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How do different distributions of random input have an effect on output results in a simulated physical model?

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Abstract: Statistical methods are practical and unavoidable in analysis of physical and engineering results. Study of anufacturing errors and uncertainties in construction of radio frequency structures is one of cases which statistical quantities is used. In this paper, we quantify uncertainty in the cutoff frequency of a waveguide using the chaos polynomial expansion method. Different distributions for uncertainty in the waveguide width are considered. We then investigate the effect of the distributions on the waveguide cutoff frequency. Using statistical quantities, we determine the amount of acceptable error during the construction of the waveguide such that it does not affect the wavequide performance.

Keywords: Generalized polynomial chaos; Random distributions; Uncertainty Quantification; Waveguide.
Mathematics Subject Classification (2010): 62M99.

1 Introduction

During the last few decades, computer simulations and statistical methods are usually used for the design and analysis of many radio frequency (RF) structures. These structures have an uncertainty in their dimensional parameters due to manufacturing errors (Mesogitis et al., 2014). It is essential to study how these uncertainties affect on the results of the structure, in other words, determination of uncertainty quantification (UQ) is necessary to achieve an acceptable level of reliability in the results. The most important step in the UQ process is choosing the random distribution for the uncertain parameter. In some cases, the determination of these distributions can be carried out

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using experimental results. Here we attempt to show the effect of choosing different distribution of an uncertainty for waveguide width on the cut off frequency (f_c). For this purpose, UQ process must be carried out using an appropriate method, in some cases, which is the function of outputs are theoretically defined, monte carlo (MC) method is used. MC method needs a lot of random samples of inputs to achieve results with the desired accuracy which causes the result's computation time to be increased. One of the effective UQ methods is the generalized Polynomial Chaos expansion (gPC) (Ghanem and Spanos, 1991). The gPC method has been used extensively in the last decade to model uncertainty in many domains: stochastic elastic materials, finite deformations, heat conduction, incompressible flows, reacting flows, finding the resolution of the uncertain linear boltzmann equation, approximating of kinetic equations for kinetic flocking models with Uncertainties, error evaluating in chromatography methods and solving stochastic equations (Acharjee and Zabaras, 2006; Wan et all., 2006; Le Maitre and Knio, 2007; Poette, 2019, 2022; Carrillo and Zanella, 2019; Pollock et al., 1979; Xiu, 2009).

The geometry parameters of a superconducting radio frequency cavity were modeled as uniformly distributed random variables (Schmidt et al., 2014). Hadj Kacem et al. (2022) deal with the problem of the uncertainties interaction in the design parameters of a two-stage gearbox of a wind turbine. For this purpose, the gPC was applied to simulate the dynamic behaviour of the wind turbine gearbox, using uniform distribution for uncertain parameters. Another study, DeGennaro et al. (2015) demonstrated the utility of gPC methods as a fast and accurate method for quantifying the effects of ice shape uncertainty on the performance of airfoil. The results being in good agreement with MC method also confirm that gPC methods are much more efficient for the canonical airfoil icing uncertainty quantification problem than MC methods.

In most physical problems, the distribution of random variables is considered normal according to the central limit theorem (Barany and Vu, 2007). Or, for simplicity, these distributions are assumed to be uniform, but so far, the effect of choosing these distributions on the results has not been investigated. In this study, we investigate the effect of choosing these distributions on the results. Here, we use the gPC method instead of a MC method to determine the uncertainty of f_c, in which its theoretical equation is defined. The reason for using this method is that we can determine which types of those distributions have a better agreement in comparison with the theoretical result, f_c. Another important subject in designing and building waveguides is determining how much manufacturing error is acceptable, which is called as dimensional tolerance (Ameta et al., 2015). In this paper, the gPC method are presented in Section 2. The Uncertainty quantification of cutoff frequency of the waveguide are expressed in Section 3. In Section 4 the validation results of the gPC method in comparison with MC method are presented, after that, the effect of the different input distributions is discussed, and in the final part of this section, the dimensional tolerance of the waveguide is determined

2 The univariate gPC method

In the gPC method, the uncertainty influence of random model parameter, x, on a desired quantity Y of the model is investigated. The quantity Y can be expressed as a convergence series of orthogonal polynomials

$$Y = \sum_{i=1}^{N} c_i \psi_i(\xi_i),$$

where $\psi_i(\xi_i)$s are the orthogonal polynomials which are dependent on the random variables ξ having some standard distributions, N is the degree of polynomial expansion and c_i are the expansion coefficients. Orthogonal polynomial types suitable for each standard distribution of ξ are listed in Table 1.

Table 1: Distributions types and orthogonal basis polynomial support ranges I (Schmidt et al., 2012).

Distribution type	Orthogonal basis polynomial	I
Normal	Hermit	$[-1, 1]$
Uniform	Legendre	$[-1, 1]$
Beta	Jacobi	$[1, 1]$
Exponential	Laguerre	$[0, 1]$
Gamma	Generalised Laguerre	$[0, 1]$

The coefficients c_i are determined by projecting the truncated expansion of Y on each basis polynomial and exploiting its orthogonality in the domain I in Table 1.

$$c_i = \langle \frac{1}{\psi_i(\xi)\psi_j(\xi)} \rangle \int_I Y(\xi)\psi(\xi)f(\xi), \tag{1}$$

where $f(\xi)$ is probability density function(PDF) of the random variables ξ. Since the quantity Y depends on the parameter x, a transformation has to be defined, mapping the standard random variable ξ_i on the random variable x_i. We used the inverse transformation method that relies on the principle that continuous cumulative distribution functions (CDFs) are uniformly bounded in the interval $[0, 1]$ (Schmidt et al., 2012).

• Transfer from the standard uniform, $[-1, 1]$, to the uniform, $U[a, b]$,

$$x(\xi_i) = \frac{b-a}{2}\xi_i + \frac{b+a}{2}\xi_i, \tag{2}$$

where ξ_i is a random sample with standard uniform distribution, $[-1, 1]$, and x is a transformed random variable of the uniform interval $[a, b]$ (Adelmann, 2015).

• Transfer from the standard normal, $N(0, 1)$, to the normal, $N(\mu, \sigma)$,

$$x(\xi_i) = \mu + \sigma\xi_i, \tag{3}$$

where ξ_i is a standard random sample in $N(0, 1)$ and x is a transformed normal random variable with mean μ and standard deviation σ (Adelmann, 2015). While the denominator in (1) can be computed analytically, the integral in the nominator is computed by a numerical evaluation using a multidimensional cubature. Calculating the nodes and weights for the numerical evaluation of the integral are described in Stoer (2006), so expansion coefficients gain by (4),

$$c_N = k \sum_{i=1}^{N} f(x(t_i))\psi_i(t_i)\omega(t_i), \tag{4}$$

where t_i and $\omega(t_i)$ represent the nodes and their weights for the numerical evaluation of the integral. $x(t_i)$, represents the random samples of x that transformed in terms of nodes, $f(x(t_i))$ is the transformed function in terms of node points, $\psi_i(t_i)$ are Legendre polynomial ($p_i(t_i)$). In the case of standard uniformly distributed and the expansion coefficients are calculated numerically using the Clenshaw-Curtis or Gauss Legendre numerical integration method. For example, in the Clenshaw-Curtis method, nodes are defined as follows (Stoer, 2006),

$$t_i = \cos\left(\frac{(i-1)\pi}{N-1}\right), i = 1, 2, \ldots, N.$$

If the distribution of random variable is normal, $\psi_i(t_i)$ are hermit polynomial ($H_i(t_i)$) and Gauss hermit numerical integration method is used for calculating the expansion coefficients (Stoer, 2006). k is a coefficient relating to integral numerical solution method.

To ensure sufficient accuracy in the stochastic moments of the quantity of interest, the relative error in its estimated moments is controlled for an increasing number of random samples. If for the expansion order N, expansion coefficients are equal to $\vec{C}_N$ and for expansion order $N+1$ the coefficients are equal to $\vec{C}_{N+1}$, then, according to the definition of the second order norm, the error of the gPC method is obtained in terms of expansion coefficients using (5) (Mostajeran et al., 2021).

$$Error = \frac{\left\|\vec{C}_{N+1} - \vec{C}_N\right\|}{\left\|\vec{C}_{N+1}\right\|} = \frac{\sqrt{\sum\limits_{\alpha\in\chi(M,N+1)} \left|c_{\alpha_{N+1}} - c_{\alpha_N}\right|^2}}{\sqrt{\sum\limits_{\alpha\in\chi(M,N+1)} \left|c_{\alpha_{N+1}}\right|^2}}, \tag{5}$$

where C_N is the expansion coefficient, N is the polynomial expansion degree and M is the number of random variables (here $M = 1$). α is related to expansion coefficient that is calculated in Schmidt et al. (2012).

3 Uncertainty quantification of cutoff frequency of the waveguide

The model

A waveguide is a transmission line that transmits RF waves produced by klystron to other structures of a particle accelerator. Waveguides are classified into WR and WG type, according to their structure dimensions and frequency ranges (Pozar, 2011). The f_c of a waveguide is an important electromagnetic parameter. Frequencies below the f_c are attenuated by the waveguide and waves are not transmitted by a waveguide. Therefore, the operating frequency of the waveguide must be higher than the f_c. The waveguide which is studied here is known as the WR187 rectangular waveguide (Pozar, 2011). Figure 1 shows the dimensions of waveguide WR187 with 4.755 cm and 2.215 cm as width and high of the waveguide respectively.

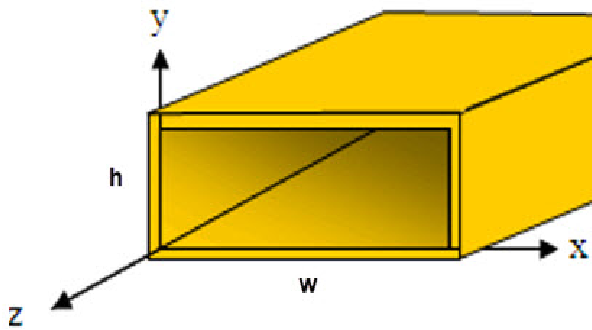

Figure 1: Rectangular Waveguide.

The f_c of a rectangular waveguide can be calculated using (6),

$$(f_c)_{mn} = \frac{1}{2\pi}\sqrt{\left(\frac{m\pi}{w}\right)^2 + \left(\frac{n\pi}{h}\right)^2}, \tag{6}$$

where c is the speed of the light, w and h are the width and height of the waveguide, respectively; m and n are the quantum numbers associated with the wave that in this study we computed T_{10} mode. To quantify the magnitude of uncertainty in the parameter w and the quantity f_c, the coefficient of variation, σ_r, was used by (7),

$$\sigma_r = \frac{\sigma}{\mu}, \tag{7}$$

where μ is the mean and σ is the standard deviation. Here we considered a relative standard deviation of $\sigma_r = 5\%$ for $\mu = w = 4.755cm$ i.e. σ is $0.23cm$. For calculation of expansion coefficients, we first converted the standard range of integration to the deviation interval of w. These transformations for both uniform and normal distributions are shown in Table 2, using (2) and (3).

Table 2: Standard range of nodes(t) and changes interval for $\sigma_r = 5\%$ in $w = 4.755cm$.

Distribution type	Standard range of nodes	Divination interval of w (cm)
Normal	$t_i \sim U[-1, 1]$	$x \sim U[a = 4.51725, \mu = 4.755, b = 4.99275]$
Uniform	$t_j \sim N(0, 1)$	$x \sim N[\mu = 4.755, \sigma = 0.23775]$

4 Results

To determine which order of expansion to use for each distribution, we calculated the relative error using (5).The respective expansion degree, N, with the lowest error is chosen in each case for further analysis. Figure 2.a and Figure 2.b illustrate the

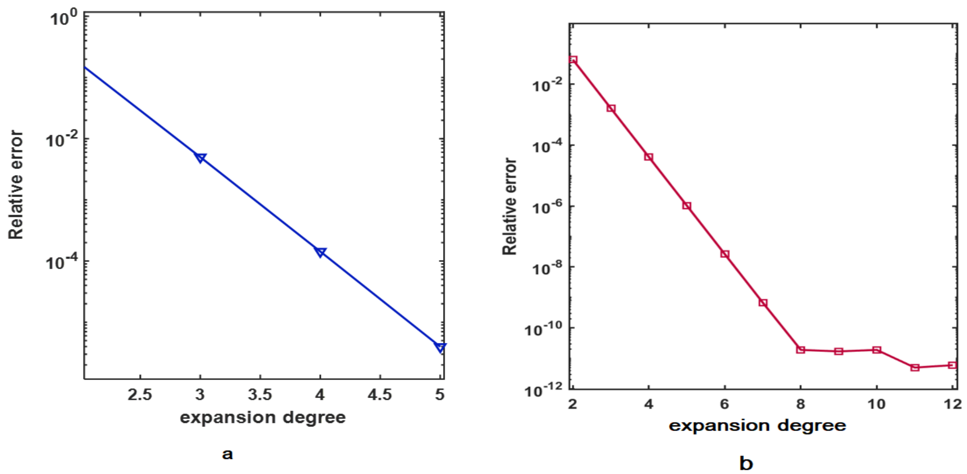

Figure 2: Relative error estimated of output function versus expansion degree in the case of $\sigma_r = 5\%$ in waveguide width, w, a. inputs with uniform distribution, b. inputs with normal distribution.

relative error estimated of the output function versus the expansion degree, N, for two distributions, uniform and normal, respectively.

Figure 2.a shows that the relative error for uniform distribution decreases linearly with polynomial expansion increasing. The relative error for normal distributions is shown in Figure 2.b. The relative error approaches convergence after the 8th degree of expansion. The respective expansion degree, N, with the lowest error is chosen in each case for further analysis (i.e. $N = 5$ for uniform distribution and $N = 11$ for normal distribution) and the PDFs of f_c are then calculated. We validated the gPC method comparing with the MC method for three different distributions of inputs separately. (6) is used for calculating f_c in MC method for the TE_{10} mode; $m = 0, n = 0$. After obtaining the fc via MC method, we calculated the PDF of the f_c. To compare gPC and MC methods, the PDFs of f_c calculated by both methods with uniform inputs, w, are shown in Figure 5. The PDF provides an overview of how data is distributed, so for a more accurate comparison, statistical values such as mean (f_μ), standard deviation(σ), and variance (Var) are also presented in the Table 3.

Table 3: Calculated statistical quantities of f_c for uniform distribution of random inputs calculated by gPC and MC methods.

Distribution type / UQ method	f_μ (GHz)	σ (GHz)	Var (GHz)
Uniform / gPC	3.16109400	0.10637118	0.01131482
Uniform / MC	3.16109400	0.10636041	0.01131253

Heavy overlap can be observed between the PDFs of f_c calculated by both gPC and MC methods in Figure 3. According to Table 3, the statistical quantities of f_c are mostly the same, for instance, the variances calculated for f_c in both methods, are the same up to 5 decimals.

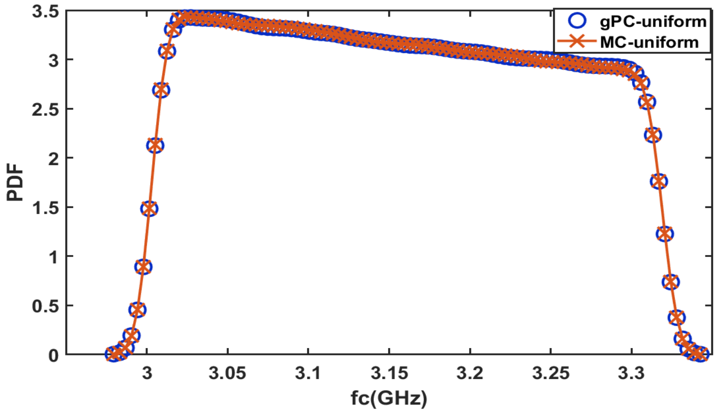

Figure 3: Comparison of PDFs of f_c for uniform distribution of random inputs by gPC and MC methods.

In Figure 4 and Table 4, we presented the PDFs and desired statistical quantities for f_c with normal distribution of inputs obtained by MC and gPC methods, respectively. Heavy overlap of f_c PDFs with normal inputs is shown in Figure 4.

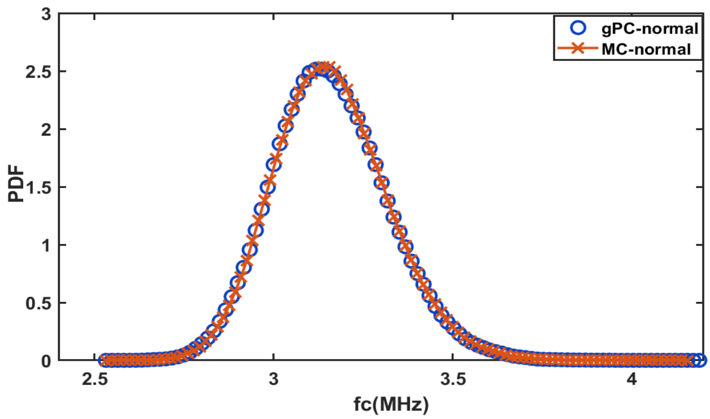

Figure 4: Comparison of PDFs of f_c for normal distribution of random inputs by gPC and MC methods.

Table 4: Calculated statistical quantities of f_c for normal distribution of random inputs calculated by gPC and MC methods.

Distribution type / UQ method	f_μ(GHz)	σ(GHz)	Var (GHz)
Normal / gPC	3.3400012	0.49377452	0.24381327
Normal / MC	3.3400409	0.50955681	0.25964814

As an example, the standard deviation, σ, of f_c calculated with method is about 6% lower than that of calculated with MC method. It seems that choosing truncated normal distributions for the input parameter seems to produce a smaller difference in comprising the results of both methods; gPC and MC methods.

Figure 5 and Table 5 show the statistical results for truncated normal distribution.Figure 5 and the statistical results presented in Table 5 show that the gPC method is more agreeing with the MC method considering the truncated normal distribution than the two previous distributions. the next step was to examine the results related to the comparison of input parameter distributions by gPC method.

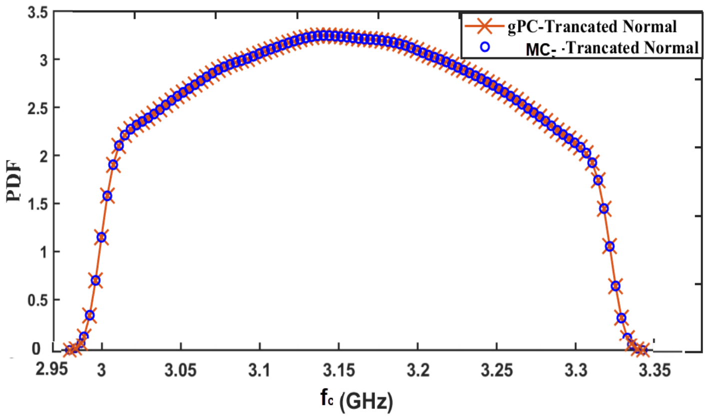

Figure 5: Calculated statistical quantities of f_c for truncated normal distribution of random inputs by gPC and MC methods.

Table 5: Calculated statistical quantities of f_c for truncated normal distribution of random inputs calculated by gPC and MC methods.

Distribution type / UQ method	f_μ (GHz)	σ (GHz)	Var(GHz)
Truncated Normal / gPC	3.16109430	0.1047784	0.0109785
Truncated Normal / MC	3.34004090	0.1047772	0.0109783

In order to obtain the mean, f_μ, and bias values of the f_c obtained for $\sigma_r = 5\%$ in w, We will compare three different distributions of random inputs in Table 6. Calculating the percentile quantity allowed us to determine another index to evaluate f_c deviations. First, we calculated which percentile of the cutoff frequencies, f_c, is equal to the exact value of cutoff frequency, f_E ,this quantity, which we call Q, shows what percentage of f_c, are lower than the f_E. Second, we then calculated the percentage of frequencies which is greater than f_E i.e. $(100 - Q)\%$. If this value is equal to 50, it means that $f_c = f_E$ and there is no deviation in the results, when it exceeds 50, it means that the accumulation of f_c exceeds f_E and such deviation is not allowed because affects the performance of the structure.

Mean, f_μ, and standard deviation of results, σ, in cases of uniform and truncated normal distribution show less dispersion than normal inputs case. Based on the results, the bias of the f_c with normal distribution is almost several times that of uniform and truncated normal inputs. For all three distributions in Table 6, more than 50% of the f_c exceed the f_c exceeding frequencies of 50% with normal inputs is more than cases with uniform and truncated normal inputs. By calculating the quantity the relative deviation, σ_r, for f_c obtained for three distributions, the amount of uncertainty

Table 6: Statistical quantities of f_c for three different distribution of random inputs in the case of $\sigma_r = 5\%$ in waveguide width ,w.

Distribution type / UQ method	f_μ (GHz)	bias (GHz)	$(100 - Q)\%$ (GHz)
Uniform	3.16109400	0.00790286	52.1550
Normal	3.34000129	0.18681015	62.3635
Truncated normal	3.16109430	0.00790316	52.1880

created in f_c due to each of these distributions has been determined (Figure 6). Figure 6 in which the waveguide width ($w = 4.755$ cm) has a deviation of $\sigma_r = 5\%$ with truncated normal distribution has lees σ_r in f_c i.e. 3.31% . Based on the results, the truncated normal distribution is more appropriate for use with gPC than the other two distributions.

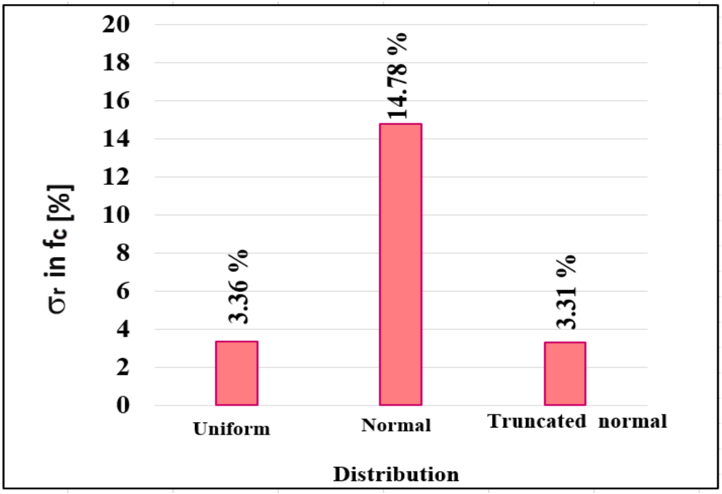

Figure 6: The amount of uncertainty in the fc due to the uncertainty, $\sigma_r = 5\%$, in w, for three different distributions.

Figure 7 shows the PDFs of f_c for three different distributions of w. The dashed line indicates the f_E. All three output PDFs of f_c are almost symmetrical. The normal distribution has led to a greater dispersion in the f_c values than two other distributions and it is also slightly skewed to the right. The results have shown that a deviation of 5% in the waveguide width causes cutoff variation, which is unacceptable and affects waveguide performance. In order to determine the dimensional tolerance, we considered different σ_r in which $(100 - Q)\%$ was less than 50%. We reduced the uncertainty of w, to $\sigma_r = 2\%$. we calculated the statistical quantities based on three distributions of inputs (Table 7). f_μ and positive values of bias show that 2% uncertainty of w also increases f_c from f_E. The percentage $((100 - Q)\%$ shows that more than 50% of the amount of f_c have exceeded the f_E.

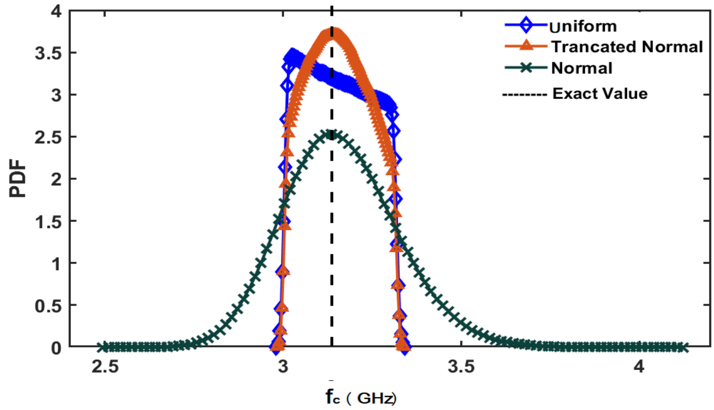

Figure 7: Comparison of cutoff frequency PDFs for three different distributions for random inputs.

Table 7: Statistical quantities of f_c for three different distribution of random inputs in the case of $\sigma_r = 2\%$ in waveguide width ,w.

Distribution type / UQ method	f_μ (GHz)	bias (GHz)	$(100 - Q)\%$ (GHz)
Uniform	3.1544529	0.0012618	50.8607
Normal	3.1952275	0.0420364	56.1638
Truncated normal	3.1544533	0.0012622	50.8750

To obtain tolerance, we considered $\sigma_r = 1\%$ calculated statistical quantities of f_c (Table 8). As shown in Table 8, bias values are negative and close to zero. This shows thatthe $\sigma_r = 1\%$ for w, does not increase the f_c from f_E so there is no change in the waveguide's performance. In addition, the accumulation of f_c greater than f_E is less than 50%, which confirms the results. It can be concluded that a deviation of 1%, in w is allowed. This deviation is equal to 4.755mm. that means, a manufacturer is allowed to consider the waveguide width to be 4.755mm smaller or larger than its actual width. This number is called the dimensional tolerance of this type of waveguide (relative to sensitivity in f_c).

Table 8: Statistical quantities of f_c for three different distribution of random inputs in the case of $\sigma_r = 1\%$ in waveguide width ,w.

Distribution type / UQ method	f_μ (GHz)	bias (GHz)	$(100 - Q)\%$ (GHz)
Uniform	3.1531065	-8.464×10^{-5}	49.9925
Normal	3.1520012	-0.001190	49.6150
Truncated normal	3.1533099	-0.0001187	49.8925

5 Conclusion

Radio Frequency (RF) structures have an uncertainty in their dimensional parameters due to manufacturing errors which causes an uncertainty in the output results. Therefore, the study of these uncertainties improve the results of the structure and the quality of the structure design. The distribution type of random inputs is important in the uncertainty quantification of the results. In this article, we considered the manufacturing error in the WR178 waveguide width, (w), as uncertain variable. We chose three different distributions include uniform, normal and truncated normal distributions for samples of w and investigated its effect on the cutoff frequency using univariable gPC method separately. The simulation of the waveguide was carried out using CST Microwave Studio software. It was shown that the results of two methods are in a good agreement using comparison the results of gPC method with MC method. From the statistical quantities results, it was concluded that the truncated normal distribution has lower contribution and the normal distribution has high contribution in the uncertainty of f_c. Furthermore, we found that the dimensional tolerance of the waveguide width is about 1%. This deviation in the waveguide dimension does not affect on the waveguide performance.

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