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Research Paper

Bayesian estimation of reliability for Rayleigh distribution and its application

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Abstract: Rayleigh distribution is one of the statistical distributions in real data modeling and is mostly used in reliability. In this paper, we derive Bayesian estimation and the reliability of Rayleigh distribution under the entropy loss function, and the risk of Bayesian estimator of its under the entropy loss function. Finally, we describe our results with a Monte Carlo numerical simulation.

Keywords: Bayes estimator; Entropy loss function; Rayleigh distribution; Reliability function; Risk function.
Mathematics Subject Classification (2010): 62C20

1 Introduction

The Rayleigh distribution is a proper model for trial and clinical studies. Polovko(1986) and Dyer and Whisenand (1973) expressed the importance of using this distribution in electrical vacuum devices and its connection with engineering sciences. Hirano (1986) also examined the properties of this distribution. Zellner (1986) obtained the Bayesian estimator prediction using the asymptotic distribution function. Howlader and Hossain (1995) derived the Bayesian estimator for the scale parameter and the Rayleigh function ($R(t)$) in the type-II censored sampling. Dey (2009) obtained and compared Bayesian estimators of the Rayleigh distribution under the square error and Linex loss functions.

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The probability density function and reliability function of the Rayleigh distribution are as follows:

$$\begin{aligned} f(s|\sigma^2) &= \frac{s}{\sigma^2}\exp(\frac{-s}{2\sigma^2}), \qquad s \geq 0,\ \sigma > 0, \qquad (1) \\ R(t) &= \overline{F}(t) = P(X > t) = \exp(\frac{-t^2}{2\sigma^2}), \qquad t \geq 0, \sigma > 0. \end{aligned}$$

In this paper, we obtain the Bayesian estimation of reliability of Rayleigh distribution under the scale invariant loss

$$L(\theta, d) = \frac{d}{\theta} - \log(\frac{d}{\theta}) - 1,$$

called the entropy loss function.

In this paper, we first introduce the entropy loss functions in Section 2 and then obtain the posterior distribution of the Rayleigh distribution and the Bayesian estimator of σ under the entropy loss function. In Section 3, we calculate the risk of this estimator under the entropy loss function. In Section 4, we obtain the Bayesian estimator of $\bar{F}$. Finally, we discuss the numerical studies of the previously obtained estimators in Section 5. We conclude the paper in Section 6.

2 Bayesian estimation

Here, we want to estimate the unknown parameter of the Rayleigh distribution based on the complete sample of size n. The likelihood function is given by

$$L(\sigma|s) = \prod_{i=1}^{n} x_i \sigma^{-2n} \exp(\frac{-s}{2\sigma^2}), \qquad \sigma > 0.$$

Consider prior distribution of the natural conjugate heat to the posterior distribution

$$\Pi(\sigma) \propto \frac{1}{\sigma^{\alpha+1}}\exp(\frac{-\beta}{2\sigma^2}), \qquad \beta, \sigma > 0.$$

If $\alpha = 0$, $\beta = 0$, the prior distribution is called non-informative (Jeffreys, 1998). Also, if $\alpha = 2$, $\beta = 0$, the prior distribution of the asymptotic invariant introduced by Hartigan (1964) is obtained. The posterior density σ using Bayes theorem is as follows

$$\begin{aligned} \Pi(\sigma|s) &= \frac{L(s|\sigma)\Pi(\sigma)}{\int_0^\infty L(s|\sigma)\Pi(\sigma)dx} \\ &= \frac{\prod_{i=1}^{n} x_i \sigma^{-2n} \exp(\frac{-s}{2\sigma^2}) \frac{1}{\sigma^{\alpha+1}} \exp(\frac{-\beta}{2\sigma^2})}{\int_0^\infty \prod_{i=1}^{n} x_i \sigma^{-2n} \exp(\frac{-s}{2\sigma^2}) \frac{1}{\sigma^{\alpha+1}} \exp(\frac{-\beta}{2\sigma^2}) dx} \\ &= \frac{2(\frac{s+\beta}{2})^{\frac{2n+\alpha}{2}} \exp(-\frac{s+\beta}{2\sigma^2})}{\Gamma(\frac{2n+\alpha}{2})\sigma^{2n+\alpha+1}}, \qquad \sigma, \alpha, \beta > 0. \qquad (2) \end{aligned}$$

Now, the Bayes estimator of σ under the entropy loss function which we show with $\hat{\sigma}_{en}$, is calculated as follows

$$E[L(\sigma, \hat{\sigma})] = \int_0^\infty (\frac{\hat{\sigma}}{\sigma} - \log\frac{\hat{\sigma}}{\sigma} - 1)\Pi(\sigma|s)d\sigma$$

$$
\begin{aligned}
&= \int_0^\infty (\frac{\hat{\sigma}}{\sigma} - \log\frac{\hat{\sigma}}{\sigma} - 1)\frac{2(\frac{s+\beta}{2})^{\frac{2n+\alpha}{2}}\exp(-\frac{s+\beta}{2\sigma^2})}{\Gamma(\frac{2n+\alpha}{2})\sigma^{2n+\alpha+1}}d\sigma \\
&= \int_0^\infty \frac{\hat{\sigma}}{\sigma}\frac{2(\frac{s+\beta}{2})^{\frac{2n+\alpha}{2}}\exp(-\frac{s+\beta}{2\sigma^2})}{\Gamma(\frac{2n+\alpha}{2})\sigma^{2n+\alpha+1}}d\sigma \\
&\quad - \int_0^\infty \log(\frac{\hat{\sigma}}{\sigma})\frac{2(\frac{s+\beta}{2})^{\frac{2n+\alpha}{2}}\exp(-\frac{s+\beta}{2\sigma^2})}{\Gamma(\frac{2n+\alpha}{2})\sigma^{2n+\alpha+1}}d\sigma - 1 \\
&= frac2\hat{\sigma}(\frac{s+\beta}{2})^{\frac{2n+\alpha}{2}}\Gamma(\frac{2n+\alpha}{2})\int_0^\infty \frac{\exp(-\frac{s+\beta}{2\sigma^2})}{\sigma^{2n+\alpha+2}}d\sigma - \log\hat{\sigma} \\
&\quad +2\frac{(\frac{s+\beta}{2})^{\frac{2n+\alpha}{2}}}{\Gamma(\frac{2n+\alpha}{2})}\int_0^\infty \frac{\log(\sigma)\exp(-\frac{s+\beta}{2\sigma^2})}{\sigma^{2n+\alpha+1}}d\sigma - 1 \\
&= \frac{\hat{\sigma}\Gamma(\frac{2n+\alpha+1}{2})}{\Gamma(\frac{2n+\alpha}{2})(s+\beta)^{\frac{1}{2}}} - \log\hat{\sigma} - 1 \\
&\quad - \frac{\Gamma(\frac{2n+\alpha+1}{2})}{2\Gamma(\frac{2n+\alpha}{2})(s+\beta)^{\frac{1}{2}}}[\Psi(\frac{2n+\alpha+1}{2}) - \log(s+\beta)],
\end{aligned}
$$

where Ψ is a degenerate hypergeometric function which is defined as follows (Magnus and Oberhettinger and Soni, 1966)

$$
\Psi(\alpha,\gamma,z) = 1 + \frac{\alpha}{\gamma}\frac{z}{1!} + \frac{\alpha}{\gamma}\frac{(\alpha+1)}{(\gamma+1)}\frac{z^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{z^3}{3!} + \dots .
$$

Now, we can obtain the value of $\hat{\sigma}_{en}$ by minimizing the above expression as

$$
\hat{\sigma}_{en} = \frac{\Gamma(\frac{2n+\alpha}{2})\sqrt{\frac{(s+\beta)}{2}}}{\Gamma(\frac{2n+\alpha+1}{2})}. \tag{3}
$$

3 The risk of $\hat{\sigma}_{en}$

In this section, we obtain the risk of Bayesian estimator of $\hat{\sigma}_{en}$ in (3) under the entropy loss function which is denoted by $R(\hat{\sigma}_{en})$. It can be calculted as

$$
\begin{aligned}
R(\hat{\sigma}_{en}) &= E[L(\sigma,\hat{\sigma}_{en})] \\
&= \int_0^\infty (\frac{\hat{\sigma}_{en}}{\sigma} - \log\frac{\hat{\sigma}_{en}}{\sigma} - 1)k(s)ds \\
&= \int_0^\infty (\frac{\hat{\sigma}_{en}}{\sigma} - \log\frac{\hat{\sigma}_{en}}{\sigma} - 1)\frac{1}{(2\sigma^2)^n\Gamma(n)}s^{n-1}\exp(-\frac{s}{2\sigma^2})ds \\
&= \frac{\Gamma(\frac{2n+\alpha}{2})}{2^{n+\frac{1}{2}}\sigma^{2n+1}\Gamma(n)\Gamma(\frac{2n+\alpha+1}{2})}\int_0^\infty (s+\beta)^{\frac{1}{2}}s^{n-1}\exp(-\frac{s}{2\sigma^2})ds + \log\sigma \\
&\quad - \frac{1}{2(2\sigma^2)^n\Gamma(n)}\int_0^\infty \log(s+\beta)s^{n-1}\exp(-\frac{s}{2\sigma^2})ds
\end{aligned}
$$

$$
\begin{aligned}
&-\log(\frac{\Gamma(\frac{2n+\alpha)}{2})}{\Gamma(\frac{2n+\alpha+1}{2})})-1+\log 2\\
=\;&\frac{\Gamma(\frac{2n+\alpha}{2})}{\sigma^{2n+1}2^{n+\frac{1}{2}}\Gamma(\frac{2n+\alpha+1}{2})}(\frac{1}{2\sigma^2})^{-\frac{(n+\frac{3}{2})}{2}}\beta^{\frac{2n-1}{4}}\\
&\times\exp(-\frac{3\beta}{4\sigma^2})W_{(\frac{3-2n}{4},-\frac{2n+1}{4})}(\frac{\beta}{2\sigma^2})+\log\sigma\\
&-\log(\frac{\Gamma(\frac{2n+\alpha}{2})}{\Gamma(\frac{2n+\alpha+1}{2})})-1+\log 2-\frac{P}{2(2\sigma^2)^n\Gamma(n)}
\end{aligned}
$$

where

$$P=\int_0^\infty \log(s+\beta)s^{n-1}\exp(-\frac{s}{2\sigma^2})ds, \tag{4}$$

and $W_{\mu,\lambda}(z)$ is Wittakers function (Magnus and Oberhettinger and Soni, 1966) given as

$$
\begin{aligned}
W_{\lambda,\mu}(z) &= \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda)}M_{\lambda,\mu}(z)+\frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda)}M_{\lambda,-\mu}(z),\\
M_{\lambda,\mu}(z) &= z^{\mu+\frac{1}{2}}\exp(-\frac{z}{2})\Phi(\mu-\lambda+\frac{1}{2},2\mu+1;z),\\
M_{\lambda,-\mu}(z) &= z^{-\mu+\frac{1}{2}}\exp(-\frac{z}{2})\Phi(-\mu-\lambda+\frac{1}{2},-2\mu+1;z),\\
\Phi(a,c;z) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(z-a)}\int_0^1 \exp(zx)x^{a-1}(1-x)^{c-a-1}dx.
\end{aligned}
$$

To solve the integral in , we use the following method. Suppose that $s{=}\sigma^2y^2$, $ds{=}\,2\sigma^2ydy$. Therefore,

$$
\begin{aligned}
P &= \int_0^\infty \log(\sigma^2y^2+\beta)(y^2\sigma^2)^{n-1}\exp(-\frac{y^2}{2})2ydy\\
&= \sigma^{2n}\sqrt{2\pi}\int_0^\infty \log(\sigma^2y^2+\beta)y^{2n-1}\frac{2\exp(-\frac{y^2}{2})}{\sqrt{2\pi}}dy\\
&= \sigma^{2n}\sqrt{2\pi}\int_0^\infty \log(\sigma^2y^2+\beta)y^{2n-1}f_Y(y)dy\ ,
\end{aligned}
$$

where $f_Y(y)=\frac{2}{\sqrt{2\pi}}\exp(-\frac{y^2}{2})I_{(0,\infty)}(y)$, is a probability density function. Therefore

$$P=\sigma^{2n}\sqrt{2\pi}E[\log(\sigma^2Y^2+\beta)Y^{2n-1}].$$

The cdf of Y is given by $F_Y(y)=2(\Phi(y)-.5)=2\Phi(y)-1$, where Φ is a normal distribution function. Suppose that $U{\sim}u(0,1)$. Therefore, $Y=\Phi^{-1}(\frac{u+1}{2})$.

Suppose that $u_1,u_2\ldots,u_m$ is random sample from $u(0,1)$, then

$$\hat{P}{=}\frac{\sigma^{2n}\sqrt{2n}}{m}\sum_{i=1}^{m}\log(\sigma^2(\Phi^{-1}(\frac{u_i+1}{2}))^2+\beta)(\Phi^{-1}(\frac{u_i+1}{2}))^{2n-1},$$

is an estimator for P. Therefore, we have

$$\begin{aligned} E[L(\sigma,\hat{\sigma}_{en})] &= \frac{\Gamma(\frac{2n+\alpha}{2})}{\sigma^{2n+1}2^{n+\frac{1}{2}}\Gamma(\frac{2n+\alpha+1}{2})}(\frac{1}{2\sigma^2})^{-\frac{n+\frac{3}{2}}{2}}\beta^{\frac{2n-1}{4}} \\ &\times\exp(-\frac{3\beta}{4\sigma^2})W_{(\frac{3-2n}{4},-\frac{2n+1}{4})}(\frac{\beta}{2\sigma^2})+\log\sigma \\ &-\log(\frac{\Gamma(\frac{2n+\alpha}{2})}{\Gamma(\frac{2n+\alpha+1}{2})})-1+\log 2-\frac{\sqrt{2\pi}}{2^{2n+1}m\Gamma(n)} \\ &\times\sum_{i=1}^{m}\log(\sigma^2(\Phi^{-1}(\frac{u_i+1}{2}))^2+\beta)(\Phi^{-1}(\frac{u_i+1}{2}))^{2n-1}. \end{aligned}$$

4 Bayesian estimator of $\bar{F}(t)$

Let $\gamma=\bar{F}(t)$ be the probability of a system after time t. If $\sigma^2=-\frac{t^2}{2\log\gamma}$, the posterior density function for γ in (2) can be written as

$$\begin{aligned} \Pi^*(\gamma|s) &= \frac{(\frac{s+\beta}{2})^{\frac{2n+\alpha}{2}}\exp(-\frac{s+\beta}{\frac{t^2}{\log\gamma}})}{\Gamma(\frac{2n+\alpha}{2})(-\frac{t^2}{2\log\gamma})}(\frac{t^2}{2})(\log\gamma)^{-2}\frac{1}{\gamma} \\ &= \frac{1}{\Gamma(\frac{2n+\alpha}{2})}(\frac{s+\beta}{t^2})^{\frac{2n+\alpha}{2}}\gamma^{\frac{2n+\alpha}{2}-1}(-\log\gamma)^{\frac{2n+\alpha}{2}-1}, \quad 0<\gamma<1. \end{aligned}$$

Using the posterior density function γ, we have

$$\begin{aligned} E(L(\gamma,\hat{\gamma})) &= \int_0^1(\frac{\hat{\gamma}}{\gamma}-\log\frac{\hat{\gamma}}{\gamma}-1)\Pi^*(\gamma|x)d\gamma \\ &= \hat{\gamma}E_\gamma(\frac{1}{\gamma})-\log\hat{\gamma}+E_\gamma(\log\gamma)-1. \end{aligned} \tag{5}$$

By minimizing the expression (5), the Bayesian estimator, $\hat{\gamma}_{en}$, is obtained. With some algebraic operation, the value of $\hat{\gamma}_{en}$ is obtained as

$$\hat{\gamma}_{en}=(\frac{t^2(2n+\alpha-2)}{2(s+\beta)})^{\frac{2n+\alpha}{2}}.$$

5 Numerical study

Here, we examine the obtained estimators numerically. $N=2000$ samples of sizes $n=10,20,30$ are generated from (1) with $\sigma=1$. The estimators $\hat{\sigma}_{en}$, $R_{en}(\hat{\sigma}_{en})$ and $\hat{\gamma}_{en}$ are calculated for the different values of α andβ, and $t=1$. The results are given in Table 1.

Also, we consider two sets of data from the Netherlands and Italy, which are related to the mortality rate of COVID-19. (see https://covid19.who.int/). The first dataset related to the mortality rate from COVID-19 Netherlands is 30 days from 31 March to 31 April 2020. These data as follows:

Table 1: The estimators $\hat{\sigma}_{en}$, $R_{en}(\hat{\sigma}_{en})$ and $\hat{\gamma}_{en}$ for the different values of α and β, and $t = 1$.

n	α	β	$\hat{\sigma}_{en}$	$\hat{R}_{en}$	$\hat{\gamma}_{en}$
10	1	1	1.0031	7.1996	0.0975
20	1	1	0.9978	10.4476	0.9302
30	1	1	0.9992	73.2208	0.0213
10	1	2	1.0237	3.2475	0.0676
20	1	2	1.0148	54.2988	0.0025
30	1	2	1.0097	27.1625	0.0539
10	0	1	1.0253	7.3350	0.0494
20	0	1	1.0121	10.5705	0.0043
30	0	1	1.0076	73.8112	0.0012
10	0	2	1.0550	3.2827	0.0391
20	0	2	1.0259	54.9541	0.0126
30	0	2	1.0159	27.3594	0.0382
10	2	2	1.0035	3.2146	0.0914
20	2	2	0.9980	53.6684	0.0121
30	2	2	0.9993	26.9657	0.1855

14.918, 10.656, 12.274, 10.289, 10.332, 7.099, 5.928, 13.211, 7.968, 7.534 5.555, 6.027, 4.097, 3.611, 4.960, 7.498, 6.940, 5.307, 5.043, 2.357, 2.254, 5.431, 4.462, 3.333, 3.461, 3.647, 1.974, 1.273, 1.416, 4.235.

The second dataset related to the mortality rate from COVID-19 Italy is 59 days from 27 February to 27 April 2020. These data as follows:
4.571, 7.201, 3.606, 8.479, 11.410, 8.961, 10.919, 10.908, 5.503, 18.474, 11.010, 17.337, 16.561, 13.226, 15.137, 8.697, 15.878, 13.333, 11.822, 14.242, 11.273, 14.330, 16.046, 11.950, 10.282, 11.775, 10.138, 9.037, 12.396, 10.644, 8.646, 8.905, 8.906, 7.407, 7.445, 7.214, 6.194, 4.640, 5.452, 5.073, 4.416, 4.859, 4.408, 4.639, 3.148, 4.040, 4.253, 4.011, 3.564, 3.827, 3.134, 2.780, 2.881, 3.341, 2.686, 2.814, 2.508, 2.450, 1.518.

For each data set, we obtain $\hat{\sigma}_{en}$, $R_{en}(\hat{\sigma}_{en})$ and $\hat{\gamma}_{en}$. The results are given in Table 2.

Table 2: The Estimators $\hat{\sigma}_{en}$, $R_{en}(\hat{\sigma}_{en})$ and $\hat{\gamma}_{en}$for the different values of α and β, and $t = 1$.

Country	α	β	σ	$\hat{\sigma}_{en}$	$\hat{R}_{en}$	$\hat{\gamma}_{en}$
Netherlands	1	1	4.88	4.9495	51.4788	0.1067
	1	2	4.88	4.9512	27.1625	0.0021
	0	1	4.88	4.9909	73.8112	0.0274
	0	2	4.88	4.9926	27.3594	0.0257
	2	2	4.88	4.9107	26.9659	0.0811
Italy	1	1	6.49	6.4957	6412.1956	0.0333
	1	2	6.49	6.5643	2364.0398	0.0035
	0	1	6.49	6.5915	6433.1172	0.0474
	0	2	6.49	6.5922	2374.5002	0.0333
	2	2	6.49	6.5368	2667.3901	0.0196

6 Concluding remark

Due to the application of Rayleigh distribution in the issues of reliability and modeling of many practical models with entropy loss function, Bayesian estimators and the reliability of this distribution is effective in explaining and solving these problems. The numerical results obtained from estimating the parameter of this distribution under the entropy loss function show the high accuracy of this method.

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