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Research Paper

Prediction for the future system failures based on Type-II censored coherent systems data under a proportional hazard model

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Abstract: In this paper, we consider a k-component coherent system while the system lifetimes are observed, the system structure is known and the component lifetime follows the proportional hazard rate model. We discuss the prediction problem based on Type-II censored coherent system lifetime data. For predicting the future system failures, we obtain the maximum likelihood predictor, the best unbiased predictor, the conditional median predictor and the Bayesian predictors. As it seems that the integrals of the Bayes prediction do not possess closed forms, the Metropolis-Hastings method is applied to approximating these integrals. Different interval predictors based on classical and Bayesian approaches are derived. A numerical example is presented to illustrate the prediction methods used in this paper. A Monte Carlo simulation study is performed to evaluate and compare the performances of different prediction methods.

Keywords: Bayesian predictor; Best unbiased predictor; Conditional median predictor; Maximum likelihood predictor; Prediction intervals.
Mathematics Subject Classification (2010): 62N05, 90B25.

1 Introduction

Prediction of unobserved or censored observations is an interesting topic, especially in the viewpoint of acturial, medical and engineering sciences. In reliability and system lifetime data analysis, the study of coherent systems is one of the important topics. Researchers and experimenters are interested in learning the lifetime characteristic of the system as well as the lifetime characteristic of the components that make up the system. There are numerous situations that the lifetimes of k-component coherent

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systems can be observed but not the lifetimes of the components and the prediction of the future failures is of interest. Hence, in this paper, we consider the prediction of future system failures based on Type-II censored system lifetime data.

Let T be the lifetime of a coherent system with independent and identically distributed (i.i.d.) component lifetimes $X_1, X_2, \ldots, X_k$ with common absolutely continuous cumulative distribution function $F_X(\cdot)$, probability density function $f_X(\cdot)$, and survival function $\bar{F}_X(\cdot) = 1 - F_X(\cdot)$. We denote the corresponding order statistics of the lifetimes of the k components as $X_{1:k} < X_{2:k} < \ldots < X_{k:k}$. Further, we denote the survival function of the i-th order statistic by $\bar{F}_{i:k}(\cdot)$. Suppose n independent k-component systems with the same structure are placed on a life-test with the corresponding lifetimes $T_1, T_2, \ldots, T_n$ being identically distributed as T with cumulative distribution function $F_T(\cdot)$, probability density function $f_T(\cdot)$ and survival function $\bar{F}_T(\cdot) = 1 - F_T(\cdot)$. Samaniego (1985) showed that the probability density function and survival function of the system lifetime T can be written as

$$\begin{aligned} f_T(t) &= \sum_{i=1}^{k} p_i \binom{k}{i} i f_X(t)[F_X(t)]^{i-1}[\bar{F}_X(t)]^{k-i}, \\ \bar{F}_T(t) &= \sum_{i=1}^{k} p_i \sum_{j=0}^{i-1} \binom{k}{j} [F_X(t)]^{j}[\bar{F}_X(t)]^{k-j}, \end{aligned}$$

respectively, where p_i is the i-th component in the system signature vector for the k-component system such that $p_i = \Pr(T = X_{i:k})$, and $\sum_{i=1}^{k} p_i = 1$ (see, Kochar et al. (1999) and Samaniego (2007)).

Navarro et al. (2007) showed that the survival function and probability density function of a coherent system can be expressed as

$$\bar{F}_T(t) = \sum_{i=1}^{k} a_i \bar{F}_{1:i}(t) = \sum_{i=1}^{k} a_i [\bar{F}_X(t)]^{i}, \tag{1}$$

where $\bar{F}_{1:i}(\cdot)$ is the survival function of the series system lifetime with i components. The vector $\boldsymbol{a} = (a_1, a_2, \ldots, a_k)$ is called the minimal signature of the system and $a_1, a_2, \ldots, a_k$ are some negative and nonnegative integers that do not depend on F_X, and satisfy $\sum_{i=1}^{k} a_i = 1$.

In this paper, we consider the popular proportional hazard rate (PHR) model for the common distribution of the i.i.d lifetimes of the components, i.e., we assume that the survival function of X_i is

$$\bar{F}_X(t) = [\bar{F}_0(t)]^{\theta}, \tag{2}$$

for $i = 1, 2, \ldots, k$, where $\theta > 0$ is the unknown parameter and $\bar{F}_0(t)$ is the baseline survival function of a lifetime distribution with support $[0, \infty)$, whose form is completely specified and it does not depend on θ. The PHR model covers some commonly used statistical lifetime distributions which are applicable to modeling component lifetimes. The following are some examples:

i) Exponential distribution: $EXP(\theta)$ with $\bar{F}_0(t) = e^{-t}$, $t > 0$, and the survival function is given by $\bar{F}_X(t) = e^{-\theta t}$, $t > 0$, $\theta > 0$.

ii) Type-II standard Pareto distribution: $Pa(\theta)$ with $\bar{F}_0(t) = \frac{1}{1+t}$, $t > 0$, and the survival function is given by $\bar{F}_X(t) = \left(\frac{1}{1+t}\right)^{\theta}$, $t > 0$, $\theta > 0$.
iii) Weibull distribution with known shape parameter: $We(\theta)$ with $\bar{F}_0(t) = e^{-t^{\beta}}$, $t > 0$, and known shape parameter β. The survival function is given by $\bar{F}_X(t) = e^{-\theta t^{\beta}}$, $t > 0$, $\theta > 0$.

Based on the model in (2) and from (1), the probability density function and survival function of the coherent system lifetime are given by

$$f_T(t) = \theta f_0(t) \sum_{j=1}^{k} j a_j \bar{F}_0^{\,j\theta-1}(t), \tag{3}$$

$$\bar{F}_T(t) = \sum_{j=1}^{k} a_j \bar{F}_0^{\,j\theta}(t), \tag{4}$$

respectively, where $f_0(t) = \frac{d}{dt} F_0(t)$ is the baseline probability density function.

In a Type-II censored experiment, n independent k-components systems with the same system structure are placed on a life-testing experiment and the experiment is terminated when the m-th (where $m \le n$ is pre-fixed) system fails. In other words, only the first m failures out of the n systems in the life-test will be observed. The ordered system data obtained from such a life-test, denoted as $T_{1:n} < T_{2:n} < \ldots < T_{m:n}$, is referred to as a Type-II censored system lifetime data. Based on the observed Type-II censored coherent system lifetime data, we aim at predicting the future system failures $T' = T_{s+m:n}(s = 1, 2, \ldots, n-m)$ when the system signature (or, equivalently, the minimal signature) is available. Under the assumption that the component lifetime is modeled by the proportional hazard rate (PHR) model, we derive the maximum likelihood predictor, the best unbiased predictor, the conditional median predictor and the Bayesian point predictors for future system failures $T' = T_{s+m:n}(s = 1, 2, \ldots, n-m)$. Furthermore, we present the prediction intervals (PIs) for future failures $T' = T_{s+m:n}(s = 1, 2, \ldots, n-m)$. To compare the performances of different point and interval prediction methods, a Monte Carlo simulation study is used.

In recent years, many authors studied the statistical inference of the component lifetime distribution and prediction for the future system failures based on system lifetime data when the system signature is known; for example, Bhattacharya and Samaniego (2010) studied the nonparametric estimation of the component lifetime distribution from system lifetime data under the assumption that the component lifetime are i.i.d. Balakrishnan et al. (2011a) discussed the linear inference for Type-II censored system lifetime data of reliability systems with known signature. Balakrishnan et al. (2011b) considered signature-based nonparametric inferential methods for component lifetime characteristics based on system lifetime data. Ng et al. (2012) discussed the statistical inference for the component lifetime distribution from system lifetime data under a proportional hazard rate model when the system signature is known. Chahkandi et al. (2014) developed the non-parametric prediction intervals for the lifetime of coherent systems. Zhang et al. (2015a) considered the maximum likelihood estimation method and a regression-based method for statistical inference of the component lifetime distribution based on Type-II censored system lifetime data. Zhang et al. (2015b) discussed the problem of testing the homogeneity of distributions of component lifetimes based on system lifetime data when the system signatures are known. MirMostafaee et al. (2016) studied the Bayesian prediction of minimal repair times of a series system based on hybrid censored sample of components' lifetimes under the Rayleigh distribution. Yang et al. (2016) derived the expectation-maximization-type algorithms for parameter estimation of the component lifetime distribution based on system lifetime data when the system structures are known and unknown. Tavangar and Asadi (2020) studied the component reliability estimation based on system failure-time data. Fallah et al. (2021a) discussed the statistical inference for component lifetime distribution from coherent system lifetimes under a proportional reversed hazard model. Fallah et al. (2021b) discussed the prediction based on Type-II censored coherent system lifetime data under a proportional reversed hazard rate model.

This paper is organized as follows. In Section 2, we obtain the maximum likelihood estimator of the exponentiated parameter. In Section 3, we provide different point predictors for the future system failures. Based on Type-II censored data, we obtain the maximum likelihood predictor, the best unbiased predictor, the conditional median predictor and Bayes predictors. Different prediction intervals for the system failures are provided in Section 4. An illustrative example and a Monte Carlo simulation study are presented in Section 5. Recommendations are provided based on the simulation results in Section 6. Finally a conclusion is presented in Section 7.

2 Maximum likelihood estimation

Suppose $T_{1:n}, T_{2:n}, \cdots, T_{m:n}$ are ordered Type-II censored system lifetime data from a population with probability density function $f_T(t)$ and survival function $\bar{F}_T(t)$. To simplify the notation, we will use $(T_1, T_2, \cdots, T_m)$ in place of $T_{1:n}, T_{2:n}, \cdots, T_{m:n}$. Under the Type-II censored sample $\boldsymbol{T} = (T_1, T_2, \cdots, T_m)$, the likelihood function for θ is given by

$$L(\boldsymbol{t};\theta) = \frac{n!}{(n-m)!}\prod_{i=1}^{m} f(t_i;\theta)\left\{1 - F(t_m;\theta)\right\}^{n-m}. \tag{5}$$

where $\boldsymbol{t} = (t_1, t_2, \ldots, t_m)$ is the vector of observations. From (3), (4) and (5), the likelihood function can be expressed as

$$L(\boldsymbol{t};\theta) = \frac{n!}{(n-m)!}\theta^m \prod_{i=1}^{m}\frac{f_0(t_i)}{\bar{F}_0(t_i)}\prod_{i=1}^{m}\left\{\sum_{j=1}^{k} j a_j \bar{F}_0^{j\theta}(t_i)\right\}\left\{\sum_{j=1}^{k} a_j \bar{F}_0^{j\theta}(t_m)\right\}^{n-m}. \tag{6}$$

The log-likelihood function is obtained as

$$\log L(\boldsymbol{t};\theta) = C_1 + m\log\theta + \sum_{i=1}^{m}\log\left\{\sum_{j=1}^{k} j a_j \bar{F}_0^{j\theta}(t_i)\right\} + (n-m)\log\left\{\sum_{j=1}^{k} a_j \bar{F}_0^{j\theta}(t_m)\right\},$$

where $C_1 = \log\{\frac{n!}{(n-m)!}\prod_{i=1}^{m}\frac{f_0(t_i)}{\bar{F}_0(t_i)}\}$ is a constant independent of the parameter θ. Then, we can obtain the log-likelihood equation as

$$\begin{aligned}\frac{d\log L(\boldsymbol{t};\theta)}{d\theta} &= \frac{m}{\theta}+\sum_{i=1}^{m}\left[\frac{\sum_{j=1}^{k} j^2 a_j \bar{F_0}^{j\theta}(t_i)\log\bar{F}_0(t_i)}{\sum_{j=1}^{k} j a_j \bar{F_0}^{j\theta}(t_i)}\right] \\ &\quad +(n-m)\left[\frac{\sum_{j=1}^{k} j a_j \bar{F_0}^{j\theta}(t_m)\log\bar{F}_0(t_m)}{\sum_{j=1}^{k} a_j \bar{F_0}^{j\theta}(t_m)}\right]=0. \qquad (7)\end{aligned}$$

Therefore, the maximum likelihood estimator (MLE) of θ, say $\hat{\theta}_{MLE}$ can be obtained by maximizing (7) with respect to θ. Since the likelihood equation is a non-linear equation, therefore, the MLE of θ needs to be obtained by using numerical methods.

3 Point predictors

In this section, we address prediction of future system failures $T' = T_{s+m:n}(s = 1,2,\ldots,n-m)$ under classical and Bayesian approaches. Because of the Markov property of the conditional order statistics, the conditional distribution of T' given $\boldsymbol{t} = (t_1,t_2,\ldots,t_m)$ is equal to the conditional distribution of T' given $T_{m:n} = t_m$. This implies the density of T' given $T_{m:n} = t_m$ is the same as the density of the s-th order statistic from a sample of size $n-m$ from the population with the right truncated density $f(t';\theta)/[1-F(t_m;\theta)], t' \geq t_m$. Therefore, the conditional probability density function of T' given $T_{m:n} = t_m$ is

$$\begin{aligned}h(t'|t_m;\theta) = {}& s\binom{n-m}{s} f(t';\theta)\left[F(t';\theta)-F(t_m;\theta)\right]^{s-1} \\ &\times\left[1-F(t';\theta)\right]^{n-m-s}\left[1-F(t_m;\theta)\right]^{-(n-m)}, \quad t' \geq t_m. \qquad (8)\end{aligned}$$

Substituting (3) and (4), in (8), the conditional probability density function for $t' \geq t_m$, is

$$\begin{aligned}h(t'|t_m;\theta) &= s\binom{n-m}{s}\theta\frac{f_0(t')}{\bar{F}_0(t')}\left\{\sum_{j=1}^{k} j a_j \bar{F_0}^{j\theta}(t')\right\}\left\{\sum_{j=1}^{k} a_j \bar{F_0}^{j\theta}(t')\right\}^{n-m-s} \\ &\quad\times\left[\sum_{j=1}^{k} a_j \bar{F_0}^{j\theta}(t_m)-\sum_{j=1}^{k} a_j \bar{F_0}^{j\theta}(t')\right]^{s-1}\left\{\sum_{j=1}^{k} a_j \bar{F_0}^{j\theta}(t_m)\right\}^{-(n-m)} \qquad (9)\end{aligned}$$

Using (9) and the binomial expansion, we have

$$\begin{aligned}&\left[\sum_{j=1}^{k} a_j \bar{F_0}^{j\theta}(t_m)-\sum_{j=1}^{k} a_j \bar{F_0}^{j\theta}(t')\right]^{s-1} \\ &\qquad = \sum_{l=0}^{s-1}\left[\binom{s-1}{l}(-1)^l\left\{\sum_{j=1}^{k} a_j \bar{F_0}^{j\theta}(t')\right\}^{l}\left\{\sum_{j=1}^{k} a_j \bar{F_0}^{j\theta}(t_m)\right\}^{s-l-1}\right].\end{aligned}$$

The conditional density of T' given $T_m = t_m$ is given by

$$h(t'|t_m;\theta) = s\binom{n-m}{s}\theta\frac{f_0(t')}{\bar{F}_0(t')}\left\{\sum_{j=1}^{k} ja_j\bar{F}_0^{j\theta}(t')\right\}\sum_{l=0}^{s-1}\left[\binom{s-1}{l}(-1)^l \times\left\{\sum_{j=1}^{k} a_j\bar{F}_0^{j\theta}(t')\right\}^{n-m-s+l}\left\{\sum_{j=1}^{k} a_j\bar{F}_0^{j\theta}(t_m)\right\}^{s-l-1-n+m}\right]. \quad (10)$$

Several different point predictors for $T' = T_{s+m:n}$ $(s = 1, 2, \dots, n-m)$ are discussed in the following subsection.

3.1 Maximum likelihood approach

In this subsection, the likelihood approach is used to obtain the maximum likelihood predictor (MLP) for $T' = T_{s+m:n}$ $(s = 1, 2, \dots, n-m)$. The likelihood approach, introduced by Kaminsky and Rhodin (1985), has become a very useful tool to predict the future order statistics and estimate the parameters involved in the model. Given the informative sample $\boldsymbol{t} = (t_1, \dots, t_m)$, the predictive likelihood function (PLF) of T' and θ is considered and maximized simultaneously with regard to the future observation T' and the parameter θ. The PLF of T' and θ, is given by

$$L(T', \theta|\boldsymbol{t}) = h(T'|\boldsymbol{t};\theta)L(\boldsymbol{t};\theta) = h(T'|t_m;\theta)L(\boldsymbol{t};\theta). \quad (11)$$

Suppose $\widehat{T'} = u(\boldsymbol{T})$ and $\widehat{\theta} = \nu(\boldsymbol{T})$ are statistics for which

$$L(u(\boldsymbol{t}), \nu(\boldsymbol{t})|\boldsymbol{t}) = \sup_{(t',\theta)} L(t', \theta|\boldsymbol{t}).$$

Then, we call $u(\boldsymbol{T})$ the MLP of T' and $\nu(\boldsymbol{T})$ the predictive maximum likelihood estimator (PMLE) of θ. By substituting (6) and (9) into (11), the PLF of T' and θ can be obtained as:

$$\begin{aligned} L(t', \theta|\boldsymbol{t}) \quad = \quad & \frac{n!}{(s-1)!(n-m-s)!}\prod_{i=1}^{m}\frac{f_0(t_i)}{\bar{F}_0(t_i)}\theta^{m+1}\frac{f_0(t')}{\bar{F}_0(t')}\left\{\sum_{j=1}^{k} ja_j\bar{F}_0^{j\theta}(t')\right\} \\ & \times\left[\sum_{j=1}^{k} a_j\bar{F}_0^{j\theta}(t_m) - \sum_{j=1}^{k} a_j\bar{F}_0^{j\theta}(t')\right]^{s-1}\prod_{i=1}^{m}\left\{\sum_{j=1}^{k} ja_j\bar{F}_0^{j\theta}(t_i)\right\} \\ & \times\left\{\sum_{j=1}^{k} a_j\bar{F}_0^{j\theta}(t')\right\}^{n-m-s}. \end{aligned}$$

The predictive log-likelihood function is given by

$$\log L(t', \theta|\boldsymbol{t}) \quad = \quad C_2 + (m+1)\log\theta + \log f_0(t') - \log\bar{F}_0(t') + \sum_{i=1}^{m}\log\left\{\sum_{j=1}^{k} ja_j\bar{F}_0^{j\theta}(t_i)\right\}$$

$$+\log\left\{\sum_{j=1}^{k} ja_j\bar{F_0}^{j\theta}(t')\right\}+(n-m-s)\log\left\{\sum_{j=1}^{k} a_j\bar{F_0}^{j\theta}(t')\right\}$$
$$+(s-1)\log\left[\sum_{j=1}^{k} a_j\bar{F_0}^{j\theta}(t_m)-\sum_{j=1}^{k} a_j\bar{F_0}^{j\theta}(t')\right]. \tag{12}$$

where $C_2=\log\{\frac{n!}{(s-1)!(n-m-s)!}\prod_{i=1}^{m}\frac{f_0(t_i)}{\bar{F}_0(t_i)}\}$ is a constant independent of the parameter θ and t'. Using (12), the predictive likelihood equations (PLEs) for $T'=T_{s+m:n}$ $1\le s\le n-m$ and θ are given by

$$\begin{aligned}\frac{\partial\log L(t',\theta|\boldsymbol{t})}{\partial t'}=&\frac{f_0'(t')}{f_0(t')}+\frac{f_0(t')}{\bar{F}_0(t')}-\frac{f_0(t')}{\bar{F}_0(t')}\left(\frac{\sum_{j=1}^{k}\theta j^2a_j\bar{F_0}^{j\theta}(t')}{\sum_{j=1}^{k}ja_j\bar{F}_0^{j\theta}(t')}\right)\\&-(n-m-s)\frac{f_0(t')}{\bar{F}_0(t')}\left(\frac{\sum_{j=1}^{k}\theta ja_j\bar{F_0}^{j\theta}(t')}{\sum_{j=1}^{k}a_j\bar{F}_0^{j\theta}(t')}\right)\\&+(s-1)\frac{f_0(t')}{\bar{F}_0(t')}\left(\frac{\sum_{j=1}^{k}\theta ja_j\bar{F_0}^{j\theta}(t')}{\sum_{j=1}^{k}a_j\bar{F_0}^{j\theta}(t_m)-\sum_{j=1}^{k}a_j\bar{F_0}^{j\theta}(t')}\right)=0,\end{aligned}\tag{13}$$

$$\begin{aligned}\frac{\partial\log L(t',\theta|\boldsymbol{t})}{\partial\theta}=&\frac{m+1}{\theta}+(n-m-s)\left(\frac{\sum_{j=1}^{k}ja_j\bar{F_0}^{j\theta}(t')\log\bar{F}_0(t')}{\sum_{j=1}^{k}a_j\bar{F_0}^{j\theta}(t')}\right)\\&+\sum_{i=1}^{m}\left(\frac{\sum_{j=1}^{k}j^2a_j\bar{F_0}^{j\theta}(t_i)\log\bar{F}_0(t_i)}{\sum_{j=1}^{k}ja_j\bar{F_0}^{j\theta}(t_i)}\right)+\left(\frac{\sum_{j=1}^{k}j^2a_j\bar{F_0}^{j\theta}(t')\log\bar{F}_0(t')}{\sum_{j=1}^{k}ja_j\bar{F_0}^{j\theta}(t')}\right)\\&+(s-1)\left(\frac{\sum_{j=1}^{k}ja_j\bar{F_0}^{j\theta}(t_m)\log\bar{F}_0(t_m)-\sum_{j=1}^{k}ja_j\bar{F_0}^{j\theta}(t')\log\bar{F}_0(t')}{\sum_{j=1}^{k}a_j\bar{F_0}^{j\theta}(t_m)-\sum_{j=1}^{k}a_j\bar{F_0}^{j\theta}(t')}\right)\\=&0.\end{aligned}\tag{14}$$

By solving (13) and (14) with respect to t' and θ simultaneously, the MLP of T', $\widehat{T'}_{MLP}$, and PMLE of θ can be obtained. Numerical methods can be used to solve these PLEs and obtain the MLP $\widehat{T'}_{MLP}$ and PMLE of θ.

Example 3.1. *For k-component systems with exponential distributed components and baseline survival function $\bar{F}_0(t)=e^{-t}$, the MLP for $T'=T_{s+m:n}(s=1,2,\dots,n-m)$ and the PMLE of θ can be computed by solving the* (13) *and* (14) *which can be expressed as*

$$\begin{aligned}\frac{\partial\log L(t',\theta|\boldsymbol{t})}{\partial t'}&=-\left(\frac{\sum_{j=1}^{k}\theta j^2a_je^{-j\theta t'}}{\sum_{j=1}^{k}ja_je^{-j\theta t'}}\right)-(n-m-s)\left(\frac{\sum_{j=1}^{k}\theta ja_je^{-j\theta t'}}{\sum_{j=1}^{k}a_je^{-j\theta t'}}\right)\\&\quad+(s-1)\left(\frac{\sum_{j=1}^{k}\theta ja_je^{-j\theta t'}}{\sum_{j=1}^{k}a_je^{-j\theta t_m}-\sum_{j=1}^{k}a_je^{-j\theta t'}}\right)=0,\\\frac{\partial\log L(t',\theta|\boldsymbol{t})}{\partial\theta}&=\frac{m+1}{\theta}-(n-m-s)t'\left(\frac{\sum_{j=1}^{k}ja_je^{-j\theta t'}}{\sum_{j=1}^{k}a_je^{-j\theta t'}}\right)\end{aligned}$$

$$
\begin{aligned}
&-\sum_{i=1}^{m} t_i\left(\frac{\sum_{j=1}^{k} j^2 a_j e^{-j\theta t_i}}{\sum_{j=1}^{k} j a_j e^{-j\theta t_i}}\right)-t'\left(\frac{\sum_{j=1}^{k} j^2 a_j e^{-j\theta t'}}{\sum_{j=1}^{k} j a_j e^{-j\theta t'}}\right)\\
&-(s-1)\left(\frac{t_m\sum_{j=1}^{k} j a_j e^{-j\theta t_m}-t'\sum_{j=1}^{k} j a_j e^{-j\theta t'}}{\sum_{j=1}^{k} a_j e^{-j\theta t_m}-\sum_{j=1}^{k} a_j e^{-j\theta t'}}\right)=0.
\end{aligned}
$$

3.2 Best unbiased predictor

A statistic $\widehat{T'}$ which is used to predict $T' = T_{s+m:n}$ is called a best unbiased predictor (BUP) of T' if the predictor error $\widehat{T'} - T'$ has a mean zero, and its prediction error variance, $\mathrm{Var}(\widehat{T'} - T')$ is less than or equal to that of any other unbiased predictor of T'. Now observe that here the conditional density of T' given the observed data $\boldsymbol{t} = (t_1, t_2, \ldots, t_m)$ is just the density of T' given the observed lifetime t_m. Therefore the BUP of T' is $\widehat{T'}_{BUP} = E[T'|\boldsymbol{t}]$. For known parameter, the BUP of T' is obtained as

$$
\widehat{T'}_{BUP} = E[T'|\boldsymbol{t}] = E[T'|t_m] = \int_{t_m}^{\infty} t' h(t'|t_m;\theta)dt'. \tag{15}
$$

Using (10) and (15), the BUP of T' can be obtained as

$$
\begin{aligned}
\widehat{T'}_{BUP} \;=\; & s\binom{n-m}{s}\theta\sum_{l=0}^{s-1}\binom{s-1}{l}(-1)^l\left(\sum_{j=1}^{k} a_j \bar{F}_0^{\,j\theta}(t_m)\right)^{s-l-1-n+m}\\
&\times\int_{t_m}^{\infty} t'\frac{f_0(t')}{\bar{F}_0(t')}\left(\sum_{j=1}^{k} j a_j \bar{F}_0^{\,j\theta}(t')\right)\left(\sum_{j=1}^{k} a_j \bar{F}_0^{\,j\theta}(t')\right)^{n-m-s+l} dt'. \quad (16)
\end{aligned}
$$

If the parameter θ is unknown, the BUP of T' can be approximated by replacing θ by its MLE.

Example 3.2. *For k-component systems with exponential distributed components, the BUP can be obtained from* (16) *as*

$$
\begin{aligned}
\widehat{T'}_{BUP} \;=\; & s\binom{n-m}{s}\theta\sum_{l=0}^{s-1}\binom{s-1}{l}(-1)^l\left(\sum_{j=1}^{k} a_j e^{-j\theta t_m}\right)^{s-l-1-n+m}\\
&\times\int_{t_m}^{\infty} t'\left(\sum_{j=1}^{k} j a_j e^{-j\theta t'}\right)\left(\sum_{j=1}^{k} a_j e^{-j\theta t'}\right)^{n-m-s+l} dt'.
\end{aligned}
$$

3.3 Conditional median predictor

Another conditional predictor is the conditional median predictor (CMP). This predictor was first proposed by Raqab and Nagaraja (1995) in the context of order statistics. Notice that a predictor $\widehat{T'}$ is called the CMP of T', if it is the median of the conditional density of T' given t_m observation, that is

$$
\Pr_\theta(T' \le \widehat{T'}|T_{m:n} = t_m) = \Pr_\theta(T' \ge \widehat{T'}|T_{m:n} = t_m).
$$

For the probability density function and survival function of the system lifetime presented in (3) and (4), we can obtain

$$
\begin{aligned}
&\Pr_\theta(T' \le \widehat{T'} \mid T_{m:n} = t_m) \\
&\quad = \Pr_\theta\left(\frac{F(T') - F(t_m)}{1 - F(t_m)} \le \frac{F(\widehat{T'}) - F(t_m)}{1 - F(t_m)} \middle| T_{m:n} = t_m \right) \\
&\quad = \Pr_\theta\left(1 - \frac{\sum_{j=1}^k a_j \bar{F}_0^{\,j\theta}(T')}{\sum_{j=1}^k a_j \bar{F}_0^{\,j\theta}(t_m)} \le 1 - \frac{\sum_{j=1}^k a_j \bar{F}_0^{\,j\theta}(\widehat{T'})}{\sum_{j=1}^k a_j \bar{F}_0^{\,j\theta}(t_m)} \middle| T_{m:n} = t_m \right). \quad (17)
\end{aligned}
$$

Note that the conditional distribution of random variable Z,

$$
Z = 1 - \frac{\sum_{j=1}^k a_j \bar{F}_0^{\,j\theta}(T')}{\sum_{j=1}^k a_j \bar{F}_0^{\,j\theta}(t_m)},
$$

given t_m is a beta distribution with parameters s and $n - m - s + 1$ (denoted as $Beta(s, n-m-s+1)$) using an application of probability integral transformation and the fact that the i-th smallest order statistics from a random sample of size n from the standard uniform distribution is distributed as $Beta(i, n-i+1)$. Therefore, by (17), the CMP of T' can be obtained by solving the equation

$$
1 - \frac{\sum_{j=1}^k a_j \bar{F}_0^{\,j\theta}(\widehat{T'})}{\sum_{j=1}^k a_j \bar{F}_0^{\,j\theta}(t_m)} = Med(B), \tag{18}
$$

where B is a random variable that follows $Beta(s, n-m-s+1)$ distribution and $Med(B)$ stands for the median of B. Now, from (18), the CMP of T', $\widehat{T'}_{CMP}$, is computed by solving the nonlinear equation:

$$
\left(\sum_{j=1}^k a_j \bar{F}_0^{\,j\theta}(t_m) \right)(1 - Med(B)) - \left(\sum_{j=1}^k a_j \bar{F}_0^{\,j\theta}(\widehat{T'}) \right) = 0. \tag{19}
$$

When θ is unknown, we can substitute θ with its MLE and obtain an approximate CMP of T'.

Example 3.3. *For k-component systems with exponential distributed components, since $\bar{F}_0(t) = e^{-t}$, the CMP for $T' = T_{s+m:n} (s = 1, 2, \ldots, n-m)$ can be obtained from (19) by solving the nonlinear equation*

$$
\left(\sum_{j=1}^k a_j e^{-j\theta t_m} \right)(1 - Med(B)) - \left(\sum_{j=1}^k a_j e^{-j\theta \widehat{T'}} \right) = 0.
$$

3.4 Bayesian sample-based prediction

In the Bayesian inference procedures, we specify a loss function $L(\phi, \hat{\phi})$ that describes the loss incurred by making an estimate $\hat{\phi}$ when the true value of the parameter is

$\phi = \phi(\theta)$. In the literature, the most commonly used loss function is the squared error loss (SEL) function, $L(\phi, \hat{\phi}) = (\hat{\phi} - \phi)^2$. The symmetric nature of this function gives equal weights to overestimation and underestimation, while in the estimation of parameters of lifetime model, overestimation may by more serious than underestimation or vice-versa. The Bayes estimator of ϕ under the SEL function ($\hat{\phi}_{BS}$) is the posterior mean of ϕ given the data. One of the most popular asymmetric loss functions is the linear-exponential (LINEX) loss function. This loss function was introduced by Varian (1975) and was extensively discussed by Zellner (1986). Under the assumption that the minimal loss occurs at $\hat{\phi} = \phi$, the LINEX loss function for $\phi = \phi(\theta)$ can be expressed as

$$L_2(\phi, \hat{\phi}) \propto \exp(c(\hat{\phi} - \phi)) - c(\hat{\phi} - \phi) - 1, \quad c \neq 0, \tag{20}$$

where $\hat{\phi}$ is an estimate of ϕ. The sign and magnitude of the shape parameter c represents the direction and degree of symmetry, respectively. (If $c > 0$, the overestimation is more serious than underestimation, and vice-versa.) For c close to zero, the LINEX loss is approximately SEL and therefore almost symmetric. The posterior expectation of the LINEX loss function (20) is

$$E_\phi[L_2(\phi, \hat{\phi})] \propto \exp(c\hat{\phi})E_\phi[\exp(-c\phi)] - c(\hat{\phi} - E_\phi(\phi)) - 1, \tag{21}$$

where $E_\phi(.)$ denotes the posterior expectation with respect to the posterior density of ϕ. The Bayes estimator of ϕ, denoted by $\hat{\phi}_{BL}$ under the LINEX loss function is the value $\hat{\phi}$ which minimizes (21). It is

$$\hat{\phi}_{BL} = -\frac{1}{c} \log\left\{E_\phi[\exp(-c\phi)]\right\},$$

provided that the expectation $E_\phi[\exp(-c\phi)]$ exists and is finite.

In this subsection, we consider the Bayesian point prediction for the future system failures $T' = T_{s+m:n}(s = 1, 2, \ldots, n - m)$, based on the observed Type-II censored sample $\boldsymbol{t} = (t_1, t_2, \cdots, t_m)$. Under the Bayesian paradigm, θ is considered as a random variable following a specific prior distribution. Since the parameter θ is positive, for computational and mathematical case, we consider the gamma prior with shape parameter α and rate parameter β, denoted as $G(\alpha, \beta)$, for θ which has the probability density function

$$\pi(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\, \theta^{\alpha-1} e^{-\beta\theta}, \quad \theta > 0, \quad (\beta > 0, \alpha > 0). \tag{22}$$

where α and β are chosen to reflect the prior knowledge about θ. The gamma prior is flexible because informative and non-informative priors can be considered with suitable choices of the parameters α and β. By combining (6) and (22), we obtain the posterior density function of θ as

$$\pi(\theta|\boldsymbol{t}) = \frac{1}{m(\boldsymbol{t})}\theta^{m+\alpha-1}\, e^{-\beta\theta} \prod_{i=1}^{m} \left\{\sum_{j=1}^{k} j a_j \bar{F}_0^{j\theta}(t_i)\right\} \left\{\sum_{j=1}^{k} a_j \bar{F}_0^{j\theta}(t_m)\right\}^{n-m}, \tag{23}$$

where

$$m(\boldsymbol{t}) = \int_0^\infty \theta^{m+\alpha-1}\, e^{-\beta\theta} \prod_{i=1}^{m} \left\{ \sum_{j=1}^{k} j a_j \bar{F}_0^{j\theta}(t_i) \right\} \left\{ \sum_{j=1}^{k} a_j \bar{F}_0^{j\theta}(t_m) \right\}^{n-m} d\theta.$$

The Bayes predictive density function of T' given $T_m = t_m$ is given by

$$h^*(t'|t_m;\theta) = \int_0^\infty h(t'|t_m;\theta)\pi(\theta|\boldsymbol{t})d\theta. \tag{24}$$

Substituting (10) and (23) into (24), the Bayes predictive density function, for $t' > t$, is

$$\begin{aligned} h^*(t'|t_m;\theta) \quad = \quad & s\binom{n-m}{s} \frac{f_0(t')}{m(\boldsymbol{t})\bar{F}_0(t')} \int_0^\infty \frac{\theta^{m+\alpha}}{e^{\beta\theta}} \prod_{i=1}^{m} \left\{ \sum_{j=1}^{k} j a_j \bar{F}_0^{j\theta}(t_i) \right\} \\ & \times \left\{ \sum_{j=1}^{k} j a_j {\bar{F}_0}^{j\theta}(t') \right\} \sum_{l=0}^{s-1} \left[\binom{s-1}{l} (-1)^l \left(\sum_{j=1}^{k} a_j {\bar{F}_0}^{j\theta}(t') \right)^{n-m-s+l} \right. \\ & \left. \times \left(\sum_{j=1}^{k} a_j {\bar{F}_0}^{j\theta}(t_m) \right)^{s-l-1} \right] d\theta. \end{aligned}$$

Then the Bayesian point predictors of T' under the SEL function, $\hat{T'}_{SEP}$, and under the LINEX loss function, $\widehat{T'}_{LEP}$ are

$$\widehat{T'}_{SEP} \quad = \quad \int_{t_m}^\infty t' h^*(t'|t_m;\theta)dt', \tag{25}$$

$$\widehat{T'}_{LEP} = -\frac{1}{c}\log\left[\int_{t_m}^\infty e^{-ct'} h^*(t'|t_m;\theta)\right] dt'. \tag{26}$$

respectively. Due to the complicated form of $h^*(t'|t_m;\theta)$, the Bayesian point predictors in (25) and (26) cannot be computed explicitly. Here, we propose using the Metropolis-Hastings algorithm (see e.g., Robert and Casella (2004)) with a normal proposal distribution to find a simulation-based consistent estimator of $h^*(t'|t_m;\theta)$. For our situation, we first generate a Markov Chain Monte Carlo (MCMC) sample of size N, $\theta_1, \ldots, \theta_N$, from the posterior distribution $\pi(\theta|\boldsymbol{t})$, using the Metropolis-Hastings algorithm.

Algorithm 3.4. *The Metropolis-Hastings algorithm for generating the MCMC sample of size N from $\pi(\theta|\boldsymbol{t})$ is described as follows:*
Step 1: Start with an initial guess $\theta^{(0)}$ that can be the MLE of θ.
Step 2: Set $i = 1$.
Step 3: Using the Metropolis-Hastings algorithm, generate θ_t from $\pi(\theta^{(i-1)}|\boldsymbol{t})$ with $N(\theta^{(i-1)}, S_\theta^2)$ as a proposal distribution, where S_θ^2 is inverse of the Fisher information.
Step 4: Set $i = i + 1$.
Step 5: Repeat Steps 3 and 4, N times to arrive at the MCMC sample $\theta_1, \ldots, \theta_N$.

Based on an MCMC sample $\theta_{M+1}, \ldots, \theta_N$, the simulation consistent estimator of $h^*(t'|t_m;\theta)$ can be obtained as

$$\widehat{h}^*(t'|t_m;\theta) = \frac{1}{N-M}\sum_{i=M+1}^{N} h(t'|t_m;\theta_i), \tag{27}$$

where M is the burn-in period and $h(t'|t_m;\theta_i)$ is given in (10) with $\theta=\theta_i$. By using (25) and (26), the Bayes predictors of the future failure under the SEL function, $\widehat{T'}_{s,SEP}$ and under the LINEX loss function $\widehat{T'}_{s,LEP}$ can be obtained as

$$\begin{aligned}
\widehat{T'}_{s,SEP} &= \int_{t_m}^{\infty} t'\widehat{h}^*(t'|t_m;\theta)dt' = \frac{1}{N-M}\sum_{i=M+1}^{N}\int_{t_m}^{\infty} t'h(t'|t_m;\theta_i)dt' \\
&= \frac{1}{N-M}\sum_{i=M+1}^{N}\sum_{l=0}^{s-1} s\binom{n-m}{s}\binom{s-1}{l}(-1)^l\theta_i \\
&\quad \times \left(\sum_{j=1}^{k} a_j \bar{F_0}^{j\theta_i}(t_m)\right)^{s-l-n+m-1} \Psi_1(t_m;\theta_i),
\end{aligned} \tag{28}$$

where

$$\Psi_1(t_m;\theta) = \int_{t_m}^{\infty} t'\frac{f_0(t')}{\bar{F_0}(t')}\left(\sum_{j=1}^{k} ja_j\bar{F_0}^{j\theta}(t')\right)\left(\sum_{j=1}^{k} a_j\bar{F_0}^{j\theta}(t')\right)^{n-m-s+l} dt',$$

and

$$\begin{aligned}
\widehat{T'}_{s,LEP} &= -\frac{1}{c}\log\left(\int_{t_m}^{\infty} e^{-ct'}\widehat{h}^*(t'|t_m;\theta)dt'\right) \\
&= -\frac{1}{c}\log\left(\frac{1}{N-M}\sum_{i=M+1}^{N}\int_{t_m}^{\infty} e^{-ct'}h(t'|t_m;\theta_i)dt'\right) \\
&= -\frac{1}{c}\log\Bigg[\frac{1}{N-M}\sum_{i=M+1}^{N}\sum_{l=0}^{s-1} s\binom{n-m}{s}\binom{s-1}{l}(-1)^l\theta_i \\
&\quad \times \left(\sum_{j=1}^{k} a_j \bar{F_0}^{j\theta_i}(t_m)\right)^{s-l-n+m-1} \Psi_2(t_m;\theta_i)\Bigg],
\end{aligned} \tag{29}$$

where

$$\Psi_2(t_m;\theta) = \int_{t_m}^{\infty} \exp(-ct')\frac{f_0(t')}{\bar{F_0}(t')}\left(\sum_{j=1}^{k} ja_j\bar{F_0}^{j\theta}(t')\right)\left(\sum_{j=1}^{k} a_j\bar{F_0}^{j\theta}(t')\right)^{n-m-s+l} dt',$$

respectively.

Example 3.5. *For exponential distributed components, the Bayesian point predictors of the future failure* $T' = T_{s+m:n}(s = 1, 2, \cdots, n-m)$ *can be obtained from* (28) *and* (29) *as*

$$\begin{aligned}
\widehat{T'}_{s,SEP} &= \frac{1}{N-M}\sum_{i=M+1}^{N}\sum_{l=0}^{s-1} s\binom{n-m}{s}\binom{s-1}{l}(-1)^l\theta_i \\
&\times \left(\sum_{j=1}^{k} a_j e^{-j\theta_i t_m}\right)^{s-l-n+m-1} \Psi_1(t_m;\theta_i),
\end{aligned}$$

where

$$\begin{aligned}
\Psi_1(t_m;\theta) &= \int_{t_m}^{\infty} t' \left(\sum_{j=1}^{k} j a_j e^{-j\theta t'}\right)\left(\sum_{j=1}^{k} a_j e^{-j\theta t'}\right)^{n-m-s+l} dt', \\
\widehat{T'}_{s,LEP} &= -\frac{1}{c}\log\Bigg[\frac{1}{N-M}\sum_{i=M+1}^{N}\sum_{l=0}^{s-1} s\binom{n-m}{s}\binom{s-1}{l}(-1)^l\theta_i \\
&\times \left(\sum_{j=1}^{k} a_j e^{-j\theta_i t_m}\right)^{s-l-n+m-1} \Psi_2(t_m;\theta_i)\Bigg], \\
\Psi_2(t_m;\theta) &= \int_{t_m}^{\infty} e^{-ct'} \left(\sum_{j=1}^{k} j a_j e^{-j\theta t'}\right)\left(\sum_{j=1}^{k} a_j e^{-j\theta t'}\right)^{n-m-s+l} dt'.
\end{aligned}$$

4 Prediction intervals

In this section, based on the observed Type-II censored system lifetime data $\boldsymbol{t} = (t_1, t_2, \ldots, t_m)$, we want to obtain the prediction intervals for the s-th censored system lifetime data $T' = T_{s+m:n}(s = 1, 2, \ldots, n-m)$ based on classical and Bayesian approaches.

4.1 Non-Bayesian prediction Intervals

In Subsection 3.3, we have already discussed that the distribution of random variable Z,

$$Z = 1 - \frac{\sum_{j=1}^{k} a_j \bar{F_0}^{j\theta}(T')}{\sum_{j=1}^{k} a_j \bar{F_0}^{j\theta}(t_m)}, \tag{30}$$

given $T_{m:n} = t_m$ is a beta distribution with parameters s and $n-m-s+1$. Therefore, we can consider Z as a pivotal quantity to obtain the prediction interval for T'. Taking $1-\gamma$ $(0 < \gamma < 1)$ as the prediction coefficient and using (30), we have

$$\Pr(B_{\frac{\gamma}{2}} < Z < B_{1-\frac{\gamma}{2}} | T_{m:n} = t_m) = 1 - \gamma,$$

where B_γ is the 100γ-th upper percentile of $Beta(s, n-m-s+1)$ distribution. Therefore, $100(1-\gamma)\%$ prediction interval for T' $(L_1(\boldsymbol{T}), U_1(\boldsymbol{T}))$, are the solutions of

$$\left(\sum_{j=1}^{k} a_j \bar{F_0}^{j\theta}(t_m)\right)(1 - B_{1-\frac{\gamma}{2}}) - \left(\sum_{j=1}^{k} a_j \bar{F_0}^{j\theta}(T')\right) = 0,$$
$$\left(\sum_{j=1}^{k} a_j \bar{F_0}^{j\theta}(t_m)\right)(1 - B_{\frac{\gamma}{2}}) - \left(\sum_{j=1}^{k} a_j \bar{F_0}^{j\theta}(T')\right) = 0,$$

respectively. When parameter θ is unknown, the prediction interval $(L_1(\boldsymbol{T}), U_1(\boldsymbol{T}))$ can be approximated by replacing θ with its corresponding MLE.

Let us now consider the highest conditional density (HCD) method for constructing the prediction interval of T'. The distribution of Z given $T_{m:n} = t_m$ is a $Beta(s, n-m-s+1)$ distribution with the probability density function

$$f(z|t_m) = \frac{(n-m)!}{(s-1)!(n-m-s)!} z^{s-1}(1-z)^{n-m-s}, \ 0 < z < 1,$$

which is a unimodal function of z for $(s = 2, \dots, n-m-1)$. The $100(1-\gamma)\%$ HCD prediction interval for T' is $(L_2(\boldsymbol{T}), U_2(\boldsymbol{T}))$, where the observed values of $L_2(\boldsymbol{T})$ and $U_2(\boldsymbol{T})$ are given by w_1 and w_1, respectively, where w_1 and w_1 satisfy

$$\left(\sum_{j=1}^{k} a_j \bar{F_0}^{j\theta}(t_m)\right)(1 - w_1) - \left(\sum_{j=1}^{k} a_j \bar{F_0}^{j\theta}(T')\right) = 0,$$
$$\left(\sum_{j=1}^{k} a_j \bar{F_0}^{j\theta}(t_m)\right)(1 - w_2) - \left(\sum_{j=1}^{k} a_j \bar{F_0}^{j\theta}(T')\right) = 0,$$

and at the same time w_1 and w_1 are the simultaneous solutions of the following equations:

$$\int_{w_1}^{w_2} f(z|t_m)dz = 1 - \gamma, \tag{31}$$
$$f(w_1|t_m) = f(w_2|t_m). \tag{32}$$

(31) and (32) can be simplified as

$$B_{w_2}(s, n-m-s+1) - B_{w_1}(s, n-m-s+1) = 1 - \gamma,$$
$$\left(\frac{1-w_2}{1-w_1}\right)^{n-m-s} = \left(\frac{w_1}{w_2}\right)^{s-1},$$

where $B_t(a, b)$ is incomplete beta function defined as

$$B_t(a, b) = \frac{1}{B(a, b)} \int_0^t x^{a-1}(1-x)^{b-1}\, dx,$$

with $B(a,b) = \frac{[\Gamma(a)\Gamma(b)]}{\Gamma(a+b)}$ being the incomplete beta function for real values of a and b. When parameter θ is unknown, the prediction interval $(L_2(\boldsymbol{T}), U_2(\boldsymbol{T}))$ can be approximated by replacing θ with its corresponding MLE. It should be mentioned here that for the case that $s = 1$ or $s = n - m$, the function $f(z|t_m)$ is not unimodal and the HCD prediction interval cannot be obtained in these cases.

4.2 Bayesian prediction intervals

A $100(1-\gamma)\%$ Bayesian prediction interval for the s-th censored system lifetime data $T' = T_{s+m:n}(s = 1, 2, \ldots, n-m)$ is given by $(L(t_m), U(t_m))$, where $L(t_m)$ and $U(t_m)$ can be obtained by solving the following two nonlinear equations simultaneously

$$\begin{aligned}
\Pr(T' > L(t_m)|t_m) &= \int_{L(t_m)}^{\infty} h^*(t'|t_m;\theta)dt' = 1 - \frac{\gamma}{2}, \\
\Pr(T' > U(t_m)|t_m) &= \int_{U(t_m)}^{\infty} h^*(t'|t_m;\theta)dt' = \frac{\gamma}{2}.
\end{aligned}$$

By using $\widehat{h}^*(t'|t_m;\theta)$ defined in (27) to approximate $h^*(t'|t_m;\theta)$, and using the MCMC sample $\{\theta_i : i = M+1, M+2, \ldots, N\}$, from $\pi(\theta|\boldsymbol{t})$, we can compute the lower and upper bounds $L(t_m)$ and $U(t_m)$ from the relations

$$\begin{aligned}
1 - \frac{\gamma}{2} &= \frac{1}{N-M} \sum_{i=1M+1}^{N} \sum_{l=0}^{s-1} s \binom{n-m}{s} \binom{s-1}{l} (-1)^l \theta_i \\
&\quad \times \left(\sum_{j=1}^{k} a_j \bar{F_0}^{j\theta_i}(t_m) \right)^{s-l-n+m-1} \Psi_3(L(t_m);\theta_i), \\
\frac{\gamma}{2} &= \frac{1}{N-M} \sum_{i=M+1}^{N} \sum_{l=0}^{s-1} s \binom{n-m}{s} \binom{s-1}{l} (-1)^l \theta_i \\
&\quad \times \left(\sum_{j=1}^{k} a_j \bar{F_0}^{j\theta_i}(t_m) \right)^{s-l-n+m-1} \Psi_3(U(t_m);\theta_i),
\end{aligned}$$

where

$$\Psi_3(Len, \theta) = \int_{Len}^{\infty} \frac{f_0(t')}{\bar{F_0}(t')} \left(\sum_{j=1}^{k} j a_j \bar{F_0}^{j\theta}(t') \right) \left(\sum_{j=1}^{k} a_j \bar{F_0}^{j\theta}(t') \right)^{n-m-s+l} dt'.$$

For the exponential distribution considered in Section 4, different PIs can be obtained as described in this section by taking $\bar{F_0}(t) = e^{-t}$.

5 Numerical study

In this section, a numerical example is considered for illustrative purposes and a Monte Carlo simulation study is performed to compare the point and interval prediction methods presented in Sections 3 and 4.

5.1 Algorithm to Generate System Lifetimes

Here, we assume that the lifetimes of the components are i.i.d. exponentially distributed with $\bar{F}_X(t) = e^{-\theta t}$, $t > 0$, $\theta > 0$. This model is equivalent to setting $\bar{F}_0(t) = e^{-t}$ in (2). Before progressing further, first we describe how we can generate a sample $T_1, T_2, \cdots, T_n$ of i.i.d. system lifetimes for systems with exponentially distributed components. The following algorithm is used to generate the system lifetime $T_1, T_2, \cdots, T_n$ with system signature $\boldsymbol{p} = (p_1, p_2, \cdots, p_k)$ $(0 < p_j < 1,\ \sum_{j=1}^{k} p_j = 1)$ with exponentially distributed components.

Algorithm 5.1. *Algorithm to generate system signature lifetimes:*
Step 1: Generate $U, U_1, U_2, \ldots, U_k$ *independently from a uniform distribution in* $[0, 1]$*;*
Step 2: Set $X_j = F_0^{-1}\left(1 - (1 - U_j)^{1/\theta}\right)$, $j = 1, 2, \ldots, k$*;*
Step 3: Sort $X_1, X_2, \cdots, X_k$ *in ascending order to obtain* $X_{1:k} < X_{2:k} < \cdots < X_{k:k}$.
Step 4: Take $T = X_{j:k}$ *for* $\sum_{i=1}^{j-1} p_i < U < \sum_{i=1}^{j} p_i, (j = 1, 2, \cdots, k)$, *i.e.,*

$$T = \begin{cases} X_{1:k} & 0 < U < p_1 \\ X_{2:k} & p_1 < U < p_1 + p_2 \\ X_{3:k} & p_1 + p_2 < U < p_1 + p_2 + p_3 \\ \vdots & \vdots \\ X_{k:k} & \sum_{j=1}^{k-1} p_j < U < \sum_{j=1}^{k} p_j \end{cases}$$

Step 5: Repeat Steps $1 - 4$, n *times, to generate lifetimes* $T_1, T_2, \cdots, T_n$.

Suppose that n items are placed on a life-testing experiment and it is planned that the experiment will be terminated as soon as the m-th (where m is pre-fixed) failure is observed. Then, only the first m failures out of n units under the test will be observed. The data obtained from such a life-test, are denoted as $T_{1:n} < T_{2:n} < \ldots < T_{m:n}$.

5.2 Numerical example

To illustrate all the methods presented in the preceding sections, a sample of $n = 30$ is generated from a 4-component system with lifetime

$$T = \min\{X_1, \max\{X_2, X_3, X_4\}\}.$$

The corresponding system signature is $\boldsymbol{P} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)$ and the corresponding minimal signature is $\boldsymbol{a} = (0, 3, -3, 1)$. The component lifetimes follow the exponential distribution with parameter $\theta = 1$. The simulated system lifetimes are presented in Table 1.

Table 1: Simulated 4-component system lifetimes with system signature $P = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)$.

0.038	0.091	0.115	0.144	0.169	0.209	0.225	0.228	0.257	0.284
0.331	0.364	0.369	0.399	0.475	0.504	0.715	0.735	0.782	0.822
0.922	1.053	1.140	1.252	1.313	1.371	1.624	1.803	1.920	2.499

Table 2: Point predictors and 95% prediction intervals for T'.

Point prediction

	True Value	PMLE	MLP	BUP	CMP	Bayesian predictor (SEL)	(LINEX) $c=-0.1$	$c=0.05$
$s=1$	1.371	1.033	1.313	1.428	1.393	1.433	1.434	1.433
$s=2$	1.624	1.032	1.439	1.569	1.529	1.583	1.585	1.584
$s=3$	1.803	1.030	1.599	1.753	1.706	1.775	1.779	1.777
$s=4$	1.920	1.029	1.819	2.023	1.961	2.052	2.060	2.056
$s=5$	2.499	1.028	2.187	2.548	2.431	2.597	2.623	2.610

Prediction intervals

	True Value	Pivot Method	HCD Method	Bayes Method
$s=1$	1.371	(1.316,1.731)	—	(1.316,1.765)
$s=2$	1.624	(1.344 ,2.016)	(1.328,1.934)	(1.344,2.085)
$s=3$	1.803	(1.405,2.367)	(1.398,2.298)	(1.403,2.483)
$s=4$	1.920	(1.505,2.898)	(1.543,2.898)	(1.500,3.078)
$s=5$	2.499	(1.683,4.092)	—	(1.670,4.361)

We consider the case when we observe the first 25 observations and the rest are censored, i.e., a Type-II censored sample with $n = 30$ and $m = 25$ is observed. With this Type-II censored sample, we compute the point and interval predictions for future system failures $T' = T_{s+25:30}(s = 1, 2, 3, 4, 5)$ as described in Sections 3 and 4. Specifically, we compute the MLP, BUP, CMP, Bayesian predictions and we also compute the 95% prediction intervals for $T' = T_{s+25:30}(s = 1, 2, 3, 4, 5)$ based on the pivotal quantity method, HCD method and Bayesian method. It should be mentioned here that for the case that $s = 1$ or $s = n - m = 5$, the function $f(z|t_m)$ is not unimodal and the HCD prediction interval cannot be obtained in these cases. The results are presented in 2.

For the Bayesian predictions, we use the Metropolis-Hastings method to compute the Bayesian point predictions of $T' = T_{s+25:30}(s = 1, 2, 3, 4, 5)$ under the SEL and LINEX loss functions. We have generated $N = 50000$ values by Metropolis-Hastings algorithm with $S^2_\theta = 0.0283$, $\hat{\theta}_{MLE} = 0.9990$ and the acceptance rate is about 70%. We discard the initial $M = 5000$ as burn-in samples and compute the Bayesian predictions based on the remaining 45000 samples. For computing the Bayesian predictions and corresponding prediction intervals, it is assumed that the prior of θ is almost improper, i.e., $\alpha = \beta = 0.0001$. The LINEX loss function is used for computing Bayes predictions under different values of c ($c = -0.1, 0.05$). The convergence of Metropolis-Hastings samples can be verified through graphical inspection. The histogram of the Metropolis-Hastings sequence of θ after burn-in is presented in Figure 1. The plot of the posterior probability density function of θ in Figure 1 shows that the choice of normal distribution as a proposal distribution is quite appropriate. The trace plot for the Metropolis-Hastings sequence of values of θ is also presented in 1. From Figure 1, the trace plot shows the values of θ are randomly scattered around the average. To check the sensitivity of the prior parameters on the convergence of Metropolis-Hastings algorithm, we have taken another Metropolis-Hastings simulation using a proper prior. Let us consider the case where $\alpha = 2, \beta = 4$. Figure 2 shows the trace and posterior probability density function plots for the parameter θ. In this case, we have $S_\theta^2 = 0.0243$ and the acceptance rate is about 70%. From Figures 1 and 2, it can be seen that both priors maintain similar acceptance rates and that the simulated posterior distributions of θ under the proper and almost improper priors converge to the normal distribution as a desired proposal. Moreover, these figures also show the convergence of the algorithm.

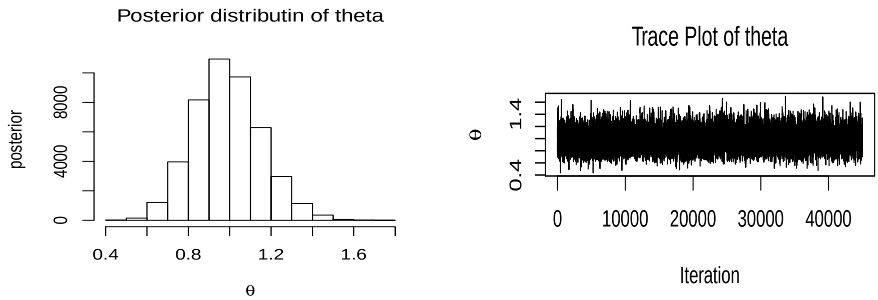

Figure 1: Plots of Metropolis-Hastings Markov chains for θ using improper prior.

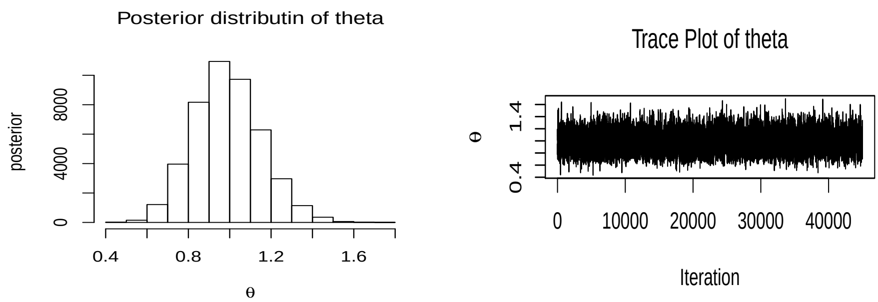

Figure 2: Plots of Metropolis-Hastings Markov chains for θ using proper prior.

Table 3: System signatures and minimal signatures of the 4-component systems.

System no.	System lifetime T	$\boldsymbol{p}$	$\boldsymbol{a}$
1	$T = \min(X_1, X_2, X_3, X_4)$	$(1, 0, 0, 0)$	$(0, 0, 0, 1)$
2	$T = \min(X_1, \max(X_2, X_3, X_4))$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)$	$(0, 3, -3, 1)$
3	$T = \min(X_1, \max(X_2, X_3), \max(X_2, X_4))$	$(\frac{1}{4}, \frac{7}{12}, \frac{1}{6}, 0)$	$(0, 1, 1, -1)$
4	$T = \max(\min(X_1, X_2), \min(X_3, X_4))$	$(0, \frac{2}{3}, \frac{1}{3}, 0)$	$(0, 2, 0, -1)$
5	$T = \max(\min(X_1, X_2, X_3), \min(X_2, X_3, X_4))$	$(\frac{1}{2}, \frac{1}{2}, 0, 0)$	$(0, 0, 2, -1)$

5.3 Monte Carlo simulation study

In this subsection, a Monte Carlo simulation study is employed to evaluate the performances of the different proposed predictors for systems with exponentially distributed components. We compare the performances of the point predictors MLP, BUP, CMP and the Bayesian point predictors in terms of their estimated biases and estimated mean square prediction errors (MSPEs). The predictive intervals obtained by different methods are compared by means of their estimated average widths (AWs) and coverage probabilities (CPs). For notational convenience, Table 3 lists the different systems and their signatures and minimal signatures used in the simulation study.

For different choices of sample size n and effective sample size m, we generated 1000 sets of Type-II censored sample $T_{1:n}, \ldots, T_{m:n}$ from coherent systems with exponentially distributed component lifetimes with $\theta = 2$ using Algorithm 5.1. We then obtained the point predictors MLP, BUP, CMP for the sth future system failure time. Then, we have computed different point predictors for the sth future failure time $T' = T_{s+m:n}(s = 1, 2, \cdots, n-m)$. We also obtained Bayesian point predictions under a gamma prior with $\alpha = 0.0001$ and $\beta = 0.0001$, denoted as $G(0.0001, 0.0001)$. For various choices of $(n = 15,\ m = 10)$, $(n = 25,\ m = 20)$ and $(s = 1, \ldots, 5)$, Tables 4 and 5 present the estimated biases and estimated MSPEs of different predictors obtained from this simulation study.

The estimated biases and estimated MSPEs are computed as follows. Suppose $\hat{T'}_i$ is the prediction of T' obtained in the i-th replication, then we compute the estimated bias and estimated MSPE as

$$Bias = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{T'}_i - T'_i), \quad MSPE = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{T'}_i - T'_i)^2.$$

We also computed the 95% non-Bayesian and Bayesian prediction intervals for T' by using the results given in Section 4. The average lengths and the corresponding coverage probabilities of 95% prediction intervals are also computed and reported in Table 6. If (L_i, U_i) is the 95% prediction interval of T' obtained in the i-th replication, then the coverage probability (CP) is computed as

$$\mathrm{CP} = \frac{1}{1000}\sum_{i=1}^{1000} I\left\{L_i < T'_i < U_i\right\},$$

where $I\{\cdot\}$ denotes the indicator function. The results for sample sizes $(n = 15,\ m = 10)$, $(n = 25,\ m = 20)$ and $(s = 1, \ldots, 5)$ are reported in Table 6.

The computations are performed in R (R Core Team, 2019) with the MHadaptive package Chivers (2012).

6 Discussions

For point prediction, from Table 4, we observe that BUP performs better that the MLP and CMP in terms of both estimated biases and estimated MSPEs. The estimated MSPEs of CMP and the estimated MSPEs of BUP are close to each other in the most

Table 4: Estimated biases and estimated MSPEs of classic point predictions for T'.

n	m	no.	s	MLP Bias(MSPE)	BUP Bias(MSPE)	CMP Bias(MSPE)
15	10	1	1	-0.026(0.002)	-0.001(0.001)	-0.008(0.001)
			2	-0.030(0.004)	-0.001(0.002)	-0.009(0.002)
			3	-0.042(0.007)	-0.003(0.003)	-0.013(0.003)
			4	-0.059(0.012)	-0.004(0.008)	-0.018(0.009)
			5	-0.099(0.036)	-0.006(0.023)	-0.026(0.024)
		2	1	-0.061(0.008)	0.004(0.005)	-0.015(0.005)
			2	-0.085(0.019)	-0.006(0.011)	-0.027(0.012)
			3	-0.092(0.032)	-0.010(0.026)	-0.033(0.026)
			4	-0.108(0.049)	-0.014(0.044)	-0.044(0.045)
			5	-0.209(0.171)	-0.036(0.134)	-0.092(0.135)
		3	1	-0.050(0.004)	-0.001(0.002)	-0.014(0.002)
			2	-0.064(0.012)	-0.003(0.007)	-0.018(0.007)
			3	-0.075(0.028)	-0.018(0.018)	-0.024(0.020)
			4	-0.110(0.041)	-0.021(0.025)	-0.033(0.026)
			5	-0.197(0.113)	-0.034(0.075)	-0.054(0.078)
		4	1	-0.057(0.008)	-0.002(0.003)	-0.018(0.004)
			2	-0.060(0.013)	-0.004(0.007)	-0.021(0.008)
			3	-0.073(0.023)	-0.006(0.014)	-0.022(0.015)
			4	-0.104(0.046)	-0.007(0.031)	-0.030(0.032)
			5	-0.179(0.128)	-0.010(0.093)	-0.049(0.096)
		5	1	-0.038(0.003)	-0.001(0.001)	-0.011(0.001)
			2	-0.046(0.006)	-0.002(0.003)	-0.013(0.004)
			3	-0.061(0.011)	-0.005(0.006)	-0.014(0.006)
			4	-0.085(0.034)	-0.007(0.020)	-0.018(0.021)
			5	-0.165(0.088)	-0.014(0.052)	-0.033(0.055)
25	20	1	1	-0.023(0.001)	0.000(0.001)	-0.006(0.001)
			2	-0.025(0.003)	-0.001(0.001)	-0.008(0.001)
			3	-0.043(0.004)	-0.002(0.002)	-0.009(0.002)
			4	-0.050(0.012)	-0.003(0.006)	-0.014(0.007)
			5	-0.093(0.033)	-0.005(0.022)	-0.021(0.023)
		2	1	-0.061(0.007)	-0.001(0.003)	-0.019(0.004)
			2	-0.071(0.013)	-0.003(0.008)	-0.023(0.009)
			3	-0.081(0.021)	-0.004(0.014)	-0.023(0.015)
			4	-0.102(0.044)	-0.007(0.030)	-0.024(0.031)
			5	-0.171(0.120)	-0.024(0.091)	-0.034(0.092)
		3	1	-0.045(0.004)	-0.001(0.002)	-0.014(0.002)
			2	-0.052(0.010)	-0.002(0.006)	-0.016(0.007)
			3	-0.066(0.017)	-0.006(0.012)	-0.018(0.013)
			4	-0.087(0.031)	-0.013(0.023)	-0.025(0.024)
			5	-0.178(0.111)	-0.032(0.074)	-0.048(0.077)
		4	1	-0.050(0.006)	-0.001(0.003)	-0.015(0.003)
			2	-0.059(0.011)	-0.003(0.007)	-0.017(0.007)
			3	-0.070(0.020)	-0.004(0.013)	-0.021(0.014)
			4	-0.101(0.041)	-0.005(0.030)	-0.030(0.031)
			5	-0.177(0.132)	-0.009(0.093)	-0.049(0.095)
		5	1	-0.037(0.003)	0.000(0.001)	-0.011(0.001)
			2	-0.045(0.005)	-0.002(0.003)	-0.015(0.003)
			3	-0.046(0.009)	-0.003(0.006)	-0.011(0.006)
			4	-0.064(0.017)	-0.006(0.012)	-0.013(0.013)
			5	-0.116(0.049)	-0.009(0.033)	-0.031(0.034)

Table 5: Estimated biases and estimated MSPEs of Bayes point predictions for T'.

n	m	no.	s	$G(0.0001, 0.0001)$ SEL Bias(MSPE)	LINEX Bias(MSPE) c=-0.05	LINEX Bias(MSPE) c=1
15	10	1	1	0.003(0.001)	0.003(0.001)	0.003(0.001)
			2	0.007(0.002)	0.007(0.002)	0.008(0.003)
			3	0.009(0.005)	0.009(0.005)	0.010(0.005)
			4	0.015(0.010)	0.016(0.011)	0.017(0.012)
			5	0.035(0.032)	0.036(0.033)	0.037(0.035)
		2	1	0.009(0.005)	0.010(0.005)	0.011(0.006)
			2	0.013(0.014)	0.013(0.014)	0.014(0.016)
			3	0.018(0.027)	0.020(0.028)	0.023(0.030)
			4	0.032(0.046)	0.032(0.046)	0.033(0.048)
			5	0.066(0.143)	0.071(0.145)	0.076(0.147)
		3	1	0.001(0.002)	0.001(0.002)	0.002(0.003)
			2	0.005(0.009)	0.005(0.009)	0.007(0.010)
			3	0.015(0.021)	0.015(0.021)	0.016(0.022)
			4	0.020(0.035)	0.021(0.036)	0.027(0.037)
			5	0.024(0.086)	0.025(0.087)	0.026(0.090)
		4	1	0.001(0.004)	0.001(0.004)	0.001(0.004)
			2	0.012(0.008)	0.013(0.009)	0.015(0.010)
			3	0.019(0.019)	0.019(0.019)	0.022(0.020)
			4	0.020(0.042)	0.021(0.042)	0.023(0.045)
			5	0.043(0.109)	0.043(0.109)	0.044(0.110)
		5	1	0.006(0.001)	0.006(0.001)	0.006(0.001)
			2	0.008(0.004)	0.008(0.004)	0.008(0.004)
			3	0.009(0.008)	0.010(0.008)	0.011(0.009)
			4	0.037(0.026)	0.038(0.026)	0.042(0.027)
			5	0.052(0.077)	0.054(0.078)	0.055(0.080)
25	20	1	1	0.002(0.001)	0.002(0.001)	0.002(0.001)
			2	0.003(0.002)	0.003(0.002)	0.003(0.003)
			3	0.009(0.003)	0.009(0.003)	0.010(0.004)
			4	0.006(0.009)	0.007(0.009)	0.008(0.011)
			5	0.017(0.026)	0.019(0.027)	0.020(0.030)
		2	1	0.000(0.004)	0.000(0.004)	0.001(0.004)
			2	0.002(0.010)	0.002(0.010)	0.003(0.011)
			3	0.009(0.016)	0.009(0.016)	0.009(0.017)
			4	0.021(0.037)	0.022(0.037)	0.028(0.038)
			5	0.039(0.101)	0.043(0.102)	0.060(0.105)
		3	1	0.001(0.002)	0.001(0.002)	0.001(0.002)
			2	0.002(0.007)	0.002(0.007)	0.003(0.008)
			3	0.012(0.014)	0.012(0.014)	0.013(0.016)
			4	0.019(0.026)	0.020(0.027)	0.021(0.030)
			5	0.020(0.085)	0.021(0.086)	0.025(0.087)
		4	1	0.001(0.003)	0.001(0.003)	0.002(0.004)
			2	0.005(0.008)	0.005(0.008)	0.006(0.009)
			3	0.006(0.015)	0.006(0.016)	0.007(0.017)
			4	0.010(0.033)	0.010(0.033)	0.012(0.035)
			5	0.017(0.106)	0.017(0.107)	0.019(0.109)
		5	1	0.001(0.001)	0.001(0.001)	0.001(0.001)
			2	0.004(0.003)	0.004(0.003)	0.004(0.004)
			3	0.009(0.007)	0.009(0.007)	0.010(0.008)
			4	0.018(0.014)	0.019(0.014)	0.021(0.015)
			5	0.032(0.040)	0.033(0.040)	0.042(0.041)

Table 6: Simulated average widths (AW) and coverage probabilities (CP) of 95% PIs.

n	m	System no.	s	Pivot method AW(CP)	HCD method AW(CP)	Bayesian PIs $G(0.0001, 0.0001)$ AW(CP)
15	10	1	1	0.091(0.941)	—	0.111(0.950)
			2	0.152(0.925)	0.136(0.927)	0.189(0.952)
			3	0.221(0.936)	0.205(0.923)	0.290(0.949)
			4	0.326(0.926)	0.324(0.923)	0.439(0.943)
			5	0.581(0.910)	—	0.778(0.948)
		2	1	0.223(0.918)	—	0.276(0.940)
			2	0.357(0.938)	0.324(0.933)	0.469(0.957)
			3	0.495(0.916)	0.461(0.904)	0.671(0.956)
			4	0.721(0.895)	0.715(0.908)	0.983(0.949)
			5	1.209(0.889)	—	1.621(0.942)
		3	1	0.179(0.933)	—	0.212(0.952)
			2	0.293(0.942)	0.264(0.932)	0.354(0.956)
			3	0.419(0.922)	0.389(0.920)	0.524(0.951)
			4	0.622(0.906)	0.619(0.918)	0.798(0.955)
			5	1.114(0.916)	—	1.399(0.950)
		4	1	0.198(0.926)	—	0.223(0.943)
			2	0.318(0.930)	0.287(0.923)	0.369(0.950)
			3	0.456(0.935)	0.423(0.918)	0.539(0.947)
			4	0.665(0.929)	0.661(0.911)	0.778(0.948)
			5	1.183(0.928)	—	1.390(0.952)
		5	1	0.139(0.927)	—	0.170(0.945)
			2	0.227(0.922)	0.205(0.920)	0.285(0.966)
			3	0.321(0.914)	0.298(0.914)	0.428(0.958)
			4	0.469(0.925)	0.468(0.915)	0.639(0.937)
			5	0.805(0.901)	—	1.084(0.950)
25	20	1	1	0.091(0.930)	—	0.100(0.926)
			2	0.150(0.926)	0.135(0.927)	0.172(0.939)
			3	0.217(0.930)	0.201(0.912)	0.253(0.946)
			4	0.322(0.924)	0.280(0.910)	0.379(0.954)
			5	0.577(0.924)	—	0.673(0.942)
		2	1	0.212(0.945)	—	0.234(0.951)
			2	0.342(0.945)	0.309(0.941)	0.387(0.943)
			3	0.487(0.924)	0.453(0.920)	0.560(0.943)
			4	0.699(0.927)	0.694(0.922)	0.811(0.954)
			5	1.194(0.918)	—	1.369(0.941)
		3	1	0.176(0.942)	—	0.190(0.950)
			2	0.287(0.933)	0.258(0.939)	0.317(0.943)
			3	0.418(0.929)	0.388(0.926)	0.472(0.947)
			4	0.617(0.943)	0.615(0.946)	0.699(0.946)
			5	1.111(0.932)	—	1.257(0.954)
		4	1	0.193(0.937)	—	0.203(0.944)
			2	0.311(0.924)	0.280(0.925)	0.330(0.944)
			3	0.448(0.932)	0.416(0.932)	0.488(0.944)
			4	0.661(0.949)	0.658(0.934)	0.724(0.950)
			5	1.164(0.936)	—	1.263(0.945)
		5	1	0.136(0.944)	—	0.150(0.949)
			2	0.221(0.929)	0.199(0.928)	0.248(0.945)
			3	0.319(0.932)	0.297(0.925)	0.364(0.939)
			4	0.459(0.922)	0.456(0.933)	0.544(0.945)
			5	0.798(0.927)	—	0.924(0.947)

systems. We also observe that the CMP and BUP compare very well with the MLP (in terms of both estimated biases and estimated MSPEs) in all the considered cases. Comparing the Bayesian point predictions, the Bayesian point predictors under the SEL function perform better than the Bayesian point predictors under the LINEX loss function in terms of estimated biases and estimated MSPEs. We also observe that the CMP and BUP perform better than the Bayesian point predictors in term of estimated MSPEs. We also observe that the MLP does not work well because it gives the largest absolute values of estimated biases and estimated MSPEs among all the predictors considered here. For fixed value of n and m and the system structure, the estimated MSPEs of all the point predictors are increasing with respect to s as expected.

For prediction intervals, we observe from Table 6, that the simulated coverage probabilities are close to the nominal level 95% in the most cases. It can be seen that Bayesian prediction intervals are wider than the prediction intervals obtained by the pivotal quantity method and the HCD method. Among the pivotal quantity method and the HCD method, the simulated average widths of prediction intervals based on HCD method are smaller. For fixed values of n and m and the system structure, the average widths of different prediction intervals increase as s increases.

7 Concluding remarks

In this paper, we consider a k-component coherent system while the system lifetimes are observed and the system structure is known. We discuss the prediction problem based on Type-II censored system lifetime data when the component lifetime follows the proportional hazard model. For predicting the future system failures, different point predictors including the maximum likelihood predictor, best unbiased predictor, conditional median predictor and Bayesian predictors are developed. We also computed the associated predictive interval estimates using pivotal quantity method, highest conditional density method and Bayesian method. A comprehensive Monte Carlo simulation is performed to assess the performance of the prediction methods.

Overall speaking, based on the simulation results here, we would recommend using the best unbiased predictor for the point prediction of the future system failures and using the prediction interval based on HCD method for interval prediction.

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