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Research Paper

A new extended alpha power transformed family of distributions: properties and applications

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Abstract: In this paper, a new method has been proposed to introduce an extra parameter to a family of lifetime distributions for more flexibility. A special sub-case has been considered in details namely; two parameters Weibull distribution. Various mathematical properties of the proposed distribution, including explicit expressions or the moments, quantile, moment generating function, residual life, mean residual life and order statistics are derived. The maximum likelihood estimators of unknown parameters cannot be obtained in explicit forms, and they have to be obtained by solving non-linear equations only. A simulation study is conducted to evaluate the performances of these estimators. For the illustrative purposes, two data sets have been analyzed to show how the proposed model work in practice.

Keywords: Alpha power transformation; Family of distributions; Maximum likelihood estimation; Moments; Order statistic; Residual life function; Weibull distribution.
Mathematics Subject Classification (2010): 60E05, 62F10.

1 Introduction

Broadly speaking, classical distributions are widely used for modelling real life data in several areas such as engineering, environmental, actuarial, and medical sciences, biological studies, economics, finance and insurance. However, in many applied areas such as lifetime analysis, engineering, finance and insurance, there is a clear need for the extended versions of these classical distributions. Furthermore, in many practical fields, classical distributions do not provide an adequate fit to real data. For instance,

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if the data are skewed, the normal distribution will not be a good candidate model and vice verse. So, in such situations we need modified forms of the existing distributions. In this regard, a serious attempt have been made that is quite rich and still growing rapidly. After the massive work by Pearson (1895) for proposing statistical distributions via the system of differential equation approach, several general methods have been introduced for generating family of lifetime distributions. Another prominent approach based on differential equation was developed by Burr (1942). The quantile method to develop new distributions of Hastings et al. (1947), Johnson (1949) translation approach and Tukey (1960) are other milestone set in this credit. Since 1990, methodologies of generating new statistical distributions shifted to introduce additional parameters to the existing distributions or combining existing distributions. Often adding extra parameter(s) brings more flexibility to a class of distribution functions, and it can be very useful for statistical modelling. For example, Marshal and Olkin (1997) introduced the Marshal-Olkin generated (MO-G) family by introducing an extra parameter to the Weibull distribution to bring more flexibility to the Weibull model takes the following form

$$G\left(x;\sigma,\xi\right)=\frac{F\left(x;\xi\right)}{1-\left(1-\sigma\right)\left(1-F\left(x;\xi\right)\right)},\qquad \sigma,\xi>0,\ x\in\mathbb{R}. \tag{1}$$

Mudholkar and Srivastava (1993) proposed another method to introduce an extra parameter to a two-parameter Weibull distribution. The cumulative distribution function (cdf) of the Mudholkar and Srivastava (1993)'s proposed exponentiated Weibull (EW) model has the following form

$$G\left(x;a,\theta,\gamma\right)=\left(1-e^{-\gamma x^{\theta}}\right)^{a},\qquad x\geq 0,\ a,\theta,\gamma>0. \tag{2}$$

The model (2) has two shape parameters and one scale parameter. Due to the presence of an extra shape parameter, the proposed EW distribution is more flexible than the traditional two-parameter Weibull model. Eugene et al. (2002) introduced the beta generated method that uses the beta distribution with parameters a and b as the generator to develop the beta generated distributions. The distribution of a beta-generated random variable X is defined as

$$G\left(x;a,b,\xi\right)=\int_{0}^{F(x;\xi)} r(t;a,b)dt,\qquad a,b,\xi>0. \tag{3}$$

where $r(t)$ is the probability density function (pdf) of a beta random variable and $F\left(x;\xi\right)$ is the cdf of any random variable X. Alzaatreh et al. (2013) proposed another prominent method for generating families of continuous distributions called T-X family by replacing the beta pdf with a pdf, *b(t)*, of a continuous random variable and applying a function $W\left\{F\left(x;\xi\right)\right\}$ that satisfies some specific conditions. Other families belong to the T-X family including: Gamma-G Type-1 of Zografos and Balakrishnan (2009), Gamma-G Type-2 of Risti´c and Balakrishnan (2012), Gamma-G Type-3 of Torabi and Montazari (2012), McDonald-G (Mc-G) of Alexander et al. (2012), Logistic-G of Torabi and Montazari (2014), Logistic-X Family of Tahir et al. (2016), and a new Weibull-X family of Ahmad et al. (2018). Recently Aljarrah et al. (2014) used quantile functions to generate T-X family of distributions. Shaw and Buckley (2009) proposed another useful approach known as the quadratic rank transmutation map defined by

$$G(x;\lambda,\xi) = (1+\lambda)F(x;\xi) - \lambda F(x;\xi)^2 \qquad \xi > 0,\ |\lambda| \leq 1,\ x \in \mathbb{R}. \tag{4}$$

where λ is a transmuted parameter. Cordeiro and deCastro (2011) proposed another prominent approach known as the Kumaraswamy-G (Ku-G) family of distributions by

$$G(x;a,b,\xi) = 1 - \{1 - (1 - F(x;\xi)^a)\}^b \qquad a,b,\xi > 0,\ x \in \mathbb{R}. \tag{5}$$

where $a>0$ and $b>0$ are the additional shape parameters. Mahdavi and Kundu (2017) proposed a new method for introducing new lifetime distributions by the cdf given by

$$G(x;\alpha,\xi) = \frac{\alpha^{F(x;\xi)} - 1}{\alpha - 1}, \qquad \alpha,\xi > 0,\ \alpha \neq 1,\ x \in \mathbb{R}. \tag{6}$$

Using (6), Mahdavi and Kundu (2017) and Dey et al. (2017) introduced the alpha power exponential (APE) and alpha power transformed Weibull (APTW) distribution, respectively. Elbatala et al. (2018) proposed a new power transformation to extend the existing distributions. The distribution function of the Elbatala et al. (2018)'s new alpha power transformed family of distributions is given by

$$G(x;\alpha,\xi) = \frac{F(x;\xi)\,\alpha^{F(x;\xi)}}{\alpha}, \qquad \alpha,\xi > 0,\ \alpha \neq 1,\ x \in \mathbb{R}. \tag{7}$$

Using $F(x;\xi)$ as the cdf of the Weibull model, Elbatala et al. (2018) proposed a three-parameter new alpha power transformed Weibull (NAPTW) distribution. Recently, Ahmad (2018) proposed a new family of lifetime distributions, called the Zubair-G family by the cdf

$$G(x;\alpha,\xi) = \frac{e^{\alpha F(x;\xi)^2} - 1}{e^{\alpha} - 1}, \qquad \alpha,\xi > 0,\ x \in \mathbb{R}. \tag{8}$$

For more survey about methods to generating distributions see Lee et al. (2013) and Jones (2015). The key aim of this research is to introduce an extra parameter to a family of lifetime distribution functions to bring more flexibility to the given family. We call this new method as a new extended alpha power transformation (NEAPT) method. The proposed NEAPT method is very easy to deploy, hence it can be used quite effectively for data analysis purposes. The rest of this article is organized as follows. In Section 2, we define the proposed method. A special sub-model of the proposed family along with the graphical sketching of its pdf and cdf is discussed in Section 3. Some mathematical properties are obtained in Section 4. Maximum likelihood estimates of the model parameters are obtained in Section 5. A simulation study is conducted in Section 6. Section 7, is devoted to analyze two real life applications. Finally, concluding remarks are provided in Section 8.

2 Proposed method and motivation

In this section, we define the propose class, a new extended alpha power transformed family. Let $F(x;\xi)$ be the cdf of a continuous random variable X depending upon the vector parameter $(\xi)^T$, then the new extended alpha power transformation of $F(x;\xi)$ for $x \in \mathbb{R}$, is defined as follows

$$G(x;\alpha,\xi) = \frac{\alpha^{F(x;\xi)} - e^{\alpha F(x;\xi)}}{\alpha - e^{\alpha}}, \qquad \alpha,\xi > 0,\ \alpha \neq e,\ x \in \mathbb{R}, \tag{9}$$

where,$F(x;\xi)$ is cdf of the baseline random variable depending on the vector parameter ξ and α is an additional parameter. The probability density function (pdf), survival function (sf), hazard rate function (hrf), reverse hazard rate function (rhrf) and cumulative hazard rate function (chrf) of the NEAPT family, respectively, are given by

$$g(x;\alpha,\xi) = \frac{f(x;\xi)\left((\log\alpha)\,\alpha^{F(x;\xi)} - \alpha e^{\alpha F(x;\xi)}\right)}{\alpha - e^{\alpha}}, \tag{10}$$

$$S(x;\alpha,\xi) = \frac{\alpha - e^{\alpha} - \alpha^{F(x;\xi)} + e^{\alpha F(x;\xi)}}{\alpha - e^{\alpha}}, \tag{11}$$

$$h(x;\alpha,\xi) = \frac{f(x;\xi)\left((\log\alpha)\,\alpha^{F(x;\xi)} - \alpha e^{\alpha F(x;\xi)}\right)}{\alpha - e^{\alpha} - \alpha^{F(x;\xi)} + e^{\alpha F(x;\xi)}}, \tag{12}$$

$$H(x;\alpha,\xi) = -\log\left(\frac{\alpha - e^{\alpha} - \alpha^{F(x;\xi)} + e^{\alpha F(x;\xi)}}{\alpha - e^{\alpha}}\right). \tag{13}$$

The new pdf is most tractable when $F(x,\xi)$ and $f(x,\xi)$ have simple analytical expressions. Henceforth, a random variable X with pdf (10) is denoted by$X \sim NEAPT(x;\alpha,\xi)$. Furthermore, for the sake of simplicity, the dependence on the vector of the parameters is omitted and simply $G(x)$= $G(x;\alpha,\xi)$ will be used. Moreover, the key motivations for using the NEAPT family in practice are the following:

1. A very simple and convenient method of adding additional parameters to modify the existing distributions.

2. To improve the characteristics and flexibility of the existing distributions.

3. To introduce the extended version of the baseline distribution having closed form for cdf, sf as well as hrf.

4. To provide better fits than the other modified models.

3 Sub-model description

Let X be the Weibull random variable with cdf $F(x;\xi) = 1 - e^{-\gamma x^{\theta}}, \quad x \geq 0,\ \gamma,\theta > 0$, where $\xi = (\gamma,\theta)$. Then, the cdf of the new extended alpha power transformed Weibull (NEAPTW) distribution has the following expression

$$G(x;\alpha,\xi) = \frac{\alpha^{\left(1-e^{-\gamma x^{\theta}}\right)} - e^{\alpha\left(1-e^{-\gamma x^{\theta}}\right)}}{\alpha - e^{\alpha}}, \qquad \alpha,\xi > 0,\ \alpha \neq e,\ x \geq 0, \tag{14}$$

The pdf, sf and hrf of the NEAPTW distribution are given, respectively, by

$$g(x;\alpha,\xi) = \frac{\gamma\theta x^{\theta-1}e^{-\gamma x^{\theta}}}{\alpha-e^{\alpha}}\left((\log\alpha)\,\alpha^{\left(1-e^{-\gamma x^{\theta}}\right)}-\alpha e^{\alpha\left(1-e^{-\gamma x^{\theta}}\right)}\right),\quad x\geq 0, \tag{15}$$

$$S(x;\alpha,\xi) = \frac{\alpha-e^{\alpha}-\alpha^{\left(1-e^{-\gamma x^{\theta}}\right)}+e^{\alpha\left(1-e^{-\gamma x^{\theta}}\right)}}{\alpha-e^{\alpha}},\qquad x\geq 0, \tag{16}$$

$$h(x;\alpha,\xi) = \frac{\gamma\theta x^{\theta-1}e^{-\gamma x^{\theta}}\left((\log\alpha)\,\alpha^{\left(1-e^{-\gamma x^{\theta}}\right)}-\alpha e^{\alpha\left(1-e^{-\gamma x^{\theta}}\right)}\right)}{\alpha-e^{\alpha}-\alpha^{\left(1-e^{-\gamma x^{\theta}}\right)}+e^{\alpha\left(1-e^{-\gamma x^{\theta}}\right)}},\quad x\geq 0. \tag{17}$$

For different values of the model parameters, plots for the pdf of the NEAPTW distribution are sketched in Figure 1. For the selected values of parameters, some possible shapes for the hrf of the NEAPTW model are drawn in Figure 2.

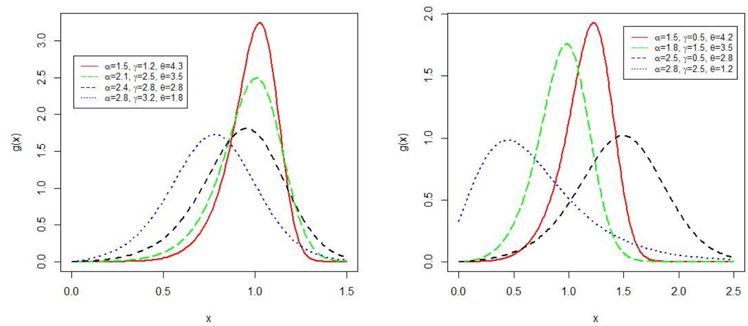

Figure 1: Different plots for the pdf of the NEAPTW distribution.

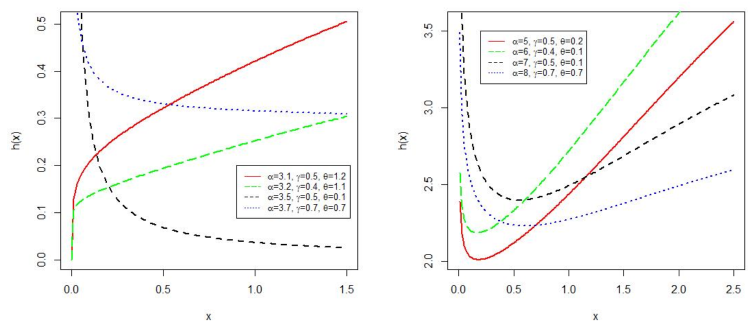

Figure 2: Different plots for the hrf of the NEAPTW distribution.

4 Basic mathematical properties

In this section, we derive some general properties of the new model, such as the quantile function, moments, moment generating function, residual, reverse residual life and order statistic.

4.1 Quantile function

Let X be the NEAPTW random variable with pdf (10), then, the quantile function of X is derived as

$$x = Q(u) = G^{-1}(u) = F^{-1}(t), \tag{18}$$

where t is the solution of the expression $\alpha t - \log(\alpha^t - u(\alpha - e^\alpha))$, and u has the uniform distribution on interval (0,1). The expression (18) does not have a closed form, therefore, computer software can be used to obtain a closed form solution of the quantile function.

4.2 Moments

Moments are very important and helps to describe the important characteristics of the distribution (e.g., central tendency, dispersion, skewness and kurtosis). The r^{th} moment of the NEAPT family of distributions are derived as follows

$$\mu_r^{/} = \frac{1}{\alpha - e^\alpha}\left((\log\alpha)\int_{-\infty}^{\infty} x^r f(x;\xi)\,\alpha^{F(x;\xi)}dx - \alpha\int_{-\infty}^{\infty} x^r f(x;\xi)\,e^{\alpha F(x;\xi)}dx\right). \tag{19}$$

Using the series representation in the form $\alpha^v = \sum_{i=0}^{\infty}\frac{(\log\alpha)^i}{i!}v^i$, so the expression (19) can be re-write as

$$\begin{aligned}
\mu_r^{/} &= \frac{1}{\alpha - e^\alpha}\left(\sum_{i=0}^{\infty}\frac{(\log\alpha)^{i+1}}{i!}\int_{-\infty}^{\infty} x^r f(x;\xi)\,F(x;\xi)^i\,dx\right.\\
&\quad \left.-\sum_{i=0}^{\infty}\frac{\alpha^{i+1}}{i!}\int_{-\infty}^{\infty} x^r f(x;\xi)\,F(x;\xi)^i\,dx\right), \qquad (20)\\
&= \frac{1}{\alpha - e^\alpha}\left(\sum_{i=0}^{\infty}\frac{(\log\alpha)^{i+1}}{i!}\eta_{r,i} - \sum_{i=0}^{\infty}\frac{\alpha^{i+1}}{i!}\eta_{r,i}\right),
\end{aligned}$$

where $\eta_{r,i} = \int_{-\infty}^{\infty} x^r f(x;\xi)\,F(x;\xi)^i\,dx$. Furthermore, a general expression for moment generating function (mgf) of the NEAPT random variable X is

$$M_X(t) = \frac{1}{\alpha - e^\alpha}\left(\sum_{i,r=0}^{\infty}\frac{(\log\alpha)^{i+1}t^r}{r!i!}\eta_{r,i} - \sum_{i,r=0}^{\infty}\frac{\alpha^{i+1}t^r}{r!i!}\eta_{r,i}\right). \tag{21}$$

4.3 Residual and reverse residual life

The residual life offer wider applications in reliability theory and risk management. The residual lifetime of NEAPT random variable X denoted by $R_{(t)}$is derived as

$$\begin{aligned}
R_{(t)}(x) &= \frac{S(x+t)}{S(t)},\\
R_{(t)}(x) &= \frac{\alpha - e^\alpha - \alpha^{F(x+t;\xi)} + e^{\alpha F(x+t;\xi)}}{\alpha - e^\alpha - \alpha^{F(t;\xi)} + e^{\alpha F(t;\xi)}}. \qquad (22)
\end{aligned}$$

Additionally, the reverse residual lifetime of the NEAPT random variable denoted by $\bar{R}_{(t)}$is

$$\bar{R}_{(t)} = \frac{S(x-t)}{S(t)},$$
$$\bar{R}_{(t)}(x) = \frac{\alpha - e^{\alpha} - \alpha^{F(x-t;\xi)} + e^{\alpha F(x-t;\xi)}}{\alpha - e^{\alpha} - \alpha^{F(t;\xi)} + e^{\alpha F(t;\xi)}}. \tag{23}$$

4.4 Order statistics

Let $X_1, X_2, \ldots, X_k$ be a random sample of size k taken independently from the NEAPT distribution with parameters α and ξ. Let $X_{1:k}, X_{2:k}, \ldots, X_{k:k}$ be the corresponding order statistics. Then, from David (1981), the density of $X_{r:k}$ for (r=1, 2, . . . , k) is given by

$$g_{r:k}(x) = \frac{g(x;\alpha,\xi)}{B(r,\ k-r+1)} \sum_{i=0}^{k-r} \binom{k-r}{i} (-1)^i \left[G(x;\alpha,\xi)\right]^{i+r-1}. \tag{24}$$

Using (9) and (10), in (24), we get the density of the r^{th} order statistic.

5 Estimation

In this section, the maximum likelihood estimators of the parameters α and ξ of NEAPT family from complete samples are derived. Let $X_1, X_2, \ldots, X_k$ be a simple random sample from NEAPT family with observed values $x_1, x_2, \ldots, x_k$. The log-likelihood function of this sample is

$$\log L(x;\alpha,\xi) = -k\log(\alpha - e^{\alpha}) + \sum_{i=1}^{k} \log\left[f(x_i;\xi)\right] + \sum_{i=1}^{k} \log\left[(\log\alpha)\,\alpha^{F(x_i;\xi)} - \alpha e^{\alpha F(x_i;\xi)}\right], \tag{25}$$

Obtaining the partial derivatives of (25), we get

$$\frac{\partial}{\partial\alpha}\log L(x;\alpha,\xi) = -\frac{k(1-e^{\alpha})}{\alpha - e^{\alpha}} + \sum_{i=1}^{k} \frac{\left((\log\alpha)F(x_i;\xi)+1\right)\alpha^{F(x_i;\xi)-1} - \alpha e^{\alpha F(x_i;\xi)}\left(1+F(x_i;\xi)\right)}{(\log\alpha)\,\alpha^{F(x_i;\xi)} - e^{F(x_i;\xi)}}, \tag{26}$$

$$\frac{\partial}{\partial\xi}\log L(x;\alpha,\xi) = \sum_{i=1}^{k} \frac{\partial f(x_i;\xi)/\partial\xi}{f(x_i;\xi)} + \sum_{i=1}^{k} \frac{(\log\alpha)^2\alpha^{F(x_i;\xi)}\partial F(x_i;\xi)/\partial\xi - \alpha^2 e^{\alpha F(x_i;\xi)}\partial F(x_i;\xi)/\partial\xi}{(\log\alpha)\,\alpha^{F(x_i;\xi)} - \alpha e^{\alpha F(x_i;\xi)}}. \tag{27}$$

Setting $\frac{\partial}{\partial\alpha}\log L(x;\alpha,\xi)$ and $\frac{\partial}{\partial\xi}\log L(x;\alpha,\xi)$ equal to zero and solving numerically these expressions simultaneously, yields the maximum likelihood estimates (MLEs) of (α,ξ).

6 Simulation study

In this section, we assess the performance of the maximum likelihood estimators in terms of the sample size n. A numerical evaluation is carried out to examine the performance of maximum likelihood estimators for NEAPTW model (as particular case from the family). The evaluation of estimates is performed based on the following quantities for each sample size; the biases and the empirical mean square errors (MSEs) using the mathematical package. The numerical steps are listed as follows:

1. A random sample $X_1,\ X_2,\ .\ .\ .\ .\ ,\ X_n$ of sizes; *n=30* and *50* are considered, these random samples are generated from the NEAPTW distribution by using inversion method.
2. Eight sets of the parameters are considered. The MLEs of NEAPTW model are evaluated for each parameter value and for each sample size.
3. 1000 repetitions are made to calculate the biases and mean square error (MSE) of these estimators.
4. Formulas used for calculating bias and MSE are given by $Bias\left(\hat{\alpha}\right)=\frac{1}{1000}\sum_{i=1}^{1000}\left(\hat{\alpha}-\alpha\right)\text{and}MSE\left(\hat{\alpha}\right)=\frac{1}{1000}\sum_{i=1}^{1000}\left(\hat{\alpha}-\alpha\right)^2$, respectively.
5. Step (iii) is also repeated for the other parameters(θ,γ).

Empirical results are reported in Table 1. We can detect from these tables that the estimates are quite stable and are close to the true value of the parameters as the sample sizes increase.

7 Applications

To prove the flexibility of the proposed family two applications to real data sets are analyzed. The goodness of fit test of the NEAPTW distribution have been compared with the other lifetime models including exponentiated Weibull, Marshall-Olkin Weibull and Kumaraswamy Weibull distributions. The distribution functions of the competing models are as:

1. The exponentiated Weibull is given by

$$G\left(x;\alpha,\theta,\gamma\right)=\left(1-e^{-\gamma x^{\theta}}\right)^{a},\qquad x,a,\theta,\gamma>0.$$

2. The Marshall-Olkin Weibull (MOW) is

$$G\left(x;\sigma,\xi\right)=\frac{\left(1-e^{-\gamma x^{\theta}}\right)^{a}}{1-\left(1-\sigma\right)\left(1-\left(1-e^{-\gamma x^{\theta}}\right)^{a}\right)},\qquad x,\sigma,\xi>0.$$

Table 1: Simulation results: MLEs, biases and MSEs

		$\alpha=0.5,\gamma=0.5,\theta=0.5$			$\alpha=0.6,\gamma=0.5,\theta=1$		
n	Parameters	MLE	Bias	MSE	MLE	Bias	MSE
30	α	0.3173	-0.1827	0.0400	0.4458	-0.1542	0.0395
	γ	0.5775	0.0775	0.1840	0.6468	0.1468	0.1472
	θ	1.4039	0.9039	1.0334	3.3607	2.3607	6.1295
50	α	0.4214	-0.0786	0.0107	0.5923	-0.1077	0.0271
	γ	0.5690	0.0690	0.0187	0.6522	0.1522	0.1021
	θ	0.9435	0.4435	0.2094	2.0076	1.0076	1.6355
		$\alpha=0.5$, γ= 0.5, θ= 0.2			$\alpha=0.3$, γ= 1, θ= 0.5		
30	α	0.4139	-0.0861	0.0080	0.2488	-0.1212	0.0149
	γ	0.3796	-0.1204	0.0206	0.9839	-0.0161	0.0113
	θ	1.1505	0.9505	0.9371	1.7008	1.2008	1.4631
50	α	0.5453	0.0453	0.0052	0.3126	-0.0474	0.0024
	γ	0.3656	-0.1344	0.0193	0.9933	-0.0067	0.0064
	θ	0.7862	0.5862	0.4112	1.0624	0.5624	0.3292
		$\alpha=0.7$, γ= 1.5, θ= 0.5			$\alpha=0.5$, γ= 2, θ= 0.5		
30	α	0.4823	-0.1177	0.0140	0.3835	-0.1165	0.0138
	γ	1.4461	-0.0539	0.0216	2.0013	0.0013	0.0636
	θ	1.6326	1.1326	1.3134	1.7260	1.2260	1.5385
50	α	0.6600	-0.0400	0.0018	0.4526	-0.0474	0.0027
	γ	1.4300	-0.0700	0.0174	1.9423	-0.0577	0.0138
	θ	0.9927	0.4927	0.2490	1.0227	0.5227	0.2860
		$\alpha=1$, γ= 0.5, θ= 0.5			$\alpha=0.7$, γ= 0.5, θ= 0.5		
30	α	0.6805	-0.6195	0.3854	0.5617	-0.3383	0.1185
	γ	0.5353	0.0353	0.0193	0.5744	0.0744	0.0362
	θ	1.3352	0.8352	0.8101	1.4132	0.9132	0.9261
50	α	0.9796	-0.5204	0.2709	0.6819	-0.2181	0.0482
	γ	0.5005	0.0005	0.0028	0.4916	-0.0084	0.0012
	θ	0.8493	0.3493	0.1277	0.9416	0.4416	0.2124

3. The Kumaraswamy Weibull (Ku-W) is given by

$$G\left(x;a,b,\theta,\gamma\right)=1-\left(1-\left(1-e^{-\gamma x^{\theta}}\right)^{a}\right)^{b},\qquad x,a,b,\theta,\gamma>0.$$

The analytical measures of goodness of fit including the Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Baysian information criterion (BIC), Hannan-Quinn information criterion (HQIC) and Kolmogorov-Smirnov (KS) statistic are considered to compare the proposed method with the other fitted models. In general, a model with smaller values of these analytical measures indicate better fit to the data. All the required computations have been carried out through the statistical software R using the package (AdequacyModel) with ”SANN” algorithm.

Data 1:

The data set represents survival times of guinea pigs injected with the different amount of tubercle bacilli studied by Bjerkedal (1960). Table 2 shows the estimated parameters along with the corresponding standard errors. Whereas, the considered statistical tools for the fitted density functions are provided in Table 3. The NEAPTW distribution provides a better fit than the other distributions in terms of AIC, BIC, CAIC, HQIC and KS test statistic. These distributions are some of the most widely used for this kind of data. The MOW is a common alternative for positive data and the EW and Ku-W are very flexible distributions.

Corresponding to data 1, the estimated pdf and cdf are provided in Figure 3, the PP plot and Kaplan-Meier survival plot are sketched in Figure 4, whereas, the QQ and box plots are given in Figure 5.

Table 2: Maximum likelihood estimates of the fitted distributions using data 1.

Dist.	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\sigma}$	$\hat{a}$	$\hat{b}$
NEAPTW	0.017	0.074	1.989			
	(0.4132)	(0.03817)	(0.1604)			
MOW		0.210	0.698	1.770		
		(0.0255)	(0.3153)	(0.0771)		
EW		0.708	1.171		1.994	
		(0.3530)	(0.2830)		(0.9831)	
Ku-W		0.641	1.062		2.310	1.432
		(0.5713)	(0.6322)		(2.4604)	(1.0107)

Table 3: The statistics of the fitted models using data 1.

Dist.	KS	AIC	BIC	CIAC	HQIC
NEAPTW	0.094	209.53	216.36	209.88	212.25
MOW	0.106	213.45	220.28	213.80	216.17
EW	0.100	211.62	218.45	211.97	214.341
Ku-W	0.097	213.63	222.73	214.22	217.25

Data 2:

The second data set taken from Dey et al. (2017), representing the life of fatigue fracture of Kevlar 373/epoxy that are subject to constant pressure at the 90 stress level until all had failed. The data set was initially presented by Barlow et al. (1984) and later studied by Andrews and Herzberg (2012). For this data set, the NEAPTW distribution has the overall best performance when compared to the same distributions of the previous example. The estimated parameters along with the standard errors and goodness-of-fit statistics for the fitted models are given in Tables 4 and 5, respectively. Corresponding to data 2, the estimated pdf and cdf are proved in Figure 6, the PP plot and Kaplan Meier survival plot are sketched in Figure 7, whereas, the QQ and box plots are given in Figure 8.

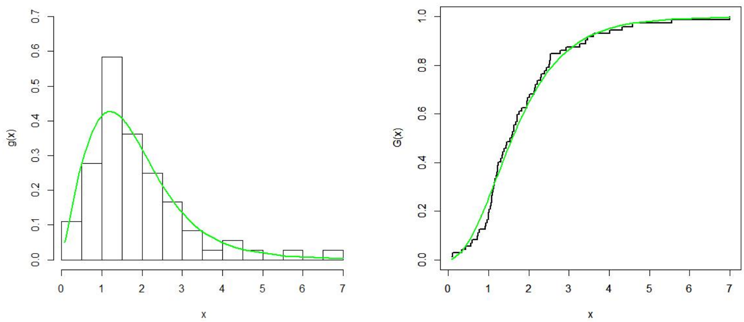

Figure 3: Plots of the estimated pdf and cdf of the NEAPTW distribution for data 1.

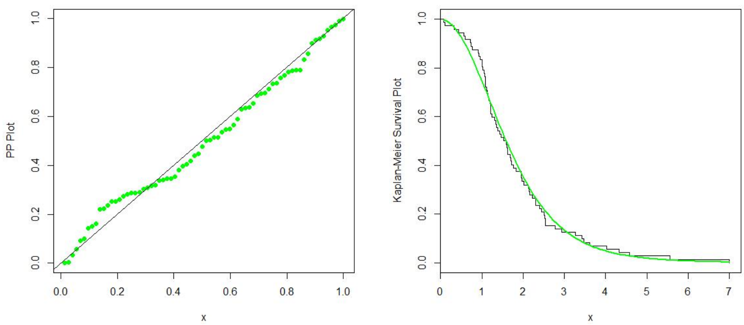

Figure 4: PP and Kaplan-Meier survival plots of the NEAPTW distribution for data 1.

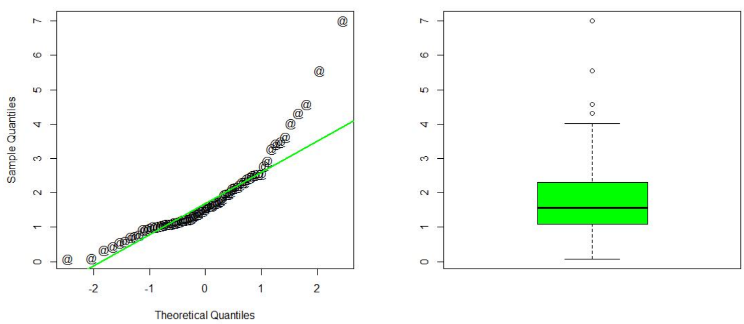

Figure 5: QQ-plot of the NEAPTW distribution and box plot for data 1.

Conclusions

In this article, a new method is adopted to add an additional parameter to the existing distributions. This effort leads to a new family of lifetime distributions, called the NEAPT family of distributions. General expressions for some of mathematical properties of the new family are derived. The estimation of the of model parameters through maximum likelihood method is discussed and a simulation study is carried out. There

Table 4: Maximum likelihood estimates of the fitted distributions using data 2.

Dist.	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\sigma}$	$\hat{a}$	$\hat{b}$
NEAPTW	3.538	0.373	0.753			
	(0.3369)	(0.4041)	(0.7029)			
MOW		0.616	1.622	5.371		
		(0.8930)	(0.8006)	(54.074)		
EW		0.023	1.046		3.3627	
		(0.0749)	(0.9743)		(0.6778)	
Ku-W		0.012	2.917		1.235	4.1479
		(0.5073)	(0.4676)		(0.2678)	(2.7288)

Table 5: Analytical measures of the fitted distributions using data 2.

Dist.	KS	AIC	BIC	CIAC	HQIC
NEAPTW	0.058	173.596	179.165	173.983	176.192
MOW	0.061	175.439	181.999	175.817	178.026
EW	0.080	178.246	184.815	178.633	180.842
Ku-W	0.0827	180.483	189.242	181.139	183.944

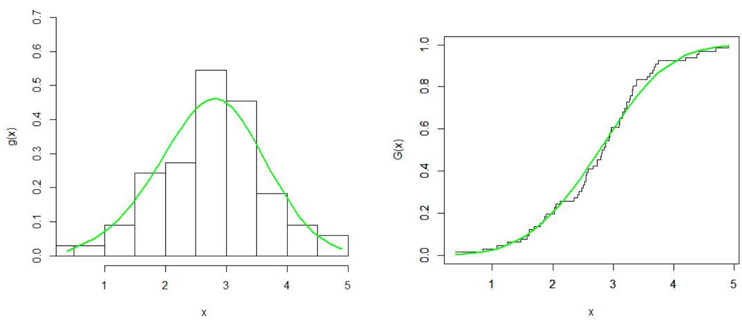

Figure 6: Plots of the estimated pdf and cdf of the NEAPTW distribution for data 2.

are certain advantages of using the proposed method like its cdf has a closed form solution and facilitating data modeling with monotonic and non-monotonic failure rates. A special sub-model of the new family, called NEAPTW distribution is considered and two real applications are analyzed. For these data sets, some certain accuracy measures are calculated to compare the goodness of fit of the proposed model to the other competing distributions. These measures reveal that the proposed distribution provides best fit to these data than the other considered distributions. To support these accuracy measures, empirical pdf, cdf, PP, QQ and Kaplan–Meier plots are also sketched which show that the NEAPTW model fits the data well. We hope that the proposed model will attract wider applications in numerous applied fields..

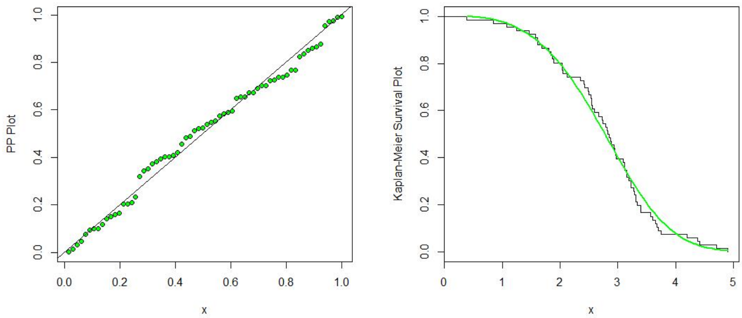

Figure 7: PP and Kaplan-Meier survival plots of the NEAPTW distribution for data 2.

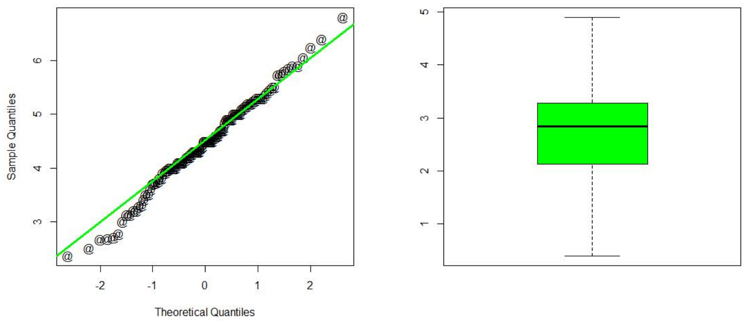

Figure 8: QQ-plot of the NEAPTW distribution and box plot for data 2.

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