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Application of smoothing spline in sinusoidal modeling

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Abstract: The sinusoidal model has many applications in time series analysis, signal processing, regression, and other phenomena that are repeated periodically. On the other hand, smoothing spline is a flexible and useful method in many fields. In this article, smoothing spline is applied to interpolate data generated from the sinusoidal model. Therefore, a sinusoidal model is considered in three general forms. Then, in a simulation study, data sets are generated from each of the sinusoidal model forms, and the effect of changing the model components is assessed. Besides, the smoothing spline method is applied to estimate the related sinusoidal model, and the performance of the smoothing spline for fitting a proper model to the sinusoidal data is studied. Furthermore, by fitting a proper sinusoidal model to each generated data set, the performance of smoothing spline is compared with the sinusoidal model. The sum of squares error criterion is applied to compare the performance of models. The simulation results illustrate that smoothing spline has better performance for model fitting to sinusoidal data.

Keywords: Amplitude; Angular frequency; Phase; Sine model; Spline.
Mathematics Subject Classification (2010): 62Q99.

1 Introduction

It is well known that time series analysis is studied in two domains; time domain and frequency domain. ARIMA models are commonly used models in the time domain. In the frequency domain, Fourier transform and spectral analysis are the known methods. One of the applications of the sinusoidal model is in time series analysis, especially in the frequency domain. Box and Jenkins (1970), Pfaff (2008), Cryer and Chan (2008),

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and Mongomery et al. (2015) discussed the methods of time series analysis. On the other hand, some nonparametric regression methods such as splines have many applications in various fields. Rykala et al. (2022) applied a smoothing spline to determine the unmanned ground vehicles route based on ultra-wideband distance measurements. Lyudmila and Lyudmila (2022) studied spline-based phase analysis of macroeconomic dynamics. Alimohammadi (2009) compared spline with kriging in an epidemiological problem, and Alimohammadi (2014) applied splines to estimate ARIMA models.

In this article, the smoothing spline is applied to estimate a sinusoidal model in some special forms, and the performance of smoothing spline is investigated under some assumptions. In Section 2, the sinusoidal model is reviewed in three general forms and the smoothing spline method is applied to interpolate sinusoid waves. In Section 3, a simulation study is performed to evaluate the fit of smoothing spline to time series data generated by the sinusoidal model. Finally, the concluding remarks are presented in Section 4.

2 Sinusoidal model and smoothing spline

The sinusoidal model has many applications in signal processing such as speech or sound signal analysis, regression, time series, etc. Indeed, any variable that is periodic, seasonal, cyclical, harmonic, or oscillating in nature could be modeled by sine or cosine waves. Therefore, the sinusoidal model may be applied in various fields.

Analysis of time series fluctuations based on frequency is known as time series in the frequency domain. Usually, a time series in the time domain could be converted to a time series in the frequency domain with Fourier transform. Time series fluctuations according to sinusoidal (cosinusoidal) behavior at different frequencies exhibit time series in the frequency domain. The sinusoidal model is reviewed as follows.

Suppose that time series contain a component of periods with a known frequency. In this case, the natural model is:

$$x_t = a\cos(\omega t + \theta) + \epsilon_t, \tag{1}$$

where ω is the angular frequency, a, the amplitude of variation, θ is the phase and ϵ_t is a stationary random process with mean function 0, and variance σ^2. The angle $\omega_t + \theta$, in relation (1) is measured in radian.

A generalized form of the sinusoidal model (1) may be given as:

$$x_t = \Sigma_{j=1}^{k} a_j \cos(\omega t + \theta_j) + \epsilon_t,$$

where a_j is the amplitude of jth frequency, ω_t is the angular frequency of jth frequency, θ_j is the phase of jth frequency and k is the number of frequencies.

For more information about the sinusoidal model could be referred to Pankratz (1991) and Smith (2023).

Minimizing the penalized sum of squares criterion is the base of the smoothing spline. The criterion includes two components, goodness of fit and smoothness as follows:

$$S(g) = \sum_{i=1}^{n} (y_i - g(t_i))^2 + \lambda \int_a^b (g''(t))^2 dt, \tag{2}$$

where g is any twice-differentiable function on the interval $[a, b]$, g'' is the second derivative of g, λ is smoothing parameter and n is the sample size.

In (2), the penalized least squares estimator, $\hat{g}$ is defined as the minimizer of $S(g)$ over the class of all twice differentiable functions g. This minimizer function is called smoothing spline (Green and Silverman, 1994).

Choosing the proper value of smoothing parameter is an important concept. Two major methods are cross validation (CV) and generalized cross validation (GCV). When data is measured in upper dimension, that is R^D (instead of real line $\mathbb{R}$ in (2)), smoothing spline is discussed by Wang (2011).

3 Simulation study

In this section, a simulation study is carried out to assess the fit of smoothing spline to time series data generated by the sinusoidal model and its performance is computed. Three sinusoidal models in specified forms are considered. The first model is

$$Z_t = a \sin(\omega t + \theta) + \epsilon_t, \tag{3}$$

where a is amplitude, θ is phase, ω is angular frequency, ϵ is random error, and t is time. The second model is a cosinusoidal model which is written as:

$$Z_t = a \cos(\omega t + \theta) + \epsilon_t. \tag{4}$$

A combination of models (3) and (4) is considered as:

$$Z_t = a \cos(\omega_1 t + \theta_1) + b \sin(\omega_2 t + \theta_2) + \epsilon_t. \tag{5}$$

Here, the performance of the smoothing spline and sinusoidal models (3), (4), and (5) is assessed when the sinusoidal components; amplitude (a), phase (θ), and angular frequency (ω) vary. The simulation is iterated 100 times in each case and the average of SSE is given.In the tables, SM and SP indicate sinusoidal model and smoothing spline respectively.

First, the effect of changing amplitude (a) in models (3), (4) and (5) is investigated when $\omega = \frac{\pi}{6}$ and $\theta = \frac{\pi}{3}$. The results of the simulation are given in Table 1. When amplitude increases, in the model (4) and (5), SSE increases most of the time, but the model (3) does not exhibit this pattern. Furthermore, in all of the cases smoothing spline has better performance.

Table 1: SSE of Spline and sinusoidal models (3) to (5) when a changes.

		a						
Sine model	Model	8	10	12	14	16	18	20
	SP	48	47	46	44.5	45	44	42
(3)	SM(3)	52	138	38	71	95	42	149
	SP	42	44	46	49	53	57	63
(4)	SM(4)	40	56	72	45	65	69	53
	SP	40	42	44	48	51	56	59
(5)	SM(5)	39	43	53	55	75	53	71

Now, by fixing amplitude $a = 10$ and phase $\theta = \frac{\pi}{3}$, variations of angular frequency (ω) is assessed. The results are given in Table 2. The results show that SSE increases for ω greater than $\frac{\pi}{2}$ and again decreases for ω greater than π. Besides, smoothing spline has better performance than the other method.

Table 2: SSE of fitting Spline and sinusoidal model (3) to (5) when ω changes.

		ω							
Sine model	Model	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$2\frac{\pi}{3}$	π	$3\frac{\pi}{2}$	2π
(3)	SP	75	49	43	307	4980	7393	309	82
	SM(3)	167	150	150	4290	4500	5000	511	171
(4)	SP	87	49	46	477	7065	6427	388	87
	SM(4)	149	146	200	495	6900	6056	530	149
(5)	SP	86	49	44	396	6202	11914	388	86
	SM(5)	84	53	128	486	5500	9826	529	84

Now, by fixing amplitude $a = 10$ and phase $\omega = \frac{\pi}{6}$, the effect of changing phase (θ) is assessed. The results are given in Table 3. The results show that SSE increases for θ greater than $\frac{\pi}{2}$ and decreases for θ greater than π. Furthermore, smoothing spline has a better performance for θs except $\theta = \frac{\pi}{2}, 2\frac{\pi}{3}$.

Table 3: SSE of fitting Spline and sinusoidal model (3) to (5) when θ changes.

		θ							
Sine model	Model	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$2\frac{\pi}{3}$	π	$3\frac{\pi}{2}$	2π
	SP	75	49	43	307	4980	7393	309	82
(3)	SM(3)	167	150	150	290	4500	5000	511	171
	SP	87	49	46	477	7065	6427	388	87
(4)	SM(4)	149	146	200	495	6900	6056	530	149
	SP	86	49	44	396	6202	11914	388	86
(5)	SM(5)	84	53	128	486	5500	9826	529	84

Now, changes of angular frequency (θ) in models (3) to (5) is studied for two amounts of amplitude, $a = 5$ and $a = 10$. The results of the simulation are shown in Table 4 when $\omega_1 = \omega_2 = \frac{\pi}{6}$ and $a = b = 5$. The results show that SSE increases somewhat when amplitude increases for all models. Furthermore, smoothing spline has better performance in all of the cases for models (3) and (4). However, in model 3, the SSE of the smoothing spline is less than the sinusoid model in most cases.

Table 4: SSE of fitting Spline and sinusoidal model (3), (4) and (5).

		θ							
Range (a)	Model	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$2\frac{\pi}{3}$	π	$3\frac{\pi}{2}$	2π
$a = 5$	SP	41	49	41.3	41	41.2	41.5	41	41
	SM(3)	48	56	80	73	39.7	68	71	48
	SM(4)	39	52	56	61	35	69	72	39
	SM(5)	41	44	48	49	50	56	38	41
$a = 10$	SP	44	45	45	46	47	47	46	44
	SM(3)	52	62	98	103	46	73	79	52
	SM(4)	41.5	59	62	83	39	75	84	41.5
	SM(5)	42	44	51	42	52	58	44	42

Finally, the results of the simulation study indicate that smoothing spline is an appropriate method for sinusoidal data.

4 Discussion and conclusions

Sinusoidal models have many applications to explain phenomena that exhibit periodic behavior such as time series, and signal processing. In this article, the smoothing spline method is applied to interpolate sinusoid waves. Indeed, in a simulation study, the performance of the smoothing spline is compared with the sinusoidal model for different amounts of amplitude, angular frequency, and phase according to the sine, cosine, and sine-cosine considered models. The simulation results indicate that smoothing spline has a smaller error, and has better performance than the sinusoidal model in most cases. Therefore, the application of smoothing spline is supposed to fit the model to sinusoidal data.

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