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The odd generalized half-logistic Weibull-G family of distributions: properties and applications

Fastel Chipepa*[,1,2], Broderick Oluyede[1], Boikanyo Makubate[1]
[1]Department of Mathematics and Statistical Sciences, BIUST, Palapye, BW
[2]Department of Applied Mathematics and Statistics, MSU, Gweru, ZW

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Abstract: We propose a new generalized family of distributions called the odd generalized half logistic Weibull-G family of distributions. We also considered some special cases when the baseline distribution are uniform, Weibull and normal distributions. Structural properties of the new family of distributions including expansion of density, distribution of order statistics, Rényi entropy, moments, probability weighted moments, quantile and generating functions, and maximum likelihood estimates were derived. A characterization based on conditional expectation is presented. A simulation study to examine efficiency of the maximum likelihood estimates is also conducted. Finally, a real data example is presented to illustrate the applicability and usefulness of the proposed model.

Keywords: Half Logistic Distribution; Half Logistic-G Distribution; Weibull-G Distribution; Maximum Likelihood Estimation.
Mathematics Subject Classification (2010): 60E05 62E15 62F30.

1 Introduction

Balakrishnan (1985) developed the half-logistic distribution by transforming the logistic distribution. This distribution was well received by physicists, statisticians and hydrologists and can be applied in many areas of science and research. The half logistic density function has uni-modal or reversed J-shaped and this is a limitation since in most cases real data exhibits hazard rate functions that are often non-monotonic. Due to the challenges faced mainly in lifetime data analysis, where data may be highly skewed or heavy tailed, and where the hazard rate function is non-monotonic (uni-modal, bathtub, upside bathtub or upside bathtub followed by bathtub), there is an obvious need to add extra shape parameters to classical models so as to add flexibility

*Corresponding author: `chipepa.fastel@studentmail.biust.ac.bw`

in data fitting. Work on the generalization of classical distributions has received great interest in areas of biology, engineering, hydrology, medicine, economics and finance.

Many generators are suggested in literature for adding extra parameters to classical distributions and these includes work by Eugene et al. (2002), Cordeiro and de Castro (2011), Alexander et al. (2012), Zografos and Balakrishnan (2009), Ristić and Balakrishnan (2012), Torabi and Montazari (2012), Alzaatreh et al (2013), Alzaghal et al. (2013), Cordeiro et al. (2013), Bourguignon et al. (2014), Cordeiro et al. (2014) and Gomes-Silva et al. (2017), to mention a few. Generalization of the half logistic model includes the work by Afify et al. (2017), Cordeiro et al. (2014a), Cordeiro et al (2016), El-sayed and Mahmoud (2019).

Cordeiro et al (2016) developed the type 1 half-logistic family of distributions with the cdf and pdf given by

$$\begin{aligned} G(x;\lambda,\theta) &= \int_0^{-\ln(1-F(x;\theta))} \frac{2\lambda\exp\{-\lambda x\}}{(1+\exp\{-\lambda x\})^2}dx \\ &= \frac{1-[1-F(x;\theta)]^\lambda}{1+[1-F(x;\theta)]^\lambda}, \end{aligned} \quad (1)$$

$$g(x;\lambda,\theta) = \frac{2\lambda f(x;\theta)[1-F(x;\theta)]^{\lambda-1}}{\{1+[1-F(x;\theta)]^\lambda\}^2}, \quad (2)$$

respectively, where $F(x;\theta)$ is the cdf of the baseline distribution and $\lambda > 0$, is the shape parameter. We obtain a special case, namely, half-logistic-G (HL-G) model, with cdf

$$G(x;\theta) = \frac{F(x;\theta)}{1+\overline{F}(x;\theta)}, \quad (3)$$

by setting $\lambda = 1$ in Equation (1). The corresponding pdf of the HL-G model is given by

$$g(x;\theta) = \frac{2f(x;\theta)}{(1+\overline{F}(x;\theta))^2}. \quad (4)$$

Bourguignon et al. (2014) applied the Weibull generator to the odds ratio $G(x;\xi)/\overline{G}(x;\xi)$, using the ideas from Gurvich et al. (1997), and Zografos and Balakrishnan (2009). They developed the Weibull-G family of probability distributions with cdf and pdf given by

$$F(x;\alpha,\beta,\xi) = 1-\exp\Big\{-\alpha\Big[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\Big]^\beta\Big\}, \quad (5)$$

and

$$f(x;\alpha,\beta,\xi) = \alpha\beta g(x;\xi)\frac{G(x;\xi)^{\beta-1}}{\overline{G}(x;\xi)^{\beta+1}}\exp\Big\{-\alpha\Big[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\Big]^\beta\Big\}, \quad (6)$$

respectively. This distribution can also be expressed as a linear combination of exponentiated -G (Exp-G) distribution. In this paper, we develop the odd generalized half logistic Weibull-G (OGHLW-G) family of distributions using the generalization studied by Cordeiro et al (2016) and Bourguignon et al. (2014) (see Equations (3), (4), (5) and (6)), respectively.

This paper is organized as follows: In Section 2, we present the generalized family of distributions and the expansion of its density. Some special cases and sub-models of the new family are presented in Section 3. Structural properties including the distribution of order statistics, Rényi entropy, moments, probability weighted moments, quantile and generating functions are presented in Section 4. A characterization based on truncated conditional expectation is presented in Section 5. In Section 6, we present the maximum likelihood estimates. Monte Carlo simulation study is conducted to examine the bias and mean square errors of the maximum likelihood estimators for each parameter in Section 7. Application of the proposed model to real data is given in Section 8, followed by concluding remarks.

2 The odd generalized half-logistic Weibull-G family of distributions

In this section, we derive a new family of distributions, namely, the odd generalized half logistic Weibull-G (OGHLW-G) distribution. We also derive the series representation of this new distribution. The cdf and pdf of the OGHLW-G family of distributions are respectively, given by

$$F(x;\alpha,\beta,\xi)=\frac{1-\exp\left\{-\alpha\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{\beta}\right\}}{1+\exp\left\{-\alpha\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{\beta}\right\}}, \tag{7}$$

$$f(x;\alpha,\beta,\xi)=\frac{2\alpha\beta g(x;\xi)G(x;\xi)^{\beta-1}\exp\left\{-\alpha\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{\beta}\right\}}{\overline{G}(x;\xi)^{\beta+1}\left(1+\exp\left\{-\alpha\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{\beta}\right\}\right)^{2}}, \tag{8}$$

where $G(x;\xi)$ is the baseline cdf and $g(x;\xi)$ is the derivative of the baseline cdf, α, $\beta>0$ are parameters and ξ is a vector of parameters from the baseline distribution. If X is a random variable with density (8), we can write $X\sim OGHLW-G(\alpha,\beta,\xi)$. The survival function $S(x)$, hazard rate function (hrf) and cumulative hazard rate function, respectively, are given by

$$S(x;\alpha,\beta,\xi)=1-\left[\frac{1-\exp\left\{-\alpha\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{\beta}\right\}}{1+\exp\left\{-\alpha\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{\beta}\right\}}\right],$$

$$h(x;\alpha,\beta,\xi)=\frac{\alpha\beta g(x;\xi)G(x;\xi)^{\beta-1}}{\overline{G}(x;\xi)^{\beta+1}\left(1+\exp\left\{-\alpha\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{\beta}\right\}\right)},$$

$$H(x;\alpha,\beta,\xi)=-\log\left(1-\left[\frac{1-\exp\left\{-\alpha\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{\beta}\right\}}{1+\exp\left\{-\alpha\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{\beta}\right\}}\right]\right).$$

Interpretation of the OGHLW-G family of distributions can be as follows: Let X be a random variable, whose cdf is G(.) and describing a stochastic process. Let Y be the odds, the risk that the system following the lifetime X will not be working at time

x which is given by $G(x;\xi)/(1-G(x;\xi))$. The randomness of the odds can be modeled by the OGHLW-G distribution. In this case, the cdf of X is given by

$$Pr(X \leq x) = \frac{1-\exp\left\{-\alpha\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{\beta}\right\}}{1+\exp\left\{-\alpha\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{\beta}\right\}}.$$

2.1 Expansion of Density

We provide a linear representation of the OGHLW-G family of distributions. Equation (8) can be written as

$$f(x;\alpha,\beta,\xi) = \frac{2\alpha\beta g(x;\xi)G(x;\xi)^{\beta-1}\exp\left\{-\alpha\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{\beta}\right\}\left(1+\exp\left\{-\alpha\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{\beta}\right\}\right)^{-2}}{\overline{G}(x;\xi)^{\beta+1}}.$$

Using the series expansion $(1-x)^{-2} = \sum_{n=1}^{\infty} nx^{n-1}$, we have

$$\left(1+\exp\left\{-\alpha\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{\beta}\right\}\right)^{-2} = \sum_{n=1}^{\infty}(-1)^{n-1}n\left[\exp\left\{-\alpha\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{\beta}\right\}\right]^{n-1},$$

so that

$$f(x;\alpha,\beta,\xi) = \sum_{n=1}^{\infty}(-1)^{n-1}n\frac{2\alpha\beta g(x;\xi)G(x;\xi)^{\beta-1}\exp\left\{-\alpha n\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{\beta}\right\}}{\overline{G}(x;\xi)^{\beta+1}}.$$

Using series expansion

$$\exp\left\{-\alpha n\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{\beta}\right\} = \sum_{q=0}^{\infty}\frac{(-1)^q(\alpha n)^q}{q!}\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{\beta q},$$

we can write

$$f(x;\alpha,\beta,\xi) = \sum_{q=0}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{q+n-1}2\beta(\alpha n)^{q+1}}{q!}\frac{g(x;\xi)G(x;\xi)^{\beta(1+q)-1}}{\overline{G}(x;\xi)^{\beta(1+q)+1}},$$

and applying the following generalized binomial expansion

$$[\overline{G}(x;\xi)]^{-(\beta(1+q)+1)} = \sum_{p=0}^{\infty}\frac{\Gamma(p+\beta(1+q)+1)}{\Gamma(\beta(1+q)+1)p!}G^{p}(x;\xi),$$

we have

$$\begin{aligned} f(x;\alpha,\beta,\xi) &= \sum_{p,q=0}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{q+n-1}2\beta(\alpha n)^{q+1}}{p!q!}\frac{\Gamma(p+\beta(1+q)+1)}{\Gamma(\beta(1+q)+1)} \\ &\quad \times g(x;\xi)G(x;\xi)^{\beta(1+q)+p-1}. \end{aligned}$$

We can therefore write the linear representation of Equation (8) as

$$f(x;\alpha,\beta,\xi)=\sum_{p,q=0}^{\infty}v_{p,q}g_{\beta(1+q)+p}(x;\xi), \tag{9}$$

where

$$v_{p,q}=\sum_{n=1}^{\infty}\frac{(-1)^{q+n-1}2\beta(\alpha n)^{q+1}}{p!q!(\beta(1+q)+p)}\frac{\Gamma(p+\beta(1+q)+1)}{\Gamma(\beta(1+q)+1)} \tag{10}$$

and $g_{\beta(1+q)+p}(x;\xi)=(\beta(1+q)+p)g(x;\xi)[G(x;\xi)]^{\beta(1+q)+p-1}$ is an Exp-G with power parameter $(\beta(1+q)+p)>0$. The OGHLW-G distribution, therefore, is an infinite linear combination of Exp-G densities. Thus, the mathematical and statistical properties of the OGHLW-G distribution can be obtained directly from that of the Exp-G family of distributions.

3 Some special cases

In this section, we present three special cases of the OGHLW-G family of distributions. We considered baseline distributions that have simple analytic expressions so as to maintain the tractability property.

3.1 The Odd generalized half logistic Weibull-uniform distribution

Consider the uniform distribution as the baseline distribution with pdf and cdf given by $g(x;\theta)=1/\theta$ and $G(x;\theta)=x/\theta$, for $0<x<\theta$, respectively. We can define the cdf and pdf of the odd generalized half logistic Weibull-uniform (OGHLW-U) distribution as

$$F_{OGHLW-U}(x;\alpha,\beta,\theta)=\frac{1-\exp\left(-\alpha\left(\frac{x}{\theta-x}\right)^{\beta}\right)}{1+\exp\left(-\alpha\left(\frac{x}{\theta-x}\right)^{\beta}\right)},$$

$$f_{OGHLW-U}(x;\alpha,\beta,\theta)=\frac{2\alpha\beta(1/\theta)(x/\theta)^{\beta-1}\exp\left(-\alpha\left(\frac{x}{\theta-x}\right)^{\beta}\right)}{(1-x/\theta)^{\beta+1}\left[1+\exp\left(-\alpha\left(\frac{x}{\theta-x}\right)^{\beta}\right)\right]^{2}},$$

respectively, for $\alpha,\beta,\theta>0$.

3.1.1 Sub-models of OGHLW-U distribution

- We obtain the odd half logistic exponential-uniform distribution from the OGHLW-U distribution by setting $\alpha=\beta=1$.
- When $\alpha=1$, we obtain a sub-model with the pdf given by

$$f(x;\beta,\theta)=\frac{2\beta(1/\theta)(x/\theta)^{\beta-1}\exp\left(-\left(\frac{x}{\theta-x}\right)^{\beta}\right)}{(1-x/\theta)^{\beta+1}\left[1+\exp\left(-\left(\frac{x}{\theta-x}\right)^{\beta}\right)\right]^{2}}.$$

- We also obtain the pdfs

$$f(x;\alpha,\theta)=\frac{2\alpha(1/\theta)\exp\left(-\alpha\left(\frac{x}{\theta-x}\right)\right)}{(1-x/\theta)^2\left[1+\exp\left(-\alpha\left(\frac{x}{\theta-x}\right)\right)\right]^2}, \tag{11}$$

$$f(x;\alpha,\beta)=\frac{2\alpha\beta(x)^{\beta-1}\exp\left(-\alpha\left(\frac{x}{1-x}\right)^{\beta}\right)}{(1-x)^{\beta+1}\left[1+\exp\left(-\alpha\left(\frac{x}{1-x}\right)^{\beta}\right)\right]^2},$$

  by setting $\beta=1$ and $\theta=1$, respectively.

- Furthermore, other sub-models may be obtained by setting $\beta=\theta=1$ and $\alpha=\theta=1$.

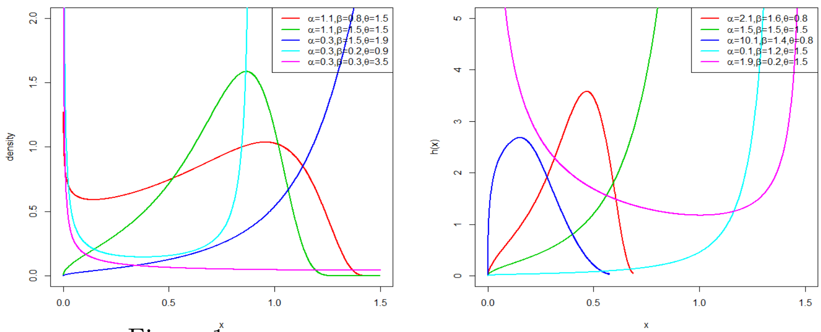

Figure 1: The pdf and hrf plots of OGHLW-U distribution

Figures 1 shows the plots of pdfs and hrfs of OGHLW-U distribution for selected parameters values. The pdf can take various shapes including uni-modal, left and right skewed. Graphs of the hazard function shows the flexibility of the new family and exhibit increasing, bathtub and upside down bathtub shapes.

3.2 The odd generalized half logistic Weibull-Weibull distribution

Consider the Weibull distribution as the baseline distribution with pdf and cdf given by $g(x;\lambda,\gamma)=\lambda\gamma x^{\gamma-1}e^{-\lambda x^{\gamma}}$ and $G(x;\lambda,\gamma)=1-e^{-\lambda x^{\gamma}}$, respectively. We can define the cdf and pdf of the odd generalized half logistic Weibull-Weibull (OGHLW-W) distribution as

$$F_{OGHLW-W}(x;\alpha,\beta,\lambda,\gamma)=\frac{1-\exp\left\{-\alpha\left[\frac{1-e^{-\lambda x^{\gamma}}}{e^{-\lambda x^{\gamma}}}\right]^{\beta}\right\}}{1+\exp\left\{-\alpha\left[\frac{1-e^{-\lambda x^{\gamma}}}{e^{-\lambda x^{\gamma}}}\right]^{\beta}\right\}},$$

$$f_{OGHLW-W}(x;\alpha,\beta,\lambda,\gamma)=\frac{2\alpha\beta\lambda\gamma x^{\gamma-1}e^{-\lambda x^{\gamma}}(1-e^{-\lambda x^{\gamma}})^{\beta-1}\exp\left\{-\alpha\left[\frac{1-e^{-\lambda x^{\gamma}}}{e^{-\lambda x^{\gamma}}}\right]^{\beta}\right\}}{e^{-(\beta+1)\lambda x^{\gamma}}\left(1+\exp\left\{-\alpha\left[\frac{1-e^{-\lambda x^{\gamma}}}{e^{-\lambda x^{\gamma}}}\right]^{\beta}\right\}\right)^2},$$

respectively, for $\alpha, \beta, \lambda, \gamma > 0$.

3.2.1 Sub-models of OGHLW-W distribution

- We obtain the odd half logistic exponential-Weibull distribution from the OGHLW-W distribution by setting $\alpha = \beta = 1$.
- When $\alpha = 1$, we obtain a sub-model with the pdf given by

$$f(x;\beta,\lambda,\gamma) = \frac{2\alpha\beta\lambda\gamma x^{\gamma-1}e^{-\lambda x^{\gamma}}(1-e^{-\lambda x^{\gamma}})^{\beta-1}\exp\left\{-\left[\frac{1-e^{-\lambda x^{\gamma}}}{e^{-\lambda x^{\gamma}}}\right]^{\beta}\right\}}{e^{-(\beta+1)\lambda x^{\gamma}}\left(1+\exp\left\{-\left[\frac{1-e^{-\lambda x^{\gamma}}}{e^{-\lambda x^{\gamma}}}\right]^{\beta}\right\}\right)^{2}}.$$

- We also obtain the pdfs

$$f(x;\alpha,\lambda,\gamma) = \frac{2\alpha\lambda\gamma x^{\gamma-1}e^{-\lambda x^{\gamma}}\exp\left\{-\alpha\left[\frac{1-e^{-\lambda x^{\gamma}}}{e^{-\lambda x^{\gamma}}}\right]\right\}}{e^{-2\lambda x^{\gamma}}\left(1+\exp\left\{-\alpha\left[\frac{1-e^{-\lambda x^{\gamma}}}{e^{-\lambda x^{\gamma}}}\right]\right\}\right)^{2}},$$

$$f(x;\alpha,\beta,\gamma) = \frac{2\alpha\beta\gamma x^{\gamma-1}e^{-x^{\gamma}}(1-e^{-x^{\gamma}})^{\beta-1}\exp\left\{-\alpha\left[\frac{1-e^{-x^{\gamma}}}{e^{-x^{\gamma}}}\right]^{\beta}\right\}}{e^{-(\beta+1)x^{\gamma}}\left(1+\exp\left\{-\alpha\left[\frac{1-e^{-x^{\gamma}}}{e^{-x^{\gamma}}}\right]^{\beta}\right\}\right)^{2}},$$

  by setting $\beta = 1$ and $\lambda = 1$, respectively.

- Furthermore, other sub-models may be obtained by setting $\beta = \lambda = 1$, $\alpha = \lambda = 1$, $\alpha = \gamma = 1$ and $\beta = \gamma = 1$.

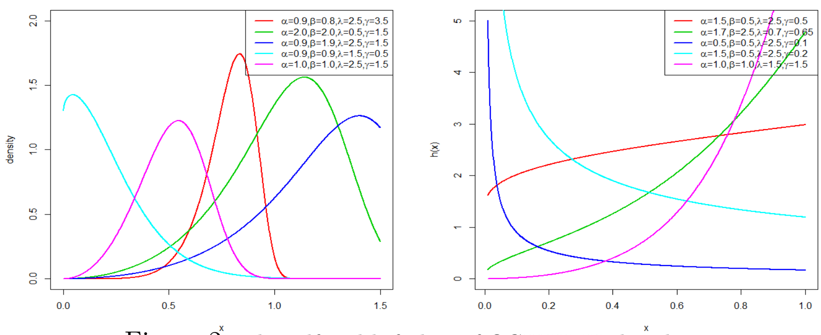

Figure 2: The pdf and hrf plots of OGHLW-W distribution

Figure 2 shows the plots of pdfs and hrfs of OGHLW-W distribution for selected parameters values. The pdf can take various shapes including uni-modal, left and right skewed. Graphs of the hazard rate function exhibit increasing and decreasing shapes.

3.3 The odd generalized half logistic Weibull-normal distribution

By taking the normal distribution with pdf $g(x;\mu,\sigma)=\sigma^{-1}\phi\big(\frac{x-\mu}{\sigma}\big)$ and cdf $G(x;\mu,\sigma)=\Phi\big(\frac{x-\mu}{\sigma}\big)$, for $\mu\in\Re$ and $\sigma>0$, as the baseline distribution, we obtain the odd generalized half logistic Weibull-normal (OGHLW-N) distribution with cdf and pdf given by

$$F_{OGHLW-N}(x;\alpha,\beta,\mu,\sigma)=\frac{1-\exp\left\{-\alpha\left[\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]^{\beta}\right\}}{1+\exp\left\{-\alpha\left[\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]^{\beta}\right\}},$$

$$f_{OGHLW-N}(x;\alpha,\beta,\mu,\sigma)=\frac{2\alpha\beta\phi\left(\frac{x-\mu}{\sigma}\right)\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\beta-1}}{\sigma\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\beta+1}}\frac{\exp\left\{-\alpha\left[\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]^{\beta}\right\}}{\left(1+\exp\left\{-\alpha\left[\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]^{\beta}\right\}\right)^{2}},$$

for α, β, $\sigma>0$ and $-\infty<\mu<\infty$.

3.3.1 Sub-models of OGHLW-N distribution

- We obtain the odd half logistic exponential-normal distribution from the OGHLW-N distribution by setting $\alpha=\beta=1$.
- When $\alpha=1$, we obtain a sub model with the pdf given by

$$f(x;\beta,\mu,\sigma)=\frac{2\beta\phi\left(\frac{x-\mu}{\sigma}\right)\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\beta-1}}{\sigma\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\beta+1}}\frac{\exp\left\{-\left[\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]^{\beta}\right\}}{\left(1+\exp\left\{-\left[\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]^{\beta}\right\}\right)^{2}}.$$

- Furthermore, other sub-models may be obtained by setting $\beta=1$, $\beta=\mu=1$, $\alpha=\mu=1$, $\alpha=\sigma=1$ and $\beta=\sigma=1$.

Figure 3 shows the plots of the pdfs and hrfs of OGHLW-N distribution for selected parameters values. The pdf can take various shapes including J-shaped, uni-modal and left skewed. Graphs of the hazard rate function exhibit increasing, bathtub, upside down bathtub followed by bathtub and upside down bathtub shapes.

4 Some properties

We study some properties of the OGHLW-G family of distributions, which includes order statistics, entropy, moments, incomplete moments, probability weighted moments (PWMs), quantiles and moment generating function (mgf).

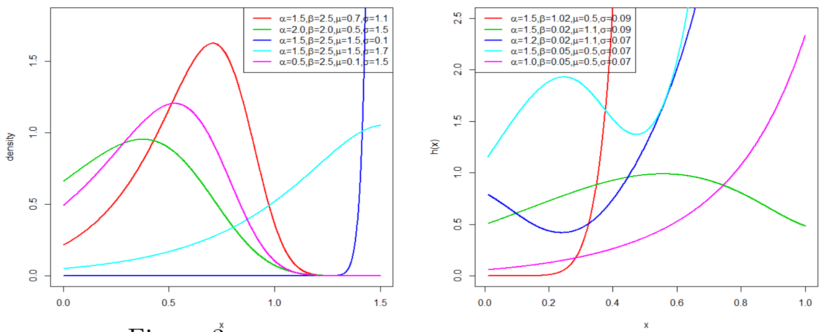

Figure 3: The pdf and hrf plots of OGHLW-N distribution

4.1 Distribution of order statistics

The pdf of the i^{th} order statistic can be written as

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-j} \binom{n-i}{j} F(x)^{j+i-1},$$

where $B(.,.)$ is the beta function. Substituting Equations (7) and (8), we get

$$\begin{aligned} f(x)F(x)^{j+i-1} &= \frac{2\alpha\beta g(x;\xi)G(x;\xi)^{\beta-1}\exp\Big\{-\alpha\Big[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\Big]^{\beta}\Big\}}{\overline{G}(x;\xi)^{\beta+1}\Big(1+\exp\Big\{-\alpha\Big[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\Big]^{\beta}\Big\}\Big)^{j+i+1}} \\ &\quad \times\Big(1-\exp\Big\{-\alpha\Big[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\Big]^{\beta}\Big\}\Big)^{j+i-1}. \end{aligned}$$

Expanding

$$\Big(1+\exp\Big\{-\alpha\Big[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\Big]^{\beta}\Big\}\Big)^{-(j+i+1)} = \sum_{w=0}^{\infty}\binom{-(j+i+1)}{w}\exp\Big\{-\alpha w\Big[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\Big]^{\beta}\Big\},$$

$$\Big(1-\exp\Big\{-\alpha\Big[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\Big]^{\beta}\Big\}\Big)^{(j+i-1)} = \sum_{z=0}^{\infty}(-1)^{z}\binom{j+i-1}{z}\exp\Big\{-\alpha z\Big[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\Big]^{\beta}\Big\},$$

yields

$$\begin{aligned} f(x)F(x)^{j+i-1} &= \sum_{w,z=0}^{\infty}(-1)^{z}2\alpha\beta\binom{-(j+i+1)}{w}\binom{j+i-1}{z}\frac{g(x;\xi)G(x;\xi)^{\beta-1}}{\overline{G}(x;\xi)^{\beta+1}} \\ &\quad \times\exp\Big\{-\alpha(1+w+z)\Big[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\Big]^{\beta}\Big\}. \end{aligned}$$

Also, expanding

$$\exp\Big\{-\alpha(1+w+z)\Big[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\Big]^{\beta}\Big\} = \sum_{m=0}^{\infty}\frac{(-1)^{m}\alpha^{m}(1+w+z)^{m}}{m!}\Big[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\Big]^{\beta m},$$

and

$$[\overline{G}(x;\xi)]^{-(\beta(m+1)+1)} = \sum_{d=0}^{\infty} \frac{\Gamma(d+\beta(m+1)+1)}{\Gamma(\beta(m+1)+1)d!}[G(x;\xi)]^{d},$$

concludes that

$$\begin{aligned} f(x)F(x)^{j+i-1} &= \sum_{d,m,w,z=0}^{\infty} \frac{(-1)^{m+z}2\beta\alpha^{m+1}(1+w+z)^{m}}{d!m!}\binom{-(j+i+1)}{w} \\ &\times \binom{j+i-1}{z}\frac{\Gamma(d+\beta(m+1)+1)}{\Gamma(\beta(m+1)+1)}g(x;\xi)G(x;\xi)^{\beta(m+1)+d-1}. \end{aligned} \tag{12}$$

Therefore, the i^{th} order statistic from the OGHLW-G distribution can be expressed as

$$\begin{aligned} f_{i:n}(x) &= \frac{1}{B(i,n-i+1)}\sum_{d,m=0}^{\infty}\sum_{w,z=0}^{\infty}\sum_{j=0}^{n-i}\frac{(-1)^{j+m+z}2\beta\alpha^{m+1}(1+w+z)^{m}}{d!m!(\beta(m+1)+d)} \\ &\times\binom{n-i}{j}\binom{-(j+i+1)}{w}\binom{j+i-1}{z}\frac{\Gamma(d+\beta(m+1)+1)}{\Gamma(\beta(m+1)+1)} \\ &\times(\beta(m+1)+d)g(x;\xi)G(x;\xi)^{\beta(m+1)+d-1} \\ &= \sum_{d,m=0}^{\infty} v_{d,m}^{*}g_{\beta(m+1)+d}(x;\xi), \end{aligned} \tag{13}$$

where $g_{\beta(m+1)+d}(x;\xi) = (\beta(m+1)+d)g(x;\xi)[G(x;\xi)]^{\beta(m+1)+d-1}$ is an Exp-G distribution with power parameter $\beta(m+1)+d > 0$ and

$$\begin{aligned} v_{d,m}^{*} &= \frac{1}{B(i,n-i+1)}\sum_{w,z=0}^{\infty}\sum_{j=0}^{n-i}\frac{(-1)^{j+m+z}2\beta\alpha^{m+1}(1+w+z)^{m}}{d!m!(\beta(m+1)+d)} \\ &\times\binom{n-i}{j}\binom{-(j+i+1)}{w}\binom{j+i-1}{z}\frac{\Gamma(d+\beta(m+1)+1)}{\Gamma(\beta(m+1)+1)}. \end{aligned} \tag{14}$$

It follows that the pdf of the i^{th} order statistic from the OGHLW-G distribution can be expressed as an infinite linear combination of Exp-G densities.

4.2 Entropy

Rényi entropy is defined by

$$I_R(\nu) = (1-\nu)^{-1}\log\left[\int_0^{\infty} f_{\theta}^{\nu}(x)dx\right], v \neq 1, v > 0. \tag{15}$$

There are two main types of entropy, Shannon entropy and Rényi entropy. Shannon entropy is due to Shannon (1951) and Rényi entropy is due to Rényi (1960). Shannon entropy is given by $E\{-\log[f(X)]\}$. In view of the fact that Shannon entropy is a special case of Rényi entropy, we only derive expressions for Rényi entropy for the OGHLW-G distribution. Using Equation (8), $f^{\nu}(x)$, can be written as

$$f^{\nu}(x)=\frac{2^{\nu}\alpha^{\nu}\beta^{\nu}g^{\nu}(x;\xi)G(x;\xi)^{(\beta-1)\nu}\exp\Big\{-\alpha\nu\Big[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\Big]^{\beta}\Big\}}{\overline{G}(x;\xi)^{(\beta+1)\nu}\Big(1+\exp\Big\{-\alpha\Big[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\Big]^{\beta}\Big\}\Big)^{2\nu}}.$$

Expanding

$$\Big(1+\exp\Big\{-\alpha\Big[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\Big]^{\beta}\Big\}\Big)^{-2\nu}=\sum_{i=0}^{\infty}\binom{-2\nu}{i}\exp\Big\{-\alpha i\Big[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\Big]^{\beta}\Big\},$$

$$\exp\Big\{-\alpha(\nu+i)\Big[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\Big]^{\beta}\Big\}=\sum_{j=0}^{\infty}\frac{(-1)^{j}(\nu+i)^{j}\alpha^{j}}{j!}\Big[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\Big]^{\beta j},$$

yields

$$f^{\nu}(x)=\sum_{i,j=0}^{\infty}\frac{(-1)^{j}(\nu+i)^{j}(2\beta)^{\nu}\alpha^{\nu+j}}{j!\overline{G}(x;\xi)^{(\beta(\nu+j)+\nu)}}\binom{-2\nu}{i}g^{\nu}(x;\xi)[G(x;\xi)]^{(\beta(\nu+j)-\nu)}.$$

Applying the generalized binomial expansion as

$$[\overline{G}(x;\xi)]^{-(\beta(\nu+j)+\nu)}=\sum_{k=0}^{\infty}\frac{\Gamma(k+\beta(\nu+j)+\nu)}{\Gamma(\beta(\nu+j)+\nu)k!}G(x;\xi)^{k},$$

concludes that

$$\begin{aligned}f^{\nu}(x) &= \sum_{i,j,k=0}^{\infty}\frac{(-1)^{j}(\nu+i)^{j}(2\beta)^{\nu}\alpha^{\nu+j}}{j!k!}\frac{\Gamma(k+\beta(\nu+j)+\nu)}{\Gamma(\beta(\nu+j)+\nu)}\binom{-2\nu}{i}\\ &\quad\times g^{\nu}(x;\xi)[G(x;\xi)]^{(\beta(\nu+j)-\nu+k)}.\end{aligned}$$

Rényi entropy of OGHLW-G distribution can be written as

$$I_R(\nu)=(1-\nu)^{-1}\log\left[\sum_{i,j,k=0}^{\infty}w_{i,j,k}e^{(1-\nu)I_{REG}}\right],v\neq 1,v>0, \tag{16}$$

where

$$\begin{aligned}w_{i,j,k} &= \frac{(-1)^{j}(\nu+i)^{j}(2\beta)^{\nu}\alpha^{\nu+j}}{j!k!}\frac{\Gamma(k+\beta(\nu+j)+\nu)}{\Gamma(\beta(\nu+j)+\nu)}\binom{-2\nu}{i}\\ &\quad\times\Big(\frac{\nu}{\beta(\nu+j)-\nu+k}+1\Big)^{\nu},\end{aligned} \tag{17}$$

and $I_{REG}=\int_0^{\infty}\Big[\big(\frac{\beta(\nu+j)-\nu+k}{\nu}+1\big)g(x;\xi)[G(x;\xi)]^{\frac{\beta(\nu+j)-\nu+k}{\nu}}\Big]^{\nu}dx$ is Rényi entropy of Exp-G distribution with parameter $\frac{\beta(\nu+j)-\nu+k}{\nu}$. It follows that Rényi entropy of OGHLW-G family of distributions can be derived directly from Rényi entropy of Exp-G distribution.

4.3 Moments

The s^{th} ordinary moment is derived from Equation (9) as

$$\mu'_s = E(X^s) = \sum_{p,q=0}^{\infty} v_{p,q} E(Y^s_{\beta(1+q)+p}), \tag{18}$$

where $Y_{\beta(1+q)+p}$ follows an Exp-G distribution with power parameter $(\beta(1+q)+p)$ and $v_{p,q}$ is as defined in Equation (10). The n^{th} central moment of X is given by

$$\mu_n = \sum_{s=0}^{n} \binom{n}{s} (-\mu'_1)^{n-s} E(X^s) = \sum_{s=0}^{n} \sum_{p,q=0}^{\infty} v_{p,q} \binom{n}{s} (-\mu'_1)^{n-s} E(Y^s_{\beta(1+q)+p}).$$

The cumulants of X follow recursively from

$$k_n = \mu'_n - \sum_{s=0}^{n-1} \binom{n-1}{s-1} k_s \mu'_{n-s},$$

where $k_1 = \mu'_1$, $k_2 = \mu'_2 - \mu'^2_1$, $k_3 = \mu'_3 - 3\mu'_2\mu'_1 + \mu'^3_1$, etc. Ordinary moments may also be used to calculate the measures of dispersion, namely, variance, skewness and kurtosis.

The r^{th} incomplete moment of X is given by

$$\phi_r(z) = \int_{-\infty}^{z} x^r f(x) dx = \sum_{p,q=0}^{\infty} v_{p,q} \int_{-\infty}^{z} x^r g_{\beta(1+q)+p}(x;\xi) dx. \tag{19}$$

The incomplete moment can be used to derive quantities which have wide application in demography, economics, insurance, medicine and reliability. These quantities are the Lorenz and Bonferroni curves for a given probability p, $L(p) = \phi_1(q)/\mu'_1$ and $B(p) = \phi_1(q)/(p\mu'_1)$, respectively, where μ'_1 is given by Equation (18), with $s = 1$ and $q = Q(p)$ is the quantile function of X at p. Equation (19) can be expressed as

$$\phi_r(z) = \sum_{p,q=0}^{\infty} v_{p,q} H_{\beta(1+q)+p}(z), \tag{20}$$

where $H_{\beta(1+q)+p}(z) = \int_{-\infty}^{z} x^r g_{\beta(1+q)+p}(x;\xi) dx$ is the r^{th} incomplete moment of the Exp-G distribution.

Table 1 lists the first five moments together with the standard deviation (SD or σ), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) of the OGHLW-N distribution for selected values of the parameters.

4.4 Probability weighted moments

Probability Weighted Moments (PWMs) is mainly used to estimate parameters of a distribution whose inverse form cannot be expressed explicitly. The (j,i)th PWM, say $\eta_{j,i}$ of $X \sim$ OGHLW-G$(\alpha, \beta; \xi)$ is defined by

$$\eta_{j,i} = E(X^j F(X)^i) = \int_{-\infty}^{\infty} x^j f(x) F(x)^i dx.$$

Table 1: Moments of the OGHLW-N distribution for some parameters values

	(0.9,1.5,0.5,0.5)	(0.5,1,1.5,0.5)	(1,0.5,0.5,1.5)	(1,1.5,0.5,0.5)	(1.1,0.5,1.1,0.5)
EX	0.50445	0.03573	0.10925	0.48371	0.22290
EX^2	0.29872	0.02920	0.07443	0.27735	0.16146
EX^3	0.18910	0.02477	0.05653	0.17045	0.12704
EX^4	0.12536	0.02155	0.04559	0.10990	0.10486
EX^5	0.08610	0.01908	0.03821	0.07350	0.08933
SD	0.21035	0.16711	0.24999	0.20825	0.33432
CV	0.41699	4.67730	2.28816	0.43053	1.49988
CS	-0.66927	4.65791	2.22378	-0.62663	1.10306
CK	2.87012	23.37119	6.60431	2.77412	2.58679

Using Equation (12), we can write

$$\begin{aligned} f(x)F(x)^i &= \sum_{d,m,w,z=0}^{\infty} \frac{(-1)^{m+z} 2\beta\alpha^{m+1}(1+w+z)^m}{d!m!} \binom{-(i+1)}{w} \\ &\quad \times \binom{(i-1)}{z} \frac{\Gamma(d+\beta(m+1)+1)}{\Gamma(\beta(m+1)+1)} g(x;\xi)[G(x;\xi)]^{\beta(m+1)+d-1}, \end{aligned}$$

which can be expressed as

$$f(x)F(x)^i = \sum_{d,m,w,z=0}^{\infty} \gamma^*_{d,m,w,z} g_{\beta(m+1)+d}(x;\xi),$$

where

$$\begin{aligned} \gamma*_{d,m,w,z} &= \frac{(-1)^{m+z} 2\beta\alpha^{m+1}(1+w+z)^m}{d!m!(\beta(m+1)+d)} \binom{-(i+1)}{w} \\ &\quad \times \binom{(i-1)}{z} \frac{\Gamma(d+\beta(m+1)+1)}{\Gamma(\beta(m+1)+1)}. \end{aligned}$$

Therefore, the PWM is given by

$$\begin{aligned} \eta_{j,i} &= \sum_{d,m,w,z=0}^{\infty} \gamma^*_{d,m,w,z} \int_{-\infty}^{\infty} x^j g_{\beta(m+1)+d}(x;\xi)dx \\ &= \sum_{d,m,w,z=0}^{\infty} \gamma^*_{d,m,w,z} E(T^j_{(\beta(m+1)+d)}), \end{aligned}$$

where $T^j_{(\beta(m+1)+d)}$ is j^{th} power of an Exp-G distributed random variable with power parameter $(\beta(m+1)+d) > 0$.

4.5 Quantile and generating functions

We obtain the quantile function by inverting Equation (7). We invert the function

$$\frac{1-\exp\left\{-\alpha\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{\beta}\right\}}{1+\exp\left\{-\alpha\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{\beta}\right\}}=u.$$

After simplifying, we find that

$$1-u=(1+u)\exp\left\{-\alpha\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{\beta}\right\},$$

which can be written as

$$\left[\frac{\ln(1+u)-\ln(1-u)}{\alpha}\right]^{1/\beta}=\frac{G(x;\xi)}{\overline{G}(x;\xi)}.$$

The equation simplifies to

$$G(x;\xi)=\frac{\left[\frac{\ln(1+u)-\ln(1-u)}{\alpha}\right]^{1/\beta}}{1+\left[\frac{\ln(1+u)-\ln(1-u)}{\alpha}\right]^{1/\beta}}.$$

The quantiles of the OGHLW-G family of distributions may be determined by solving the equation

$$x(u)=G^{-1}\left[\frac{\left[\frac{\ln(1+u)-\ln(1-u)}{\alpha}\right]^{1/\beta}}{1+\left[\frac{\ln(1+u)-\ln(1-u)}{\alpha}\right]^{1/\beta}}\right], \tag{21}$$

using iterative methods.

The moment generating function (mgf) is given by

$$M_x(t)=E(e^{tX})=\sum_{p,q=0}^{\infty}v_{p,q}M_{\beta(1+q)+p}(t),$$

where $M_{\beta(1+q)+p}(t)$ is the mgf of Exp-G with power parameter $\beta(1+q)+p>0$. Therefore, the mgf of OGHLW-G distribution can be derived from that of the Exp-G distribution.

Table 2 shows some quantiles for selected parameters values for the OGHLW-N distribution.

5 Characterization

We characterize the OGHLW-G family of distributions via truncated conditional expectation in this Section.

Table 2: Quantiles for Selected Parameters of OGHLW-N Distribution

u	(1.5,1.5,1.5,0.5)	(0.5,1,1.5,0.2)	(1.1,0.5,1.5,0.5)	(0.5,1.5,1.5,0.5)	(1.1,0.9,1,0.1)
0.1	1.09216	1.38721	0.57535	1.31054	0.88793
0.2	1.23031	1.47376	0.91154	1.45624	0.93194
0.3	1.31626	1.52675	1.14770	1.54460	0.96024
0.4	1.38100	1.56577	1.33722	1.60993	0.98185
0.5	1.43500	1.59760	1.49921	1.66363	0.99992
0.6	1.48352	1.62555	1.64441	1.71128	1.01607
0.7	1.53037	1.65192	1.78137	1.75678	1.03155
0.8	1.57965	1.67900	1.92009	1.80410	1.04768
0.9	1.64039	1.71148	2.08135	1.86176	1.06719

5.1 Characterization based on conditional expectation

Proposition 5.1. *Let $X : \Omega \to (0, \infty)$ be a continuous random variable with cdf $F(x)$, $(0 < F(x) < 1 \ \text{for} \ x \geq 0)$, then X belongs to the family with the cdf in Equation (7) if and only if*

$$E(Y|X < t) = -2\left(1 + \exp\left\{ - \alpha\left[\frac{G(x)}{\overline{G}(x)}\right]^{\beta}\right\}\right). \tag{22}$$

where $Y = \left(1 + \exp\left\{ - \alpha\left[\frac{G(X)}{\overline{G}(X)}\right]^{\beta}\right\}\right)^2$.

We replace $G(x;\xi)$ from (7) by $G(x)$ for computational purposes.

Proof. If pdf of X is defined by Equation (8), then

$$\begin{aligned}
E(Y|X < t) &= \frac{1}{F(t)}\int_0^t \left(1 + \exp\left\{ - \alpha\left[\frac{G(x)}{\overline{G}(x)}\right]^{\beta}\right\}\right)^2 f(x)dx \\
&= \frac{1}{F(t)}\int_0^t \left(1 + \exp\left\{ - \alpha\left[\frac{G(x)}{\overline{G}(x)}\right]^{\beta}\right\}\right)^2 \\
&\quad \times \frac{2\alpha\beta g(x)G(x)^{\beta-1}\exp\left\{ - \alpha\left[\frac{G(x)}{\overline{G}(x)}\right]^{\beta}\right\}}{\overline{G}(x)^{\beta+1}\left(1 + \exp\left\{ - \alpha\left[\frac{G(x)}{\overline{G}(x)}\right]^{\beta}\right\}\right)^2}dx \\
&= \frac{1}{F(t)}\int_0^t \frac{2\alpha\beta g(x)G(x)^{\beta-1}}{\overline{G}(x)^{\beta+1}}\exp\left\{ - \alpha\left[\frac{G(x)}{\overline{G}(x)}\right]^{\beta}\right\}dx \\
&= -2\left(1 + \exp\left\{ - \alpha\left[\frac{G(x)}{\overline{G}(x)}\right]^{\beta}\right\}\right),
\end{aligned}$$

after integration by parts. Conversely, if (22) holds, then

$$\int_0^t \left(1 + \exp\left\{ - \alpha\left[\frac{G(x)}{\overline{G}(x)}\right]^{\beta}\right\}\right)^2 f(x)dx = -2F(t)\left(1 + \exp\left\{ - \alpha\left[\frac{G(t)}{\overline{G}(t)}\right]^{\beta}\right\}\right). \tag{23}$$

Differentiation (23) with respect to t, we obtain

$$\left(1 + \exp\left\{ - \alpha\left[\frac{G(t)}{\overline{G}(t)}\right]^{\beta}\right\}\right)^2 f(t) \quad = \quad -2f(t)\left(1 + \exp\left\{ - \alpha\left[\frac{G(t)}{\overline{G}(t)}\right]^{\beta}\right\}\right)$$

$$+2F(t)\frac{\beta\alpha g(t)G(t)^{\beta-1}\exp\Big\{-\alpha\Big[\frac{G(t)}{\bar{G}(t)}\Big]^{\beta}\Big\}}{\bar{G}(t)^{\beta+1}},$$

which after simplification and integration, results in

$$F(t)=\frac{1-\exp\{-\alpha\big[\frac{G(t)}{\bar{G}(t)}\big]^{\beta}\}}{1+\exp\{-\alpha\big[\frac{G(t)}{\bar{G}(t)}\big]^{\beta}\}}.$$

□

6 Maximum likelihood estimation

Let $X_i \sim OGHLW-G(\alpha,\beta;\xi)$ and $\Delta=(\alpha,\beta;\xi)^T$ be the parameters vector. The log-likelihood $\ell=\ell(\Delta)$ based on a random sample of size n is given by

$$\begin{aligned}\ell &= n\log(2\alpha)+n\log\beta+\sum_{i=1}^{n}\log[g(x;\xi)]+(\beta-1)\sum_{i=1}^{n}\log[G(x;\xi)]\\&\quad -\alpha\sum_{i=1}^{n}\Big[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\Big]^{\beta}-(\beta+1)\sum_{i=1}^{n}\log[\overline{G}(x;\xi)]\\&\quad -2\sum_{i=1}^{n}\log\Big(1+\exp\Big\{-\alpha\Big[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\Big]^{\beta}\Big\}\Big).\end{aligned}$$

Elements of the score vector $U=(\frac{\partial\ell}{\partial\alpha},\frac{\partial\ell}{\partial\beta},\frac{\partial\ell}{\partial\xi_k})$ are given by:

$$\begin{aligned}\frac{\partial\ell}{\partial\alpha} &= \frac{n}{\alpha}-\sum_{i=1}^{n}\Big[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\Big]^{\beta}+2\sum_{i=1}^{n}\frac{G(x;\xi)^{\beta}\exp\Big\{-\alpha\Big[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\Big]^{\beta}\Big\}}{\Big(1+\exp\Big\{-\alpha\Big[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\Big]^{\beta}\Big\}\Big)[\overline{G}(x;\xi)]^{\beta}},\\ \frac{\partial\ell}{\partial\beta} &= \frac{n}{\beta}+\sum_{i=1}^{n}\log[G(x;\xi)]-\sum_{i=1}^{n}\log[\overline{G}(x;\xi)]-\sum_{i=1}^{n}\alpha\Big[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\Big]^{\beta}\log\Big[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\Big]\\&\quad +2\sum_{i=1}^{n}\frac{\alpha\log\Big[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\Big]\exp\Big\{-\alpha\Big[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\Big]^{\beta}\Big\}}{\Big[1+\exp\Big\{-\alpha\Big[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\Big]^{\beta}\Big\}\Big][\overline{G}(x;\xi)]^{\beta}},\\ \frac{\partial\ell}{\partial\xi_k} &= \sum_{i=1}^{n}\frac{1}{g(x;\xi)}\frac{\partial g(x;\xi)}{\partial\xi_k}+(\beta-1)\sum_{i=1}^{n}\frac{1}{G(x;\xi)}\frac{\partial G(x;\xi)}{\partial\xi_k}\\&\quad -(\beta+1)\sum_{i=1}^{n}\frac{1}{\overline{G}(x;\xi)}\frac{\partial\overline{G}(x;\xi)}{\partial\xi_k}\\&\quad -\alpha\sum_{i=1}^{n}\frac{\beta\overline{G}(x;\xi)^{\beta}G(x;\xi)^{\beta-1}+\beta G'(x;\xi)G(x;\xi)^{\beta}\overline{G}(x;\xi)}{\overline{G}(x;\xi)^{2\beta}}\end{aligned}$$

$$-2\sum_{i=1}^{n}\frac{1}{\left(1+\exp\left\{-\alpha\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{\beta}\right\}\right)}\frac{\partial\left(1+\exp\left\{-\alpha\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{\beta}\right\}\right)}{\partial\xi_k}.$$

The values of the parameters α, β, ξ_k are found using MATLAB or NLMIXED in SAS or R software via iterative methods.

7 Simulation study

A simulation study was conducted to evaluate efficiency of the maximum likelihood estimators for the OGHLW-N distribution. The simulation study was repeated for N=1000 times with sample size n= 30, 60, 120, 240, 480, 960 and 1920. Tables 3 and 4 indicate the mean MLEs of the model parameters along with the respective root mean square errors (RMSE) and average bias for the OGHLW-N distribution for selected parameters values. From the results, we can verify that as the sample size increases, the mean estimates of the parameters tend to be closer to the true parameters values, since RMSEs and average bias decays toward zero for all the parameters.

Table 3: Monte Carlo Simulation Results for OGHLW-N Distribution: Mean, RMSE and Average Bias (Av. Bias).

		$\alpha = 1.5, \beta = 1.5, \mu = 0.5, \sigma = 0.1$			$\alpha = 1.0, \beta = 1.0, \mu = 1.0, \sigma = 0.1$		
Parameter	n	Mean	RMSE	Av. Bias	Mean	RMSE	Av. Bias
α	30	4.569059	16.349983	3.069059	2.757701	7.378323	1.757701
	60	3.540066	9.560091	2.040066	2.945819	7.885353	1.945819
	120	3.088283	7.796252	1.588283	2.080138	5.107825	1.080138
	240	2.593596	5.817122	1.093596	1.461129	2.205485	0.461129
	480	2.331828	5.126737	0.831828	1.330378	1.643822	0.330378
	960	1.878588	2.540492	0.378588	1.099201	0.442613	0.099201
	1920	1.633484	0.701397	0.133484	1.057133	0.274805	0.057133
β	30	5.127208	14.088604	3.627208	3.193842	15.095538	2.193842
	60	3.849348	9.505456	2.349348	3.537458	11.132416	2.537458
	120	2.982137	5.577696	1.482137	2.003886	5.262557	1.003886
	240	2.543898	5.285319	1.043898	1.502349	3.489703	0.502349
	480	2.079795	3.328097	0.579795	1.257670	1.824428	0.257670
	960	1.732601	1.404187	0.232601	1.054419	0.375433	0.054419
	1920	1.600485	0.514960	0.100485	1.038597	0.218928	0.038597
μ	30	0.485765	0.072142	-0.014236	1.003237	0.085955	0.003237
	60	0.495944	0.057353	-0.004056	1.012316	0.070020	0.012316
	120	0.498752	0.046781	-0.001248	1.006380	0.061478	0.006380
	240	0.499040	0.041580	-0.000960	1.003917	0.041803	0.003917
	480	0.503818	0.025855	0.003818	1.005272	0.032034	0.005272
	960	0.501681	0.020089	0.001681	1.002184	0.019844	0.002184
	1920	0.501540	0.011683	0.001540	1.001741	0.013472	0.001741
σ	30	0.290333	0.693584	0.190333	0.301832	1.471941	0.201832
	60	0.237549	0.542196	0.137548	0.322692	0.971631	0.222692
	120	0.189905	0.331462	0.089905	0.189306	0.463920	0.089306
	240	0.165326	0.328750	0.065325	0.145136	0.305232	0.045136
	480	0.136807	0.209965	0.036807	0.124337	0.169925	0.024337
	960	0.115129	0.089107	0.015129	0.105341	0.034552	0.005341
	1920	0.106623	0.032778	0.006623	0.103721	0.020092	0.003721

Table 4: Monte Carlo Simulation Results for OGHLW-N Distribution: Mean, RMSE and Average Bias (Av. Bias).

		$\alpha=2.0,\beta=0.5,\mu=2.0,\sigma=0.4$			$\alpha=0.5,\beta=0.3,\mu=0.5,\sigma=0.1$		
Parameter	n	Mean	RMSE	Av. Bias	Mean	RMSE	Av. Bias
α	30	5.204730	14.927740	3.204730	0.642515	0.368557	0.142515
	60	4.474557	11.503408	2.474557	0.570638	0.214631	0.070638
	120	3.059420	7.110755	1.059420	0.552686	0.151035	0.052686
	240	2.595238	5.533821	0.595238	0.534412	0.107953	0.034412
	480	2.119376	0.852056	0.119376	0.539224	0.080650	0.039224
	960	1.999315	0.381351	-0.000685	0.530025	0.058517	0.030025
	1920	2.017128	0.250004	0.017128	0.474874	0.025126	-0.025126
β	30	1.777097	11.811309	1.277097	0.483465	1.376860	0.183465
	60	0.449129	0.273661	-0.050871	0.318877	0.179377	0.018877
	120	0.465599	0.206981	-0.034401	0.306559	0.140396	0.006559
	240	0.471970	0.154216	-0.028030	0.300277	0.093176	0.000277
	480	0.475200	0.113472	-0.024800	0.293343	0.072947	-0.006657
	960	0.479810	0.079473	-0.020190	0.282012	0.046941	-0.017988
	1920	0.483180	0.056793	-0.016820	0.286733	0.013267	-0.013267
μ	30	2.096376	0.524046	0.096376	0.535403	0.081549	0.035403
	60	2.094982	0.474805	0.094982	0.521969	0.054883	0.021969
	120	2.047267	0.336500	0.047267	0.519046	0.042421	0.019046
	240	2.018797	0.254908	0.018797	0.513182	0.029317	0.013182
	480	2.005962	0.137615	0.005962	0.515552	0.023221	0.015552
	960	1.992404	0.083035	-0.007596	0.510715	0.015819	0.010715
	1920	2.000390	0.056328	0.000390	0.495734	0.004266	-0.004266
σ	30	1.083896	6.345428	0.683896	0.124526	0.204818	0.024526
	60	0.379241	0.202829	-0.020759	0.099830	0.035445	-0.000170
	120	0.383363	0.150307	-0.016637	0.098023	0.028744	-0.001977
	240	0.385034	0.112777	-0.014966	0.098122	0.019431	-0.001878
	480	0.384296	0.080770	-0.015704	0.096130	0.015631	-0.003870
	960	0.385967	0.054994	-0.014033	0.094712	0.010495	-0.005288
	1920	0.388835	0.039230	-0.011165	0.096875	0.003125	-0.003125

8 Applications

We applied the OGHLW-W distribution to real data sets in order to assess the flexibility of the new family of distributions. Model performance was assessed by the use of goodness-of-fit statistics that includes: -2loglikelihood (-2ℓ), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), Cramer von Mises (W^*), Andersen-Darling (A^*), Kolmogorov Smirnov (KS) and sum of squares (SS) from probability plots.

We present two applications of the OGHLW-W distribution. We compared the new distribution with its nested models and non-nested models. We considered competing non-nested models, namely, exponentiated Weibull (EW), Beta-Weibull (BW) by Cordeiro et al. (2013), Kumaraswamy-Weibull (KwW) by Cordeiro et al. (2010), odd log-logistic exponentiated Weibull (OLLEW) by Afify et al. (2018) and odd exponentiated half logistic-Burr XII (OEHLBXII) by Aldahlan et al. (2018) distributions.

We estimated the model parameters using the subroutine NLMIXED in SAS and the function nlm in R. Parameter estimates (standard error in parenthesis) for the two data sets are given in Tables 5 and 8. Likelihood ratio (LR) test was used to compare the fit of the OGHLW-W distribution with it's sub-models for a given data set. Plots of the fitted densities, the histogram of the data and probability plots (Chambers et al., 1983) are given in Figures 4 and 5.

The goodness-of-fit statistics W^* and A^*, described by Chen and Balakrishnan (1985) are also presented in Tables 6 and 9. These statistics can be used to verify that the distribution in question gives a satisfactory fit to the data. In general, smaller values of W^*, A^*, KS and SS, indicate a better fit.

8.1 Kevlar 49/epoxy strands failure at 90% data

The data set consist of 101 observations representing the stress-rupture life of kevlar 49/epoxy strands which are subjected to constant sustained pressure at the 90% stress level until all had failed. The failure times are in hours and are shown below (see Andrews and Herzberg (2012) or Barlow et al. (1984), for details). 0.01, 0.01, 0.02, 0.02, 0.02, 0.03, 0.03, 0.04, 0.05, 0.06, 0.07, 0.07, 0.08, 0.09, 0.09, 0.10, 0.10, 0.11, 0.11, 0.12, 0.13, 0.18, 0.19, 0.20, 0.23, 0.24, 0.24, 0.29, 0.34, 0.35, 0.36, 0.38, 0.40, 0.42, 0.43, 0.52, 0.54, 0.56, 0.60, 0.60, 0.63, 0.65, 0.67, 0.68, 0.72, 0.72, 0.72, 0.73, 0.79, 0.79, 0.80, 0.80, 0.83, 0.85, 0.90, 0.92, 0.95, 0.99, 1.00, 1.01, 1.02, 1.03, 1.05, 1.10, 1.10, 1.11, 1.15, 1.18, 1.20, 1.29, 1.31, 1.33, 1.34, 1.40, 1.43, 1.45, 1.50, 1.51, 1.52, 1.53, 1.54, 1.54, 1.55, 1.58, 1.60, 1.63, 1.64, 1.80, 1.80, 1.81, 2.02, 2.05, 2.14, 2.17, 2.33, 3.03, 3.03, 3.34, 4.20, 4.69, 7.89.

Table 5: Parameter estimates for various models fitted for Kevlar data set.

Model	α	β	λ	γ
OGHLW-W	$4.9604e{-5}$	$5.9916e{-2}$	$1.7139e2$	$7.8775e{-2}$
	$(1.1862e{-4})$	$(1.3985e{-2})$	$(3.069e{-6})$	$(1.930e{-2})$
OGHLW-W$(1,\beta,\lambda,\gamma)$	1	$4.5354e{-3}$	$1.6542e2$	$5.6603e{-1}$
	-	$(3.1520e{-4})$	$(8.2054e{-9})$	$(4.3548e{-2})$
OGHLW-W$(\alpha,\beta,1,1)$	0.9361	0.4232	1	1
	(0.0999)	(0.0243)	-	-
OGHLW-W$(1,\beta,1,\gamma)$	1	1.7879	1	0.2265
	1	(0.1039)	1	(0.0215)
OGHLW-W$(\alpha,1,\lambda,1)$	1.6719	1	0.3980	1
	(0.2771)	-	(0.0272)	-
OGHLW-W$(1,1,\lambda,\gamma)$	1	1	1.0782	0.3696
	-	-	(0.0449)	(0.0292)
	α	β	δ	-
EW	4.4795	0.1966	33.3692	-
	(0.6721)	(0.0400)	(22.6988)	-
	a	b	λ	k
BW	0.7491	15.3390	0.0614	1.1079
	(0.3063)	$(4.5347e{-5})$	(0.0178)	(0.2942)
	a	b	α	β
KwW	0.7019	53.7926	0.0129	1.3065
	(0.7932)	(0.0018)	(0.0044)	(1.4548)
	α	β	γ	θ
OLLEW	1.6251	1.1121	0.6080	1.1686
	(2.6169)	(0.4425)	(0.9013)	(0.9465)
	α	λ	a	b
OEHLBXII	0.1451	0.7319	5.1144	0.2036
	(0.0421)	(0.8593)	(1.4689)	(0.0954)

Table 6: Goodness-of-fit statistics for various models fitted for Kevlar data set

Model	$-2\log L$	AIC	$CAIC$	BIC	W^*	A^*	KS	$P-value$	SS
OGHLW-W	205.9	213.9	214.4	224.4	0.1004	0.6903	0.0699	0.7058	0.1044
OGHLW-W$(1,\beta,\lambda,\gamma)$	336.3	342.3	342.6	350.2	0.2703	2.1685	0.4842	$< 2.2\times 10^{-16}$	7.6216
OGHLW-W$(\alpha,\beta,1,1)$	595.9	599.9	600.1	605.2	0.5117	3.4046	0.1676	0.0069	0.5039
OGHLW-W$(1,\beta,1,\gamma)$	292.2	296.2	296.3	301.4	0.1341	0.8296	0.3662	3.421×10^{-12}	5.3424
OGHLW-W$(\alpha,1,\lambda,1)$	649.2	653.2	653.3	658.4	0.3253	2.2920	0.2198	0.0001	2.5996
OGHLW-W$(1,1,\lambda,\gamma)$	349.6	353.6	353.7	358.8	0.1094	0.7359	0.2412	1.574×10^{-05}	2.0814
EW	239.3	245.3	245.5	253.1	0.7401	3.9791	0.1805	0.0027	0.7967
BW	205.5	213.5	213.9	223.9	0.1614	0.9412	0.0833	0.4845	0.1597
KwW	205.9	213.9	214.3	224.4	0.1947	1.0932	0.0905	0.3799	0.1891
OLLEW	205.5	213.5	213.9	223.9	0.1616	0.9432	0.0798	0.5407	0.1548
OEHLBXII	215.6	223.6	224.1	234.1	0.1636	0.9641	0.0964	0.3042	0.2291

The variance-covariance matrix is given by

$$\begin{bmatrix} 1.4070\times 10^{-08} & -1.6577\times 10^{-06} & -3.6132\times 10^{-10} & 2.1499\times 10^{-06} \\ -1.6577\times 10^{-06} & 1.9559\times 10^{-04} & 4.2546\times 10^{-08} & -2.5497\times 10^{-04} \\ -3.6132\times 10^{-10} & 4.2546\times 10^{-08} & 9.4185\times 10^{-12} & -5.2904\times 10^{-08} \\ 2.1499\times 10^{-06} & -2.5497\times 10^{-04} & -5.2904\times 10^{-08} & 3.7250\times 10^{-04} \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\alpha \in [4.9604\times 10^{-05} \pm 2.3248\times 10^{-04}]$, $\beta \in [5.9916\times 10^{-02} \pm 2.7411\times 10^{-02}]$, $\lambda \in [1.7139\times 10^{02} \pm 6.0152\times 10^{-06}]$ and $\gamma \in [2.1499\times 10^{-06} \pm 3.7829\times 10^{-02}]$.

8.1.1 Likelihood ratio test for Kevlar 49/epoxy strands failure at 90% data

We present results of the likelihood ratio (LR) test for testing if the OGHLW-W model for kevlar 49/Epoxy Strands Failure at 90% data is the best fit compared to its nested models. The results are shown in Table 7.

Table 7: Likelihood ratio test results for Kevlar data

Model	Chi-Square	df	p-value
OGHLW-W$(1,\beta,\lambda,\gamma)$	130.4	1	< 0.00001
OGHLW-W$(\alpha,\beta,1,1)$	390.0	2	< 0.00001
OGHLW-W$(1,\beta,1,\gamma)$	86.3	2	< 0.00001
OGHLW-W$(\alpha,1,\lambda,1)$	443.3	2	< 0.00001
OGHLW-W$(1,1,\lambda,\gamma)$	143.7	2	< 0.00001

From the results of the likelihood ratio test, we conclude that there is a significant difference between the OGHLW-W model and its nested models for the kevlar 49/Epoxy strands failure at 90% data. The OGHLW-W model performs better than the non-nested EW, BW, KwW, OLLEW and OEHLBXII distributions based on the goodness-of-fit statistics (W^*, A^*, KS and its P-value) presented in Table 6. Also, we can conclude from Figure 4 that the OGHLW-W model fit the kevlar data better than the nested models.

8.2 Breaking stress of carbon fibres of 50 mm length data

The second data set is on breaking stress of carbon fibres of 50 mm length (GPa). Cordeiro and Lemonte (2011) also analyzed the same data set. The data are as follows:

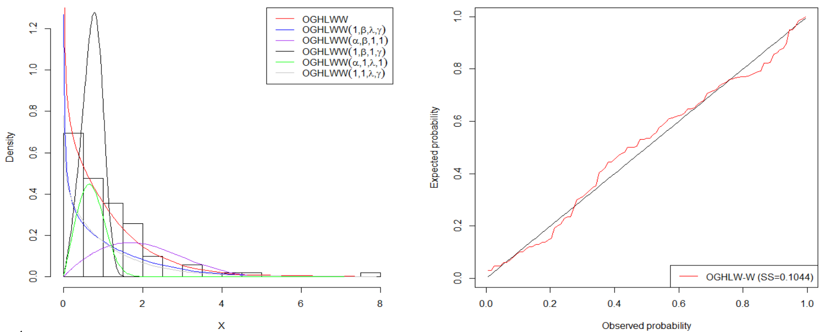

Figure 4: Fitted densities (left) and probability plots (right) for OGHLW-W distribution on kevlar data set

0.39, 0.85, 1.08 ,1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09 ,3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90.

Table 8 show the parameter estimates (standard errors in parentheses) for the OGHLW-W model on carbon fibres data set. Goodness-of-fit statistics for the carbon data set are presented in Table 9.

Table 8: Parameter estimates for various models fitted for carbon fibres data set

Model	α	β	λ	γ
OGHLW-W	$2.0903e{-}5$	0.4108	20.0470	0.2686
	$(9.6968e{-}6)$	$(9.5277e{-}3)$	$(1.9100e{-}4)$	$(1.3083e{-}2)$
OGHLW-W$(\alpha, \beta, \lambda, 1)$	$6.7703e^{-2}$	$4.7190e^{-2}$	20.5840	1
	$(2.3071e{-}2)$	$(4.3646e{-}3)$	$(9.9124e{-}6)$	
OGHLW-W$(\alpha, \beta, 1, 1)$	0.0697	0.9731	1	1
	(0.0236)	(0.0895)	-	-
OGHLW-W$(\alpha, 1, \lambda, 1)$	0.0579	1	1.0230	1
	(0.0209)	-	(0.0912)	-
OGHLW-W$(1, 1, \lambda, \gamma)$	1	1	0.2319	1.3716
	-	-	(0.0412)	(0.1337)
	α	β	δ	-
EW	1.3185	0.9022	14.2572	-
	(0.4391)	(0.1896)	(7.3551)	-
	a	b	α	β
KwW	130.98	5666.1	645.16	0.1336
	$(2.3367e{-}07)$	$(6.4716e{-}10)$	$(1.8198e{-}08)$	$(6.7393e{-}04)$
	α	λ	a	b
OEHLBXII	0.2814	0.0001	7.9391	0.8501
	(0.0811)	(0.0003)	(0.0169)	(0.1948)

Table 9: Goodness-of-fit statistics for various models fitted for carbon fibres data set

Model	$-2\log L$	AIC	$CAIC$	BIC	W^*	A^*	KS	$P-value$	SS
OGHLW-W	171.4	179.4	180.1	188.2	0.0644	0.4438	0.0882	0.6827	0.0678
OGHLW-W$(\alpha,\beta,\lambda,1)$	181.6	187.6	188.0	194.2	0.1525	1.1182	0.1115	0.3840	0.1483
OGHLW-W$(\alpha,\beta,1,1)$	192.3	196.3	196.5	200.6	0.1322	0.9494	0.1109	0.3908	0.1203
OGHLW-W$(\alpha,1,\lambda,1)$	192.3	196.3	196.5	200.7	0.1445	1.0301	0.1168	0.3292	0.1487
OGHLW-W$(1,1,\lambda,\gamma)$	265.7	269.7	269.9	274.1	0.0720	0.4874	0.2537	0.0004	1.4278
EW	203.2	209.2	209.6	215.8	0.3958	2.1888	0.1794	0.0286	0.4827
KwW	174.8	182.8	183.5	191.6	0.1349	0.7297	0.0974	0.5578	0.1177
OEHLBXII	194.4	202.4	203.1	211.2	0.1948	1.2538	0.1613	0.0646	0.3210

The variance-covariance matrix for carbon fibres model is given by

$$\begin{bmatrix} 9.402730\times10^{-11} & -9.007179\times10^{-08} & -1.805615\times10^{-09} & -1.236798\times10^{-07} \\ -9.007179\times10^{-08} & 9.077654\times10^{-05} & 1.819742\times10^{-06} & 1.246475\times10^{-04} \\ -1.805615\times10^{-09} & 1.819742\times10^{-06} & 3.647927\times10^{-08} & 2.498733\times10^{-06} \\ -1.236798\times10^{-07} & 1.246475\times10^{-04} & 2.498733\times10^{-06} & 1.711566\times10^{-04} \end{bmatrix}$$

and the 95% confidence intervals for the parameters are given by $\alpha \in [2.0903\times10^{-05} \pm 1.9006\times10^{-05}]$, $\beta \in [0.4108 \pm 1.8674\times10^{-02}]$, $\lambda \in [20.0470 \pm 3.7435\times10^{-04}]$ and $\gamma \in [0.2686 \pm 2.5642\times10^{-02}]$.

8.2.1 Likelihood ratio test for carbon fibres data

Likelihood ratio test results to assess if the the OGHLW-W model fit the carbon fibres data better than its nested models are shown in Table 10. We conclude that our new model fit the carbon fibres data better than its nested models. Also, using results from Table 9, we conclude that the proposed new model performs better than the competing non-nested models EW, KwW and OEHLBXII on carbon fibres data set. Furthermore, we conclude from Figure 5 that the OGHLW-W model fit the carbon fibres data better than the nested models.

Table 10: Likelihood ratio test results for carbon fibres data

Model	Chi-Square	df	p-value
OGHLW-W$(\alpha,\beta,\lambda,1)$	10.2	1	0.00140
OGHLW-W$(\alpha,\beta,1,1)$	20.9	2	0.00003
OGHLW-W$(\alpha,1,\lambda,1)$	20.9	2	0.00003
OGHLW-W$(1,1,\lambda,\gamma)$	94.3	2	< 0.00001

9 Concluding Remarks

We have developed a new family of distributions called the odd generalized half-logistic Weibull-G (OGHLW-G) family of distributions and three special cases, OGHLW-U, OGHLW-W and OGHLW-N distributions. These distributions have interesting hazard rate function shapes which includes unimodal, bathtub and upside down bathtub. We also derived some structural properties of the new family of distributions. A characterization based on the conditional expectation was also given. The model was applied

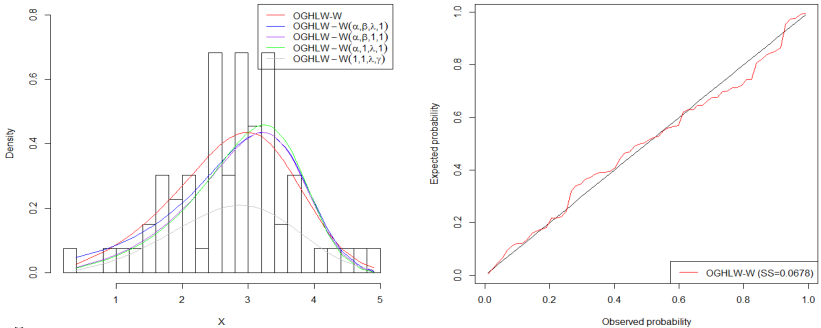

Figure 5: Fitted densities (left) and probability plots for OGHLW-W distribution on carbon fibres data set

to a real data set in order to illustrate the applicability and usefulness of the proposed family of distributions. The OGHLW-W distribution was compared to several non-nested models.

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