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The fractional discrete Weibull distribution

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Abstract: The two-parameter discrete Weibull distribution is an important model especially in reliability studies when the data are reported on a discrete scale. The hazard rate function of a discrete Weibull distribution is monotonically increasing and decreasing. The present paper provides a family of parametric discrete distributions which is an infinite mixture of exponentiated discrete Weibull distributions, and versatile in fitting increasing, decreasing, and bathtub-shaped failure rate models to different discrete life-test data. Some important distributional properties of the model such as the moments, order statistics, and infinite divisibility are investigated and the parameters of the distribution are estimated by the maximum likelihood method. In addition, a real data set is analyzed to show the effectiveness of the model. Finally we conclude the paper.

Keywords: Discrete univariate model; Infinite divisibility; Maximum likelihood estimation; Order statistics.
Mathematics Subject Classification (2010): 62F10, 62H10.

1 Introduction

Recently, Nekoukhou et al. (2021), as a consequence of their study, introduced the class of univariate fractional discrete (FD) distributions whose cumulative distribution function (CDF) is given by

$$F(x;\alpha) = \frac{G^{\alpha}(x)}{2 - G^{\alpha}(x)}, \quad x \in \mathbb{N}_0 = \{0, 1, \ldots\}, \tag{1}$$

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where $\alpha > 0$, and G is an arbitrary discrete CDF. Using the geometric CDF, they investigated the fractional geometric (FG) distribution with CDF

$$F_{FG}(x;\alpha,p) = \frac{(1-p^{x+1})^{\alpha}}{2-(1-p^{x+1})^{\alpha}}, \quad x \in \mathbb{N}_0; \alpha > 0, 0 < p < 1, \tag{2}$$

and study some of its important properties.

The discrete Weibull (DW) distribution, introduced by Nakagawa and Osaki (1975), plays a key role in connection with some situations where lifetimes are recorded on a discrete scale. The hazard rate function of a DW distribution is monotonically increasing and decreasing. Jayakumar and Sankaran (2018) introduced a generalization of the DW distribution. Gillariose et al. (2021) studied an extension of the DW distribution. Ma and Yan (2022) considered the discrete Weibull-Rayleigh distribution.

The aim of the present paper is to study a generalization of the FG distribution. Indeed, the class of fractional discrete Weibull (FDW) distributions is introduced. In this way we have an opportunity to investigate some new aspects of the class of FD distributions. For example, a new representation of (1) is provided which will state that it can be considered as an infinite mixture distribution of some exponentiated discrete distributions. Hence, some of important distributional properties of an FDW distribution, such as its moments, can be obtained in a convenient manner.

The failure rate function of an object, when the failures are reported on a discrete scale, may be bathtub-shaped or unimodal. Therefore, the DW distribution cannot be a suitable choice in dealing with these situations. The failure rate function of the FDW distribution is found to be increasing, decreasing and bathtub-shaped. Hence, the FDW distributions can analyze more failure rate data with respect to the geometric and DW distributions whose hazard rate functions are constant and monotone, respectively. In addition, in the application section we will see that the new model can provide satisfactory fits to some real data and that is competitive with traditional and also newly developed discrete distributions.

The rest of the paper is organized as follows. In Section 2 we provide the FDW distribution and study some of its important distributional properties. For example, the CDF, probability mass function (PMF) and the moments of an FDW distribution are investigated. Moreover, the hazard rate function is studied and the order statistics are discussed. In Section 3, the parameters of an FDW distribution are estimated by means of the maximum likelihood method, and for illustrative purposes, analysis of a real data set is provided in this section. Finally we conclude the paper in Section 4.

2 The FDW distribution

2.1 Basic properties

Nakagawa and Osaki (1975) introduced a discrete counterpart of the well-known Weibull distribution. The CDF and PMF of the discrete Weibull (DW) distribution are given, respectively, by

$$\begin{aligned} F_{DW}(x;\gamma,p) &= 1-p^{(\lfloor x \rfloor+1)^{\gamma}}, \quad x \geq 0, \\ f_{DW}(x;\gamma,p) &= p^{x^{\gamma}} - p^{(x+1)^{\gamma}}, \quad x \in \mathbb{N}_0, \end{aligned}$$

where $\gamma > 0$ and $0 < p < 1$ are the model parameters, and $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

The hazard rate function of a DW distribution is monotone; increasing for $\gamma > 1$, and decreasing for $0 < \gamma < 1$. If $\gamma = 1$, the DW distribution reduces to the geometric distribution and the hazard rate function will be constant.

When the CDF of the DW distribution with parameters γ and p, denoted by $\mathrm{DW}(\gamma, p)$ is inserted into the FD family of distributions, (1), the CDF of the FDW distribution is achieved. For non-negative integer values of x, such CDF is given by

$$F_{FDW}(x;\gamma,\alpha,p) = \frac{\left(1-p^{(x+1)^\gamma}\right)^\alpha}{2-\left(1-p^{(x+1)^\gamma}\right)^\alpha}.$$

The notation $X \sim \mathrm{FDW}(\gamma, \alpha, p)$ represents a discrete random variable X follows the FDW distribution with parameters γ, α and p.

The PMF of the $\mathrm{FDW}(\gamma, \alpha, p)$ distribution, denoted by f_{FDW}, can be obtained as $F(x;\gamma,\alpha,p) - F(x-1;\gamma,\alpha,p)$. More precisely, we have the following result:

$$f_{FDW}(x;\gamma,\alpha,p) = \frac{2\left\{\left(1-p^{(x+1)^\gamma}\right)^\alpha - \left(1-p^{x^\gamma}\right)^\alpha\right\}}{\left\{2-\left(1-p^{(x+1)^\gamma}\right)^\alpha\right\}\left\{2-\left(1-p^{x^\gamma}\right)^\alpha\right\}}, \quad x \in \mathbb{N}_0. \tag{3}$$

Result 2.1 If $\gamma = 1$, then the FDW distribution reduces to the FG distribution, given by (2).

Figure 1 illustrates the PMF plots of FDW distributions for some possible values of the parameters.

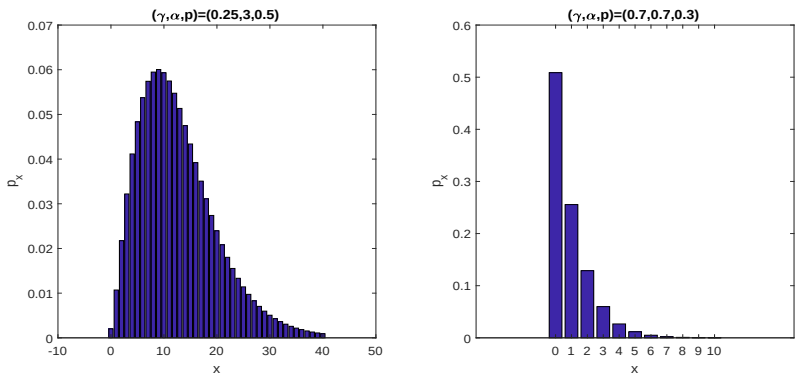

Figure 1: PMF plots of $\mathrm{FDW}(\gamma, \alpha, p)$ distributions.

Nekoukhou and Bidram (2015) introduced the exponentiated DW (EDW) distribution whose CDF, for non-negative integer values of x, is given by

$$F_{EDW}(x;\gamma,\alpha,p) = \left(1-p^{(x+1)^\gamma}\right)^\alpha,$$

where $\gamma > 0$, $\alpha > 0$ and $0 < p < 1$ are the associated parameters. If $\alpha = 1$, then the EDW distribution reduces to the DW distribution. The PMF of an EDW distribution

is also given by

$$f_{EDW}(x;\gamma,\alpha,p) = \left(1-p^{(x+1)^\gamma}\right)^\alpha - \left(1-p^{x^\gamma}\right)^\alpha,$$

where can also be written as

$$\begin{aligned} f_{EDW}(x;\gamma,\alpha,p) &= \sum_{j=1}^{\infty}(-1)^{j+1}\binom{\alpha}{j}\{p^{jx^\gamma} - p^{j(x+1)^\gamma}\} \\ &= \sum_{j=1}^{\infty}(-1)^{j+1}\binom{\alpha}{j} f_{DW}(x;\gamma,p^j), \end{aligned} \tag{4}$$

in which $\binom{\alpha}{j} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-j)j!}$. For an integer α, the sum in (4) stops at α; see Nekoukhou and Bidram (2015).

Result 2.2 The PMF of an FDW distribution, given by (3), can be viewed as the weighted EDW probability mass functions with the weight function

$$w(x) = \frac{2}{\left\{2-\left(1-p^{(x+1)^\gamma}\right)^\alpha\right\}\left\{2-(1-p^{x^\gamma})^\alpha\right\}}. \tag{5}$$

It is interesting to note that the CDF of the FDW distribution has an alternative expression as an infinite mixture of EDW cumulative distribution functions. More precisely, we have the following result.

Theorem 2.1. *The CDF of the FDW(γ,α,p) distribution can be written as an infinite mixture of EDW distributions.*

Proof. By using the series representation $(1-z)^{-1} = \sum_{k=0}^{\infty} z^k, \quad \text{for} \quad |z|<1$, we see that

$$\begin{aligned} F_{FDW}(x;\gamma,\alpha,p) &= \frac{1}{2}\left(1-p^{(x+1)^\gamma}\right)^\alpha \left(1-\frac{\left(1-p^{(x+1)^\gamma}\right)^\alpha}{2}\right)^{-1} \\ &= \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^{k+1}\left(1-p^{(x+1)^\gamma}\right)^{(k+1)\alpha} \\ &= \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^{k+1} F_{EDW}(x;\gamma,(k+1)\alpha,p). \end{aligned} \tag{6}$$

□

As an immediate result, an alternative expression for the PMF of the FDW distribution is given by:

$$f_{FDW}(x;\gamma,\alpha,p) = \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^{k+1} f_{EDW}(x;\gamma,(k+1)\alpha,p).$$

Using (4), the PMF of the FDW distribution can also be described as

$$f_{FDW}(x;\gamma,\alpha,p) = \sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{2}\right)^{k+1}(-1)^{j+1}\binom{(k+1)\alpha}{j} f_{DW}(x;\gamma,p^j). \tag{7}$$

By means of (7) the rth moment of the random variable $X \sim \mathrm{FDW}(\gamma, \alpha, p)$ can be obtained from those of $\mathrm{DW}(\gamma, p^j)$ distributions. In other words, we see that

$$E(X^r) = \sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\sum_{x=1}^{\infty}\left(\frac{1}{2}\right)^{k+1}(-1)^{j+1}\binom{(k+1)\alpha}{j}\left\{x^r-(x-1)^r\right\}p^{jx^\gamma}.$$

In particular, the first and second moments become

$$\begin{aligned} E(X) &= \sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\sum_{x=1}^{\infty}\left(\frac{1}{2}\right)^{k+1}(-1)^{j+1}\binom{(k+1)\alpha}{j}p^{jx^\gamma}, \\ E(X^2) &= \sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\sum_{x=1}^{\infty}\left(\frac{1}{2}\right)^{k+1}(-1)^{j+1}(2x-1)\binom{(k+1)\alpha}{j}p^{jx^\gamma}, \end{aligned}$$

respectively.

The mean and variance of the FDW distribution, for different values of the parameters, have been calculated in Table 1.

Table 1: Mean (Variance) of the $\mathrm{FDW}(\gamma, \alpha, p)$ distributions.

	$\gamma = 0.25$				
α \ p	0.005	0.10	0.25	0.50	0.75
0.05	0.040 (0.065)	0.157 (0.484)	0.312 (1.487)	0.720 (6.465)	1.909 (39.406)
0.5	0.345 (0.485)	1.196 (2.971)	2.235 (8.470)	4.880 (34.557)	12.379 (202.321)
1.0	0.602 (0.742)	1.932 (4.095)	3.514 (11.357)	7.507 (45.526)	18.778 (264.451)
3.0	1.228 (1.069)	3.484 (5.282)	6.118 (14.416)	12.737 (57.406)	31.394 (332.857)
	$\gamma = 0.50$				
0.05	0.007 (0.008)	0.052 (0.093)	0.120 (0.316)	0.312 (1.487)	0.891 (9.517)
0.5	0.074 (0.080)	0.438 (0.668)	0.934 (2.010)	2.235 (8.470)	5.967 (50.292)
1.0	0.142 (0.145)	0.752 (0.996)	1.529 (2.813)	3.514 (11.357)	9.143 (66.088)
3.0	0.361 (0.299)	1.489 (1.388)	2.808 (3.671)	6.118 (14.416)	15.447 (83.277)
	$\gamma = 0.75$				
0.05	0.001 (0.002)	0.023 (0.032)	0.062 (0.120)	0.181 (0.609)	0.557 (4.085)
0.5	0.019 (0.019)	0.208 (0.258)	0.516 (0.838)	1.363 (3.674)	3.836 (22.194)
1.0	0.037 (0.037)	0.376 (0.417)	0.877 (1.230)	2.188 (5.029)	5.934 (29.354)
3.0	0.107 (0.100)	0.823 (0.660)	1.703 (1.682)	3.911 (6.456)	10.131 (37.059)

One can easily investigate that $\frac{\partial}{\partial\alpha}E(X) > 0$, $\frac{\partial}{\partial\gamma}E(X) < 0$ and $\frac{\partial}{\partial p}E(X) > 0$. Table 1 confirms these results and we see that the mean increases when α increases or p increases. In addition, the mean decreases when γ increases.

The variance has the same manner, too. In addition, the variance can be larger or greater than the mean. Therefore, the parameters of an FDW distribution can be adjusted to suit over- and under-dispersed data sets.

The mth quantile of an FDW distribution, say q_m, is given by

$$q_m = \left\{\frac{\log\left(\frac{1-m}{1+m}\right)^{1/\alpha}}{\log p}\right\}^{1/\gamma} - 1.$$

Particularly, the median is immediately achieved by setting $m = 0.5$ in the above equation.

Numerous stochastic orderings between random variables X_1 and X_2 or, equivalently, their distribution functions F and G, have been introduced in the literature. One of the most basic and oldest stochastic order in Probability and Statistics is the simple stochastic order. More precisely, X_1 is said to be stochastically smaller than X_2 (written as $X_1 \prec_{st} X_2$) if for all x, $F(x) \geq G(x)$. In this case, if G is simpler than F, $G(x)$ may provide a useful lower bound for $F(x)$. For more details, we refer, e.g., to Shaked and Shanthikumar (2007).

Now, we have the following results regarding the stochastic ordering of the family of FDW distributions.

Result 2.3 If $X_1 \sim \text{FDW}(\gamma_1, \alpha, p)$, $X_2 \sim \text{FDW}(\gamma_2, \alpha, p)$ and $\gamma_1 < \gamma_2$, then $X_2 \prec_{st} X_1$.

Result 2.4 If $X_1 \sim \text{FDW}(\gamma, \alpha_1, p)$, $X_2 \sim \text{FDW}(\gamma, \alpha_2, p)$ and $\alpha_1 > \alpha_2$, then $X_2 \prec_{st} X_1$.

Result 2.5 If $X_1 \sim \text{FDW}(\gamma, \alpha, p_1)$, $X_2 \sim \text{FDW}(\gamma, \alpha, p_2)$ and $p_1 > p_2$, then $X_2 \prec_{st} X_1$.

The proofs of Results 2.3-2.5 can be obtained by using (6) and hence the details are avoided.

2.2 Survival and hazard rate functions

The survival function of the $\text{FDW}(\gamma, \alpha, p)$ distribution, for non-negative integer vales of x, is given by

$$S_{FDW}(x; \gamma, \alpha, p) = P(X \geq x) = \frac{2\left\{1 - (1 - p^{x^\gamma})^\alpha\right\}}{2 - (1 - p^{x^\gamma})^\alpha}.$$

Discrete hazard rates arise in several common situations in Reliability theory where clock time is not the best scale on which to describe lifetime. For example, in weapons reliability, the number of rounds fired until failure is more important than age in failure. This is the case also when a piece of equipment operates in cycles and the observation is the number of cycles successfully completed prior to failure. In other situations a device is monitored only once per time period and the observation then is the number of time periods successfully completed prior to the failure of the device (cf. Shaked et al. (1995)).

The hazard rate function of the $\text{FDW}(\gamma, \alpha, p)$ distribution is given by

$$\begin{aligned} h_{FDW}(x; \gamma, \alpha, p) &= \frac{f(x; \gamma, \alpha, p)}{S(x; \gamma, \alpha, p)} \\ &= \frac{(1 - p^{(x+1)^\gamma})^\alpha - (1 - p^{x^\gamma})^\alpha}{\left\{2 - (1 - p^{(x+1)^\gamma})^\alpha\right\}\left\{1 - (1 - p^{x^\gamma})^\alpha\right\}}, \quad x \in \mathbb{N}_0. \end{aligned}$$

Figure 2 illustrates the hazard rate function of the $\text{FDW}(\gamma, \alpha, p)$ distribution for different values of the parameters. The hazard rate function of the FDW distribution can be increasing, decreasing and bathtub-shaped. Hence, FDW distributions are more flexible than other discrete distributions such as the geometric and DW, whose hazard rate functions are constant and monotone, respectively.

2.3 Rényi and Shanon entropies

The entropy of a random variable X is a measure of uncertainty variation. The Rényi and Shanon entropies are important in statistics, reliability and quantum information theory.

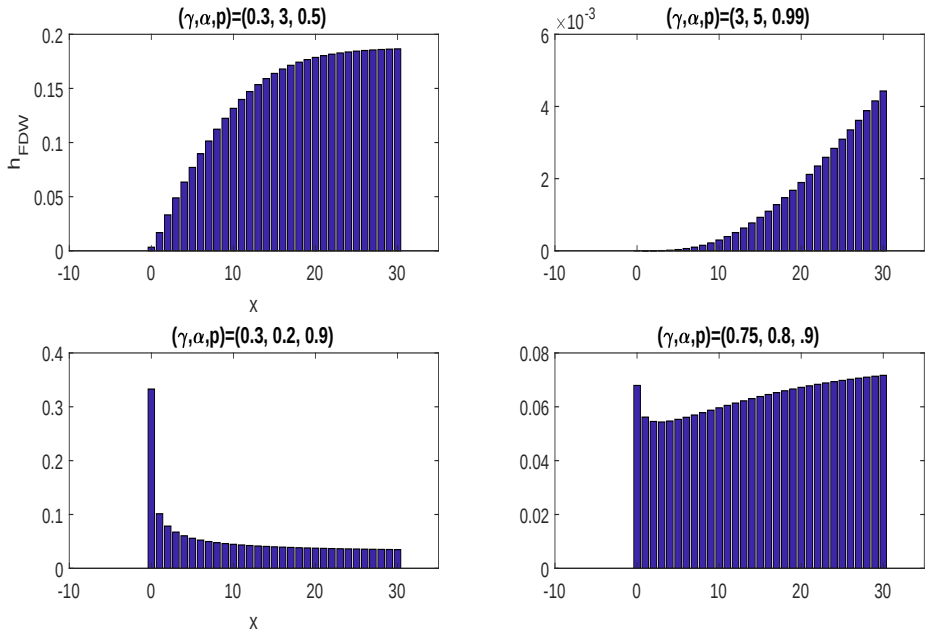

Figure 2: Hazard rate function plots of FDW(γ, α, p) distributions.

The Rényi and Shannon entropies of a discrete random variable X, whose PMF is p_x, in general, are given by:

$$\begin{aligned} R_\eta &= \frac{1}{1-\eta} \log \sum_{x=0}^{\infty} p_x^\eta, \quad \eta \neq 1, \\ S &= -\sum_{x=0}^{\infty} p_x \log p_x, \end{aligned}$$

respectively.

In view of the fact that the PMF of an FDW distribution can be written as

$$p_x = w(x) \left\{ \left(1 - p^{(x+1)^\gamma}\right)^\alpha - \left(1 - p^{x^\gamma}\right)^\alpha \right\},$$

where $w(x)$ is given by (5), and using the binomial expansion, we find that

$$R_\eta = \frac{1}{1-\eta} \log \sum_{x=0}^{\infty} \sum_{j=0}^{\infty} w^\eta(x)(-1)^j \binom{\eta}{j} \left(1 - p^{x^\gamma}\right)^{j\alpha} \left(1 - p^{(x+1)^\gamma}\right)^{(\eta-j)\alpha}, \quad \eta \neq 1.$$

It must be mentioned that for an integer value of η, the interior summation stops at η in the above relation.

The Shanon entropy of the FDW distribution is also given by

$$S = -\sum_{x=0}^{\infty} w(x) \log w(x) \left\{ \left(1 - p^{(x+1)^\gamma}\right)^\alpha - \left(1 - p^{x^\gamma}\right)^\alpha \right\}$$

$$-\sum_{x=0}^{\infty} w(x)\left\{\left(1-p^{(x+1)^{\gamma}}\right)^{\alpha}-\left(1-p^{x^{\gamma}}\right)^{\alpha}\right\}\log\left\{\left(1-p^{(x+1)^{\gamma}}\right)^{\alpha}-\left(1-p^{x^{\gamma}}\right)^{\alpha}\right\}.$$

The Rényi and Shanon entropies of the FDW distribution have been calculated in Table 2 for some values of the parameters.

Table 2: Rényi and Shanon entropies of the FDW distribution.

η	(γ,α,p)	Rényi	Shanon
0.50	(0.25, 0.5, 0.5)	3.1059	2.6563
0.995	(0.25, 0.9, 0.75)	3.8926	3.8909
0.998	(0.8, 1.1, 0.8)	3.7330	3.7323
1.005	(0.5, 0.75, 0.8)	3.3542	3.3564
1.50	(0.8, 1.0, 0.75)	4.3697	4.4941
2.0	(0.75, 0.8, 0.98)	5.1272	4.8112
2.5	(1.2, 0.3, 0.9)	1.6389	2.5301

According to the results of Table 2, we see that when $\eta \to 1$, the values of Rényi entropy are approximately equivalent to that of Shanon entropy. Therefore, the known relation between these two entropies, i.e. $S = \lim_{\eta\to 1} R_{\eta}$, is investigated in FDW distributions, too.

2.4 Order statistics

Order statistics are among the most fundamental tools in Non-parametric statistics and Inference. They usually enter the problems of estimation and hypothesis testing. Here, we want to establish some general relations regarding the FDW distributions. Let $F_{i:n}(x;\gamma,\alpha,p)$ and $f_{i:n}(x;\gamma,\alpha,p)$ denote the CDF and PMF of the ith order statistic of a random sample of size n from the FDW distribution.

As we know, the CDF of the ith order statistic is

$$F_i(x;\gamma,\alpha,p)=\sum_{\ell=i}^{n}\binom{n}{\ell}\{F_{FDW}(x;\gamma,\alpha,p)\}^{\ell}\{1-F_{FDW}(x;\gamma,\alpha,p)\}^{n-\ell},$$

see, e.g., David and Nagaraja (2003).

Expansion of $\{1-F_{FDW}(x;\gamma,\alpha,p)\}^{n-\ell}$ yields that

$$F_i(x;\gamma,\alpha,p)=\sum_{\ell=i}^{n}\sum_{j=0}^{n-\ell}\binom{n}{\ell}\binom{n-\ell}{j}(-1)^j\{F_{FDW}(x;\gamma,\alpha,p)\}^{\ell+j}.$$

By using (6) and the expansion

$$(\sum_{i=0}^{\infty}a_i)^{\ell}=\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\ldots\sum_{m_\ell=0}^{\infty}\prod_{j=1}^{\ell}a_{m_j},\quad \ell=1,2,\ldots,$$

discussed by Erdelyi et al. (1953), we can write

$$[F_{FDW}(x;\gamma,\alpha,p)]^{\ell+j} = \left[\sum_{t=0}^{\infty}\left(\frac{1}{2}\right)^{t+1}\left(1-p^{(x+1)^{\gamma}}\right)^{(t+1)\alpha}\right]^{\ell+j}$$

$$= \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\cdots\sum_{m_{\ell+j}=0}^{\infty}\prod_{r=1}^{\ell+j}\left(\frac{1}{2}\right)^{m_r+1}\left(1-p^{(x+1)^\gamma}\right)^{(m_r+1)\alpha}.$$

Therefore, the CDF of the ith order statistic becomes

$$\begin{aligned}
F_{i:n}(x;\gamma,\alpha,p) &= \sum_{\ell=i}^{n}\sum_{j=0}^{n-\ell}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\cdots\sum_{m_{\ell+j}=0}^{\infty}\left(\frac{1}{2}\right)^{\sum_{r=1}^{\ell+j} m_r+\ell+j} \\
&\quad\times\left(1-p^{(x+1)^\gamma}\right)^{\alpha\left(\sum_{r=1}^{\ell+j} m_r+\ell+j\right)} \\
&= \sum_{\ell=i}^{n}\sum_{j=0}^{n-\ell}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\cdots\sum_{m_{\ell+j}=0}^{\infty}\left(\frac{1}{2}\right)^{\sum_{r=1}^{\ell+j} m_r+\ell+j} \\
&\quad\times F_{EDW}\left(x;\gamma,\alpha(\sum_{r=1}^{\ell+j} m_r+\ell+j),p\right).
\end{aligned}$$

The PMF of the ith order statistic for non-negative integer values of x, $f_{i:n}(x) = F_{i:n}(x) - F_{i:n}(x-1)$, is given by

$$\begin{aligned}
f_{i:n}(x;\gamma,\alpha,p) &= \sum_{\ell=i}^{n}\sum_{j=0}^{n-\ell}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\cdots\sum_{m_{\ell+j}=0}^{\infty}\left(\frac{1}{2}\right)^{\sum_{r=1}^{\ell+j} m_r+\ell+j} \\
&\quad\times f_{EDW}\left(x;\gamma,\alpha(\sum_{r=1}^{\ell+j} m_r+\ell+j),p\right).
\end{aligned}$$

It is seen that the PMF of the ith order statistic of an FDW distribution can be written as a linear combination of the EDW PMFs. This is a useful advantage in view of the fact that some characteristics of the order statistics can be obtained from those of EDW PMFs. For example, different moments of the order statistics, which is widely used in, e.g., L-moments, can be obtained in a convenient manner; see, e.g., David and Nagaraja (2003). For instance, the mean of the $i$the order statistic, say $\mu_{i:n}$, is obtained as

$$\mu_{i:n}=\sum_{\ell=i}^{n}\sum_{j=0}^{n-\ell}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\cdots\sum_{m_{\ell+j}=0}^{\infty}\sum_{z=1}^{\infty}\sum_{x=1}^{\infty}\left(\frac{1}{2}\right)^{\sum_{r=1}^{\ell+j} m_r+\ell+j}(-1)^{z+1}\binom{\alpha(\sum_{r=1}^{\ell+j} m_r+\ell+j)}{z}p^{zx^\gamma}.$$

2.5 Infinite divisibility

Here we make the following note in regards to the famous structural property of infinite divisibility of the FDW distribution. Infinite divisibility has a close relation to the central limit theorem and waiting time distributions. Hence, it is an important question in modelling to know whether a given distribution is infinitely divisible or not.

Remember that according to Warde and Katti (1971), a PMF is infinitely divisible if p_{x+1}/p_x, for all $x \in \mathbb{N}_0$, forms a monotone increasing sequence.

First note that in an FDW(γ, α, p) distribution,

$$\begin{aligned} p_0 &= f_{FDW}(0;\gamma,\alpha,p) = \frac{(1-p)^\alpha}{2-(1-p)^\alpha} > 0, \\ p_1 &= f_{FDW}(1;\gamma,\alpha,p)) = \frac{2\left\{(1-p^{2^\gamma})^\alpha - (1-p)^\alpha\right\}}{\{2-(1-p^{2^\gamma})^\alpha\}\{2-(1-p)^\alpha\}} > 0. \end{aligned}$$

In addition, note that

$$\frac{p_{x+1}}{p_x} = \frac{f_{FDW}(x+1;\gamma,\alpha,p)}{f_{FDW}(x;\gamma,\alpha,p)} = w^*(x)\frac{f_{EDW}(x+1;\gamma,\alpha,p)}{f_{EDW}(x;\gamma,\alpha,p)},$$

where $w^*(x) = \frac{2-(1-p^{x^\gamma})^\alpha}{2-(1-p^{(x+2)^\gamma})^\alpha}$. In Table 3, p_{x+1}/p_x has been calculated for $x = 0, 1, 2, 3, 4$ and some different values of the parameters. In general, increasing trends are not seen in the sequences. So, in general, the FDW distributions are not infinitely divisible.

Table 3: The behaviour of p_{x+1}/p_x in an FDW distribution.

(γ, α, p)	p_1/p_0	p_2/p_1	p_3/p_2	p_4/p_3	p_5/p_4
(0.25, 0.5, 0.5)	0.4892	0.7882	0.8317	0.8469	0.8528
(0.75, 0.8. 0.1)	0.2729	0.1990	0.1816	0.1785	0.1779
(0.75, 0.8, 0.9)	0.7707	0.9167	0.9420	0.9518	0.9561
(0.5, 1.5, 0.75)	1.6523	1.1819	1.0881	1.0444	1.0173

Since the classes of self-decomposable and stable distributions, in their discrete concepts, are subclasses of infinitely divisible distributions, one can conclude that the FDW distributions can be neither self-decomposable nor stable; see, e.g., Steutel and Van Harn (2004).

3 Statistical inference

3.1 Maximum likelihood estimation

Let $X \sim$ FDW(γ, α, p). In addition, let us consider $\mathbf{\Omega} = (\gamma, \alpha, p)^T$. The likelihood function of a single observation x is given by

$$L(\mathbf{\Omega}) = \frac{2\left\{\left(1-p^{(x+1)^\gamma}\right)^\alpha - \left(1-p^{x^\gamma}\right)^\alpha\right\}}{\left\{2-\left(1-p^{(x+1)^\gamma}\right)^\alpha\right\}\left\{2-(1-p^{x^\gamma})^\alpha\right\}}. \tag{8}$$

The derivatives of the likelihood function with respect to the parameters involved are given by

$$\frac{\partial L}{\partial \gamma} = \frac{2\left\{-\frac{\left(1-p^{(x+1)^\gamma}\right)^\alpha \alpha p^{(x+1)^\gamma}(x+1)^\gamma \log(x+1)\log p}{1-p^{(x+1)^\gamma}} + \frac{\left(1-p^{x^\gamma}\right)^\alpha \alpha p^{x^\gamma} x^\gamma \log x \log p}{1-p^{x^\gamma}}\right\}}{\left\{2-\left(1-p^{(x+1)^\gamma}\right)^\alpha\right\}\left\{2-(1-p^{x^\gamma})^\alpha\right\}}$$

$$
\begin{aligned}
&-\frac{2\left\{\left(1-p^{(x+1)^\gamma}\right)^\alpha-\left(1-p^{x^\gamma}\right)^\alpha\right\}\left(1-p^{(x+1)^\gamma}\right)^\alpha \alpha p^{(x+1)^\gamma}(x+1)^\gamma \log(x+1)\log p}{\left\{2-\left(1-p^{(x+1)^\gamma}\right)^\alpha\right\}^2\left\{2-(1-p^{x^\gamma})^\alpha\right\}\left(1-p^{(x+1)^\gamma}\right)}\\
&-\frac{2\left\{\left(1-p^{(x+1)^\gamma}\right)^\alpha-\left(1-p^{x^\gamma}\right)^\alpha\right\}\left(1-p^{x^\gamma}\right)^\alpha \alpha p^{x^\gamma}x^\gamma \log x\log p}{\left\{2-\left(1-p^{(x+1)^\gamma}\right)^\alpha\right\}\left\{2-(1-p^{x^\gamma})^\alpha\right\}^2\left(1-p^{x^\gamma}\right)},\\
\frac{\partial L}{\partial \alpha}=&\frac{2\left\{\left(1-p^{(x+1)^\gamma}\right)^\alpha\log\left(1-p^{(x+1)^\gamma}\right)-\left(1-p^{x^\gamma}\right)^\alpha\log(1-p^{x^\gamma})\right\}}{\left\{2-\left(1-p^{(x+1)^\gamma}\right)^\alpha\right\}\left\{2-(1-p^{x^\gamma})^\alpha\right\}}\\
&+\frac{2\left\{\left(1-p^{(x+1)^\gamma}\right)^\alpha-\left(1-p^{x^\gamma}\right)^\alpha\right\}\left(1-p^{(x+1)^\gamma}\right)^\alpha\log\left(1-p^{(x+1)^\gamma}\right)}{\left\{2-\left(1-p^{(x+1)^\gamma}\right)^\alpha\right\}^2\left\{2-(1-p^{x^\gamma})^\alpha\right\}}\\
&+\frac{2\left\{\left(1-p^{(x+1)^\gamma}\right)^\alpha-\left(1-p^{x^\gamma}\right)^\alpha\right\}\left(1-p^{x^\gamma}\right)^\alpha\log\left(1-p^{x^\gamma}\right)}{\left\{2-\left(1-p^{(x+1)^\gamma}\right)^\alpha\right\}\left\{2-(1-p^{x^\gamma})^\alpha\right\}^2},\\
\frac{\partial L}{\partial p}=&\frac{2\left\{-\frac{\left(1-p^{(x+1)^\gamma}\right)^\alpha\alpha p^{(x+1)^\gamma}(x+1)^\gamma}{p\left(1-p^{(x+1)^\gamma}\right)}+\frac{\left(1-p^{x^\gamma}\right)^\alpha\alpha p^{x^\gamma}x^\gamma}{p\left(1-p^{x^\gamma}\right)}\right\}}{\left\{2-\left(1-p^{(x+1)^\gamma}\right)^\alpha\right\}\left\{2-(1-p^{x^\gamma})^\alpha\right\}}\\
&-\frac{2\left\{\left(1-p^{(x+1)^\gamma}\right)^\alpha-\left(1-p^{x^\gamma}\right)^\alpha\right\}\left(1-p^{(x+1)^\gamma}\right)^\alpha\alpha p^{(x+1)^\gamma}(x+1)^\gamma}{\left\{2-\left(1-p^{(x+1)^\gamma}\right)^\alpha\right\}^2\left\{2-(1-p^{x^\gamma})^\alpha\right\}p\left(1-p^{(x+1)^\gamma}\right)}\\
&-\frac{2\left\{\left(1-p^{(x+1)^\gamma}\right)^\alpha-\left(1-p^{x^\gamma}\right)^\alpha\right\}\left(1-p^{x^\gamma}\right)^\alpha\alpha p^{x^\gamma}x^\gamma}{\left\{2-\left(1-p^{(x+1)^\gamma}\right)^\alpha\right\}\left\{2-(1-p^{x^\gamma})^\alpha\right\}^2p(1-p^{x^\gamma})p\left(1-p^{(x+1)^\gamma}\right)}.
\end{aligned}
$$

Now, let $x_1,\ldots,x_n$ be observations of a random sample drawn from an FDW(γ,α,p) distribution. In this case, the total likelihood function is

$$L_n(\boldsymbol{\Omega})=\prod_{k=1}^{n}L_{\lfloor k\rfloor}(\boldsymbol{\Omega}),$$

where $L_{\lfloor k\rfloor}(\boldsymbol{\Omega})$; $k=1,...,n$, is given by (8). The maximum likelihood estimate (MLE) of $\boldsymbol{\Omega}$, say $\hat{\boldsymbol{\Omega}}$, is obtained by solving the nonlinear equation

$$\boldsymbol{M}_n=(\partial L_n/\partial\gamma,\partial L_n/\partial\alpha,\partial L_n/\partial p)^T=\mathbf{0}.$$

It is obvious that a numerical method must be used in order to solve the above equation.

The Fisher information matrix is also given by

$$\boldsymbol{I}(\boldsymbol{\Omega})=[I_{\omega_i,\omega_j}]_{3\times3},\quad i,j=1,2,3,$$

whose components can be calculated, numerically, by the relation

$$I_{\omega_i,\omega_j}=E(-\frac{\partial^2L(\boldsymbol{\Omega})}{\partial\omega_i\partial\omega_j}),\quad i,j=1,2,3.$$

The total Fisher information matrix is given by $\boldsymbol{I}_n(\boldsymbol{\Omega}) = n\boldsymbol{I}(\boldsymbol{\Omega})$ which can be approximated by

$$\boldsymbol{I}_n(\hat{\boldsymbol{\Omega}}) \approx [-\frac{\partial^2 L_n(\boldsymbol{\Omega})}{\partial\omega_i\partial\omega_j} \mid_{\boldsymbol{\Omega}=\hat{\boldsymbol{\Omega}}}]_{3\times 3} \quad i,j = 1,2,3.$$

Under some regularity conditions given, e.g., by Ferguson (1996), $\hat{\boldsymbol{\Omega}}$ has an asymptotic normal distribution as $N_3(\boldsymbol{\Omega}, \boldsymbol{I}_n(\hat{\boldsymbol{\Omega}})^{-1})$. Asymptotic normal distributions are usually used for constructing approximate confidence intervals, confidence regions, and testing hypotheses of the parameters.

Table 4 presents the maximum likelihood estimates of $\boldsymbol{\Omega} = (\gamma, \alpha, p)^T$ of an FDW distribution and also contains their standard errors (SEs) for different values of n as a kind of simulated example. The SEs are attained by means of the asymptotic covariance matrix of the MLEs.

Table 4: MLEs of the FDW(0.50, 1.00, 0.75) distribution for different values of n.

n	$(\hat{\gamma}, \hat{\alpha}, \hat{p})$	$\hat{SE}(\hat{\gamma})$	$\hat{SE}(\hat{\alpha})$	$\hat{SE}(\hat{p})$
50	(0.62, 0.88, 0.69)	0.085	0.1429	0.1057
100	(0.54, 0.94, 0.78)	0.0093	0.1123	0.0101
200	(0.59, 1.09, 0.74)	0.0007	0.0973	0.0082
500	(0.53, 1.11, 0.72)	9×10^{-6}	0.0152	0.0071

3.2 Stress-strength parameter

The stress-strength parameter $R = P(X \geq Y)$ is a measure of component reliability and its estimation problem when X and Y are independent and follow a specified common distribution has been widely discussed in the literature. Suppose that the random variable X is the strength of a component which is subjected to a random stress Y. The estimation of R in regards to an FDW distribution is now considered, in view of the fact that a relatively small amount of work is devoted to discrete or categorical data.

The stress-strength parameter, in the discrete setup, is defined as $R = P(X \geq Y)$. Now, let $X \sim \text{FDW}(\boldsymbol{\Omega_1})$ and $Y \sim \text{FDW}(\boldsymbol{\Omega_2})$, where $\boldsymbol{\Omega_1} = (\gamma_1, \alpha_1, p_1)^T$ and $\boldsymbol{\Omega_2} = (\gamma_2, \alpha_2, p_2)^T$. In this case, we find that

$$\begin{aligned} R &= \sum_{x=0}^{\infty} f_{FDW}(x;\gamma_1,\alpha_1,p_1)F_{FDW}(x;\gamma_2,\alpha_2,p_2) \\ &= \sum_{x=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^{k+1} f_{EDW}(x;\gamma_1,(k+1)\alpha_1,p_1)F_{FDW}(x;\gamma_2,\alpha_2,p_2) \\ &= \sum_{x=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^{k+1}\left[\left\{1-p_1^{(x+1)^{\gamma_1}}\right\}^{(k+1)\alpha_1} - \left\{1-p_1^{x^{\gamma_1}}\right\}^{(k+1)\alpha_1}\right] \\ &\quad \times \frac{\left\{1-p_2^{(x+1)^{\gamma_2}}\right\}^{\alpha_2}}{2-\left\{1-p_2^{(x+1)^{\gamma_2}}\right\}^{\alpha_2}}. \end{aligned} \tag{9}$$

Assume that $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_m$ are independent observations from $X \sim \text{FDW}(\boldsymbol{\Omega_1})$ and $Y \sim \text{FDW}(\boldsymbol{\Omega_2})$, respectively. The total likelihood function is $L_R(\boldsymbol{\Omega^*}) = L_n(\boldsymbol{\Omega_1})L_m(\boldsymbol{\Omega_2})$, where $\boldsymbol{\Omega^*} = (\boldsymbol{\Omega_1}, \boldsymbol{\Omega_2})$. The score vector is given by

$$M_R(\boldsymbol{\Omega^*}) = (\partial L_R/\partial\gamma_1, \partial L_R/\partial\alpha_1, \partial L_R/\partial p_1, \partial L_R/\partial\gamma_2, \partial L_R/\partial\alpha_2, \partial L_R/\partial p_2),$$

and the MLE of $\boldsymbol{\Omega^*}$, say $\hat{\boldsymbol{\Omega}}^*$, may be attained from that of nonlinear equation $M_R(\boldsymbol{\Omega^*}) = \mathbf{0}$. Hence, by substituting the MLEs in (9), the stress-strength parameter R will be estimated.

3.3 Application

In this section, the FDW model will be examined for a real data set which is given by Karlis et al. (2001) on the numbers of fires in Greece for the period from 1 July 1998 to 31 August of the same year. This data set consists of 123 observation and has been presented in Table 5. Only fires in forest districts are considered. Bakouch et al. (2014) considered these data to indicate the potentiality of discrete Lindley (DL) distribution in data modeling and compared it with Poisson, geometric and discrete gamma (DG) distributions. The pmf of DG distribution which has been used first by Yang (1994) and recently considered by Chakraborty and Chakravarty (2012), for $x \in \mathbb{N}_0$, is given by

$$p_x = \frac{\gamma(\alpha, \beta(x+1)) - \gamma(\alpha, \beta x)}{\Gamma(\alpha)}, \quad \alpha > 0,\ \beta > 0,$$

where $\gamma(a, x) = \int_0^x t^{a-1}e^{-t}dt$ denotes the incomplete gamma function. Additionally, the pmf of the DL distribution for $x \in \mathbb{N}_0$ is given by

$$p_x = \frac{p^x}{1+\theta}\{\theta(1-2p) + (1-p)(1+\theta x)\}, \quad 0 < p < 1,\ \theta > 0.$$

Nekoukhou and Bidram (2015), used these data to compare the capacity of the EDW distribution and a special case of it, i.e., the generalized discrete Rayleigh (GDR) model. Now, the FDW distribution is compared with the above discrete distributions. In addition, because of the over dispersion phenomena in the data set, $\overline{x} = 5.3984$ and $s^2 = 30.0449$, the negative binomial (NB) distribution is also compared with the others. The maximum likelihood method is used to obtain the estimates of the parameters of the FDW distribution. Comparing the FDW model with its rival models is performed by using the Akaike information criterion (AIC) and Kolmogrov-Smirnov (K-S) test statistic. Table 6 indicates the fitting computations which consists of the MLEs, AICs and the values of the K-S test statistics determined by the fitting models. The MLEs and K-S test statistic values of the DL and DG distributions, given in this table, are directly reported from Table 6 of Bakouch et al. (2014).

Table 5: Numbers of fires in Greece.

Numbers	0	1	2	3	4	5	6	7	8	9	10	11	12	15	16	20	43
Frequency	16	13	14	9	11	13	8	4	9	6	3	4	6	4	1	1	1

According to the values of the K-S test statistics and AICs in Table 6, it seems that FDW model gives a satisfactory fit to this real data set.

Table 6: MLEs, AICs and K-S statistic values for the FDW distribution and its rival models.

Models	MLEs	AIC	K-S statistic
FDW	$(\hat{\gamma}, \hat{\alpha}, \hat{p}) = (0.4975, 0.9509, 0.6298)$	684.2392	0.1249
EDW	$(\hat{\alpha}, \hat{\gamma}, \hat{p}) = (1.0809, 1.0923, 0.8599)$	685.5859	0.1254
GDR	$(\hat{\gamma}, \hat{p}) = (0.3934, 0.9924)$	694.6178	0.1467
NB	$(\hat{r}, \hat{p}) = (1.3360, 0.1984)$	683.2989	0.3350
DL	$(\hat{\theta}, \hat{p}) = (0.3090, 0.7343)$	685.8067	0.1122
DG	$(\hat{\alpha}, \hat{\beta}) = (0.7525, 0.1543)$	749.7162	0.2683

To construct approximate confidence intervals for the parameters of the FDW distribution and also for evaluating the accuracy of the estimated parameters, we use the corresponding estimated SEs. More precisely, we can find $\hat{\mathrm{SE}}(\hat{\gamma}) = 0.0044$, $\hat{\mathrm{SE}}(\hat{\alpha}) = 0.1047$ and $\hat{\mathrm{SE}}(\hat{p}) = 0.0086$. Hence, for instance, 95% asymptotic confidence intervals for FDW parameters are obtained as $\gamma \in (0.5 \mp 0.0086)$, $\alpha \in (0.951 \mp 0.2052)$ and $p \in (0.63 \mp 0.0168)$.

4 Conclusions and comments

In this paper, a new representation of the class of fractional discrete distributions was considered. Indeed, it was illustrated that the CDF (1) can be considered as an infinite mixture of the exponentiated arbitrary discrete G distributions. More precisely, one can show that the CDF of the class of fractional discrete distributions can be written as $F(x;\alpha) = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{k+1} G^{\alpha(k+1)}(x)$, and its corresponding PMF, for integer values of x, is given by

$$f(x;\alpha) = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{k+1} \left\{ G^{\alpha(k+1)}(x) - G^{\alpha(k+1)}(x-1) \right\}.$$

It is obvious that $G^{\alpha(k+1)}(x) - G^{\alpha(k+1)}(x-1)$ is the corresponding PMF of $G^{\alpha(k+1)}(x)$. In the case that G is the CDF of the discrete Weibull distribution, $G^{\alpha(k+1)}$ is the CDF of the exponentiated discrete Weibull distribution, and the CDF (PMF) of the corresponding fractional discrete distribution, can be written as an infinite mixture of the CDF (PMF) of the exponentiated discrete Weibull distributions.

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