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Statistical inference of component lifetimes in a coherent system under proportional hazard rate model with known signature

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Abstract: In this paper, we discuss the statistical inference of the lifetime distribution of components in a n-component coherent system when the system structure is known and the component lifetime follows the proportional hazard rate model. Different estimation methods, the maximum likelihood estimator, approximation of the maximum likelihood estimator, and Bayes estimator for the component lifetime parameter are discussed. Because the integrals of the Bayes estimates do not possess closed forms, the Metropolis-Hastings method and Lindley's approximate method are applied to approximate these integrals. Confidence intervals based on the asymptotic distribution of the MLE, likelihood ratio test, pivotal method, and highest posterior density credible are computed. Two numerical examples are used to illustrate the methodologies developed in this paper and a Monte Carlo simulation study is used to compare the performance of these estimation methods and recommendations are made based on these results.

Keywords: Coherent system; Lindley approximation; Metropolis-Hastings algorithm; Proportional hazard rate model; Stochastic expectation-maximization algorithm.
Mathematics Subject Classification (2010): 62F10, 62F15, 62C10.

1 Introduction

Due to technological advancements in various fields, we depend on many devices and equipments to perform tasks and even daily affairs. Usually, these devices use different hardware or software systems that have smaller components. The lifetime of the devices

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and ensuring the performance of such systems depends on the lifetime and proper functioning of their components. A set including several components designed with a certain structure is called a system and a system n component is coherent when each n its components are useful for the system. In the theory of reliability, coherent systems provide a classical framework for describing the structure of technical systems.

Consider an n-component system with the component lifetimes, denoted by $X_1, X_2, \ldots, X_n$, being independent and identically distributed (i.i.d.) with common absolutely continuous cumulative distribution function $F_X(\cdot)$, probability density function $f_X(\cdot)$, and survival function $\bar{F}_X(\cdot) = 1 - F_X(\cdot)$. We further denote the lifetime of the n-component system by T, with cumulative distribution function $F_T(\cdot)$, probability density function $f_T(\cdot)$ and survival function $\bar{F}_T(\cdot) = 1 - F_T(\cdot)$.

In the study of coherent systems, system signature, introduced by Samaniego (1985) and discussed further by Kochar et al. (1999), is an index that characterizes a system with i.i.d. components in a simple and elegant probabilistic way. The system signature is an n-dimensional probability vector $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$, where the i-th element p_i is the probability that the i-th ordered component failure causes the failure of the system, i.e., $p_i = \Pr(T = X_{i:n}) \quad i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} p_i = 1$.

Samaniego (1985) showed that the probability density function and survival function of the system lifetime T can be written as

$$\begin{aligned} f_T(t) &= \sum_{i=1}^{k} p_i \binom{k}{i} i f_X(t) [F_X(t)]^{i-1} [\bar{F}_X(t)]^{k-i}, \\ \bar{F}_T(t) &= \sum_{i=1}^{k} p_i \sum_{j=0}^{i-1} \binom{k}{j} [F_X(t)]^j [\bar{F}_X(t)]^{k-j}, \end{aligned}$$

respectively.

Navarro et al. (2007) showed that the survival function of a coherent system can be expressed as

$$\bar{F}_T(t) = \sum_{i=1}^{n} a_i \bar{F}_{1:i}(t) = \sum_{i=1}^{n} a_i [\bar{F}_X(t)]^i, \tag{1}$$

where $\bar{F}_{1:i}(\cdot)$ is the survival function of the series system lifetime with i components. The vector $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ is called the minimal signature of the system and $a_1, a_2, \ldots, a_n$ are some negative and nonnegative integers that do not depend on F_X, and satisfy $\sum_{i=1}^{n} a_i = 1$.

Our purpose here is to develop statistical inference for the component lifetime distribution based on observed lifetimes of n-component coherent systems with the same structure and a known system signature. It is supposed that the common distribution of the n i.i.d. component lifetimes in a coherent system is the PHR model with survival function

$$\bar{F}_X(t) = [\bar{F}_0(t)]^{\theta}, \quad -\infty \leqslant c < x < d \leqslant \infty, \ \theta > 0, \tag{2}$$

where $\bar{F}_0(t) = 1 - F_0(t)$, and $F_0(t)$ is a baseline cumulative distribution function with $F_0(c) = 0$ and $F_0(d) = 1$. The PHR model includes the following lifetime models which can be used to model component lifetimes:

i) *Exponential distribution*: $EXP(\theta)$ with $\bar{F}_0(t) = e^{-t}$, $t > 0$, and the survival function

is given by $\bar{F}_X(t) = e^{-\theta t}$, $t > 0$, $\theta > 0$.
ii) *Type II standard Pareto distribution*: $Pa(\theta)$ with $\bar{F}_0(t) = \frac{1}{1+t}$, $t > 0$, and the survival function is given by $\bar{F}_X(t) = \left(\frac{1}{1+t}\right)^{\theta}$, $t > 0$, $\theta > 0$.
iii) *Weibull distribution with known shape parameter*: $We(\theta)$ with $\bar{F}_0(t) = e^{-t^{\beta}}$, $t > 0$, and known shape parameter β. The survival function is given by $\bar{F}_X(t) = e^{-\theta t^{\beta}}$, $t > 0$, $\theta > 0$.

In recent years, many authors studied the statistical inference of the component lifetime distribution based on system lifetime data when the system signature is known; see, for example, Tavangar and Asadi (2020), Navarro and Rubio (2010), Bhattacharya and Samaniego (2010), Balakrishnan et al. (2011a), Balakrishnan et al. (2011b), Ng et al. (2012), Chahkandi et al. (2014), Zhang et al. (2015a), Zhang et al. (2015b), Yang et al. (2016) and Yang et al. (2019).

In this paper, we discuss statistical inference for the component lifetime distribution based on ordered system lifetimes with known system structure. The rest of this paper is organized as follows. MLE, approximation of the MLE and interval estimation of the exponentiated parameter are discussed in Section 2. We discuss Bayesian estimation using sample-based and Lindley approximation in Section 3. In Section 4, statistical testing procedures to test whether the exponentiated parameter equals to a particular value are developed and different confidence intervals are presented in Section 4. In Section 5, two numerical example are used to illustrate the methodologies developed in this paper and a Monte Carlo simulation study is employed to evaluate the performances of the proposed procedures. Finally, a conclusion remark is presented in Section 6.

2 Maximum likelihood estimation

Based on the model in (2) and from (1), the survival function and probability density function of the coherent system lifetime are given by

$$\begin{aligned}
\bar{F}_T(t;\theta) &= 1 - F_T(t;\theta) = \sum_{i=1}^{n} a_i \bar{F}_0^{i\theta}(t), \\
f_T(t;\theta) &= -\frac{d\bar{F}_T(t;\theta)}{dt} = \theta f_0(t) \sum_{i=1}^{n} i a_i \bar{F}_0^{i\theta-1}(t) = \theta \left[\frac{f_0(t)}{\bar{F}_0(t)}\right] \sum_{i=1}^{n} i a_i \bar{F}_0^{i\theta}(t),
\end{aligned}$$

respectively, where $f_0(t) = -d\bar{F}_0(t)/dt$ is the baseline probability density function.

Suppose m independent n-component systems with the same distribution as T are placed on a life-test and that the corresponding lifetimes $T_1, T_2, \ldots, T_m$ are observed. The likelihood function of θ is given by

$$L(t;\theta) = \prod_{k=1}^{m} f_T(t_k;\theta) = \theta^m \prod_{k=1}^{m} \left\{ \frac{f_0(t_k)}{\bar{F}_0(t_k)} \sum_{i=1}^{n} i a_i \bar{F}_0^{i\theta}(t_k) \right\}, \tag{3}$$

where $\boldsymbol{t} = (t_1, t_2, \ldots, t_m)$ is the vector of observations. The corresponding log-likelihood

function can be written

$$\log L(t;\theta) = m\log\theta + \sum_{k=1}^{m}\log\left[\frac{f_0(t_k)}{\bar{F}_0(t_k)}\right] + \sum_{k=1}^{m}\log\left(\sum_{i=1}^{n} ia_i\bar{F}_0^{i\theta}(t_k)\right). \tag{4}$$

Therefore, the maximum likelihood estimator (MLE) of θ, say $\widehat{\theta}_{MLE}$ can be obtained by maximizing (4) with respect to θ. From the log-likelihood function in (4), we obtain

$$\frac{d\log L(t;\theta)}{d\theta} = \frac{m}{\theta} + \sum_{k=1}^{m}\left(\frac{\sum_{i=1}^{n} i^2a_i\bar{F}_0^{i\theta}(t_k)}{\sum_{i=1}^{n} ia_i\bar{F}_0^{i\theta}(t_k)}\right)\log\bar{F}_0(t_k) = 0. \tag{5}$$

It can be shown that the maximum of (4) can be obtained as a fixed point solution of the following equation:

$$g(\theta) = \theta,$$

where

$$g(\theta) = -m\left[\sum_{k=1}^{m}\left(\frac{\sum_{i=1}^{n} i^2a_i\bar{F}_0^{i\theta}(t_k)}{\sum_{i=1}^{n} ia_i\bar{F}_0^{i\theta}(t_k)}\right)\log\bar{F}_0(t_k)\right]^{-1}.$$

Notice that to solve the equation $g(\theta) = \theta$, first consider θ^0 as an initial guess of θ. Then, the successive approximations of the θ can be obtained as $\theta^{(1)} = g(\theta^{(0)}), \theta^{(2)} = g(\theta^{(1)}), \ldots, \theta^{(j+1)} = g(\theta^{(j)})$. This iterative procedure can be terminated at the jth stage if $\mid \theta^{(j+1)} - \theta^{(j)} \mid < \epsilon$, for some small pre-specified value of ϵ. Subsequently, after jth stage, the MLE of θ is given by $\widehat{\theta}_{MLE} = \theta^{(j)}$. Different computational algorithms available in R version 4.0.0 (R Core Team, 2019) can be used for solving the nonlinear equation in (5) for θ.

Ng et al. (2012) showed that, (5) has a unique positive solution, and $L(t;\theta)$ attains a maximum at that point.

Example 2.1. *Let $\bar{F}_0(t;\beta) = \exp\{-t^\beta\}$. Therefore, the component lifetimes has Weibull distribution with known shape parameter β. Using* (5)*, the MLE of θ, $\widehat{\theta}_{MLE}$, is the solution of*

$$\frac{m}{\widehat{\theta}_{MLE}} - \sum_{k=1}^{m}\left(\frac{\sum_{i=1}^{n} i^2a_it_k^\beta e^{-i\widehat{\theta}_{MLE}t_k^\beta}}{\sum_{i=1}^{n} ia_ie^{-i\widehat{\theta}_{MLE}t_k^\beta}}\right) = 0. \tag{6}$$

It must be solved by a numerical method in order to obtain the MLE of the parameter θ.

Yang et al. (2016) proposed a stochastic expectation-maximization (SEM) algorithm (see, Celeux et al. (1996) and Celeux and Diebolt (1985)) to obtain an approximation of the MLEs of the component lifetime distribution parameter based on complete system lifetimes. The SEM algorithm involves two steps, the S-step and the M-step. In the M-step, suppose m i.i.d. n-component systems are placed on a life-testing experiment and the observed data is $\boldsymbol{t} = (t_1, t_2, \ldots, t_m)$. The likelihood function based on the complete observed data $(x_{11}, x_{12}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, \ldots, x_{m1}, \ldots, x_{mn})$, can then be expressed as

$$L_C(\theta) = \prod_{k=1}^{m}\prod_{i=1}^{n} f_X(x_{ki}) \quad = \quad \prod_{k=1}^{m}\prod_{i=1}^{n}\theta f_0(x_{ki})\left[\bar{F}_0(x_{ki})\right]^{(\theta-1)}$$

$$= \quad \theta^{mn} \prod_{k=1}^{m} \prod_{i=1}^{n} f_0(x_{ki}) \left[\bar{F}_0(x_{ki})\right]^{(\theta-1)}.$$

The log-likelihood function is

$$\log L_C(\theta) = mn \log \theta + (\theta - 1) \sum_{k=1}^{m} \sum_{i=1}^{n} \log \left[\bar{F}_0(x_{ki}\right] + \sum_{k=1}^{m} \sum_{i=1}^{n} \log f_0(x_{ki}).$$

From the log-likelihood function, we obtain the likelihood equations as

$$\frac{d \log L_C(\theta)}{d\theta} = \frac{mn}{\theta} + \sum_{k=1}^{m} \sum_{i=1}^{n} \log \left[\bar{F}_0(x_{ki}\right].$$

Therefore, the MLE of the parameter θ is

$$\hat{\theta} = -(mn) / \sum_{k=1}^{m} \sum_{i=1}^{n} \log \left[\bar{F}_0(x_{ki})\right].$$

In the S-step, we consider the observed system lifetime of the k-th system among the m systems in the experiment, t_k. Assume that the δ-th component failure in the k-th system caused the failure of the system. Then, the conditional distributions of the other $(n-1)$ components are random variables with either left-truncated or right-truncated distributions. Specifically, the conditional density of the first $(\delta-1)$ ordered component lifetimes, $X_{k,1:n}, X_{k,2:n}, \ldots, X_{k,(\delta-1):n}$, given $t_k = x_{k,\delta:n}$ is a right-truncated density

$$g_R(x|t_k, \delta) = \frac{f_X(x)}{F_X(t_k)} = \frac{\theta[\bar{F}_0(x)]^{\theta-1} f_0(x)}{1 - [\bar{F}_0(t_k)]^{\theta}} \quad x < t_k, \tag{7}$$

and similarly the conditional density of the last $(n-\delta)$ ordered component lifetimes, $X_{k,(\delta+1):n}, X_{k,(\delta+2):n}, \ldots, X_{k,n:n}$, given $t_k = x_{k,\delta:n}$ is a left-truncated density

$$g_L(x|t_k, \delta) = \frac{f_X(x)}{1 - F_X(t_k)} = \frac{\theta[\bar{F}_0(x)]^{\theta-1} f_0(x)}{[\bar{F}_0(t_k)]^{\theta}}, \quad x > t_k. \tag{8}$$

Therefore, the $(h+1)$-th SEM iteration for the PHR model, given the current value of the parameter estimate $\theta^{(h)}$ and the observed system lifetime t_k, the S-step and M-step then proceed as follows:

Algorithm 2.2. *A Stochastic Expectation-Maximization (SEM) algorithm:*
Step 1: For the k-th system, generate a discrete random variable δ based on the system signature of the n-component system with probability mass function $Pr(\delta = i) = p_i$, $i = 1, 2, \ldots, n$, and we denote the realization as δ;
Step 2: Generate $\delta - 1$ random variates from the conditional distribution in (7) *with $\theta = \theta^{(h)}$, say $\tilde{x}_{k,1:n}, \tilde{x}_{k,2:n}, \ldots, \tilde{x}_{k,(\delta-1):n}$;*
Step 3: Generate $n - \delta$ random variates from the conditional distribution in (8) *with $\theta = \theta^{(h)}$, say $\tilde{x}_{k,(\delta+1):n}, \tilde{x}_{k,(\delta+2):n}, \ldots, \tilde{x}_{k,n:n}$;*
Step 4: The pseudo-complete sample for system k is then $(\tilde{x}_{k,1:n}, \tilde{x}_{k,2:n}, \ldots, \tilde{x}_{k,(\delta-1):n}$, $\tilde{x}_{k,\delta:n} = t_k$, $\tilde{x}_{k,(\delta+1):n}, \tilde{x}_{k,(\delta+2):n}, \ldots, \tilde{x}_{k,n:n})$. Repeat Steps 1–3 for $k = 1, \ldots, m$ to

obtain the pseudo-complete sample $(\tilde{x}_{11}, \tilde{x}_{12}, \ldots, \tilde{x}_{1n}, \tilde{x}_{21}, \ldots, \tilde{x}_{2n}, \ldots, \tilde{x}_{m1}, \ldots, \tilde{x}_{mn})$.
Step 5: Obtain $\theta^{(h+1)}$ *for the next cycle as*

$$\theta^{(h+1)} = -(mn)/\sum_{k=1}^{m}\sum_{i=1}^{n} \log\left[\bar{F}_0(\tilde{x}_{ki})\right].$$

Step 6: To obtain an estimate of θ*, we run the SEM algorithm to obtain a sequence of* $\theta^{(h)}, h = 1, 2, \ldots, H$*, discard the first* B *iterations for burn-in, and average over the estimates from the remaining iterations to get an estimate of* θ *(say,* $\tilde{\theta_{AMLE}}$*), i.e.,*

$$\tilde{\theta}_{AMLE} = \sum_{h=B+1}^{H} \theta^{(h)}/(H-B).$$

Following the general asymptotic theory of the MLE, the sampling distribution of $(\widehat{\theta}_{MLE} - \theta)/\sqrt{\widehat{Var}(\widehat{\theta}_{MLE})}$ can be approximated by a standard normal distribution. Hence, an asymptotic $100(1-\alpha)\%$ confidence interval for θ can be constructed as $[\hat{\theta}_l, \hat{\theta}_u] = \hat{\theta} \pm z_{1-\frac{\alpha}{2}}\sqrt{\widehat{Var}(\widehat{\theta}_{MLE})}$, where z_q is the q-th percentile of the standard normal distribution. Here, the variance of $\widehat{\theta}_{MLE}$ can be approximated by the inverse of the observed Fisher information, i.e.,

$$\widehat{Var}(\widehat{\theta}_{MLE}) = \left[-\frac{\partial^2 L(\theta)}{\partial^2\theta}\bigg|_{\theta=\widehat{\theta}_{MLE}}\right]^{-1},$$

where

$$-\frac{\partial^2 L(\theta)}{\partial^2\theta} = (\frac{m}{\theta^2}) - \sum_{k=1}^{m}\left[\log(\bar{F}_0(t_k))\right]^2 \left\{\frac{\sum_{i=1}^{n} i^3 a_i \bar{F}_0^{i\theta}(t_k)}{\sum_{i=1}^{n} i a_i \bar{F}_0^{i\theta}(t_k)} - \left[\frac{\sum_{i=1}^{n} i^2 a_i \bar{F}_0^{i\theta}(t_k)}{\sum_{i=1}^{n} i a_i \bar{F}_0^{i\theta}(t_k)}\right]^2\right\}.$$

Note, for n-component systems with Weibull distributed components, the variance of $\widehat{\theta}_{MLE}$ can be

$$\widehat{Var}(\widehat{\theta}_{MLE}) = \left[\left(\frac{m}{\widehat{\theta}^2_{MLE}}\right) - \sum_{k=1}^{m} t_k^{2\beta}\left\{\frac{\sum\limits_{i=1}^{n} i^3 a_i e^{-it_k^{\beta}\,\widehat{\theta}_{MLE}}}{\sum\limits_{i=1}^{n} i a_i e^{-it_k^{\beta}\,\widehat{\theta}_{MLE}}} - \left[\frac{\sum\limits_{i=1}^{n} i^2 a_i e^{-it_k^{\beta}\,\widehat{\theta}_{MLE}}}{\sum\limits_{i=1}^{n} i a_i e^{-it_k^{\beta}\,\widehat{\theta}_{MLE}}}\right]^2\right\}\right]^{-1}.$$

3 Bayesian approaches

In this section, we discuss the Bayesian inference of the unknown parameter θ. For parameter estimation we have considered squared error loss (SEL) and linear-exponential (LINEX) loss functions. In the Bayesian approach, θ is considered as a random variable with a prior distribution $\pi(\theta)$. Here, we consider a gamma prior for θ which has the probability density function

$$\pi(\theta) = \frac{d_2^{d_1}\theta^{d_1-1}\, e^{-d_2\theta}}{\Gamma(d_1)}, \qquad \theta > 0, \qquad d_1 > 0, \qquad d_2 > 0, \tag{9}$$

where d_1 and d_2 are positive hyperparameters. A prior knowledge of the mean and variance of θ can help us to select these hyperparameters. Setting $d_1 = d_2 = 0$, the Jeffreys' non-informative prior distribution would be recovered.

By combining (9) and (3), we obtain the posterior of θ as

$$\pi(\theta|\boldsymbol{t}) = \frac{1}{R(\mathbf{t})}\ \theta^{m+d_1-1}e^{-\theta d_2}\ \prod_{k=1}^{m}\sum_{i=1}^{n} ia_i\bar{F}_0^{i\theta}(t_k),$$

where

$$R(\mathbf{t}) = \left[\int_0^\infty \theta^{m+d_1-1}e^{-\theta d_2}\ \prod_{k=1}^{m}\sum_{i=1}^{n} ia_i\bar{F}_0^{i\theta}(t_k)d\theta\right].$$

For a parameter θ and a decision rule δ, the most commonly used loss function is SEL function $L_1(\theta,\delta) = (\delta-\theta)^2$. Another loss function is LINEX, see, Varrian (1975) and Zellner (1986). The LINEX loss function of the decision rule δ of θ can be expressed as

$$L_2(\theta,\delta) = e^{c(\delta-\theta)} - c(\delta-\theta) - 1, \quad c \neq 0.$$

where c is the shape parameter of the loss function. It controls the direction and degree of symmetry. (If $c > 0$, the overestimation is more serious than underestimation, and vice-versa). For c close to zero, the LINEX loss is approximately SEL and therefore almost symmetric. The Bayes estimate of θ under $L_2(\theta,\delta)$, is expressed as

$$\widehat{\theta}_{BL} = -\frac{1}{c}\log\left[E_\theta\ (e^{-c\theta})\right], \tag{10}$$

where the expectation $E_\phi(.)$ is taken with respect to the posterior distribution of θ. The Bayes estimator of θ under the SEL function, $\widehat{\theta}_{BS}$, is the mean of posterior distribution of θ, i.e.

$$\begin{aligned}\widehat{\theta}_{BS} = E(\theta|\mathbf{t}) &= \int_0^\infty \theta\pi(\theta|\boldsymbol{t})d\theta \\ &= \frac{1}{R(\mathbf{t})}\int_0^\infty \theta^{m+d_1}\ e^{-\theta d_2}\ \prod_{k=1}^{m}\sum_{i=1}^{n} ia_i\bar{F}_0^{i\theta}(t_k)\ d\theta. \end{aligned}\tag{11}$$

From (10), the Bayes point estimator for θ under the LINEX loss function, denoted as, $\widehat{\theta}_{BL}$, is

$$\begin{aligned}\widehat{\theta}_{BL} &= -\frac{1}{c}\log\left(E_\theta(e^{-c\theta}|\boldsymbol{t})\right) = -\frac{1}{c}\log\left(\frac{\int_0^\infty e^{-c\theta}\pi(\theta|\boldsymbol{t})d\theta}{\int_0^\infty \pi(\theta|\boldsymbol{t})d\theta}\right) \\ &= -\frac{1}{c}\log\left(\frac{\int_0^\infty\ \theta^{m+d_1-1}e^{-\theta(c+d_1)}\prod_{k=1}^{m}\sum_{i=1}^{n} ia_i\bar{F}_0^{i\theta}(t_k)\ d\theta}{\int_0^\infty \theta^{m+d_1-1}e^{-\theta d_2}\ \prod_{k=1}^{m}\sum_{i=1}^{n} ia_i\bar{F}_0^{i\theta}(t_k)\ d\theta}\right). \end{aligned}\tag{12}$$

Example 3.1. *For system with Weibull distributed components, the Bayes estimator of θ, under the SEL function and the LINEX loss function, are given respectively by*

$$\widehat{\theta}_{BS} = \frac{\int_0^\infty \theta^{m+d_1}\ e^{-\theta d_2}\prod_{k=1}^{m}\sum_{i=1}^{n} ia_i\ e^{-i\theta t_k^\beta}\ d\theta}{\int_0^\infty \theta^{m+d_1-1}e^{-\theta d_2}\ \prod_{k=1}^{m}\sum_{i=1}^{n} ia_i\ e^{-i\theta t_k^\beta}d\theta},$$

$$\widehat{\theta}_{BL} = -\frac{1}{c}\log\left\{\frac{\int_0^\infty \theta^{m+d_1-1}e^{-\theta(c+d_1)}\prod_{k=1}^m\sum_{i=1}^n ia_i\ e^{-i\theta t_k^\beta}\ d\theta}{\int_0^\infty \theta^{m+d_1-1}\ e^{-\theta d_2}\ \prod_{k=1}^m\sum_{i=1}^n ia_i\ e^{-i\theta t_k^\beta}\ d\theta}\right\}.$$

Due to the complexity of the posterior probability density function $\pi(\theta|\boldsymbol{t})$, the Bayes estimator of θ in (11) and (12) cannot be obtained, we adopt the Metropolis-Hastings method and the Lindley's approximate method to compute the Bayes estimate of θ.

3.1 Bayesian sample-based estimation

Here we adopt the Metropolis-Hastings algorithm (see, for example, Robert and Casella (2004)) with a normal distribution as proposal distribution to generate random observations from the posterior distribution.

Algorithm 3.2. *The steps for the Metropolis-Hastings algorithm are described as follows:*
Step 1: Start with an initial guess $\theta^{(0)}$ is MLE of θ.
Step 2: Set $q = 1$.
Step 3: Using Metropolis-Hastings algorithm, generate θ_t from $\pi(\theta^{(j-1)}|\boldsymbol{y})$ with the $N(\theta^{(j-1)}, S_\theta^2)$ as proposal distribution, where S_θ^2 is inverse of the Fisher information.
Step 4: Set $q = q + 1$.
Step 5: Repeat Steps 3 and 4 N times to obtain the Markov Chain Monte Carlo (MCMC) sample of size N: $\theta_1, \theta_2, \ldots, \theta_N$.

Now the approximate Bayes estimation under squared error loss function and the LINEX loss function, are given by

$$\begin{aligned}\widehat{\theta}_{BS} &= \frac{1}{N-M}\sum_{q=M+1}^{N}\theta_q,\\ \widehat{\theta}_{BL} &= -\frac{1}{c}\log\left(\frac{1}{N-M}\sum_{q=M+1}^{N}e^{-c\theta_q}\right),\end{aligned}$$

respectively, where M is the burn-in period.

Based on the above simulated values of θ, $\{\theta_q; q = M+1, \ldots, N\}$, and using the method proposed by Chen and Shao (1999), we obtain the Highest Posterior Density (HPD) credible interval for θ. We assume that $\theta_{[M+1]} < \ldots < \theta_{[N]}$ is the ordered MCMC sample of $\{\theta_q; q = M+1, \ldots, N\}$. A $100(1-\alpha)\%$ the HPD credible interval for θ is

$$\left(\theta_{[(\frac{\alpha}{2})(N-M)]}, \theta_{[(1-\frac{\alpha}{2})(N-M)]}\right),$$

where $\theta_{[\frac{\alpha}{2}(N-M)]}$ and $\theta_{[(1-\frac{\alpha}{2})(N-M)]}$ are the $[\frac{\alpha}{2}(N-M)]$-th smallest integer and the $[(1-\frac{\alpha}{2})(N-M)]$-th smallest integer of $\{\theta_q\ q = 1, 2, \cdots, N-M\}$, respectively.

3.2 Lindley approximation

The Lindley's approximation was originally introduced by Lindley (1980) to approximate the ratio of two integrals such as (11) and (12). Using Lindley's approximation,

the Bayes estimate of θ under SEL and LINEX loss functions can be approximated as:

$$\widehat{\theta}_{BLS} = \widehat{\theta}_{MLE} + \left(\frac{d_1 - 1}{\widehat{\theta}_{MLE}} - d_2\right)\sigma^2(\widehat{\theta}_{MLE}) + \frac{1}{2}\, l_3(\widehat{\theta}_{MLE})\sigma^4(\widehat{\theta}_{MLE}),$$

$$\widehat{\theta}_{BLL} = \widehat{\theta}_{MLE} - \frac{1}{c}\log\left\{1 + \left(\frac{c^2}{2} - c(\frac{d_1 - 1}{\widehat{\theta}_{MLE}} - d_2)\right)\sigma^2(\widehat{\theta}_{MLE}) - \frac{c}{2} l_3(\widehat{\theta}_{MLE})\sigma^4(\widehat{\theta}_{MLE})\right\},$$

where

$$l_3 = \frac{\partial^3 \log L(\theta)}{\partial\theta} = (\frac{2m}{\theta^3}) + \sum_{k=1}^{m}\left[\log(\bar{F}_0(t_k))\right]^3 \left\{\frac{\sum_{i=1}^n i^4 a_i \bar{F}_0^{i\theta}(t_k)}{\sum_{i=1}^n i a_i \bar{F}_0^{i\theta}(t_k)} + 2\left(\frac{\sum_{i=1}^n i^2 a_i \bar{F}_0^{i\theta}(t_k)}{\sum_{i=1}^n i a_i \bar{F}_0^{i\theta}(t_k)}\right)^3 \right.$$
$$\left. -3\left(\frac{\sum_{i=1}^n i^2 a_i \bar{F}_0^{i\theta}(t_k)\sum_{i=1}^n i^3 a_i \bar{F}_0^{i\theta}(t_k)}{\left(\sum_{i=1}^n i a_i \bar{F}_0^{i\theta}(t_k)\right)^2}\right)\right\},$$

and $[\sigma(\theta)]^2$ is the inverse of Fisher information. For n-component systems with Weibull distributed components,

$$l_3 = \left(\frac{2m}{\widehat{\theta}^3_{MLE}}\right) - \sum_{k=1}^{m} t_k^{3\beta}\left\{\frac{\sum_{i=1}^n i^4 a_i e^{-it_k^\beta\ \widehat{\theta}_{MLE}}}{\sum_{i=1}^n i a_i e^{-it_k^\beta\ \widehat{\theta}_{MLE}}} + 2\left(\frac{\sum_{i=1}^n i^2 a_i e^{-it_k^\beta\ \widehat{\theta}_{MLE}}}{\sum_{i=1}^n i a_i e^{-it_k^\beta\ \widehat{\theta}_{MLE}}}\right)^3 \right.$$
$$\left. -3\left(\frac{\sum_{i=1}^n i^2 a_i e^{-it_k^\beta\ \widehat{\theta}_{MLE}}\sum_{i=1}^n i^3 a_i e^{-it_k^\beta\ \widehat{\theta}_{MLE}}}{\left(\sum_{i=1}^n i a_i e^{-it_k^\beta\ \widehat{\theta}_{MLE}}\right)^2}\right)\right\}.$$

4 Testing of hypothesis and interval confidence

In this section, we construct statistical testing procedures to assess whether the exponentiated parameter equals to a particular value, say θ_0. We shall consider testing hypothesis

$$H_0 : \theta = \theta_0 \text{ against } H_a : \theta \neq \theta_0. \tag{13}$$

by developing a pivotal quantity and a likelihood ratio test.

4.1 Test based on pivotal quantity

Let

$$W(\theta) = -2\sum_{k=1}^{m}\log[1 - \bar{F}(T_k)] = -2\sum_{k=1}^{m}\log\left[1 - \sum_{i=1}^{n} a_i \bar{F}_0^{i\theta}(T_k)\right].$$

Since $\log[1 - \bar{F}(T_k)]$ follows a standard exponential distribution, then the random variable $W(\theta)$ has a chi-square distribution with $2m$ degrees of freedom. Also, we have

$$\frac{d}{d\theta}\left[-2\sum_{k=1}^{m}\log\left(1 - \sum_{i=1}^{n} a_i \bar{F}_0^{i\theta}(T_k)\right)\right] = 2\sum_{k=1}^{m}\left(\log\bar{F}_0(T_k)\right)\left(\frac{\sum_{i=1}^n i a_i \bar{F}_0^{i\theta}(T_k)}{1 - \sum_{i=1}^n a_i \bar{F}_0^{i\theta}(T_k)}\right) \leq 0,$$

$$\lim_{\theta\to 0} W(\theta) = +\infty, \qquad \lim_{\theta\to+\infty} W(\theta) = 0.$$

This implies that $W(\theta)$ is a continuous function of θ on $(0,\infty)$ and $W(\theta)$ is an monotonic decreasing function in θ. To test the hypotheses (13), since $W(\theta)$ is strictly decreasing in θ, small and large values of $W(\theta_0)$ lead to the rejection of H_0. Therefore, the decision rule for testing (13) is to reject H_0 if $W(\theta_0) < \chi^2_{\alpha/2}(2m)$ or $W(\theta_0) > \chi^2_{1-\alpha/2}(2m)$. The power function of this test can be obtained as

$$\Pr\left(W(\theta_0) < \chi^2_{\alpha/2}(2m)|H_a\right) + \Pr\left(W(\theta_0) > \chi^2_{1-\alpha/2}(2m)|H_a\right).$$

Now, a $100(1-\alpha)\%$ confidence interval for θ can be constructed from the relation

$$P\left(\chi^2_{\alpha/2,2m} < -2\sum_{k=1}^{m}\log\left[1-\sum_{i=1}^{n}a_i\bar{F}_0^{i\theta}(T_k)\right] < \chi^2_{1-\alpha/2,2m}\right) = 1-\alpha,$$

from which we can compute the $100(1-\alpha)\%$ confidence interval for θ as (L,U), where L and U are the solutions of

$$-2\sum_{k=1}^{m}\log\left[1-\sum_{i=1}^{n}a_i\bar{F}_0^{i\theta}(T_k)\right] = \chi^2_{1-\alpha/2}(2m),$$
$$-2\sum_{k=1}^{m}\log\left[1-\sum_{i=1}^{n}a_i\bar{F}_0^{i\theta}(T_k)\right] = \chi^2_{\alpha/2}(2m),$$

respectively, where $\chi^2_{\alpha/2,2m}$ and $\chi^2_{1-\alpha/2,2m}$ denote the lower and upper $\alpha/2$ percentage points of a chi-square distribution with $2m$ degrees of freedom.

Example 4.1. *For system with Weibull distributed components, we have*

$$W(\theta) = -2\sum_{k=1}^{m}\log\left(1-\sum_{i=1}^{n}a_i e^{-\theta i t_k^{\beta}}\right).$$

Therefore, a $100(1-\alpha)\%$ *confidence interval for* θ, (θ_L,θ_U), *are the solutions of*

$$-2\sum_{k=1}^{m}\log\left(1-\sum_{i=1}^{n}a_i e^{-\theta i t_k^{\beta}}\right) = \chi^2_{1-\alpha/2,2m},$$
$$-2\sum_{k=1}^{m}\log\left(1-\sum_{i=1}^{n}a_i e^{-\theta i t_k^{\beta}}\right) = \chi^2_{\alpha/2,2m},$$

respectively.

4.2 Likelihood ratio test

The likelihood ratio test (LRT) statistics for testing $H_0:\theta=\theta_0$ versus $H_a:\theta\neq\theta_0$ is

$$\lambda(\boldsymbol{t}) = \frac{\sup_{\theta\in\Theta_0}L(\theta|\boldsymbol{t})}{\sup_{\theta\in\Theta}L(\theta|\boldsymbol{t})} = \frac{L(\theta_0|\boldsymbol{t})}{L(\widehat{\theta}_{MLE}|\boldsymbol{t})}, \tag{14}$$

where Θ is the parameter space and $\underline{t} = (t_1, t_2, \ldots, t_m)$ is the observed value of $\underline{T} = (T_1, T_2, \ldots, T_m)$. Substituting (3) into (14), the LRT statistic is

$$\lambda(\boldsymbol{t}) = \left(\frac{\theta_0}{\widehat{\theta}_{MLE}}\right)^m \prod_{k=1}^{m} \left\{ \frac{\sum_{i=1}^{n} i a_i \bar{F}_0^{i\theta_0}(t_k)}{\sum_{i=1}^{n} i a_i \bar{F}_0^{i\widehat{\theta}_{MLE}}(t_k)} \right\}.$$

For large sample sizes, $m \to \infty$, it is possible to approximate the distribution of the statistic $-2\log\lambda(\boldsymbol{T})$ under H_0 converges to a chi squared distribution with one degree of freedom. Thus the test statistic is

$$-2\log\lambda(\boldsymbol{T}) = -2\left[m\left[\log\theta_0 - \log\widehat{\theta}_{MLE}\right] + \sum_{k=1}^{m} \log\left\{ \frac{\sum_{i=1}^{n} i a_i \bar{F}_0^{i\theta_0}(t_k)}{\sum_{i=1}^{n} i a_i \bar{F}_0^{i\widehat{\theta}_{MLE}}(t_k)} \right\}\right].$$

Rejection of $H_0 : \theta = \theta_0$ for small values of $\lambda(\boldsymbol{t})$ is equivalent to rejection for large values of $-2\log\lambda(\boldsymbol{T})$. Thus, H_0 is rejected at α level of significance if and only if $-2\log\lambda(\boldsymbol{t}) \geq \chi^2_{1-\alpha}(1)$. The power function of the LRT can be obtained as

$$\Pr\left(-2\log\lambda(\boldsymbol{T}) \geq \chi^2_{1-\alpha}(1) | H_a\right).$$

In order to obtain the confidence interval by using the likelihood ratio test, it is enough to consider the complement of the area of rejecting the hypothesis H_0, the area of acceptance or the confidence area. Therefore, $100(1-\alpha)\%$ confidence interval for θ, based on the likelihood ratio test is

$$\begin{aligned} K(\theta) &= \{\theta : \ -2\log\lambda(\boldsymbol{T}) \leq \chi^2_{1-\alpha}(1)\} \\ &= \{\theta : \ -2[\log L(\theta|\boldsymbol{T}) - \log L(\widehat{\theta}_{MLE}|\boldsymbol{T})] \leq \chi^2_{1-\alpha}(1)\} \\ &= \{\theta : \ \log L(\theta|\boldsymbol{T}) \geq \log L(\widehat{\theta}_{MLE}|\boldsymbol{T}) - \frac{1}{2}\chi^2_{1-\alpha}(1)\}. \end{aligned}$$

Since $L(\theta|\boldsymbol{t})$ is a unimodal function, So for a given α, $K(\theta)$ will be a unique confidence interval.

Example 4.2. *For system with Weibull distributed components, a $100(1-\alpha)\%$ confidence interval for θ is*

$$\begin{aligned} K(\theta) &= \Bigg\{\theta : \ \log L(\theta|\boldsymbol{T}) \geq m\log\widehat{\theta}_{MLE} + \log\beta + (\beta-1)\sum_{k=1}^{m} t_k \\ &\quad + \sum_{k=1}^{m} \log\left\{\sum_{i=1}^{n} i a_i e^{i t_k^{\beta} \widehat{\theta}_{MLE}}\right\} - \frac{1}{2}\chi^2_{1-\alpha}(1)\Bigg\}. \end{aligned}$$

5 Numerical illustration and Monte Carlo simulation study

In this section, two numerical example are considered for illustrative purposes and a Monte Carlo simulation study is performed to compare the point and interval estimation methods presented in Sections 2-4.

Here, we assume that the lifetime of the components are i.i.d. Weibull distributed with known shape parameter and cumulative distribution function $F_X(t;\theta,\beta) = 1 - \exp\{-\theta t^\beta\}$. Before progressing further, first we describe how we can generate a sample $T_1, T_2, \ldots, T_m$ of i.i.d. system lifetimes for systems with Weibull distributed components. The following algorithm is used to generate the system lifetime $T_1, T_2, \ldots, T_m$ with system signature $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ $(0 < p_i < 1, \sum_{i=1}^n p_i = 1)$ with Weibull distributed components.

Algorithm 5.1. *The steps for Algorithm to generate system lifetimes are described as follows:*
Step 1 Generate u, u_1, $u_2, \ldots, u_n$ independently from uniform distribution in $[0,1]$;
Step 2 Set $X_j = \left[\frac{1}{\theta}\left(\log(\frac{1}{1-u_j})\right)\right]^{1/\beta}$, $j = 1, 2, \ldots, n$;
Step 3 Sort $X_1, X_2, \cdots, X_n$ in ascending order to obtain $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$;
Step 4 Take $T = X_{j:n}$ for $\sum_{i=1}^{j-1} p_i < u < \sum_{i=1}^{j} p_i, (j = 1, 2, \cdots, n)$, i.e.,

$$T = \begin{cases} X_{1:n} & 0 < u < p_1 \\ X_{2:n} & p_1 < u < p_1 + p_2 \\ X_{3:n} & p_1 + p_2 < u < p_1 + p_2 + p_3 \\ \vdots & \vdots \\ X_{n:n} & \sum_{j=1}^{n-1} p_j < u < \sum_{j=1}^{n} p_j \end{cases}$$

Step 5 Repeat Steps $1-4$, m times, and Generate lifetime $T = (T_1, T_2, \cdots, T_m)$.

Example 5.2. *To illustrate all the methods presented in the preceding sections, we consider a coherent system of 4 components according to Figure 1.*

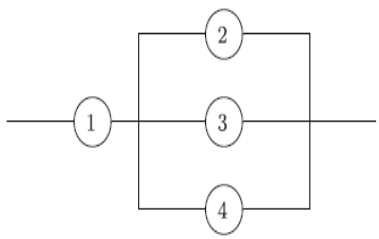

Figure 1: The coherent system of 4 components

A sample of size $m = 20$ was generated from a 4-component reliability system having lifetime $T = \min\{X_1, \max\{X_2, X_3, X_4\}\}$, system signature $\boldsymbol{P} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)$, and minimal signature $\boldsymbol{a} = (0, 3, -3, 1)$, with components following a two-parameter Weibull distribution with scale parameter $\theta = 2$, and shape parameter $\beta = 2$. The simulated system lifetimes are presented in Table 1.

Based on the system lifetime data in Table 1, we obtain the MLE, the approximate MLE (AMLE) using SEM algorithm (with $H = 20000$ and $B = 5000$) of θ. Bayes estimates are obtained using Lindley approximation and Metropolis-Hastings methods based on the SEL and LINEX loss functions. In the Metropolis-Hastings algorithm, we have generated $N = 50000$ values with $S_\theta^2 = 0.1536$, $\hat{\theta}_{MLE} = 2.063$ and the acceptance rate is about 70%. We discarded the initial $M = 5000$ burn-in sample and calculate

Table 1: Simulated 4-component system lifetimes with system signature $\boldsymbol{P} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)$ with Weibull distributed components.

m	1	2	3	4	5	6	7	8	9	10
$t_{m:20}$	0.1223	0.1503	0.2761	0.2967	0.2990	0.3136	0.3752	0.4265	0.4291	0.4578
m	11	12	13	14	15	16	17	18	19	20
$t_{m:20}$	0.5223	0.5398	0.5795	0.5839	0.6126	0.6137	0.7816	0.8879	0.9525	1.2941

the Bayes estimates based on the remaining 45000 *samples. For computing the Bayes estimates, since we do not have any prior information, we assumed that the prior of* θ *is almost improper, i.e.,* $d_1 = d_2 = 0.0001$. *The LINEX loss function is used for computing the Bayes estimates under different values of* $c, c = (-0.05, 0.05, 1)$. *The results are summarized in Table 2.*

The 95% *confidence intervals corresponding MLE, LRT, exact method and HPD credible of* θ *become* $(1.2911, 2.8350)$, $(1.3753, 2.9229)$, $(1.2845, 4.4039)$ *and* $(1.3448, 2.9020)$, *respectively.*

Figure 2 shows the trace plot and histogram plot for the parameter θ. *The trace plots show the values of* θ *is randomly scattered around the average. From the histogram of the Metropolis-Hastings sequences for* θ *in Figure 2, we observe that choosing the normal distribution as a proposal distribution is quite appropriate.*

Table 2: Point estimates of θ for Example 5.2 with Weibull distributed components.

Method		Point Estimate
MLE		2.063
Approximate MLE via SEM		1.997
Lindley's approximation		
	SEL	2.055
	LINEX ($c = -0.05$)	2.058
	LINEX ($c = 0.05$)	2.051
	LINEX ($c = 1$)	1.981
MCMC		
	SEL	2.066
	LINEX ($c = -0.05$)	2.070
	LINEX ($c = 0.05$)	2.062
	LINEX ($c = 1$)	1.991

Consider testing

$$H_0 : \theta = 1 \quad \textit{against} \quad H_1 : \theta \neq 1,$$

at the level of significance $\alpha = 0.05$. *Based on the pivotal quantity approach, since*

$$Q(\theta_0) = Q(1) = 67.9901 > \chi^2_{0.975}(40) = 59.3417,$$

then, the hypothesis $H_0 : \theta = 1$ *is rejected at* 5% *level of significance. Based on the likelihood ratio test, since*

$$-2 \log \lambda(\boldsymbol{t}) = 10.8486 \geq \chi^2_{0.95}(1) = 3.8414,$$

we reject $H_0 : \theta = 1$ *at* 5% *level of significance.*

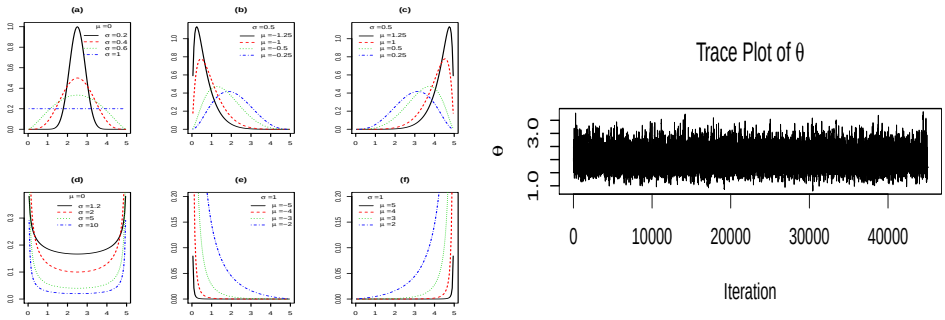

Figure 2: Plots of Metropolis-Hastings Markov chains for θ using the non-informative prior.

Example 5.3. *A sample of size $m = 25$ was generated from a 4-component reliability system having lifetime*

$$T = \min\{X_1, \max\{X_2, X_3\}, \max\{X_2, X_4\}\},$$

system signature $\boldsymbol{P} = (\frac{1}{4}, \frac{7}{12}, \frac{1}{6}, 0)$, and minimal signature $\boldsymbol{a} = (0, 1, 1, -1)$, with components following a two-parameter Weibull distribution with scale parameter $\theta = 1$, and shape parameter $\beta = 0.75$. The simulated system lifetimes are presented in Table 3.

Table 3: Simulated 4-component system lifetimes with system signature $\boldsymbol{P} = (\frac{1}{4}, \frac{7}{12}, \frac{1}{6}, 0)$ with Weibull distributed components.

m	1	2	3	4	5	6	7	8	9	10
$t_{m:25}$	0.0009	0.0101	0.0243	0.0470	0.0882	0.0919	0.0951	0.1017	0.1099	0.1247
m	11	12	13	14	15	16	17	18	19	20
$t_{m:25}$	0.1354	0.1791	0.1827	0.1938	0.3178	0.3447	0.3771	0.4877	0.5011	0.5479
m	21	22	23	24	25	26	27	28	29	30
$t_{m:25}$	0.6546	0.9910	1.0687	3.1065	5.2747					

Based on the system lifetime data in Table 3, we obtained the point and interval estimates of θ as described in Sections 2-4. For computing the Bayes estimates, it is assumed that the prior of θ is proper, $d_1 = 2$, $d_2 = 4$. The results are summarized in Table 4.

The convergence of - samples can be verified through graphical inspection. From Figure 3, we observe that choosing the normal distribution as a proposal distribution is quite appropriate. Consider testing

$$H_0 : \theta = 1 \quad \textit{against} \quad H_1 : \theta \neq 1,$$

at the level of significance $\alpha = 0.05$. Based on the pivotal quantity approach, since

$$Q(\theta_0) = 62.3684 < \chi^2_{0.975}(50) = 71.4202,$$

then, the hypothesis $H_0 : \theta = 1$ is accepted at 5% level of significance. Based on the likelihood ratio test, since

$$-2 \log \lambda(\boldsymbol{t}) = 0.0934 < \chi^2_{0.95}(1) = 3.8414,$$

Table 4: Point and interval estimates of of θ for Example 5.3 with Weibull distributed components.

Method		Point Estimate
MLE		1.0562
Approximate MLE via SEM		0.9973
Lindley's approximation		
	SEL	0.9817
	LINEX ($c = -0.05$)	0.9825
	LINEX ($c = 0.05$)	0.9810
	LINEX ($c = 1$)	0.9682
MCMC		
	SEL	0.9952
	LINEX ($c = -0.05$)	0.9959
	LINEX ($c = 0.05$)	0.9944
	LINEX ($c = 1$)	0.9808
Method		**Interval Estimate**
Exact method		(0.8334 , 2.4883)
Normal-approximation of MLE		(0.6898 , 1.4226)
LRT		(0.7298 , 1.4650)
HPD credible interval		(0.6905 , 1.3551)

we accept $H_0 : \theta = 1$ at 5% level of significance.

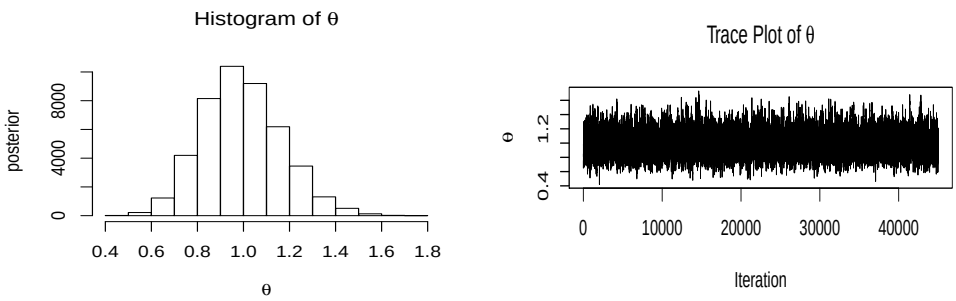

Figure 3: Plots of Metropolis-Hastings Markov chains for θ using the informative prior.

5.1 Simulation study

Here, we present some results based on Monte Carlo simulations to compare the performance of the different methods. Five different systems with different system signatures and minimal signatures are considered in the simulation study which are listed in Table 5.

For different choices of sample size m, we generated system lifetimes $T_1, \ldots, T_m$ with Weibull distributed components with the parameters $\theta = 1$ and $\beta = 1$. The MLEs and AMLEs are computed using the methods described in Section 2. The approximate Bayes estimates under the SEL and LINEX loss functions are computed for θ using Lindley's approximation and Metropolis-Hastings procedures. We compare

Table 5: System signatures and minimal signatures of the 4-component systems.

System no.	System lifetime T	$\boldsymbol{P}$	$\boldsymbol{a}$
1	$X_{1:4} = \min\{X_1, X_2, X_3, X_4\}$ (series)	$(1, 0, 0, 0)$	$(0, 0, 0, 1)$
2	$\min\{X_1, \max\{X_2, X_3, X_4\}\}$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)$	$(0, 3, -3, 1)$
3	$\max\{X_1, \min\{X_2, X_3, X_4\}\}$	$(0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4})$	$(1, 0, 1, -1)$
4	$\min\{X_1, \max\{X_2, X_3\}, \max\{X_2, X_4\}\}$	$(\frac{1}{4}, \frac{7}{12}, \frac{1}{6}, 0)$	$(0, 1, 1, -1)$
5	$\min\{\max\{X_1, X_2\}, \max\{X_3, X_4\}\}$	$(0, \frac{1}{3}, \frac{2}{3}, 0)$	$(0, 4, -4, 1)$

the performances of the MLEs, AMLEs, and the Bayes estimates in terms of biases, and mean squares errors (MSEs), which can be estimated as

$$\widehat{Bias} = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\theta}_i - \theta), \qquad \text{and} \qquad \widehat{MSE} = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\theta}_i - \theta)^2,$$

respectively, where $\hat{\theta}_i$ is the estimate of θ obtained in the i-th simulation, where $i = 1, \ldots, 1000$. In Table 6, we present the biases and MSEs of different estimator of θ for sample sizes $m = 10, 15, 20$ and 25 based on 1000 replications.

We also compare different confidence intervals, namely the confidence intervals obtained by using asymptotic distributions of the MLE, LRT, exact confidence interval and the HPD credible intervals in terms of their simulated average widths and simulated coverage probabilities based on 1000 replications. If (L_i, U_i) is the 95% confidence interval of θ obtained in the i-th simulation, then the average widths (AWs) and the coverage probabilitys (CPs) are computed as

$$\text{AW} = \frac{1}{1000}\sum_{i=1}^{1000}(U_i - L_i), \qquad \text{and} \qquad \text{CP} = \frac{1}{1000}\sum_{i=1}^{1000} I\{L_i < \theta < U_i\},$$

where $I\{\cdot\}$ denotes the indicator function. The results for sample sizes $m = 10, 15, 20$ and 25 are reported in Table 7.

For computing the Bayes estimators and HPD credible intervals, we assume two priors as follows:
Prior I: Jeffreys non-informative prior with $d_1 = d_2 = 0$.
Prior II: Informative prior with $d_1 = 2,\ d_2 = 4$.
In order to compare the power properties of the pivotal quantity approach and the LRT for testing the hypotheses $H_0 : \theta = 1$ against $H_1 : \theta \neq 1$ at the level of significance $\alpha = 0.05$, a Monte Carlo simulation study is conducted under different alternatives. The simulated rejection rates for sample sizes $m = 10, 15, 20$ and 25 are presented in Table 8. The computations are performed in R, R Core Team (2019) with the MHadaptive package Chivers (2012).

6 Discussion and conclusions

From Table 6, we observe that the MSEs of the estimators are decreasing with increase sample size m, and the performance of the estimators depend on the structure of the system. Since the serial system (System No. 1) has the minimum lifetimes, we can

Table 6: Biases and MSEs of the MLE, AMLE and Bayes estimators for $m = 10, 15, 20$ and 25.

				Prior1: $G(0,0)$				Prior2: $G(2,4)$			
				MCMC		Lindley		MCMC		Lindley	
no	$m = 10$	MLE	AMLE	BS	BL	BS	BL	BS	BL	BS	BL
1	Bias	0.109	0.157	0.109	0.105	0.109	0.106	-0.098	-0.100	-0.219	-0.217
	MSE	0.159	0.190	0.161	0.159	0.159	0.156	0.051	0.051	0.060	0.058
2	Bias	-0.178	-0.210	0.068	0.066	0.069	0.067	-0.073	-0.074	-0.123	-0.123
	MSE	0.118	0.122	0.091	0.090	0.091	0.090	0.040	0.040	0.034	0.035
3	Bias	0.063	-0.033	0.079	0.077	0.080	0.077	-0.074	-0.075	-0.130	-0.131
	MSE	0.080	0.099	0.104	0.103	0.104	0.103	0.045	0.045	0.036	0.037
4	Bias	0.086	0.068	0.084	0.082	0.085	0.083	-0.073	-0.074	-0.133	-0.133
	MSE	0.109	0.110	0.109	0.108	0.109	0.108	0.044	0.043	0.035	0.036
5	Bias	0.027	0.012	0.028	0.027	0.028	0.027	-0.063	-0.064	-0.082	-0.083
	MSE	0.050	0.048	0.050	0.050	0.050	0.050	0.032	0.032	0.029	0.029
no	$m = 15$										
1	Bias	0.069	0.099	0.069	0.067	0.069	0.067	-0.069	-0.070	-0.117	-0.118
	MSE	0.097	0.107	0.097	0.096	0.097	0.096	0.043	0.043	0.035	0.035
2	Bias	-0.201	-0.232	0.047	0.046	0.047	0.046	-0.047	-0.048	-0.068	-0.069
	MSE	0.091	0.099	0.057	0.056	0.056	0.056	0.032	0.032	0.028	0.029
3	Bias	0.046	-0.059	0.057	0.055	0.057	0.055	-0.045	-0.046	-0.069	-0.070
	MSE	0.052	0.060	0.062	0.062	0.062	0.062	0.035	0.035	0.030	0.030
4	Bias	0.042	0.010	0.047	0.046	0.047	0.046	-0.053	-0.054	-0.077	-0.078
	MSE	0.052	0.058	0.059	0.058	0.059	0.059	0.033	0.033	0.029	0.030
5	Bias	0.036	0.017	0.037	0.036	0.037	0.036	-0.028	-0.028	-0.037	-0.038
	MSE	0.038	0.036	0.039	0.038	0.038	0.038	0.026	0.026	0.024	0.024
no	$m = 20$										
1	Bias	0.044	0.066	0.043	0.041	0.044	0.042	-0.058	-0.059	-0.081	-0.082
	MSE	0.063	0.068	0.063	0.062	0.063	0.062	0.036	0.036	0.031	0.032
2	Bias	-0.225	-0.259	0.028	0.027	0.028	0.027	-0.040	-0.041	-0.051	-0.052
	MSE	0.087	0.099	0.036	0.035	0.035	0.035	0.024	0.024	0.023	0.023
3	Bias	0.016	-0.094	0.024	0.023	0.023	0.022	-0.049	-0.050	-0.061	-0.061
	MSE	0.037	0.039	0.038	0.038	0.038	0.038	0.026	0.026	0.025	0.025
4	Bias	0.034	-0.001	0.030	0.029	0.030	0.029	-0.045	-0.045	-0.057	-0.058
	MSE	0.041	0.043	0.042	0.042	0.042	0.042	0.028	0.028	0.026	0.026
5	Bias	0.026	0.004	0.026	0.026	0.026	0.026	-0.021	-0.022	-0.026	-0.027
	MSE	0.029	0.027	0.029	0.028	0.029	0.028	0.021	0.021	0.020	0.020
no	$m = 25$										
1	Bias	0.049	0.067	0.049	0.048	0.049	0.048	-0.035	-0.036	-0.050	-0.051
	MSE	0.052	0.056	0.052	0.052	0.052	0.052	0.031	0.031	0.028	0.028
2	Bias	-0.243	-0.277	0.035	0.034	0.035	0.034	-0.022	-0.022	-0.029	-0.029
	MSE	0.086	0.099	0.032	0.032	0.032	0.031	0.022	0.022	0.021	0.021
3	Bias	0.024	-0.090	0.030	0.029	0.034	0.034	-0.030	-0.030	-0.037	-0.038
	MSE	0.034	0.035	0.034	0.034	0.034	0.034	0.024	0.024	0.023	0.023
4	Bias	0.029	-0.005	0.037	0.036	0.037	0.036	-0.024	-0.025	-0.033	-0.033
	MSE	0.031	0.034	0.035	0.035	0.035	0.035	0.024	0.024	0.022	0.022
5	Bias	0.014	-0.008	0.015	0.014	0.014	0.014	-0.022	-0.023	-0.025	-0.026
	MSE	0.020	0.019	0.020	0.020	0.020	0.020	0.016	0.016	0.016	0.016

see that results based on data of lifetimes for the serial system is better than from other coherent systems (System No. 2–5). Comparing the Bayes estimators obtained, we observe that the Lindley's approximation method is better than the importance sampling method in terms of both biases and MSEs. It can also be seen that the Bayes estimators based on the proper prior perform better than the Bayes estimate based on the improper prior. From Table 6, we observe that the MSEs of the Bayes estimators

Table 7: Estimated average widths (AW) and coverage probabilities (CP) of the interval estimates.

m	no.		MLE	LRT	Exact	HPD	
						G(0,0)	G(1,2)
10	1	AW	1.395	1.421	3.177	1.387	1.021
		CP	0.957	0.954	0.943	0.949	0.952
	2	AW	1.105	1.111	2.862	1.095	0.901
		CP	0.957	0.956	0.957	0.962	0.947
	3	AW	1.145	1.166	1.645	1.150	0.896
		CP	0.958	0.956	0.957	0.954	0.949
	4	AW	1.165	1.179	2.637	1.161	0.921
		CP	0.951	0.937	0.948	0.943	0.948
	5	AW	0.893	0.899	1.332	0.889	0.769
		CP	0.946	0.936	0.951	0.939	0.939
15	1	AW	1.090	1.098	2.061	1.079	0.882
		CP	0.947	0.946	0.954	0.948	0.943
	2	AW	0.873	0.877	1.950	0.868	0.7643
		CP	0.952	0.949	0.947	0.945	0.955
	3	AW	0.921	0.932	1.261	0.921	0.779
		CP	0.955	0.945	0.954	0.948	0.951
	4	AW	0.916	0.923	1.797	0.912	0.786
		CP	0.965	0.962	0.949	0.963	0.961
	5	AW	0.711	0.715	1.053	0.707	0.644
		CP	0.947	0.951	0.952	0.945	0.943
20	1	AW	0.921	0.926	1.653	0.915	0.788
		CP	0.960	0.960	0.957	0.958	0.959
	2	AW	0.752	0.754	1.475	0.748	0.679
		CP	0.951	0.946	0.955	0.948	0.947
	3	AW	0.776	0.783	1.047	0.774	0.685
		CP	0.955	0.956	0.955	0.954	0.946
	4	AW	0.790	0.795	1.358	0.788	0.703
	4	AW	0.790	0.795	1.358	0.788	0.703
		CP	0.955	0.946	0.947	0.946	0.949
	5	AW	0.611	0.613	0.889	0.607	0.566
		CP	0.957	0.951	0.949	0.949	0.949
25	1	AW	0.816	0.819	1.446	0.811	0.719
		CP	0.957	0.957	0.950	0.950	0.952
	2	AW	0.674	0.676	1.262	0.669	0.622
		CP	0.938	0.940	0.955	0.933	0.944
	3	AW	0.695	0.700	0.923	0.693	0.626
		CP	0.956	0.953	0.950	0.953	0.947
	4	AW	0.697	0.701	1.180	0.696	0.636
		CP	0.949	0.946	0.948	0.945	0.943
	5	AW	0.549	0.550	0.777	0.546	0.517
		CP	0.956	0.949	0.960	0.950	0.959

are smaller than the MSEs of the MLE and AMLE in the most systems.

For interval estimation, from Table 7, we observe that in all of cases different methods for obtaining confidence, Bayesian credible intervals and confidence intervals based on the asymptotic distribution of the MLE work well in term of average length of confidence interval and coverage probability, respectively. For all interval estimation procedures considered here, as the sample size m increases, the simulated average width of the intervals decreases.

Table 8: Simulated rejection rates of the pivotal quantity approach and likelihood ratio test for testing $H_0 : \theta = 1$ against $H_1 : \theta \neq 1$.

	System		θ								
m	No.	Tests	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
10	1	Pivot	0.513	0.232	0.091	0.051	0.061	0.109	0.128	0.188	0.296
		LRT	0.832	0.388	0.113	0.050	0.096	0.179	0.268	0.401	0.547
	2	Pivot	0.564	0.277	0.118	0.055	0.065	0.084	0.153	0.216	0.291
		LRT	0.936	0.540	0.148	0.054	0.107	0.212	0.388	0.569	0.692
	3	Pivot	0.748	0.350	0.100	0.053	0.089	0.148	0.272	0.401	0.533
		LRT	0.911	0.493	0.131	0.051	0.105	0.240	0.390	0.566	0.719
	4	Pivot	0.615	0.328	0.101	0.056	0.055	0.109	0.182	0.248	0.348
		LRT	0.920	0.516	0.131	0.054	0.090	0.236	0.374	0.519	0.677
	5	Pivot	0.826	0.434	0.143	0.054	0.085	0.180	0.326	0.494	0.644
		LRT	0.980	0.698	0.206	0.053	0.118	0.321	0.538	0.747	0.880
15	1	Pivot	0.618	0.286	0.098	0.050	0.072	0.108	0.136	0.275	0.374
		LRT	0.939	0.565	0.145	0.050	0.107	0.223	0.289	0.596	0.758
	2	Pivot	0.719	0.348	0.117	0.052	0.053	0.125	0.224	0.322	0.480
		LRT	0.983	0.708	0.209	0.050	0.124	0.337	0.553	0.745	0.889
	3	Pivot	0.871	0.450	0.137	0.050	0.086	0.220	0.408	0.540	0.698
		LRT	0.982	0.633	0.180	0.051	0.125	0.315	0.595	0.734	0.854
	4	Pivot	0.736	0.373	0.123	0.055	0.080	0.159	0.235	0.385	0.489
		LRT	0.969	0.632	0.204	0.052	0.141	0.332	0.544	0.717	0.858
	5	Pivot	0.938	0.587	0.157	0.051	0.113	0.262	0.489	0.678	0.830
		LRT	1.000	0.820	0.271	0.051	0.159	0.435	0.731	0.905	0.978
20	1	Pivot	0.720	0.344	0.120	0.050	0.079	0.159	0.256	0.376	0.508
		LRT	0.978	0.649	0.181	0.051	0.115	0.326	0.533	0.689	0.870
	2	Pivot	0.807	0.415	0.157	0.049	0.077	0.160	0.302	0.447	0.640
		LRT	0.994	0.797	0.239	0.050	0.158	0.450	0.705	0.885	0.962
	3	Pivot	0.932	0.538	0.162	0.049	0.115	0.252	0.480	0.700	0.838
		LRT	0.994	0.749	0.203	0.051	0.150	0.408	0.686	0.851	0.955
	4	Pivot	0.835	0.435	0.141	0.053	0.081	0.175	0.340	0.486	0.639
		LRT	0.994	0.770	0.215	0.051	0.151	0.412	0.657	0.844	0.945
	5	Pivot	0.971	0.669	0.200	0.050	0.122	0.347	0.598	0.825	0.932
		LRT	1.000	0.899	0.346	0.049	0.199	0.573	0.852	0.973	0.996
25	1	Pivot	0.817	0.394	0.125	0.049	0.092	0.171	0.327	0.453	0.617
		LRT	0.993	0.735	0.207	0.048	0.165	0.359	0.641	0.828	0.942
	2	Pivot	0.857	0.482	0.145	0.049	0.0864	0.206	0.365	0.546	0.716
		LRT	1.00	0.884	0.302	0.048	0.177	0.502	0.804	0.944	0.994
	3	Pivot	0.972	0.654	0.206	0.050	0.121	0.343	0.579	0.753	0.900
		LRT	0.999	0.847	0.256	0.048	0.166	0.458	0.761	0.905	0.980
	4	Pivot	0.864	0.517	0.148	0.052	0.090	0.221	0.419	0.587	0.747
		LRT	0.999	0.861	0.237	0.046	0.183	0.422	0.780	0.906	0.985
	5	Pivot	0.989	0.757	0.215	0.049	0.126	0.403	0.689	0.914	0.980
		LRT	1.000	0.958	0.375	0.048	0.244	0.674	0.923	0.994	0.999

For hypothesis testing related to the parameter θ, from Table 8, both the pivotal quantity approach and the LRT can control the type-I error rate close to the nominal level 5%. it is clear that the LRT performs better than the pivotal quantity approach based on the simulation results.

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