1-\exp\left\{1-\left[1+\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\alpha}\right]^{\beta}\right\}\right)\right\}^{\delta_i-1} \\ &\times\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\alpha-1}\left[1+\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\alpha}\right]^{\beta-1} \\ &\times\exp\left\{1-\left[1+\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\alpha}\right]^{\beta}\right\}\frac{g(x;\xi)}{(\overline{G}(x;\xi))^2}, \quad \text{for} \quad i=1,2. \end{aligned}$$

The ratio, $\frac{f_1(x;\alpha,\beta,\delta_1,\xi)}{f_2(x;\alpha,\beta,\delta_2,\xi)}$ takes the form

$$\frac{f_1(x;\alpha,\beta,\delta_1,\xi)}{f_2(x;\alpha,\beta,\delta_2,\xi)} = \frac{\Gamma(\delta_2)}{\Gamma(\delta_1)}\left\{-\log\left(1-\exp\left\{1-\left[1+\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\alpha}\right]^{\beta}\right\}\right)\right\}^{\delta_1-\delta_2}. \quad (7)$$

If we differentiate equation (7) with respect to x, we get

$$\begin{aligned} \frac{d}{dx}\left(\frac{f_1(x;\alpha,\beta,\delta_1,\xi)}{f_2(x;\alpha,\beta,\delta_2,\xi)}\right) &= \frac{\Gamma(\delta_2)}{\Gamma(\delta_1)}(\delta_1-\delta_2)\alpha\beta \\ &\times\left\{-\log\left(1-\exp\left\{1-\left[1+\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\alpha}\right]^{\beta}\right\}\right)\right\}^{\delta_1-\delta_2-1} \\ &\times\left(1-\exp\left\{1-\left[1+\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\alpha}\right]^{\beta}\right\}\right)^{-1} \\ &\times\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\alpha-1}\left[1+\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\alpha}\right]^{\beta-1} \\ &\times\exp\left\{1-\left[1+\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\alpha}\right]^{\beta}\right\}\frac{g(x;\xi)}{(\overline{G}(x;\xi))^2}, \end{aligned}$$

which is ≤ 0 if $\delta_1 \leq \delta_2$. Therefore, $X_1 <_{lr} X_2, X_1 <_{hr} X_2$ and $X_1 <_s X_2$ and the random variables X_1 and X_2 are stochastically ordered.

3.6 Order statistics and Rényi entropy

In this subsection, we present the pdf of the i^{th} order statistic and the Rényi entropy for the GOPGW-G family of distributions.

3.6.1 Order statistics

The pdf of the i^{th} order statistic of a distribution with cdf $F(x)$ and pdf $f(x)$ is defined as

$$\begin{aligned} f_{i:n}(x) &= \frac{n!f(x)}{(i-1)!(n-i)!}[F(x)]^{i-1}[1-F(x)]^{n-i} \\ &= \frac{n!f(x)}{(i-1)!(n-i)!}\sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^j[1-F(x)]^{n-i+j}. \end{aligned}$$

Thus, the pdf of the i^{th} order statistic of the GOPGW-G family of distributions is given by

$$\begin{aligned} f_{i:n}(x) = &\frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^j b_{m,n-i+j}\frac{\Gamma(\delta(n-i+j)+\delta+m)}{\Gamma(\delta))^{n-i+j}} \\ &\times\frac{1}{\Gamma(\delta)}\alpha\beta\sum_{i,j,k,l,p,r=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\binom{\delta-1}{m}\frac{(-1)^{j+l}\Gamma(1-\beta+j)}{\Gamma(1-\beta)\Gamma(j+1)k!}\binom{\alpha(i+j+1)}{r} \\ &\times(m+s+\delta)^k\binom{k}{l}\binom{\beta l}{i}\sum_{r=0}^{\infty}\frac{\Gamma(p-\alpha(i+j+1)+r-1)}{r!\Gamma(p-\alpha(i+j+1)-1)(r+1)}[G(x;\xi)]^r g(x;\xi) \\ = &\sum_{r=0}^{\infty}d_{r+1}h_{r+1}(x;\xi), \end{aligned}$$

where

$$\begin{aligned} d_{r+1} = &\frac{n!}{(i-1)!(n-i)!}\sum_{j=1}^{i-1}\binom{i-1}{j}(-1)^j b_{m,n-i+j}\frac{\Gamma(\delta(n-i+j)+\delta+m)}{\Gamma(\delta))^{n-i+j}} \\ &\times\frac{1}{\Gamma(\delta)}\alpha\beta\sum_{i,j,k,l,p=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\binom{\delta-1}{m}\frac{(-1)^{j+l}\Gamma(1-\beta+j)}{\Gamma(1-\beta)\Gamma(j+1)k!}\binom{\alpha(i+j+1)}{r} \\ &\times(m+s+\delta)^k\binom{k}{l}\binom{\beta l}{i}\sum_{r=0}^{\infty}\frac{\Gamma(p-\alpha(i+j+1)+r-1)}{r!\Gamma(p-\alpha(i+j+1)-1)(r+1)}, \end{aligned}$$

and $h_{r+1}(x;\xi) = (r+1)[G(x;\xi)]^r g(x;\xi)$. The pdf of the i^{th} order statistics can be expressed as a linear combination of the Exp-G densities with a power parameter $r+1$ and parameter vector ξ. More derivations are given in the Appendix.

3.6.2 Rényi entropy

Rényi entropy (Rényi (1960)) for the GOPGW-G family of distributions is given by

$$\begin{aligned} I_R(\nu) &= \frac{1}{1-\nu}\log\left[\frac{1}{(\Gamma(\delta))^{\nu}}\alpha^{\nu}\beta^{\nu}\sum_{s=0}^{\infty}\sum_{m=0}^{\infty}\binom{\nu(\delta-1)}{m}b_{s,m}\right.\\ &\times\sum_{i,j,k,l,r,p=0}^{\infty}\frac{(-1)^{j+l}\Gamma((1-\beta)\nu+j)}{\Gamma((1-\beta)\nu)\Gamma(j+1)k!}\binom{\alpha(i+j+\nu)-\nu}{p}\\ &\times\frac{\Gamma(p-\alpha(i+j+\nu)+r-\nu)}{r!\Gamma(p-\alpha(i+j+\nu)-\nu)}(s+m+\nu\delta)^{k}\binom{k}{l}\binom{\beta l}{i}\left(\frac{1}{\frac{r}{\nu}+1}\right)\\ &\times\left.\int_0^{\infty}\left[\left(\frac{r}{\nu}+1\right)[G(x;\xi)]^{\frac{r}{\nu}}g(x)\right]^{\nu}dx\right]\\ &= \frac{1}{1-\nu}\log\left[\sum_{r=0}^{\infty}w_r^*(1-\nu)I_{REG}\right], \end{aligned}$$

for $\nu>0, \nu\neq 1$, where $I_{REG}=\frac{1}{1-\nu}\log\int_0^{\infty}\left[\left(\frac{r}{\nu}+1\right)[G(x;\xi)]^{\frac{r}{\nu}}g(x)\right]^{\nu}dx\Big]$, is the Rényi entropy of the Exp-G distributions with power parameter $\frac{r}{\nu}+1$, and

$$\begin{aligned} w_r^* &= \frac{1}{(\Gamma(\delta))^{\nu}}\alpha^{\nu}\beta^{\nu}\sum_{s=0}^{\infty}\sum_{m=0}^{\infty}\binom{\nu(\delta-1)}{m}b_{s,m}\\ &\times\sum_{i,j,k,l,p=0}^{\infty}\frac{(-1)^{j+l}\Gamma((1-\beta)\nu+j)}{\Gamma((1-\beta)\nu)\Gamma(j+1)k!}\binom{\alpha(i+j+\nu)-\nu}{p}\\ &\times\frac{\Gamma(p-\alpha(i+j+\nu)+r-\nu)}{r!\Gamma(p-\alpha(i+j+\nu)-\nu)}(s+m+\nu\delta)^{k}\binom{k}{l}\binom{\beta l}{i}\left(\frac{1}{\frac{r}{\nu}+1}\right). \end{aligned}$$

Derivations are given in the Appendix section.

4 Maximum likelihood estimation

In this section, the maximum likelihood technique to estimate the model parameters for the GOPGW-G family of distributions is presented. Let $X\sim GOPGW-G(\alpha,\beta,\delta,\xi)$ and $\Theta=(\alpha,\beta,\delta,\xi)^T$ be the parameter vector, then the log-likelihood function (ℓ_n) of a random sample of size n from the GOPGW-G family of distributions is given by

$$\begin{aligned} \ell_n(\Theta) &= n\log\beta+n\log\alpha-n\log(\Gamma(\delta))\\ &+(\delta-1)\sum_{i=1}^{n}\ln\left[-\log\left(1-\exp\left\{1-\left[1+\left(\frac{G(x_i;\xi)}{\overline{G}(x_i;\xi)}\right)^{\alpha}\right]^{\beta}\right\}\right)\right]\\ &+\sum_{i=1}^{n}\ln\left[\exp\left\{1-\left[1+\left(\frac{G(x_i;\xi)}{\overline{G}(x_i;\xi)}\right)^{\alpha}\right]^{\beta}\right\}\right]\\ &+(\beta-1)\sum_{i=1}^{n}\ln\left[1+\left(\frac{G(x_i;\xi)}{\overline{G}(x_i;\xi)}\right)^{\alpha}\right]+(\alpha-1)\sum_{i=1}^{n}\ln\left[\left(\frac{G(x_i;\xi)}{\overline{G}(x_i;\xi)}\right)^{\alpha}\right] \end{aligned}$$

$$+\sum_{i=1}^{n}\ln[g(x_i;\xi)]-2\sum_{i=1}^{n}\ln[1-G(x_i;\xi)].$$

Elements of the score vector $U(\mathbf{\Theta})=\left(\frac{\partial\ell_{\mathbf{n}}}{\partial\beta},\frac{\partial\ell_{\mathbf{n}}}{\partial\alpha},\frac{\partial\ell_{\mathbf{n}}}{\partial\delta},\frac{\partial\ell_{\mathbf{n}}}{\partial\xi_{\mathbf{k}}}\right)$ can be easily obtained. The maximum likelihood estimates (mle's) can be obtained by equating a system of equations $\frac{\partial\ell_n}{\partial\beta},\frac{\partial\ell_n}{\partial\alpha},\frac{\partial\ell_n}{\partial\delta}$ and $\frac{\partial\ell_n}{\partial\xi_k}$ to zero and solve them simultaneously.

The Fisher information matrix (FIM) of the GOPGW-G family of distributions is given by $\boldsymbol{I}(\Theta)=[\boldsymbol{I}_{\theta_i,\theta_j}]_{(3+q)\times(3+q)}=E\left(-\frac{\partial^2\ell}{\partial\theta_i\partial\theta_j}\right),i,j=1,2,...,(3+q)$, can be numerically obtained by mle2 package in R software or MATLAB. The expectations in the FIM can be numerically obtained and the total Fisher information matrix $n\boldsymbol{I}(\Delta)$ can be approximated by

$$\boldsymbol{J}_n(\hat{\Theta})\approx\left[-\frac{\partial^2\ell}{\partial\theta_i\partial\theta_j}\bigg|_{\Theta=\hat{\Theta}}\right]_{(3+q)\times(3+q)},i,j=1,2,....,(3+q).$$

Elements of the score vector $U(\mathbf{\Theta})$, are given in the Appendix section.

5 Some special models

In this section we present some of the special models of the GOPGW-G family of distributions. We considered the case where the baseline cdf G are the Burr XII, Weibull and Lomax distributions, respectively.

5.1 The gamma odd power generalized Weibull-Burr XII distribution

Suppose the baseline distribution is Burr XII with the cdf and pdf given by $G(x;c,k)=1-(1+x^c)^{-k}$ and $g(x;c,k)=kcx^{c-1}(1+x^c)^{-k-1}$, for $c,k,x>0$. When $c=1$ we obtain the Lomax (Lx) distribution. The cdf and pdf of the of the GOPGW-BXII distribution are given by

$$\begin{aligned}F(x;\alpha,\beta,\delta,c,k)&=1-\frac{\gamma\left(-\log\left(1-\exp\left\{1-\left[1+\left((1+x^c)^k-1\right)^{\alpha}\right]^{\beta}\right\}\right),\delta\right)}{\Gamma(\delta)},\\f(x;\alpha,\beta,\delta,\lambda)&=\frac{1}{\Gamma(\delta)}\beta\alpha\left\{-\log\left(1-\exp\left\{1-\left[1+\left((1+x^c)^k-1\right)^{\alpha}\right]^{\beta}\right\}\right)\right\}^{\delta-1}\\&\times\left((1+x^c)^k-1\right)^{\alpha-1}\left[1+\left((1+x^c)^k-1\right)^{\alpha}\right]^{\beta-1}\\&\times\exp\left\{1-\left[1+\left((1+x^c)^k-1\right)^{\alpha}\right]^{\beta}\right\}\frac{kcx^{c-1}(1+x^c)^{-k-1}}{((1+x^c)^{-k})^2},\end{aligned}$$

respectively, for $\alpha,\beta,\delta,c,k>0$.

The pdf of the GOPGW-BXII distribution can be right skewed, left skewed, reverse-J shape and decreasing shapes whereas the hazard rate function displays increasing, decreasing, bathtub, and upside down bathtub shapes. Quantiles for selected parameters values of the GOPGW-BXII distribution are given in Table 1.

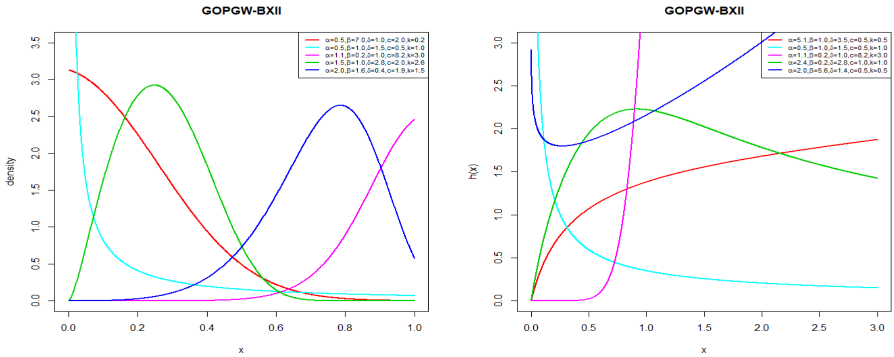

Figure 1: Plots of the pdf and hazard functions for the GOPGW-BXII Distribution.

Table 1: Some quantiles for GOPGW-BXII distribution.

	$(\alpha,\beta,\delta,c,k)$				
u	$(1, 1.5, 1.3, 0.5, 1.2)$	$(0.7, 1, 1.5, 1.5, 2.1)$	$(0.4, 1, 1, 1.3, 2)$	$(2.1, 0.5, 1.9, 0.4, 1.9)$	$(1.5, 1, 1.2, 2, 1.6)$
0.1	0.0052	0.0707	0.0078	0.0074	0.3988
0.2	0.0127	0.1097	0.0326	0.0168	0.4678
0.3	0.0253	0.1541	0.0797	0.0296	0.5263
0.4	0.0450	0.2051	0.1559	0.0466	0.5797
0.5	0.0748	0.2643	0.2711	0.0696	0.6313
0.6	0.1199	0.3348	0.4391	0.1015	0.6834
0.7	0.1909	0.4218	0.6833	0.1479	0.7389
0.8	0.3124	0.5371	1.0536	0.2232	0.8033
0.9	0.5676	0.7156	1.7093	0.3759	0.8899

5.1.1 Some sub-models of the GOPGW-BXII distribution

• If we let $\delta = 1$, we obtain the odd power generalized Burr XII (OPGW-BXII) distribution with the cdf given by

$$F(x;\alpha,\beta,c,k) = 1 - \exp\left\{1 - \left[1 + \left((1+x^c)^k - 1\right)^{\alpha}\right]^{\beta}\right\}.$$

• When $\delta = \beta = 1$, we get the Odd Weibull-Burr XII distribution (OW-BXII) with the cdf given as follows

$$F(x;\alpha,c,k) = 1 - \exp\left\{\left((1+x^c)^k - 1\right)^{\alpha}\right\}.$$

• By letting $\beta = 1$ and $\alpha = 2$ we obtain the gamma odd Rayleigh-Burr XII (GOR-BXII) distribution with the pdf given by

$$F(x;\delta,c,k) = 1 - \frac{\gamma\left(-\log\left(1 - \exp\left\{-\left((1+x^c)^k - 1\right)^2\right\}\right), \delta\right)}{\Gamma(\delta)}.$$

5.1.2 Moments

We cam obtain the q^{th} moment of the GOPGW-BXII from equation (6) as

$$E(X^q) = \int_{-\infty}^{\infty} x^q f(x)dx = \sum_{r=0}^{\infty} w_{r+1} E(Y_{r+1}^q),$$

where $E(Y_{r+1}^{q})$ is the q^{th} moment of Y_{r+1} which follows the Exp-BXII distribution with power parameter $r+1$ and w_{r+1} is defined in equation (4). The first six moments, standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) for selected parameter values of the GOPGW-BXII distribution are given in Table 2. Figure 1 displays the plots of skewness and kurtosis for the GOPGW-BXII distribution. The plots show that the skewness can either be positive and negative while kurtosis shows that GOPGW-BXII distribution is highly flexible that it can model data for different levels of skewness and kurtosis.

Table 2: Moments for selected parameters for GOPGW-BXII distribution.

	$(\alpha,\beta,\delta,c,k)$				
	(1,1.5,1.3,0.5,1.2)	(0.7,1,1.5,1.5,3)	(0.7,2,0.5,1.3,0.6)	(0.4,2.5,0.9,0.7,0.4)	(1.2,2.5,0.5,0.4,1.2)
EX	0.1345	0.2271	0.2393	0.1022	0.1856
EX^2	0.0568	0.0896	0.1523	0.0476	0.0794
EX^3	0.0338	0.0458	0.1107	0.0305	0.0461
EX^4	0.0235	0.0274	0.0867	0.0223	0.0313
EX^5	0.0179	0.0182	0.0711	0.0175	0.0233
EX^6	0.0144	0.0129	0.0602	0.0144	0.0184
SD	0.1967	0.1950	0.3082	0.1928	0.2120
CV	1.4633	0.8587	1.2877	1.8874	1.1422
CS	2.0669	1.1062	0.9835	2.5095	1.5389
CK	7.0425	3.7975	2.5665	9.0339	4.9102

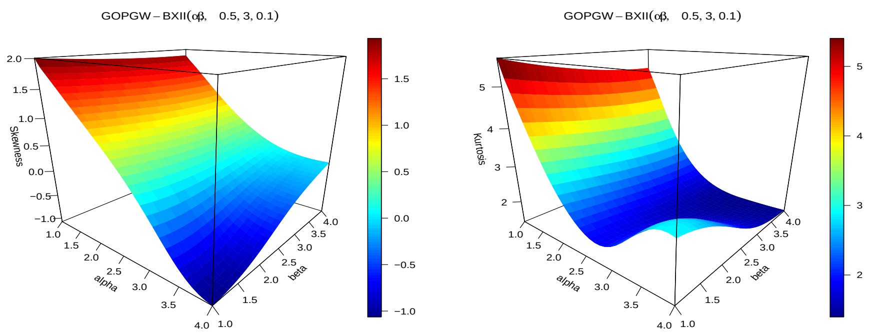

Figure 2: Skewness and kurtosis for the GOPGW-BXII distribution.

5.2 Gamma odd power generalized Weibull-Weibull distribution

Let the baseline distribution be the Weibull distribution with the cdf and pdf given by $G(x;\lambda) = 1 - e^{-x^{\lambda}}$ and $g(x;\lambda) = \lambda x^{\lambda-1}e^{-x^{\lambda}}$, for $x > 0$ and $\lambda > 0$. The cdf and the pdf of the gamma odd power generalized Weibull-Weibull (GOPGW-W) distribution are given as

$$F(x;\alpha,\beta,\delta,\lambda) = 1 - \frac{\gamma\left(-\log\left(1-\exp\left\{1-\left[1+\left(e^{x^{\lambda}}-1\right)^{\alpha}\right]^{\beta}\right\}\right),\delta\right)}{\Gamma(\delta)},$$

$$
\begin{aligned}
f(x;\alpha,\beta,\delta,\lambda) &= \frac{1}{\Gamma(\delta)}\beta\alpha\left\{-\log\left(1-\exp\left\{1-\left[1+\left(e^{x^{\lambda}}-1\right)^{\alpha}\right]^{\beta}\right\}\right)\right\}^{\delta-1}\\
&\quad\times\left(e^{x^{\lambda}}-1\right)^{\alpha-1}\left[1+\left(e^{x^{\lambda}}-1\right)^{\alpha}\right]^{\beta-1}\\
&\quad\times\exp\left\{1-\left[1+\left(e^{x^{\lambda}}-1\right)^{\alpha}\right]^{\beta}\right\}\frac{\lambda x^{\lambda-1}e^{-x^{\lambda}}}{(e^{-x^{\lambda}})^{2}},
\end{aligned}
$$

respectively, for $\alpha, \beta, \delta, \lambda > 0$.

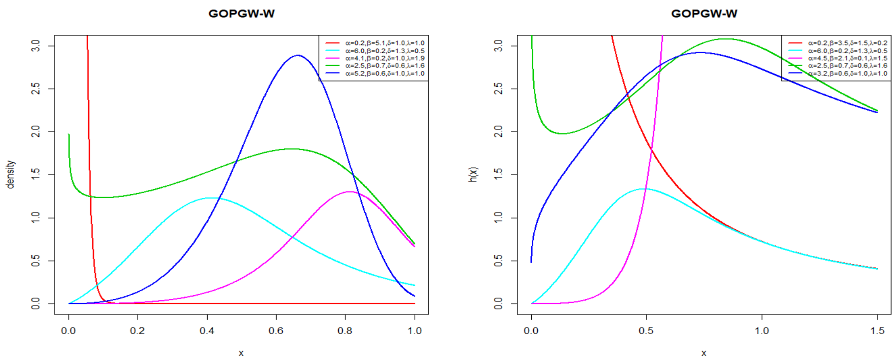

Figure 3: The density and hazard functions for the GOPGW-W distribution.

Figure 3 shows that the pdf of the GOPGW-W distribution can take several shapes including right skewed, left skewed, uni-modal, decreasing and bathtub shapes whereas the hazard rate function display increasing, decreasing, and bathtub followed by upside down bathtub, and upside down bathtub shapes. Table 3 gives the quantiles of the GOPGW-W distribution for selected parameter values. The plots of skewness and kurtosis for the GOPGW-W distribution are given in Figure 4. From the plots, we see that GOPGW-W distribution can model data for different levels of skewness and kurtosis.

Table 3: Quantiles for GOPGW-W distribution.

	$(\alpha,\beta,\delta,c,\lambda)$				
u	$(1,1.5,1.3,0.5)$	$(0.7,1,1.5,1.5)$	$(0.4,1,1,1.3)$	$(2.1,0.5,1.9,0.4)$	$(1.5,1,1.2,2)$
0.1	0.0069	0.1153	0.0132	0.0312	0.4859
0.2	0.0165	0.1778	0.0554	0.0666	0.5627
0.3	0.0314	0.2479	0.1339	0.1114	0.6255
0.4	0.0533	0.3264	0.2570	0.1671	0.6809
0.5	0.0841	0.4153	0.4326	0.2372	0.7325
0.6	0.1273	0.5171	0.6663	0.3275	0.7830
0.7	0.1893	0.6371	0.9628	0.4497	0.8350
0.8	0.2839	0.7861	1.3347	0.6301	0.8926
0.9	0.4541	0.9961	1.8354	0.9541	0.9660

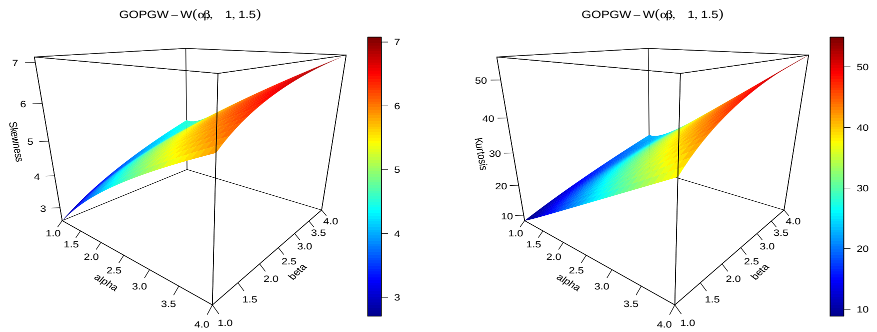

Figure 4: Skewness and kurtosis for the GOPGW-W distribution.

6 Simulation

In this section, a simulation study is carried out to examine the performance of the mean, root mean square error (RMSE) and average bias (ABIAS) of maximum likelihood estimators of the parameters of the gamma odd power generalized Weibull-Lomax (GOPGW-Lx) and GOPGW-W distributions. The experiment was conducted through various simulations for different sample sizes and different parameter values. The quantile function given in equation (5) was used to generate random samples from GOPGW-Lx and GOPGW-W distributions via the R package. The simulation was repeated for $N = 3000$ times each with sample sizes $n = 50, 100, 200, 400, 800, 1600$ and parameter values as given in Table 4. Specifically, the estimated mean, average bias and root mean square error of the parameter say $\xi = \alpha, \beta, \delta, k$ are computed as:

Mean$=\frac{\sum_{i=1}^{N}\hat{\xi}_i}{N}$, RMSE$=\sqrt{\frac{\sum_{i=1}^{N}(\hat{\xi}_i-\xi)^2}{N}}$ and ABIAS $(\hat{\xi}) = \frac{\sum_{i=1}^{N}(\hat{\xi}_i-\xi)}{N}$.

The tables list the mean MLEs of the parameters along with the respective root mean squared errors (RMSEs) and ABIAS. From the results in Table 4 and Table 5, we can see that as the sample size n increases, the mean estimates of the parameters tend to be closer to the true parameter values while the RMSEs and average bias decrease.

7 Applications

In this section, the GOPGW-Lx and GOPGW-W distributions are fitted to data sets and these fits are compared to the fits of the models, including type II general inverse exponential Lomax (TIIGIE-Lx) distribution by Hamedani et al. (2019), exponentiated modified Weibull (EMW) distribution by Elbatal (2011), the odd exponentiated half logistic-Burr XII (OEHL-BXII) Aldahlan et al. (2018), and Log-Logistic Weibull (LLW) distribution Oluyede et al. (2016).
The pdf's of the above distributions are given by

$$f_{TIIGIE-Lx}(x;\lambda,\alpha,a,b) = \lambda\alpha\frac{a}{b}\left(1+\frac{x}{b}\right)^{-(a+1)}\left(1+\frac{x}{b}\right)^{a(\alpha+1)} \times \exp\left(\lambda\left(1-\left(1+\frac{x}{b}\right)^{a\alpha}\right)\right),$$

Table 4: Simulation results for GOPGW-Lx distribution: Mean, ABIAS and RMSE.

		(1.0, 1.0, 0.4, 1.5)			(2.0, 0.4, 0.8, 0.5)		
Parameter	n	Mean	RMSE	ABIAS	Mean	RMSE	ABIAS
α	50	1.5612	1.9211	0.5612	2.8382	2.4264	0.8382
	100	1.2858	0.7639	0.2858	2.5851	1.0930	0.5851
	200	1.1421	0.3711	0.1421	2.4149	0.8278	0.4149
	400	1.0699	0.2348	0.0699	2.2408	0.6246	0.2408
	800	1.0276	0.1563	0.0276	2.1537	0.4647	0.1537
	1600	1.0155	0.1080	0.0155	2.1029	0.3624	0.1029
β	50	1.7004	2.8552	0.7004	0.5973	0.6714	0.1973
	100	1.3307	1.6846	0.3307	0.5541	0.5827	0.1541
	200	1.2889	1.3786	0.2889	0.4822	0.4582	0.0822
	400	1.1360	0.8647	0.1360	0.4695	0.3987	0.0695
	800	1.0455	0.5027	0.0455	0.4276	0.2697	0.0276
	1600	1.0347	0.3631	0.0347	0.4153	0.1797	0.0153
δ	50	0.8407	0.9589	0.4407	1.4829	1.3475	0.6829
	100	0.7578	0.8329	0.3578	1.4235	1.2738	0.6235
	200	0.6320	0.5901	0.2320	1.2955	1.0176	0.4955
	400	0.5172	0.3610	0.1172	1.0974	0.7641	0.2974
	800	0.4507	0.2259	0.0507	0.9954	0.5678	0.1954
	1600	0.4320	0.1495	0.0320	0.9357	0.4441	0.1357
k	50	2.0623	2.1385	0.5623	0.4195	0.3648	-0.0805
	100	2.0962	2.4326	0.5962	0.4332	0.3259	-0.0668
	200	1.9169	1.8017	0.4169	0.4290	0.2464	-0.0709
	400	1.8118	1.3710	0.3118	0.4574	0.2145	-0.0426
	800	1.7004	0.9556	0.2004	0.4721	0.1752	-0.0279
	1600	1.5729	0.6500	0.0729	0.4801	0.1306	-0.0199

for $x > 0, \lambda, \alpha, a, b > 0$,

$$f_{EMW}(x; \gamma, \delta, \lambda, \theta) = \gamma[\delta + \lambda\theta^{\lambda}x^{\lambda-1}]\exp(-\delta x + (\theta x)^{\lambda})[1 - \exp(-(\delta x + (\theta x)^{\lambda}))]^{\gamma-1},$$

for $x > 0, \gamma, \delta, \lambda, \theta > 0$,

$$f_{OEHLBXII}(x) = \frac{2\alpha\lambda abx^{a-1}\exp(\lambda[1-(1+x^a)^b])(1-\exp(\lambda[1-(1+x^a)^b]))^{\alpha-1}}{(1+x^a)^{-b-1}(1+\exp(\lambda[1-(1+x^a)^b]))^{\alpha+1}},$$

for $x > 0, \alpha, \lambda, a, b > 0$, and

$$f_{LLW}(x; c, \alpha, \beta) = (1 + x^c)^{-1}e^{-\alpha x^{\beta}}[(1 + x^c)^{-1}cx^{c-1} + \alpha\beta x^{\beta-1}],$$

for $x > 0, c, \alpha, \beta > 0$.

Plots of the fitted densities, the histogram of the data and probability plots Chambers et al. (1983) are given in Figures 5 and 6. For the probability plot, we plotted $F(x_{(j)}; \hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\xi})$ against $\frac{j - 0.375}{n + 0.25}$, $j = 1, 2, \cdots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measure of closeness is given by the sum of squares

$$SS = \sum_{j=1}^{n}\left[F(x_{(j)}; \hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\xi}) - \left(\frac{j - 0.375}{n + 0.25}\right)\right]^2.$$

The goodness-of-fit statistics W^* and A^*, described by Chen and Balakrishnan (1995) and Kolmogorov-Smirnov (KS) statistic are also presented in the tables. In general, the smaller the values of W^* and A^* and KS the better the fit.

Table 5: Simulation results for GOPGW-W distribution: Mean, ABIAS and RMSE

		(1.5, 1.5, 0.4, 0.7)			(0.4, 0.4, 2.0, 3.0)		
Parameter	n	Mean	RMSE	ABIAS	Mean	RMSE	ABIAS
α	50	1.1831	2.7143	-0.3169	0.3627	0.1381	-0.0373
	100	1.1551	0.8479	-0.3449	0.3664	0.0897	-0.0336
	200	1.2221	0.6046	-0.2779	0.3788	0.0625	-0.0212
	400	1.2991	0.4519	-0.2009	0.3850	0.0440	-0.0149
	800	1.3644	0.3362	-0.1356	0.3915	0.0316	-0.0085
	1600	1.4204	0.2551	-0.0796	0.3943	0.0219	-0.0057
β	50	1.1311	0.8449	-0.3689	0.5759	0.5952	0.1759
	100	1.1284	0.7907	-0.3716	0.5388	0.4843	0.1388
	200	1.1415	0.6857	-0.3585	0.4623	0.3439	0.0623
	400	1.2054	0.5612	-0.2946	0.4279	0.2425	0.0278
	800	1.2265	0.4827	-0.2735	0.4099	0.1836	0.0099
	1600	1.2942	0.3799	-0.2057	0.3943	0.1222	-0.0057
δ	50	1.3804	1.6304	0.9804	2.2178	1.3855	0.2178
	100	1.2722	1.4396	0.8722	2.1277	1.1768	0.1277
	200	1.0719	1.0969	0.67192	2.1568	1.0014	0.1568
	400	0.8606	0.7724	0.4606	2.1119	0.7733	0.1119
	800	0.7534	0.5741	0.3534	2.0743	0.5745	0.0743
	1600	0.6289	0.3899	0.2289	2.0625	0.3924	0.0625
λ	50	2.5779	2.9130	1.8779	3.4001	1.1303	0.4001
	100	1.9379	1.9606	1.2379	3.2186	0.8905	0.2186
	200	1.4339	1.1852	0.7339	3.1544	0.7369	0.1544
	400	1.1325	0.7325	0.4325	3.1073	0.5811	0.1073
	800	0.9776	0.4649	0.2776	3.0641	0.4250	0.0641
	1600	0.8633	0.2979	0.1633	3.0502	0.2947	0.0502

7.1 Kevlar 49/epoxy strands failure data

The first data set consists of 101 data points of the stress-rupture life of kevlar 49/epoxy strands which were subjected to constant sustained pressure at the 90% stress level until all had failed, so that we have complete data with exact times of failure Andrews and Herzberg (1985); Barlow et al. (1984). The failure times in hours are:
0.01,0.01,0.02,0.02,0.02,0.03,0.03,0.04,0.05,0.06,0.07,0.07,0.08,0.09,0.09,0.10,0.10,0.11, 0.11,0.12,0.13,0.18,0.19,0.20,0.23,0.24,0.24,0.29,0.34,0.35,0.36,0.38,0.40,0.42,0.43,0.52, 0.54,0.56,0.60,0.60,0.63,0.65,0.67,0.68,0.72,0.72,0.72,0.73,0.79,0.79,0.80,0.80,0.83,0.85, 0.90,0.92,0.95,0.99,1.00,1.01,1.02,1.03,1.05,1.10,1.10,1.11,1.15,1.18,1.20,1.29,1.31,1.33, 1.34,1.40,1.43,1.45,1.50,1.51,1.52,1.53,1.54,1.54,1.55,1.58,1.60,1.63,1.64,1.80,1.80,1.81, 2.02,2.05,2.14,2.17,2.33,3.03,3.03,3.34,4.20,4.69,7.89.
The estimated variance-covariance matrix for GOPGW-Lx model of Kevlar data is given by

$$\begin{bmatrix} 0.0236 & -0.0041 & 0.0176 & -0.1353 \\ -0.0041 & 0.0368 & -0.0297 & -0.5979 \\ 0.0176 & -0.0297 & 0.0568 & 0.2728 \\ -0.1353 & -0.5979 & 0.2728 & 11.8747 \end{bmatrix}.$$

From Table 6 and 7, the values of the goodness-of-fit statistics: AIC, AICC and BIC for the GOPGW-Lx and GOPGW-W distributions are the smallest as compared to TIIGIE-Lx, EMW, OEHL-BXII, OEHL-Lx, LLW, Weibull and Lomax distributions hence the GOPGW-Lx and GOPGW-W distributions are better in fitting for Epoxy

Table 6: Estimates of models for Kevlar 49/epoxy strands failure at 90% stress level data.

Model	α	β	δ	k
GOPGW-Lx	0.7667	0.2579	0.9525	5.0782
	(0.1537)	(0.1917)	(0.2383)	(3.4426)
	α	β	δ	λ
GOPGW-W	2.11986	0.76310	0.5569	0.2919
	(0.9227)	(0.4673)	(0.6285)	(0.1813)
	λ	α	a	b
TIIGIE-Lx	981.0000	109.0000	0.0001	10.0000
	(1.9699×10^{-12})	(1.7755×10^{-11})	(1.2419×10^{-05})	(1.7750×10^{-10})
	γ	δ	λ	θ
EMW	0.9289	1.0000	170.0000	0.0001
	(1.1739×10^{-01})	(1.3055×10^{-01})	(9.7521×10^{-20})	(1.6581×10^{-13})
	α	λ	a	b
OEHL-BXII	0.1436	0.4582	5.0183	0.2545
	(0.0351)	(0.3221)	(1.2869)	(0.0779)
OEHL-Lx	0.4882	0.4029	1	1.4321
	(0.0685)	(0.1915)	-	(0.2741)
	c	α	β	
LLW	2.1997	0.4113	0.5412	
	(0.3761)	(0.0919)	(0.1021)	
	β	λ		
W	0.6689	0.7558		
	(0.0670)	(0.0661)		
	k			
Lx	1.6638			
	(0.1656)			

Table 7: Statistics of models for Kevlar 49/epoxy strands failure at 90% stress level data.

Model	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	KS	P-value	SS
GOPGW-Lx	204.3	212.3	212.7	222.8	0.1239	0.7661	0.0731	0.6534	0.1629
GOPGW-W	206.3	214.3	214.7	224.8	0.1644	0.9637	0.0844	0.4677	0.1639
TIIGIE-Lx	211.9	219.9	220.3	230.3	0.2078	1.1534	0.1697	0.0059	0.8378
EMW	206.5	214.5	214.9	224.9	0.1753	1.0032	0.1025	0.2389	0.2509
OEHL-BXII	216.2	224.2	224.7	234.7	0.2660	1.4870	0.5415	2.2×10^{-16}	15.2927
OEHL-Lx	251.3	257.3	257.5	265.1	0.1152	0.7708	0.2418	1.486×10^{-05}	2.4752
LLW	207.5	213.5	213.7	221.3	0.1343	0.8644	0.0998	0.2664	0.2699
W	231.5	235.5	235.6	240.7	0.2758	1.4889	0.2419	1.473×10^{-05}	1.3752
Lx	220.6	222.6	222.6	225.2	0.4702	2.5406	0.1663	0.0075	0.6549

Strands Failure data. Also the value of SS from the probability plots and the values of the statistics A^*, W^* and KS are the smallest for the GOPGW-Lx and GOPGW-W distributions, which indeed showed that this model can be useful.

7.2 Airborne communication transceiver data

The second data correspond to maintenance on active repair times (in hours) for an airborne communication transceiver with size n=46 cited by Balakrishnan et al. (2009) and Chhikara and Folks (1989). The observations are: 0.2,0.3,0.5,0.5,0.5,0.5,0.6,0.6,0.7,0.7, 0.7,0.8,0.8,1.0,1.0,1.0,1.0,1.1,1.3,1.5,1.5,1.5,1.5,2.0,2.0,2.2,2.5,2.7,3.0,3.0,3.3,3.3,4.0, 4.0, 4.5,4.7,5.0,5.4,5.4,7.0,7.5,8.8,9.0,10.3,22.0,24.5.

The estimated variance-covariance matrix for GOPGW-Lx distribution of airborne

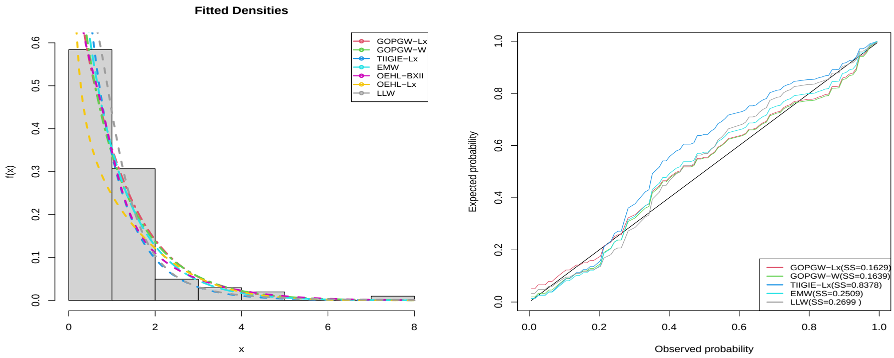

Figure 5: Fitted pdfs and cdfs plots of Epoxy strands failure data.

communication transceiver data is given by

$$\begin{bmatrix} 1.7952\times10^{+00} & -5.0520\times10^{-02} & 1.5764\times10^{-14} & 2.1629\times10^{-01} \\ -5.0520\times10^{-02} & 1.7739\times10^{-03} & -4.5536\times10^{-16} & -1.4044\times10^{-02} \\ 1.6862\times10^{-14} & -4.8636\times10^{-16} & 1.4852\times10^{-28} & 2.3406\times10^{-15} \\ 2.1629\times10^{-01} & -1.4044\times10^{-02} & 2.2053\times10^{-15} & 2.3387\times10^{-01} \end{bmatrix}.$$

Table 8: Estimates of models for airborne communication transceiver data.

Model	α	β	δ	k
GOPGW-Lx	3.2776	0.0772	1.0000	2.0930
	(1.3399)	(4.2118×10^{-02})	(1.2187×10^{-14})	(4.83601×10^{-01})
	α	β	δ	λ
GOPGW-W	141.2300	0.0100	0.3946	0.2194
	(0.0001)	(0.0022)	(0.1773)	(0.0119)
	λ	α	a	b
TIIGIE-Lx	92.3050	3.2100	0.0114	8.7477
	(3.4015×10^{-02})	(3.6148×10^{-05})	(7.0061×10^{-03})	$(6.7979\times10^{+00})$
	γ	δ	λ	θ
EMW	0.9583	0.2694	17.0010	0.0001
	(1.8975×10^{-01})	(5.4358×10^{-02})	(2.5315×10^{-18})	(4.3037×10^{-12})
	α	λ	a	b
OEHL-BXII	0.1748	0.0524	10.6125	0.1275
	(0.0627)	(0.0910)	(4.6809)	(0.0443)
OEHL-Lx	0.61034	0.3513	1	0.8601
	(0.1480)	(0.2758)	-	(0.2584)
	c	α	β	
LLW	1.1041	0.2500	0.6086	
	(0.2762)	(0.2905)	(0.2699)	
	β	λ		
W	0.3337	0.8986		
	(0.0750)	(0.0958)		
	k			
Lx	0.8338			
	(0.1229)			

Table 8 and 9 give the maximum likelihood estimates (MLE's) of the GOPGW-G distributions for parameters α, β, δ and ξ, respectively with (standard errors in parentheses), AIC, AICC, BIC, and the goodness-of-fit statistics W^*, A^*, KS and its p-value as well as SS. The values of the goodness-of-fit statistics: AIC, AICC and BIC for

Table 9: Statistics of models for airborne communication transceiver data.

Model	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	KS	P-value	SS
GOPGW-Lx	197.8	205.8	206.8	213.2	0.0248	0.1866	0.0733	0.9658	0.0289
GOPGW-W	199.2	207.2	208.2	214.5	0.0427	0.3455	0.0984	0.7645	0.0395
TIIGIE-Lx	205.9	213.9	214.9	221.2	0.0924	0.6092	0.1273	0.4449	0.0668
EMW	209.9	217.9	218.9	225.3	0.1442	1.0004	0.1519	0.2385	4.1083
OEHL-BXII	206.5	214.5	215.4	221.8	0.0806	0.4614	0.5241	2.125×10^{-11}	6.0696
OEHL-Lx	235.9	241.9	242.5	247.4	0.1391	0.9607	0.2473	0.0072	0.9749
LLW	232.3	238.3	238.9	243.8	0.0600	0.3796	0.3772	4.124×10^{-06}	2.6281
W	208.9	212.9	213.2	216.6	0.1298	0.9009	0.1204	0.5170	0.1157
Lx	219.1	221.1	221.2	222.9	0.0569	0.3427	0.2434	0.0086	0.5136

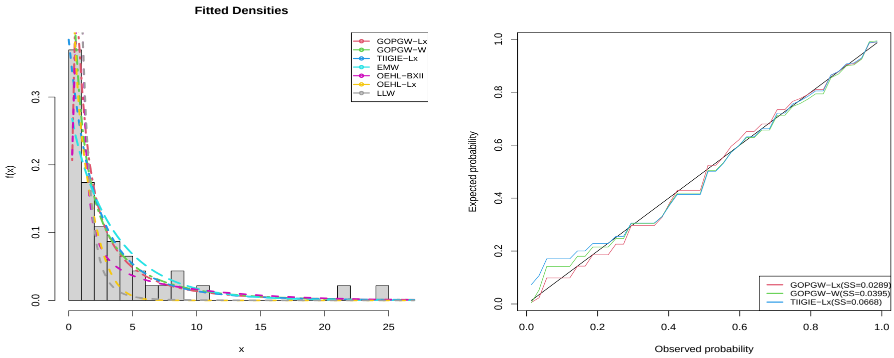

Figure 6: Fitted pdfs and cdfs plots of airborne communication transceiver data.

the GOPGW-Lx and GOPGW-W distributions are the smallest as compared to those of the models listed in the table hence the GOPGW-Lx and GOPGW-W distributions are better in fitting for the airborne communication transceiver data. Also, the value of SS from the probability plots and the values of the statistics A^*, W^* and KS are the smallest for the GOPGW-Lx and GOPGW-W distributions. Furthermore, the p-value for the KS statistic is very close to one indicating that this models are the best in fitting Airborne Communication Transceiver data.

8 Concluding Remarks

In this paper, we presented a new distribution called the Gamma Odd Power Generalized Weibull-G family of distributions. Statistical properties of the new generalized distribution such as quantile function, ordinary and incomplete moments, probability weighted moments, stochastic ordering, distribution of the order statistics and Rényi entropy were derived. The maximum likelihood technique was used to estimate the model parameters. Two special cases of the new family of distribution including Gamma Odd Power Generalized Weibull-Burr XII and Gamma Odd Power Generalized Weibull-Weibull distributions are discussed in details. The usefulness of the new family of distributions was examined by means of applications to two real data.

A

The following link contains the Appendix.
https://drive.google.com/file/d/1i7aOWcTg6Xw-iRzRj3dwv2dssDgSrxFt/view?usp=sharing

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