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Research Paper

A new power generalized Weibull-G family of distributions: Properties and applications

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Abstract: A new family of continuous distributions namely new power generalized Weibull-G family of distributions is proposed. Some special sub-models of the new family are provided. Some statistical properties of the new family of distributions are obtained including the quantile function, ordinary and incomplete moments, probability weighted moments, distribution of the order statistics and Rényi entropy. The maximum likelihood method is used for estimating model parameters. A simulation study is employed to check the consistency of the maximum likelihood estimates. The flexibility of a sub-model of the generated family is illustrated by means of two applications to real data sets.

Keywords: Generalized distribution; Maximum likelihood estimation; Power generalized Weibull distribution.
Mathematics Subject Classification (2010): 60E05; 62E15; 62F30.

1 Introduction

The limitations of standard distributions in statistical modeling has led to extension and generalization of the well-known classical distributions by statisticians to gain flexibility. There are several proposed methods for generating new distributions with desirable properties in the literature. Some notable examples include the transformed-transformer (T-X) by Alzaghal et al. (2013), the gamma-G by Zografos and Balakrishnan (2009), the Kumaraswamy-G by Cordeiro and de Castro (2011), McDonald-G (Mc-G) by Alexander et al. (2012), the exponentiated generalized (EG) class of distributions by Cordeiro et al. (2013), Kumaraswamy Marshall-Olkin family by Alizadeh et al. (2015), generalized odd log-logistic-G by Cordeiro et al. (2017), the beta Odd Lindley-G by Chipepa et al. (2019a), generalized transmuted Poisson-G by Yousof et al. (2018), the transmuted Gompertz-G by Reyad et al. (2018), the Kumaraswamy

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odd Lindley-G by Chipepa et al. (2019b), Marshall-Olkin alpha power-G by Nassar et al. (2019), the Nadarajah Haghighi Topp Leone-G family of distributions Reyad et al. (2019); among others.

Bagdonavicius and Nikulin (2002) developed an extension of the Weibull distribution namely, power generalized Weibull (PGW) distribution. Lai (2013) described the PGW as one of the extensions of Weibull distribution that can exhibit non-monotonic hazard rates. Nikulin and Haghighi (2009) obtained maximum likelihood estimates (MLEs) of the parameters and the importance of the model was illustrated using Efron's head-and-neck cancer clinical trial data (Efron, 1988) .

The new model is versatile and flexible because it applies to data sets of varying skewness and kurtosis. Moreso, the proposed distribution exhibits both monotonic and non-monotonic shapes for the hazard rate function. The distribution is not in closed form but it has a desirable property that it can be expressed as an infinite linear combination of the exponentiated-G distribution, which makes the derivation of statistical properties of this new distribution amenable. Furthermore, the new distribution is a family of distributions that contains many known and new sub-models.

The rest of the paper is organized as follows: Section 2 contain the new power generalized Weibull (NPGW-G) family of distributions and its sub-families, the density expansion, hazard function and the quantile function. Section 3 includes the special cases of (NPGW-G) family of distributions. In Section 4, we obtain the distribution of order statistics and Rényi entropy. Some basic mathematical quantities for the new family of distributions including moments and generating function, incomplete and probability weighted moments are determined in Section 5. Section 6 contain the estimation of the parameters of the NPGW-G family of distributions via the method of maximum likelihood, followed by a Monte Carlo simulation study to examine accuracy of the maximum likelihood estimates in Section 7. Some applications to real data sets are given in Section 8, followed by some concluding remarks in section 9.

2 The new model

The cumulative distribution function (cdf) and probability density function (pdf) of the power generalized Weibull (PGW) distribution, for $\alpha, \beta > 0$, repectively, are given by

$$\begin{aligned} F(t;\alpha,\beta) &= 1-\exp\left(1-[1+t^{\alpha}]^{\beta}\right), \\ f(t;\alpha,\beta) &= \alpha\beta t^{\alpha-1}(1+t^{\alpha})^{\beta-1}\exp\left(1-(1+t^{\alpha})^{\beta}\right), \end{aligned}$$

Based on the idea of T-X family of distributions by Alzaghal et al. (2013), and the PGW distribution, we define the cdf and pdf of the NPGW-G family of distributions by

$$\begin{aligned} F(x;\alpha,\beta,\xi) &= 1-\int_0^{-\log(G(x;\xi))}\alpha\beta t^{\alpha-1}(1+t^{\alpha})^{\beta-1}\exp\left(1-(1+t^{\alpha})^{\beta}\right)dt \\ &= \exp\left(1-[1+(-\log(G(x;\xi)))^{\alpha}]^{\beta}\right), \qquad (1) \\ f(x;\alpha,\beta,\xi) &= \alpha\beta\left[1+(-\log(G(x;\xi)))^{\alpha}\right]^{\beta-1}(-\log(G(x;\xi)))^{\alpha-1} \end{aligned}$$

$$\times \exp\left(1-[1+(-\log(G(x;\xi)))^{\alpha}]^{\beta}\right)\frac{g(x;\xi)}{G(x;\xi)}, \quad (2)$$

respectively, for $\alpha, \beta > 0$ and parameter vector ξ. A random variable X with cdf (1) is denoted by $X \sim NPGW - G(\alpha, \beta, \xi)$.

2.1 Linear representation

In this section, we obtain the NPGW-G density function as linear representation of the exponentiated-G distribution. By letting $y = 1 - G(x;\xi)$, and applying the following series expansions

$$e^z = \sum_{k=0}^{\infty}\frac{z^k}{k!},$$

$$(1-z)^{k-1} = \sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(k)}{\Gamma(k-j)\Gamma(j+1)}z^j,$$

$$(1+z)^{-(k+1)} = \sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(k+j+1)}{\Gamma(k+1)\Gamma(j+1)}z^j, \quad \text{for } |z|<1 \text{ and } k>0,$$

$$\Big[-\log(1-y)\Big]^{\delta-1} = y^{\delta-1}\Big[\sum_{m=0}^{\infty}\binom{\delta-1}{m}y^m\Big(\sum_{s=0}^{\infty}\frac{y^s}{s+2}\Big)^m\Big].$$

Now, applying the result on power series raised to a positive integer, with $a_s = (s+2)^{-1}$, that is,

$$\left(\sum_{s=0}^{\infty}a_s y^s\right)^m = \sum_{s=0}^{\infty}b_{s,m}y^s,$$

where $b_{s,m} = (sa_0)^{-1}\sum_{l=1}^{s}[m(l+1)-s]a_l b_{s-l,m}$, and $b_{0,m} = a_0^m$, (Gradshteyn and Ryzhik, 2000), we have

$$\begin{aligned}
f(x;\alpha,\beta,\xi) &= \alpha\beta\sum_{i,l=0}^{\infty}\binom{\beta-1}{l}(-\log(G(x;\xi)))^{\alpha(l+1)-1}\frac{g(x;\xi)}{G(x;\xi)i!}\\
&\quad\times\left(1-[1+(-\log(G(x;\xi)))^{\alpha}]^{\beta}\right)^i\\
&= \alpha\beta\sum_{i,j,l=0}^{\infty}\binom{\beta-1}{l}\binom{i}{j}\frac{(-1)^j}{i!}(-\log(G(x;\xi)))^{\alpha(l+1)-1}\\
&\quad\times[1+(-\log(G(x;\xi)))^{\alpha}]^{\beta j}\frac{g(x;\xi)}{G(x;\xi)}\\
&= \alpha\beta\sum_{i,j,l,p=0}^{\infty}\binom{\beta-1}{l}\binom{i}{j}\binom{\beta j}{p}\frac{(-1)^j}{i!}(-\log(G(x;\xi)))^{\alpha(l+p+1)-1}\\
&\quad\times\frac{g(x;\xi)}{G(x;\xi)}
\end{aligned}$$

$$
\begin{aligned}
&= \alpha\beta \sum_{i,j,l,p=0}^{\infty} \binom{\beta-1}{l}\binom{i}{j}\binom{\beta j}{p}\frac{(-1)^j}{i!} \\
&\quad \times (-\log(1-(1-G(x;\xi))))^{\alpha(l+p+1)-1}\,\frac{g(x;\xi)}{G(x;\xi)} \\
&= \alpha\beta \sum_{i,j,l,p=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s=0}^{\infty} b_{s,m}\binom{\alpha(l+p+1)-1}{m}\binom{\beta-1}{l}\binom{i}{j}\binom{\beta j}{p} \\
&\quad \times \frac{(-1)^j}{i!}\,((1-G(x;\xi))^{\alpha(l+p+1)+s+m-1}\,\frac{g(x;\xi)}{G(x;\xi)} \\
&= \alpha\beta \sum_{i,j,l,p,q=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s=0}^{\infty} b_{s,m}\binom{\alpha(l+p+1)-1}{m}\binom{\beta-1}{l}\binom{i}{j}\binom{\beta j}{p} \\
&\quad \times \binom{\alpha(l+p+1)+s+m-1}{q}\frac{(-1)^j}{i!}\,(G(x;\xi))^{q-1}\,g(x;\xi) \\
&= \alpha\beta \sum_{i,j,l,p,q=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s=0}^{\infty} b_{s,m}\binom{\alpha(l+p+1)-1}{m}\binom{\beta-1}{l}\binom{i}{j}\binom{\beta j}{p} \\
&\quad \times \binom{\alpha(l+p+1)+s+m-1}{q}\frac{(-1)^{j+q}q}{i!q}\,(G(x;\xi))^{q-1}\,g(x;\xi) \\
&= \alpha\beta \sum_{i,j,l,p,q=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s=0}^{\infty} b_{s,m}\binom{\alpha(l+p+1)-1}{m}\binom{\beta-1}{l}\binom{i}{j}\binom{\beta j}{p} \\
&\quad \times \binom{\alpha(l+p+1)+s+m-1}{q}\frac{(-1)^{j+q}}{i!q}g_{q}^{*}(x;\xi) \\
&= \sum_{q=0}^{\infty} b_{q}\, g_{q}^{*}(x;\xi), \qquad (3)
\end{aligned}
$$

where $g_{q}^{*}(x;\xi) = q[G(x;\xi)]^{q-1}g(x;\xi)$ is the exponentiated-G (E-G) pdf with the power parameter $q > 0$ and parameter vector ξ, and

$$
\begin{aligned}
b_{q} &= \alpha\beta \sum_{i,j,l,p=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s=0}^{\infty} b_{s,m}\binom{\alpha(l+p+1)-1}{m}\binom{\beta-1}{l}\binom{i}{j}\binom{\beta j}{p} \\
&\quad \times \binom{\alpha(l+p+1)+s+m-1}{q}\frac{(-1)^{j+q}}{i!q}. \qquad (4)
\end{aligned}
$$

Consequently, the mathematical and statistical properties of the NPGW-G family of distributions follow directly from those of the exponentiated-G (E-G) distribution.

2.2 Sub-families of NPGW-G family of distributions

Some new and known sub-families are presented in this subsection.

- When $\beta = 1$, we obtain the new Weibull-G (NW-G) family of distributions (Tahir et al., 2016) with the cdf

$$F(x;\alpha,\xi) = \exp\left(-\left(-\log\left(G(x;\xi)\right)\right)^{\alpha}\right), \quad \alpha > 0.$$

- If $\alpha = 1$, we obtain the new Nadarajah Haghighi-G (NNH-G) family of distributions with the cdf

$$F(x;\beta,\xi) = \exp\left(\left(1 - \left[1 + \left(-\log\left(G(x;\xi)\right)\right)\right]^{\beta}\right)\right), \ \beta > 0.$$

This is a new family of distributions.

- If $\alpha = \beta = 1$, we obtain the baseline distribution function, that is

$$F(x;\xi) = G(x;\xi).$$

- If $\beta = 1, \alpha = 2$ we obtain the new Rayleigh-G (NR-G) family of distributions with the cdf

$$F(x;\xi) = \exp\left(-\left(-\log\left(G(x;\xi)\right)\right)^{2}\right).$$

2.3 Hazard and quantile functions

Presented below are the hazard and quantile functions of the NPGW-G family of distributions. The hazard rate function of the NPGW-G family is given by

$$\begin{aligned} h_F(x;\alpha,\beta,\xi) &= \frac{f(x;\alpha,\beta,\xi)}{\overline{F}(x;\alpha,\beta,\xi)} \\ &= \alpha\beta\left[1 + \left(-\log\left(G(x;\xi)\right)\right)^{\alpha}\right]^{\beta-1}\left(-\log\left(G(x;\xi)\right)\right)^{\alpha-1} \\ &\quad \times \exp\left(1 - \left[1 + \left(-\log\left(G(x;\xi)\right)\right)^{\alpha}\right]^{\beta}\right)\frac{g(x;\xi)}{G(x;\xi)} \\ &\quad \times \left(1 - \left(1 - \left[1 + \left(-\log\left(G(x;\xi)\right)\right)^{\alpha}\right]^{\beta}\right)\right)^{-1}. \end{aligned}$$

The quantile function of the NPGW-G family of distributions is obtained by solving the non-linear equation:

$$F(x;\alpha,\beta,\xi) = \exp\left(1 - \left[1 + \left(-\log\left(G(x;\xi)\right)\right)^{\alpha}\right]^{\beta}\right) = u,$$

for $0 \leq u \leq 1$, that is,

$$\begin{aligned} 1 - \log(u) &= \left[1 + \left(-\log\left(G(x;\xi)\right)\right)^{\alpha}\right]^{\beta} \\ \left(1 - \log(u)\right)^{\frac{1}{\beta}} - 1 &= \left(-\log\left(G(x;\xi)\right)\right)^{\alpha} \\ G(x;\xi) &= \exp\left(-\left(\left(1 - \log(u)\right)^{\frac{1}{\beta}} - 1\right)^{\frac{1}{\alpha}}\right). \end{aligned}$$

Consequently, the quantile function for the NPGW-G family of distributions is given by

$$Q_G(u;\alpha,\beta,\xi) = X = G^{-1}\left[\exp\left(-\left(\left(1 - \log(u)\right)^{\frac{1}{\beta}} - 1\right)^{\frac{1}{\alpha}}\right)\right]. \tag{5}$$

It follows therefore that random numbers can be generated from the NPGW-G family of distributions based on equation (5).

3 Some sub-models

In this section, we introduce four sub-models of the NPGW-G family of distributions.

3.1 NPGW-Lindley distribution

Suppose the cdf and pdf of the baseline distribution are given by $G(x;\lambda) = 1 - \frac{1+\lambda x}{1+\lambda}\exp(-\lambda x)$ and $g(x;\lambda) = \frac{\lambda^2(1+x)\exp(-\lambda x)}{1+\lambda}$, for $\lambda > 0$, and $x > 0$. Then, the cdf and pdf of NPGW-Lindley (NPGW-L) distribution are given by

$$\begin{aligned} F(x;\alpha,\beta,\lambda) &= \exp\left(1-\left[1+\left(-\log\left(1-\frac{1+\lambda x}{1+\lambda}\exp(-\lambda x)\right)\right)^{\alpha}\right]^{\beta}\right), \\ f(x;\alpha,\beta,\lambda) &= \alpha\beta\left[1+\left(-\log\left(1-\frac{1+\lambda x}{1+\lambda}\exp(-\lambda x)\right)\right)^{\alpha}\right]^{\beta-1} \\ &\quad\times\exp\left(1-\left[1+\left(-\log\left(1-\frac{1+\lambda x}{1+\lambda}\exp(-\lambda x)\right)\right)^{\alpha}\right]^{\beta}\right) \\ &\quad\times\left(-\log\left(1-\frac{1+\lambda x}{1+\lambda}\exp(-\lambda x)\right)\right)^{\alpha-1}\frac{\frac{\lambda^2(1+x)\exp(-\lambda x)}{1+\lambda}}{1-\frac{1+\lambda x}{1+\lambda}\exp(-\lambda x)}, \end{aligned}$$

respectively, for $\alpha, \beta, \lambda > 0$. The hazard rate function is given by

$$\begin{aligned} h_F(x;\alpha,\beta,\lambda) &= \alpha\beta\left[1+\left(-\log\left(1-\frac{1+\lambda x}{1+\lambda}\exp(-\lambda x)\right)\right)^{\alpha}\right]^{\beta-1} \\ &\quad\times\exp\left(1-\left[1+\left(-\log\left(1-\frac{1+\lambda x}{1+\lambda}\exp(-\lambda x)\right)\right)^{\alpha}\right]^{\beta}\right) \\ &\quad\times\left(-\log\left(1-\frac{1+\lambda x}{1+\lambda}\exp(-\lambda x)\right)\right)^{\alpha-1}\frac{\frac{\lambda^2(1+x)\exp(-\lambda x)}{1+\lambda}}{1-\frac{1+\lambda x}{1+\lambda}\exp(-\lambda x)} \\ &\quad\times\left(1-\exp\left(1-\left[1+\left(-\log\left(1-\frac{1+\lambda x}{1+\lambda}\exp(-\lambda x)\right)\right)^{\alpha}\right]^{\beta}\right)\right)^{-1}. \end{aligned}$$

Figure 1 shows the plots of pdf and hazard functions of NPGW-Lindley distribution, respectively. The pdf can take several shapes including increasing, right skewed, left skewed, unimodal and reverse-J shapes. The NPGW-Lindley hazard displays increasing, reverse-J, bathtub and upside-down bathtub shapes.

3.2 NPGW-power distribution

The NPGW-Power (NPGW-P) distribution is defined by taking the baseline distribution to be the power distribution with the cdf and pdf given by $G(x;\theta,k) = (\theta x)^k$ and $g(x;\theta,k) = k\theta^k x^{k-1}$, for $\theta, k > 0$, and $x \in (0, \frac{1}{\theta})$. It's cdf and pdf are given by

$$F(x;\alpha,\beta,\theta,k) = \exp\left(1-\left[1+\left(-\log\left((\theta x)^k\right)\right)^{\alpha}\right]^{\beta}\right),$$

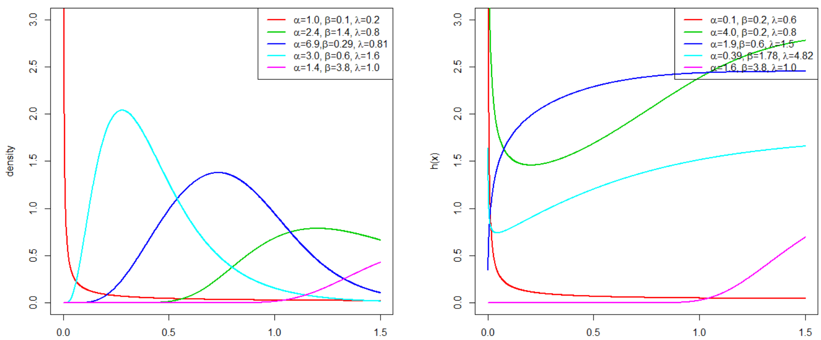

Figure 1: Density and hazard function plots for NPGW-Lindley distribution.

Table 1: The quantile for NPGW-Lindley distribution

	(α,β,λ)				
u	$(0.2,1.2,0.6)$	$(0.4,1.0,0.5)$	$(0.6,0.8,1.0)$	$(0.8,1.0,1.0)$	$(0.5,1.0,0.4)$
0.1	0.0062	0.0051	0.0072	0.1190	0.0408
0.2	0.2748	0.2157	0.0367	0.3324	0.5763
0.3	2.1089	1.0620	0.1853	0.5946	1.6193
0.4	4.5625	2.3410	0.4798	0.8936	2.9349
0.5	7.1917	3.8786	0.9023	1.2424	4.4650
0.6	10.0333	5.6631	1.4475	1.6482	6.2352
0.7	13.3186	7.7901	2.1440	2.1556	8.36401
0.8	17.5383	10.5367	3.0779	2.8324	11.1390
0.9	24.1927	14.8186	4.5622	3.9295	15.4940

and

$$\begin{aligned} f(x;\alpha,\beta,\theta,k) &= \alpha\beta\left[1+\left(-\log\left((\theta x)^k\right)\right)^{\alpha}\right]^{\beta-1}\left(-\log\left((\theta x)^k\right)\right)^{\alpha-1} \\ &\quad \times\exp\left(1-\left[1+\left(-\log\left((\theta x)^k\right)\right)^{\alpha}\right]^{\beta}\right)\frac{k\theta^k x^{k-1}}{(\theta x)^k}, \end{aligned}$$

respectively, for $\alpha,\beta,\theta,k>0$. The hazard rate function is given by

$$\begin{aligned} h_F(x;\alpha,\beta,\theta,k) &= \alpha\beta\left[1+\left(-\log\left((\theta x)^k\right)\right)^{\alpha}\right]^{\beta-1}\left(-\log\left((\theta x)^k\right)\right)^{\alpha-1} \\ &\quad \times\exp\left(1-\left[1+\left(-\log\left((\theta x)^k\right)\right)^{\alpha}\right]^{\beta}\right)\frac{k\theta^k x^{k-1}}{(\theta x)^k} \\ &\quad \times\left(1-\exp\left(1-\left[1+\left(-\log\left((\theta x)^k\right)\right)^{\alpha}\right]^{\beta}\right)\right)^{-1}. \end{aligned}$$

Figure 2 shows the plots of pdf and hazard functions of NPGW-Power distribution, respectively. The pdf can take several shapes including increasing, left skewed, right skewed, unimodal and reverse-J shapes. The NPGW-Power hazard displays increasing, reverse-J, bathtub, upside-down bathtub followed by bathtub and upside-down bathtub shapes.

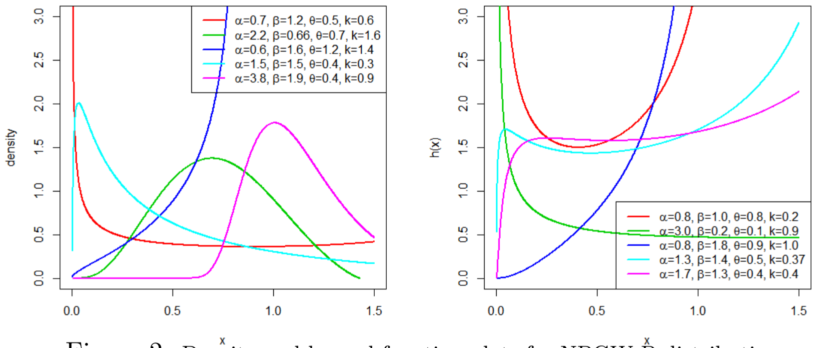

Figure 2: Density and hazard function plots for NPGW-P distribution.

Table 2: The quantile for NPGW-power distribution

	(α,β,θ,k)				
u	$(.29,1.7,1,.7)$	$(.5,.9,1,1.2)$	$(.9,1,.96,.8)$	$(1.4,2,1,.3)$	$(1,.9,1.2,.6)$
0.1	0.2171	0.0053	0.0450	0.0533	0.0078
0.2	0.5770	0.0479	0.1257	0.0937	0.0359
0.3	0.7906	0.1910	0.2248	0.1348	0.0811
0.4	0.9026	0.3900	0.3355	0.1861	0.1412
0.5	0.9560	0.5904	0.4535	0.2417	0.2218
0.6	0.9812	0.7542	0.5759	0.3124	0.3161
0.7	0.9919	0.8740	0.6978	0.4001	0.4260
0.8	0.9949	0.9503	0.8236	0.5112	0.5486
0.9	1.0000	0.9892	0.9423	0.6715	0.6848

3.3 NPGW-Weibull distribution

Let $G(x;\xi)$ be the Weibull distribution with cdf and pdf given by $G(x;\lambda,\theta) = 1 - \exp\left(-(\lambda x)^\theta\right)$ and $g(x;\lambda,\theta) = \theta\lambda^\theta x^{\theta-1}\exp\left(-(\lambda x)^\theta\right)$, for $\lambda,\theta,x>0$. Then, the NPGW-Weibull (NPGW-W) distribution has cdf and pdf given by

$$\begin{aligned}
F(x;\alpha,\beta,\lambda,\theta) &= \exp\left(1-\left[1+\left(-\log\left(1-\exp\left(-(\lambda x)^\theta\right)\right)\right)^\alpha\right]^\beta\right),\\
f(x;\alpha,\beta,\lambda,\theta) &= \alpha\beta\left[1+\left(-\log\left(1-\exp\left(-(\lambda x)^\theta\right)\right)\right)^\alpha\right]^{\beta-1}\\
&\quad\times\exp\left(1-\left[1+\left(-\log\left(1-\exp\left(-(\lambda x)^\theta\right)\right)\right)^\alpha\right]^\beta\right)\\
&\quad\times\left(-\log\left(1-\exp\left(-(\lambda x)^\theta\right)\right)\right)^{\alpha-1}\frac{\theta\lambda^\theta x^{\theta-1}\exp\left(-(\lambda x)^\theta\right)}{1-\exp\left(-(\lambda x)^\theta\right)},
\end{aligned}$$

respectively, for $\alpha,\beta,\lambda,\theta>0$. The hazard rate function is given by

$$h_F(x;\alpha,\beta,\lambda,\theta) = \alpha\beta\left[1+\left(-\log\left(1-\exp\left(-(\lambda x)^\theta\right)\right)\right)^\alpha\right]^{\beta-1}$$

$$\times \exp\left(1-\left[1+\left(-\log\left(1-\exp\left(-(\lambda x)^{\theta}\right)\right)\right)^{\alpha}\right]^{\beta}\right)$$
$$\times \left(-\log\left(1-\exp\left(-(\lambda x)^{\theta}\right)\right)\right)^{\alpha-1}\frac{\theta\lambda^{\theta}x^{\theta-1}\exp\left(-(\lambda x)^{\theta}\right)}{1-\exp\left(-(\lambda x)^{\theta}\right)}$$
$$\times \left(1-\exp\left(1-\left[1+\left(-\log\left(1-\exp\left(-(\lambda x)^{\theta}\right)\right)\right)^{\alpha}\right]^{\beta}\right)\right)^{-1}.$$

For $\theta = 1$ and $\theta = 2$ we obtain the NPGW-exponential (NPGW-E) and NPGW-Rayleigh (NPGW-R) distributions, respectively.

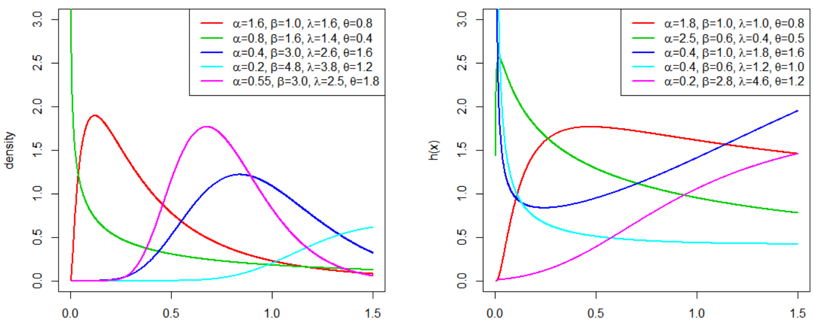

Figure 3: Density and hazard function plots for NPGW-W distribution

Figure 3 shows the plots of pdf and hazard functions of NPGW-Weibull distribution, respectively. The pdf can take several shapes including increasing, left skewed, right skewed, unimodal and reverse-J shapes. The NPGW-Weibull hazard displays increasing, decreasing, bathtub and upside-down bathtub shapes.

Table 3: The quantile for NPGW-Weibull distribution

	$(\alpha,\beta,\lambda,\theta)$				
u	$(1.2, 1.2, 1, 1.8)$	$(.2, 1.5, 2.2, 2)$	$(.8, 1.2, 2, 1.0)$	$(2, 1, 1, 2.2)$	$(1, 1.8, 2.6, 2)$
0.1	0.4490	0.1223	0.0742	0.5291	0.2704
0.2	0.5703	0.4133	0.1614	0.6033	0.3186
0.3	0.6753	0.6281	0.2551	0.6637	0.3555
0.4	0.7739	0.7992	0.3603	0.7193	0.3925
0.5	0.8728	0.9465	0.4862	0.7755	0.4279
0.6	0.9837	1.0907	0.6359	0.8361	0.4685
0.7	1.1093	1.2400	0.8266	0.9035	0.5129
0.8	1.2683	1.4170	1.0917	0.9873	0.5704
0.9	1.5096	1.6619	1.5368	1.1210	0.6552

3.4 NPGW-log-logistic distribution

Suppose a random variable X follows the log-logistic distribution with the cdf and pdf given by $G(x;s,c) = 1-\left(1+\left(\frac{x}{s}\right)^{c}\right)^{-1}$ and $g(x;s,c) = cs^{-c}x^{c-1}\left(1+\left(\frac{x}{s}\right)^{c}\right)^{-2}$, for

$s, c > 0$, and $x > 0$. Then, the NPGW-log-logistic (NPGW-LLoG) distribution has cdf and pdf given by

$$\begin{aligned}
F(x;\alpha,\beta,s,c) &= \exp\left(1-\left[1+\left(-\log\left(1-\left(1+\left(\frac{x}{s}\right)^{c}\right)^{-1}\right)\right)^{\alpha}\right]^{\beta}\right),\\
f(x;\alpha,\beta,s,c) &= \alpha\beta\left[1+\left(-\log\left(1-\left(1+\left(\frac{x}{s}\right)^{c}\right)^{-1}\right)\right)^{\alpha}\right]^{\beta-1}\\
&\quad\times\left(-\log\left(1-\left(1+\left(\frac{x}{s}\right)^{c}\right)^{-1}\right)\right)^{\alpha-1}\\
&\quad\times\exp\left(1-\left[1+\left(-\log\left(1-\left(1+\left(\frac{x}{s}\right)^{c}\right)^{-1}\right)\right)^{\alpha}\right]^{\beta}\right)\\
&\quad\times\frac{cs^{-c}x^{c-1}\left(1+\left(\frac{x}{s}\right)^{c}\right)^{-2}}{1-\left(1+\left(\frac{x}{s}\right)^{c}\right)^{-1}},
\end{aligned}$$

respectively, for $\alpha, \beta, s, c > 0$. The hazard rate function is given by

$$\begin{aligned}
h_F(x;\alpha,\beta,s,c) &= \alpha\beta\left[1+\left(-\log\left(1-\left(1+\left(\frac{x}{s}\right)^{c}\right)^{-1}\right)\right)^{\alpha}\right]^{\beta-1}\\
&\quad\times\left(-\log\left(1-\left(1+\left(\frac{x}{s}\right)^{c}\right)^{-1}\right)\right)^{\alpha-1}\\
&\quad\times\exp\left(1-\left[1+\left(-\log\left(1-\left(1+\left(\frac{x}{s}\right)^{c}\right)^{-1}\right)\right)^{\alpha}\right]^{\beta}\right)\\
&\quad\times\frac{cs^{-c}x^{c-1}\left(1+\left(\frac{x}{s}\right)^{c}\right)^{-2}}{1-\left(1+\left(\frac{x}{s}\right)^{c}\right)^{-1}}\\
&\quad\times\left(1-\exp\left(1-\left[1+\left(-\log\left(1-\left(1+\left(\frac{x}{s}\right)^{c}\right)^{-1}\right)\right)^{\alpha}\right]^{\beta}\right)\right)^{-1}.
\end{aligned}$$

Figure 4 shows the plots of pdf and hazard functions of NPGW-log-logistic distribution, respectively. The pdf can take several shapes including increasing, right skewed, left skewed, unimodal and reverse-J shapes. The NPGW-log-logistic hazard displays increasing, reverse-J, bathtub, bathtub followed by upside-down bathtub and upside-down bathtub shapes.

4 Order statistics and Rényi entropy

In this section, we obtain the distribution of the k^{th} order statistic and Rényi entropy for the NPGW-G family of distributions.

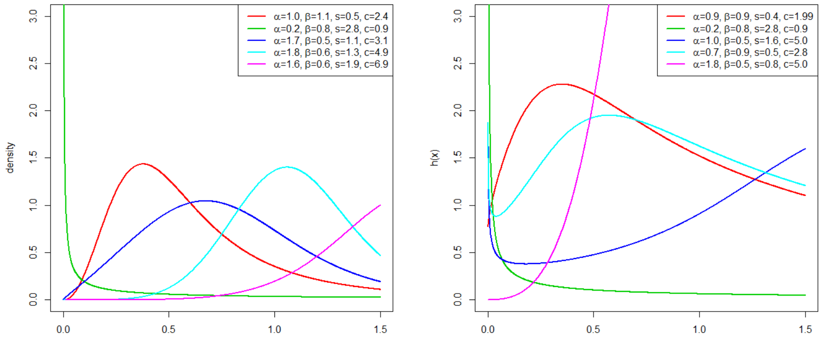

Figure 4: Density and hazard function plots for NPGW-LLoG distribution

Table 4: The quantile for NPGW-log-logistic distribution

	(α, β, s, c)				
u	$(1.2, 1.2, 1, 1.8)$	$(0.2, 1.5, 2.2, 2)$	$(0.8, 1.2, 2, 1)$	$(2, 1, 1, 2.2)$	$(1, 1.8, 2.6, 2)$
0.1	0.6713	1.6274	0.9788	0.6900	2.0000
0.2	0.7139	1.7055	1.0894	0.7271	2.0967
0.3	0.7480	1.7767	1.1876	0.7619	2.1746
0.4	0.7850	1.8437	1.2897	0.7897	2.2546
0.5	0.8203	1.9157	1.3986	0.8210	2.3342
0.6	0.8615	1.9946	1.5270	0.8599	2.4338
0.7	0.9102	2.0943	1.6902	0.9036	2.5506
0.8	0.9800	2.2331	1.9250	0.9563	2.7101
0.9	1.0969	2.4638	2.3614	1.0525	2.9850

4.1 Order statistics

Let $X_1, X_2, ..., X_n$ be a simple random sample from the NPGW-G family of distributions. Using the binomial expansion

$$(1 - F(x))^{n-k} = \sum_{w=0}^{n-k} \binom{n-k}{w} (-1)^w [F(x)]^w,$$

the pdf of the k^{th} order statistic can be written as

$$\begin{aligned} f_{k:n}(x) &= \frac{n! f(x)}{(k-1)!(n-k)!} [F(x)]^{k-1} [1 - F(x)]^{n-k} \\ &= \frac{n! f(x)}{(k-1)!(n-k)!} \sum_{w=0}^{n-k} (-1)^w \binom{n-k}{w} [F(x)]^{w+k-1}. \end{aligned} \tag{6}$$

Based on equation (1) and (2), we can write

$$f(x)F(x)^{w+k-1} = \alpha\beta \left[1 + (-\log(G(x;\xi)))^{\alpha}\right]^{\beta-1} (-\log(G(x;\xi)))^{\alpha-1}$$

$$\times \exp\left(1-\left[1+\left(-\log\left(G(x;\xi)\right)\right)^{\alpha}\right]^{\beta}\right)\frac{g(x;\xi)}{G(x;\xi)}$$
$$\times\left(\exp\left(1-\left[1+\left(-\log\left(G(x;\xi)\right)\right)^{\alpha}\right]^{\beta}\right)\right)^{w+k-1}.$$

Following the same steps of the density expansion (3), we get that

$$f(x)F(x)^{w+k-1}=\sum_{q=0}^{\infty}t_{q}g_{q}^{*}(x;\xi), \tag{7}$$

where

$$t_{q} = \alpha\beta\sum_{i,j,l,p=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}b_{s,m}\binom{\alpha(l+p+1)-1}{m}\binom{\beta-1}{l}\binom{i}{j}\binom{\beta j}{p}$$
$$\times\binom{\alpha(l+p+1)+s+m-1}{q}\frac{(-1)^{j+q}(w+k)^{i}}{i!q}.$$

Substituting (7) into (6), the pdf of the k^{th} order statistic can be expressed as

$$f_{k:n}(x)=\frac{n!}{(k-1)!(n-k)!}\sum_{q=0}^{\infty}\sum_{w=0}^{n-k}(-1)^{w}\binom{n-k}{w}t_{q}g_{q}^{*}(x;\xi),$$

where $g_{q}^{*}(x;\xi)=q[G(x;\xi)]^{q-1}g(x;\xi)$ is the exponentiated-G (E-G) pdf with the power parameter $q>0$ and parameter vector ξ. Thus, the density function of the NPGW-G order statistics is a linear combination of E-G densities.

4.2 Rényi entropy

Rényi entropy (Rényi, 1960) is an extension of Shannon entropy. Rényi entropy is defined to be

$$I_{R}(v)=\frac{1}{1-v}\log\left(\int_{0}^{\infty}[f(x;\alpha,\beta,\xi)]^{v}dx\right), v\neq 1, v>0.$$

Rényi entropy tends to Shannon entropy as $v\to 1$. Using equation (2), $[f(x;\alpha,\beta,\xi)]^{v}$ can be written as

$$[f(x;\alpha,\beta,\xi)]^{v} = \alpha^{v}\beta^{v}\left[1+\left(-\log\left(G(x;\xi)\right)\right)^{\alpha}\right]^{v(\beta-1)}\left(-\log\left(G(x;\xi)\right)\right)^{v(\alpha-1)}$$
$$\times\exp\left(v\left(1-\left[1+\left(-log\left(G(x;\xi)\right)\right)^{\alpha}\right]^{\beta}\right)\right)g(x;\xi)^{v}G(x;\xi)^{-v}.$$

Considering

$$\left[1+\left(-\log\left(G(x;\xi)\right)\right)^{\alpha}\right]^{v(\beta-1)}=\sum_{j=0}^{\infty}\binom{v(\beta-1)}{j}\left(-\log\left(G(x;\xi)\right)\right)^{\alpha j},$$

$$\exp\left(v\left(1-\left[1+\left(-\log\left(G(x;\xi)\right)\right)^{\alpha}\right]^{\beta}\right)\right)=\sum_{i,k,p=0}^{\infty}\frac{v^{i}}{i!}\binom{i}{k}\binom{\beta k}{p}(-1)^{k}\left(-\log\left(G(x;\xi)\right)\right)^{\alpha p},$$

we have

$$\begin{aligned}
[f(x;\alpha,\beta,\xi)]^v &= \sum_{i,j,k,p=0}^{\infty} \frac{v^i}{i!}\binom{v(\beta-1)}{j}\binom{i}{k}\binom{\beta k}{p}(-1)^k \\
&\quad \times(-\log\left(G(x;\xi)\right))^{\alpha(j+p+v)-v} G(x;\xi)^{-v} g(x;\xi)^v \\
&= \sum_{i,j,k,p=0}^{\infty} \frac{v^i}{i!}\binom{v(\beta-1)}{j}\binom{i}{k}\binom{\beta k}{p}(-1)^k \\
&\quad \times(-\log(1-(1-G(x;\xi))))^{\alpha(j+p+v)-v} G(x;\xi)^{-v} g(x;\xi)^v \\
&= \sum_{i,j,k,p=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s=0}^{\infty} b_{s,m}\frac{v^i}{i!}\binom{v(\beta-1)}{j}\binom{i}{k}\binom{\beta k}{p}(-1)^k G(x;\xi)^{-v} \\
&\quad \times\binom{\alpha(j+p+v)-v}{m}(1-G(x;\xi))^{\alpha(j+p+v)+s+m-v} g(x;\xi)^v \\
&= \sum_{i,j,k,p,q=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s=0}^{\infty} b_{s,m}\frac{v^i}{i!}\binom{v(\beta-1)}{j}\binom{i}{k}\binom{\beta k}{p}(-1)^{k+q} \\
&\quad \times\binom{\alpha(j+p+v)-v}{m}\binom{t^*}{q}\left(G(x;\xi)\right)^{q-v} g(x;\xi)^v.
\end{aligned}$$

where $t^* = \alpha(j+p+v)+s+m-v$. Consequently, Rényi entropy for the NPGW-G family of distributions is given by

$$\begin{aligned}
I_R(v) &= \frac{1}{1-v}\log\Bigg[\sum_{i,j,k,p,q=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s=0}^{\infty} b_{s,m}\frac{v^i}{i!}\binom{v(\beta-1)}{j}\binom{i}{k}\binom{\beta k}{p}(-1)^{k+q} \\
&\quad \times\binom{\alpha(j+p+v)-v}{m}\binom{t^*}{q}\frac{1}{\left(\frac{q-v}{v}+1\right)^v} \\
&\quad \times\int_0^{\infty}\left(\left(\frac{q-v}{v}+1\right)\left(G(x;\xi)\right)^{\frac{q-v}{v}} g(x;\xi)\right)^v dx\Bigg] \\
&= \frac{1}{1-v}\log\left[\sum_{q=0}^{\infty} w_q^* \exp((1-v)I_{REG})\right],
\end{aligned}$$

for $v>0$, $v\neq 1$, where $I_{REG} = \frac{1}{1-v}\log\int_0^{\infty}\left(\left(\frac{q-v}{v}+1\right)\left(G(x;\xi)\right)^{\frac{q-v}{v}} g(x;\xi)\right)^v dx$ is the Rényi entropy of E-G distribution with power parameter $\frac{q-v}{v}+1$, and

$$\begin{aligned}
w_q^* &= \sum_{i,j,k,p=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s=0}^{\infty} b_{s,m}\frac{v^i}{i!}\binom{v(\beta-1)}{j}\binom{i}{k}\binom{\beta k}{p}(-1)^{k+q} \\
&\quad \times\binom{\alpha(j+p+v)-v}{m}\binom{t^*}{q}\frac{1}{\left(\frac{q-v}{v}+1\right)^v}.
\end{aligned}$$

5 Moments

Here we derive some properties of the NPGW-G family of distributions, which includes moments, incomplete moments and probability weighted moments.

5.1 Moments and generating function

From now on, let $Y_q \sim Exponentiated - G(q, \xi)$. The r^{th} moment of the NPGW-G family of distributions is obtained by:

$$\mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)dx = \sum_{q=0}^{\infty} b_q E(Y_q^r).$$

The moment generating function (MGF) $M_X(t) = E(e^{tX})$ is given by

$$M_X(t) = \sum_{q=0}^{\infty} b_q M_q(t),$$

where $M_q(t)$ is the mgf of Y_q and b_q is given by equation (4). The coefficients of variation (CV), Skewness (CS) and Kurtosis (CK) can be readily obtained. The variance (σ^2), Standard deviation (SD=σ), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) are given by

$$\sigma^2 = \mu'_2 - \mu^2, \quad CV = \frac{\sigma}{\mu} = \frac{\sqrt{\mu'_2 - \mu^2}}{\mu} = \sqrt{\frac{\mu'_2}{\mu^2} - 1},$$
$$CS = \frac{E\left[(X-\mu)^3\right]}{\left[E(X-\mu)^2\right]^{3/2}} = \frac{\mu'_3 - 3\mu\mu'_2 + 2\mu^3}{(\mu'_2 - \mu^2)^{3/2}},$$
$$CK = \frac{E\left[(X-\mu)^4\right]}{\left[E(X-\mu)^2\right]^2} = \frac{\mu'_4 - 4\mu\mu'_3 + 6\mu^2\mu'_2 - 3\mu^4}{(\mu'_2 - \mu^2)^2},$$

respectively. A table of moments for some selected parameters values are given in Tables 5, 6, 7 and 8.

5.2 Incomplete moments

The s^{th} incomplete moment of X, say $\varphi_s(t)$, is given by

$$\varphi_s(t) = \int_{-\infty}^{t} x^s f(x)dx,$$

using equation (3), we obtain

$$\varphi_s(t) = \sum_{q=0}^{\infty} b_q \int_{-\infty}^{t} x^s g_q^*(x;\xi)dx. \tag{8}$$

where $g_q^*(x;\xi) = q[G(x;\xi)]^{q-1} g(x;\xi)$ is the exponentiated-G (E-G) pdf with the power parameter $q > 0$ and b_q is given by equation (4). The first incomplete moment of the NPGW-G family can be obtained by setting $s = 1$ in equation (8).

Table 5: The moments for selected parameters for NPGW-Weibull distribution

	$(\alpha,\beta,\lambda,\theta)$				
	(.1, .2, .2, .5)	(.7, 1, 1, .5)	(.2, 2, 2, .2)	(.4, 2, .5, .9)	(.2, 1, 1, .5)
$E(X)$	0.0031	0.1182	0.0134	0.0318	0.0355
$E(X^2)$	0.0017	0.0655	0.0073	0.0217	0.0199
$E(X^3)$	0.0011	0.0452	0.0050	0.0165	0.0139
EX^4	0.0009	0.0344	0.0038	0.0133	0.0106
$E(X^5)$	0.0007	0.0278	0.0031	0.0112	0.0086
$E(X^6)$	0.0006	0.0233	0.0026	0.0096	0.0073
SD	0.0005	0.0200	0.0022	0.0084	0.0063
CV	0.0004	0.0176	0.0019	0.0075	0.0055
CS	0.0004	0.0157	0.0017	0.0068	0.0049
CK	0.0003	0.0141	0.0016	0.0062	0.0044

Table 6: The moments for selected parameters for NPGW-Lindley distribution

	(α,β,λ)				
	(.5, .2, .8)	(0.1, 0.9, 1.0)	(1.4, 0.8, 1.0)	(0.7, 1.0, 0.2)	(0.3, 1.2, 0.5)
$E(X)$	0.0311	0.0230	0.2749	0.0385	0.0432
$E(X^2)$	0.0179	0.0137	0.1751	0.0241	0.0260
$E(X^3)$	0.0126	0.0099	0.1270	0.0176	0.0187
$E(X^4)$	0.0098	0.0077	0.0992	0.0139	0.0146
$E(X^5)$	0.0079	0.0063	0.0812	0.0115	0.0120
$E(X^6)$	0.0067	0.0054	0.0686	0.0098	0.0102
SD	0.0058	0.0047	0.0594	0.0085	0.0088
CV	0.0051	0.0041	0.0523	0.0075	0.0078
CS	0.0046	0.0037	0.0467	0.0067	0.0070
CK	0.0041	0.0033	0.0422	0.0061	0.0063

5.3 Probability weighted moments

The probability weighted moments (PWMs) of X is a very useful mathematical quantity. The $(p,r)^{th}$ PWMs of X denoted $Z_{p,r}$ is given by

$$Z_{p,r} = E(X^p(F(X))^r) = \int_{-\infty}^{\infty} x^p(F(x))^r f(x)dx.$$

Using equation (1) and (2), we can write

$$\begin{aligned} f(x)(F(x))^r &= \alpha\beta\left[1+(-\log\left(G(x;\xi)\right))^{\alpha}\right]^{\beta-1}(-\log\left(G(x;\xi)\right))^{\alpha-1}\frac{g(x;\xi)}{G(x;\xi)} \\ &\quad \times\exp\left(1-\left[1+(-\log\left(G(x;\xi)\right))^{\alpha}\right]^{\beta}\right) \\ &\quad \times\left(\exp\left(1-\left[1+(-\log\left(G(x;\xi)\right))^{\alpha}\right]^{\beta}\right)\right)^{r} \\ &= \alpha\beta\left[1+(-\log\left(G(x;\xi)\right))^{\alpha}\right]^{\beta-1}(-\log\left(G(x;\xi)\right))^{\alpha-1}\frac{g(x;\xi)}{G(x;\xi)} \end{aligned}$$

Table 7: The moments for selected parameters for NPGW-log-logistic distribution

	(α, β, s, c)				
	(.1, .2, .2, .5)	(.7, 1, 1.5, .5)	(.2, 2, 2, .2)	(3.4, 4, .5, .9)	(.2, 2, 1.5, .5)
$E(X)$	0.0036	0.0770	0.0056	0.4993	0.01183
$E(X^2)$	0.0019	0.0414	0.0029	0.3383	0.00667
$E(X^3)$	0.0012	0.0281	0.0020	0.2422	0.00464
$E(X^4)$	0.0009	0.0213	0.0015	0.1821	0.00356
$E(X^5)$	0.0007	0.0171	0.0012	0.1427	0.00288
$E(X^6)$	0.0006	0.0143	0.0010	0.1157	0.00242
SD	0.0005	0.0123	0.0008	0.0965	0.00209
CV	0.0004	0.0108	0.0007	0.0824	0.00184
CS	0.0004	0.0096	0.0006	0.0717	0.00164
CK	0.0003	0.0086	0.0006	0.0633	0.00148

Table 8: The moments for selected parameters for NPGW-power distribution

	$(\alpha, \beta, \theta, k)$				
	(.1, .5, .7, 1)	(2, 2, 1, 1,)	(.5, .2, .4, 2)	(1, .2, .2, .5)	(.2, .5, .9, .2)
$E(X)$	0.0257	0.5899	0.0389	0.0443	0.0708
$E(X^2)$	0.0170	0.3696	0.0231	0.0243	0.0507
$E(X^3)$	0.0129	0.2446	0.0166	0.0168	0.0407
$E(X^4)$	0.0105	0.1697	0.0130	0.0128	0.0344
$E(X^5)$	0.0089	0.1227	0.0107	0.0104	0.0300
$E(X^6)$	0.0077	0.0919	0.0091	0.0087	0.0267
SD	0.0068	0.0709	0.0079	0.0075	0.0241
CV	0.0061	0.0561	0.0070	0.0066	0.0220
CS	0.0056	0.0453	0.0063	0.0059	0.0203
CK	0.0051	0.0373	0.0057	0.0053	0.0188

$$\times \exp\left((r+1)\left(1-\left[1+\left(-\log\left(G(x;\xi)\right)\right)^{\alpha}\right]^{\beta}\right)\right).$$

Using the same steps of the density expansion (3), we obtain

$$f(x)F(x)^r = \sum_{q=0}^{\infty} w_q g_q^*(x;\xi),$$

where

$$\begin{aligned} w_q &= \alpha\beta \sum_{i,j,l,p,q=0}^{\infty} \sum_{m=0}^{\infty}\sum_{s=0}^{\infty} b_{s,m} \binom{\alpha(l+p+1)-1}{m}\binom{\beta-1}{l}\binom{i}{j}\binom{\beta j}{p} \\ &\quad \times \binom{\alpha(l+p+1)+s+m-1}{q} \frac{(-1)^{j+q}(r+1)^i}{i!q}. \end{aligned}$$

Consequently,

$$Z_{p,r} = \int_{-\infty}^{\infty} x^p \sum_{q=0}^{\infty} w_q g_q^*(x;\xi)dx = \sum_{q=0}^{\infty} w_q \int_{-\infty}^{\infty} x^p g_q^*(x;\xi)dx,$$

where $g^*_q(x;\xi) = q[G(x;\xi)]^{q-1}g(x;\xi)$ is the exponentiated-G (E-G) pdf with the power parameter $q > 0$. Finally, the $(p,r)^{th}$ PWMs of X can be obtained from the moments of the exponentiated-G (E-G) distribution.

6 Maximum likelihood estimation

In this section, the maximum likelihood estiamation technique is used to estimate the model parameters. Let $X \sim NPGW - G(\alpha,\beta,\xi)$ and $\boldsymbol{\Delta} = (\alpha,\beta,\xi)^T$ be the parameter vector. The log-likelihood function $\ell_n = \ell_n(\boldsymbol{\Delta})$ based on a random sample of size n from the NPGW-G family of distributions is given by

$$\begin{aligned}\ell_n(\boldsymbol{\Delta}) &= n\ln(\alpha\beta) + \sum_{i=1}^{n}\Big(1 - [1 + (-\log(G(x_i;\xi)))^{\alpha}]^{\beta}\Big) - \sum_{i=1}^{n}\ln(G(x_i;\xi)) \\ &+(\beta-1)\sum_{i=1}^{n}\ln[1+(-\log(G(x_i;\xi)))^{\alpha}] + \sum_{i=1}^{n}\ln(g(x_i;\xi)) \\ &+(\alpha-1)\sum_{i=1}^{n}\ln(-\log(G(x_i;\xi))).\end{aligned}$$

By differentiating the log-likelihood function with respect to each component of the parameter vector $\boldsymbol{\Delta} = (\alpha,\beta,\xi)^T$, we obtain the following

$$\begin{aligned}\frac{\partial \ell_n}{\partial \alpha} &= \frac{n}{\alpha} - \Bigg[\beta\sum_{i=1}^{n}(1+(-\log(G(x_i;\xi)))^{\alpha})^{\beta-1}(-\log(G(x_i;\xi)))^{\alpha} \\ &\times \ln(-\log(G(x_i;\xi)))\Bigg] + \sum_{i=1}^{n}\ln(-\log(G(x_i;\xi))) \\ &+(\beta-1)\sum_{i=1}^{n}\frac{(-\log(G(x_i;\xi)))^{\alpha}\ln(-\log(G(x_i;\xi)))}{[1+(-\log(G(x_i;\xi)))^{\alpha}]}, \\ \frac{\partial \ell_n}{\partial \beta} &= \frac{n}{\beta} - \sum_{i=1}^{n}(1+(-\log(G(x_i;\xi)))^{\alpha})^{\beta}\ln(1+(-\log(G(x_i;\xi)))^{\alpha}) \\ &+\sum_{i=1}^{n}\ln(1+(-\log(G(x_i;\xi)))^{\alpha}), \\ \frac{\partial \ell_n}{\partial \xi_k} &= \Bigg[\alpha\beta\sum_{i=1}^{n}(1+(-\log(G(x_i;\xi)))^{\alpha})^{\beta-1}(-\log(G(x_i;\xi)))^{\alpha-1}\frac{\partial G(x_i;\xi)}{\partial \xi_k} \\ &\times\frac{1}{G(x_i;\xi)}\Bigg] - (\beta-1)\sum_{i=1}^{n}\frac{\alpha(-\log(G(x_i;\xi)))^{\alpha-1}\frac{\partial G(x_i;\xi)}{\partial \xi_k}\frac{1}{G(x_i;\xi)}}{(1+(-\log(G(x_i;\xi)))^{\alpha})} \\ &- (\alpha-1)\sum_{i=1}^{n}\frac{\frac{\partial G(x_i;\xi)}{\partial \xi_k}\frac{1}{G(x_i;\xi)}}{(-\log(G(x_i;\xi)))} - \sum_{i=1}^{n}\frac{\frac{\partial G(x_i;\xi)}{\partial \xi_k}}{G(x_i;\xi)} + \sum_{i=1}^{n}\frac{\frac{\partial g(x_i;\xi)}{\partial \xi_k}}{g(x_i;\xi)}.\end{aligned}$$

The maximum likelihood estimates of the parameters, denoted by $\hat{\boldsymbol{\Delta}}$ is obtained by solving the nonlinear equation $(\frac{\partial \ell_n}{\partial \alpha}, \frac{\partial \ell_n}{\partial \beta}, \frac{\partial \ell_n}{\partial \xi_k})^T = \mathbf{0}$, using a numerical method such as Newton-Raphson procedure. We maximize the likelihood function using NLmixed in SAS as well as the function nlm in R (R Development Core Team, 2011). The multivariate normal distribution $N_{q+2}(\underline{0}, J(\hat{\boldsymbol{\Delta}})^{-1})$, where the mean vector $\underline{\mathbf{0}} = (0, 0, \underline{0})^T$ and $J(\hat{\boldsymbol{\Delta}})^{-1}$ is the observed Fisher information matrix evaluated at $\hat{\boldsymbol{\Delta}}$, can be used to construct confidence intervals and confidence regions for the individual model parameters.

7 Simulation study

The Monte Carlo simulation study is performed via the R package to assess the perfomance of the MLEs of NPGW-E distribution. The sample sizes considered are (n=35, 50, 100, 200, 400, 800). We simulate $N = 1000$ samples for the true parameters values given in Tables 7 and 10. The table lists the mean MLEs of the model parameters along with the respective bias and root mean squared errors (RMSEs). The bias and RMSE for the estimated parameter, say, $\hat{\theta}$, say, are given by:

$$Bias(\hat{\theta}) = \frac{\sum_{i=1}^{N} \hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{N} (\hat{\theta}_i - \theta)^2}{N}},$$

respectively. From the results in Tables 7 and 10, we can clearly verify that all the estimators reveal the consistency property, i.e., the mean estimates of the parameters tend to be closer to the true parameter values when the sample size increases. Also, the MSEs decrease with increasing sample.

8 Real data applications

In this section, the NPGW-E distribution is fitted to two data sets and these fits are compared to the fits of the non-nested models, including Marshall-Olkin log-logistic distribution (MOLLD) (Wenhao, 2013), Lindley-Weibull (LW) distribution (Asgharzadeh et al., 2016), Marshall-Olkin extended inverse Weibull (IWMO) distribution by Pakungwati et al. (2018), Marshall-Olkin extended Fréchet (MOEFr) distribution by Barreto-Souza et al. (2013), Weibull exponential (WE) distribution by Oguntunde et al. (2015). The NPGW-E distribution was also compared to the Marshall-Olkin extended generalized exponential (MOEGE) distribution that was given by Barreto-Souza et al. (2013).

The MOLLD pdf is given by

$$g_{MOLLD}(x; \alpha, \beta, \gamma) = \frac{\alpha^{\beta} \beta \gamma x^{\beta - 1}}{(x^{\beta} + \alpha^{\beta} \gamma)^2},$$

for $\alpha, \beta, \gamma > 0$, and $x > 0$. The LW distribution has the pdf given by

$$g_{LW}(x; \lambda, \alpha, \beta) = \frac{e^{-\lambda x - \alpha x^{\beta}}}{1 + \lambda} [\lambda^2 (1 + x) + (1 + \lambda + \lambda x) \alpha \beta x^{\beta - 1}],$$

Table 9: Monte Carlo simulation results

		(2.0, 1.5, 2.0)			(2.0, 1.5, 1.5)			(1.5, 2.0, 2.0)		
	n	Mean	RMSE	Bias	Mean	RMSE	Bias	Mean	RMSE	Bias
α	25	2.2076	1.9720	0.2076	3.6218	29.370	1.6218	1.2961	0.7769	-0.2038
	50	2.2074	1.3027	0.2074	2.1681	1.2105	0.1681	1.3878	0.8055	-0.1121
	100	2.1446	1.0050	0.1446	2.1415	1.0197	0.1415	1.4029	0.7082	-0.0970
	200	2.0945	0.8462	0.0945	2.0856	0.8458	0.0856	1.3782	0.6377	-0.1217
	400	2.0285	0.6801	0.0285	2.0345	0.6861	0.0345	1.4506	0.5589	-0.0493
	800	2.0087	0.5758	0.0087	2.0099	0.5763	0.0099	1.4451	0.4850	-0.0548
	1000	2.0166	0.5338	0.0166	2.0068	0.5339	0.0068	1.4800	0.4692	-0.0199
β	25	2.1680	1.3431	0.6680	2.1594	1.3523	0.6594	2.7239	1.2951	0.7239
	50	1.8329	1.0138	0.3329	1.8441	1.0174	0.3441	2.4681	1.0745	0.4681
	100	1.7314	0.8760	0.2314	1.7336	0.8754	0.2336	2.3682	1.0011	0.3682
	200	1.6740	0.7704	0.1740	1.6838	0.7785	0.1838	2.3404	0.9112	0.3404
	400	1.6620	0.6892	0.1620	1.6590	0.6906	0.1590	2.2075	0.7756	0.2075
	800	1.6301	0.6025	0.1301	1.6296	0.6044	0.1296	2.1808	0.6817	0.1808
	1000	1.6055	0.5601	0.1055	1.6158	0.5666	0.1158	2.1297	0.6524	0.1297
a	25	3.1313	2.2244	1.1313	2.3233	1.6335	0.8233	4.3646	4.9039	2.3646
	50	2.6866	1.8951	0.6866	2.0072	1.3474	0.5072	3.7969	4.1472	1.7969
	100	2.4867	1.4245	0.4867	1.8704	1.0765	0.3704	3.5439	3.7313	1.5439
	200	2.3921	1.2471	0.3921	1.8045	0.9449	0.3045	3.1959	2.8991	1.1959
	400	2.3358	1.0600	0.3358	1.7485	0.7934	0.2485	2.6394	1.6916	0.6394
	800	2.2687	0.9048	0.2687	1.7017	0.6822	0.2017	2.4788	1.2583	0.4788
	1000	2.2166	0.8115	0.2166	1.6739	0.6193	0.1739	2.3730	1.1273	0.3730

for $\lambda, \alpha, \beta > 0$ and $x > 0$. The pdf of IWMO distribution is given by

$$f_{IWMO}(x; \alpha, \theta, \lambda) = \frac{\alpha\lambda\theta^{-\lambda}x^{-\lambda-1}e^{-(\theta x)^{-\lambda}}}{[\alpha - (\alpha-1)e^{-(\theta x)^{-\lambda}}]^2},$$

for $\alpha, \theta, \lambda > 0$. The pdf of MOEFr distribution is given by

$$f_{MOEFr}(x; \alpha, \lambda, \delta) = \frac{\alpha\lambda\delta^{\lambda}x^{-(\lambda+1)}e^{-(\delta/x)^{\lambda}}}{[1 - \bar{\alpha}(1 - e^{-(\delta/x)^{\lambda}})]^2},$$

for $\alpha, \lambda, \delta > 0$. The pdf of WE distribution is given by

$$f_{WE}(x; \alpha, \beta, \lambda) = \alpha\beta(\lambda e^{-\lambda x}) \left[\frac{\left(1 - e^{-\lambda x}\right)^{\beta-1}}{\left(e^{-\lambda x}\right)^{\beta+1}}\right] e^{-\alpha\left[\frac{\left(1-e^{-\lambda x}\right)}{\left(e^{-\lambda x}\right)}\right]^{\beta}},$$

for $\alpha, \beta, \lambda > 0$, and $x > 0$. The pdf of MOEGE distribution is given by

$$f_{MOEGE}(x; \alpha, \gamma, \lambda) = \frac{\alpha\gamma\lambda e^{-\lambda x}(1 - e^{-\lambda x})^{\gamma-1}}{(1 - \bar{\alpha}[1 - e^{-\lambda x}]^{\gamma})^2},$$

for $\alpha, \gamma, \lambda > 0$.

Plots of the fitted densities, the histogram of the data and probability plots (Chambers et al., 1983) are given in Figure 5 and Figure 6. For the probability plot, we plotted

Table 10: Monte Carlo simulation results

		(1.5, 2.0, 1.5)			(1.5, 1.5, 2.0)			(1.5, 1.5, 1.5)		
	n	Mean	RMSE	Bias	Mean	RMSE	Bias	Mean	RMSE	Bias
α	25	1.3021	0.8143	-0.1978	1.3562	0.9473	-0.1437	2.0152	20.5852	0.5152
	50	1.3808	0.8128	-0.1191	1.4338	0.8476	-0.0661	1.4130	0.8151	-0.0869
	100	1.3934	0.6975	-0.1065	1.4531	0.7278	-0.0468	1.4277	0.6993	-0.0722
	200	1.3797	0.6327	-0.1202	1.4242	0.6228	-0.0757	1.4246	0.6203	-0.0753
	400	1.4418	0.5585	-0.0581	1.4512	0.5346	-0.0487	1.4536	0.5213	-0.0463
	800	1.4500	0.4836	-0.0499	1.4629	0.4728	-0.0270	1.4683	0.4624	-0.0316
	1000	1.4707	0.458	-0.0292	1.4748	0.4375	-0.0351	1.4765	0.4374	-0.0234
β	25	2.7302	1.2993	0.7302	2.0258	0.9500	0.5258	2.0383	0.9560	0.5383
	50	2.4802	1.0818	0.4802	1.8362	0.8023	0.3362	1.8537	0.8188	0.3537
	100	2.3772	1.0025	0.3772	1.7605	0.7471	0.2605	1.7788	0.7525	0.2788
	200	2.3362	0.9034	0.3362	1.7331	0.6676	0.2331	1.7312	0.6643	0.2312
	400	2.2180	0.7811	0.2180	1.6657	0.5705	0.1657	1.6598	0.5608	0.1598
	800	2.1734	0.6750	0.1734	1.6178	0.4940	0.1178	1.6191	0.4879	0.1191
	1000	2.1374	0.6430	0.1374	1.6164	0.4734	0.1164	1.6040	0.4695	0.1040
a	25	3.2835	3.6891	1.7835	4.0525	4.3569	2.0525	3.1504	3.4437	1.6504
	50	2.8782	3.1704	1.3782	3.5131	3.6467	1.5131	2.7030	2.8468	1.2030
	100	2.6867	2.8683	1.1867	3.1760	2.9266	1.1760	2.4237	2.2418	0.9237
	200	2.3722	2.1196	0.8722	2.8878	2.1949	0.8878	2.1528	1.6017	0.6528
	400	2.0141	1.3723	0.5141	2.5718	1.5052	0.5718	1.9059	1.0759	0.4059
	800	1.8472	0.9305	0.3472	2.3862	1.0916	0.3862	1.7889	0.8111	0.2889
	1000	1.7832	0.8366	0.2832	2.3608	1.0252	0.3608	1.7527	0.7588	0.2527

$F(x_{(j)}; \hat{\alpha}, \hat{\beta}, \hat{\xi})$ against $\dfrac{j-0.375}{n+0.25}$, $j = 1, 2, \cdots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measures of closeness are given by the sum of squares

$$SS = \sum_{j=1}^{n} \left[F(x_{(j)}; \hat{\alpha}, \hat{\beta}, \hat{\xi}) - \left(\frac{j-0.375}{n+0.25} \right) \right]^2 .$$

Several goodness-of-ft measure including the minus twice the loglikelihood function evaluated at the MLEs ($-2\log L$), Akaike information criteria (AIC), consistent Akaike information criteria (AIC), Bayesian information criteria (BIC), Cramér-von Mises (W^*), Anderson-Darling statistic (A^*) and Kolmogorov-Smirnov (KS) statistic. The goodness-of-fit statistics W^* and A^*, are described by Chen and Balakrishnan (1995). These statistics can be used to verify which distribution fits better to the data. In general, the smaller the values of W^* and A^*, the better the fit.

8.1 Growth hormone data

The data consists of the estimated time since growth hormone medication until the children reached the target age. The data was used by Alizadeh et al. (2017) to show the superiority of the exponentiated power Lindley power series (EPLPS) class of distributions distributions. The data are
2.15, 2.20, 2.55, 2.56, 2.63, 2.74, 2.81, 2.90, 3.05, 3.41, 3.43, 3.43, 3.84, 4.16, 4.18, 4.36,

Table 11: Estimates of models for growth hormone dataset

	Estimates			Statistics								
Model	α	β	λ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	KS	P-value	SS
NPGW-E	1.602	2.316	0.232	155.04	161.04	161.81	165.70	0.04	0.26	0.09	0.94	0.04
	(1.256)	(2.058)	(0.224)									
	α	λ	θ									
IWMO	7.1e-5	0.3780	5.7e-4	157.67	163.67	164.45	168.34	0.04	0.33	0.09	0.94	0.04
	(9.6e-5)	(2.8e-6)	(2.1e-4)									
	λ	α	β									
LW	9.6e-8	0.027	1.993	164.98	170.98	171.75	175.64	0.16	1.03	0.15	0.45	0.15
	(0.120)	(0.014)	(0.243)									
	α	β	γ									
MOLLD	3.789	3.521	1.929	158.58	164.58	165.36	169.25	0.06	0.41	0.10	0.89	0.04
	(0.252)	(0.490)	(0.140)									
	α	δ	λ									
MOEFr	184.580	1.023	3.491	158.40	164.40	165.18	169.07	12.21	70.34	0.98	2.2e-16	11.85
	(3.5e-4)	(0.227)	(0.487)									
	α	β	λ									
WE	1.1e4	1.981	1.5e-3	165.04	171.04	171.82	175.71	0.16	1.03	0.15	0.44	0.15
	(3.9e-7)	(0.231)	(7.8e-4)									
	α	γ	λ									
MOEGE	4.2e-5	8.5e-4	0.622	168.81	174.81	175.59	179.48	0.63	3.52	0.99	2.2e-16	11.62
	(1.9e-4)	(4.2e-3)	(0.098)									

4.42, 4.51,4.60, 4.61, 4.75, 5.03, 5.10, 5.44, 5.90, 5.96, 6.77, 7.82, 8.00, 8.16, 8.21, 8.72, 10.40, 13.20, 13.70.
The estimated variance-covariance matrix is given by

$$\begin{bmatrix} 1.5790 & -2.4970 & -0.2776 \\ -2.4970 & 4.2357 & 0.4575 \\ -0.2776 & 0.4575 & 0.0503 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by
$\alpha \in [1.6029 \pm 2.4629]$, $\beta \in [2.3161 \pm 4.0338]$ and $\lambda \in [0.2327 \pm 0.4396]$. Estimates of the parameters of NPGW-E distribution (standard error in parentheses), AIC, AICC, BIC, and the goodness-of-fit statistics W*, A*, Kolmogorov-Smirnov (KS) and its p-value as well as SS are given in Table 11. Plots of the fitted densities and the histogram, observed probability vs predicted probability are given in Figure 5.

It is observed, from Tables 11 above that the NPGW-E distribution has the lowest values of AIC, AICC, BIC and goodness-of-fit statistics among all fitted models. Hence, it could be termed the "best" model for the growth hormone data set. Also, the p-value of the Kolmogorov-Smirnov (KS) statistic for the NPGW-E model is larger than the other fitted models, this also support the NPGW-E as the "best" model.

8.2 Waiting times data

The data set given below represents the waiting times (in minutes) before service of 100 bank customers. The data was used by Ghitany et al. (2008) to show that Lindley distribution can be a better model than one based on the exponential distribution. The data are

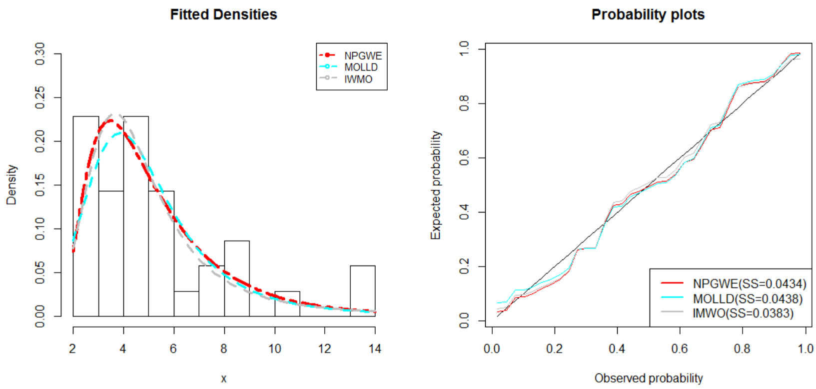

Figure 5: Fitted densities and probability plots of the growth hormone data.

Table 12: Estimates of models for waiting times data

Model	Estimates			Statistics								
	α	β	λ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	KS	P-value	SS
NPGW-E	0.713 (0.454)	1.893 (0.607)	0.210 (0.163)	633.97	639.96	640.21	647.784	0.02	0.13	.04	0.99	0.02
	α	λ	θ									
IWMO	216.040 (7.3e-4)	2.249 (0.180)	1.399 (0.288)	638.59	644.59	644.85	652.42	0.05	0.38	.05	0.96	0.03
	λ	α	β									
LW	0.136 (0.094)	0.009 (0.024)	1.514 (0.35)	637.30	643.30	643.55	651.12	0.06	0.35	.05	0.95	0.05
	α	β	γ									
MOLLD	3.892 (0.338)	2.267 (0.188)	4.859 (0.119)	638.82	644.82	645.07	652.64	0.05	0.36	.05	0.96	0.03
	α	δ	λ									
MOEFr	216.04 (1.7e-4	0.714 (0.147)	2.249 (0.180)	638.59	644.59	644.84	652.41	33.88	200.25	.99	2.2e-16	33.62
	α	β	λ									
WE	1.1e3 (9.4e-7)	1.453 (0.106)	7.4e-4 (2.4e-4)	637.564	643.56	643.81	651.38	0.06	0.40	.06	0.88	0.06
	α	γ	λ									
MOEGE	5.6e-4 (9.3e-5	3.9e-3 (8.4e-4)	0.229 (0.023)	653.85	659.85	660.10	667.67	1.34	7.47	.99	2.2e-16	33.25

0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3,4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0,11.0, 11.1, 11.2, 11.2, 11.5, 11.9,12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4,15.4,17.3, 17.3, 18.1, 18.2,18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27.0, 31.6, 33.1, 38.5.
The estimated variance-covariance matrix is given by

$$\begin{bmatrix} 3.7365e{-12} & -2.0290e{-07} & -5.2260e{-10} \\ -2.0290e{-07} & 1.1017e{-02} & 2.8378e{-05} \\ -5.2260e{-10} & 2.8378e{-05} & 7.6660e{-08} \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by
$\alpha \in [0.7130 \pm 3.7887e{-}06]$, $\beta \in [1.8929 \pm e{-}01]$ and $\lambda \in [0.2106 \pm 5.426763e{-}04]$. Estimates of the parameters of NPGW-E distribution (standard error in parentheses), AIC, AICC, BIC, and the goodness-of-fit statistics W*, A*, Kolmogorov-Smirnov (KS) and its p-value as well as SS are given in Table 12. Plots of the fitted densities and the histogram, observed probability vs predicted probability are given in Figure 6.

The values of AIC, AICC and BIC are smallest for the NPGW-E distribution, when compared to the corresponding values for the non-nested distributions. The values of the goodness-of-fit-statistics W^*, A^*, Kolmogorov-Smirnov (KS) and its p-value show that the NPGW-E distribution is the "best" fit for waiting times data.

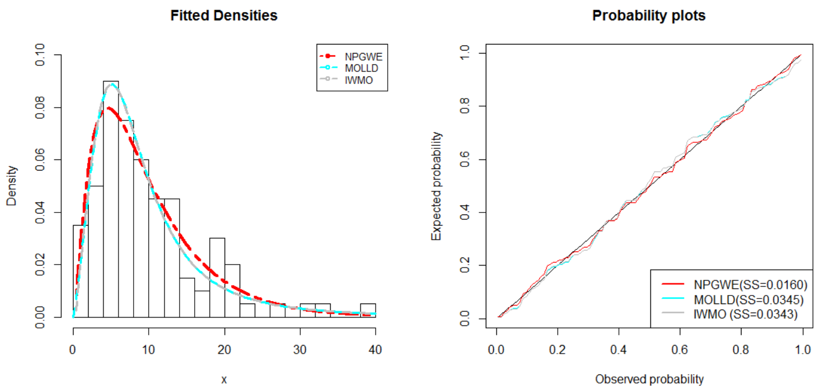

Figure 6: Fitted densities and probability plots of the waiting times data

9 Concluding remarks

A new generalized distribution referred to as the new power generalized Weibull-G (NPGW-G) family of distributions is developed and presented. The NPGW-G distribution has several new and known distributions as sub-models. The NPGW-G family of distributions possesses hazard function with flexible behavior. The proposed distribution can be expressed as an infinite linear combination of Exp-G distribution. We also obtain closed form expressions for the moments, incomplete and probability weighted moments, distribution of order statistics and entropy. Maximum likelihood estimation technique is used to estimate the model parameters. The performance of the special case of the NPGW-G family of distributions was examined by conducting various simulations for different sample sizes. Finally, the special case of the NPGW-G family of distributions is fitted to two real data sets to illustrate the applicability and usefulness of the proposed family of distributions.

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