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Research Paper

On the developments of Maxwell-Dagum distribution

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Abstract: The Maxwell-Dagum distribution is a continuous statistical distribution suitable for modeling data sets relating to various fields including finance, business, medical sciences, survival analysis, and related areas. This article aimed to propose some important properties of the Maxwell-Dagum distribution and obtained the parameters of its estimates by using different methods of estimation including maximum likelihood estimation, the maximum product of spacings, least squares estimation, and weighted least squares estimation. The first and second derivatives of this distribution are studied. We present two real data sets relating to the COVID-19 mortality rate belonging to Canada and the waiting time of bank customers to assess the performance of the proposed distribution. It is discovered that the Maxwell-Dagum distribution can be chosen as the best distribution by having a minimum value of Akaike information and Bayesian information criteria.

Keywords: Maxwell-Dagum distribution; Order statistics; q-entropy; Rényi entropy; Stress-strength.
Mathematics Subject Classification (2010): 60E05, 62F10.

1 Introduction

Dagum distribution was first introduced by Dagum Dagum (1977) to serve as an alternative to Pareto and lognormal distributions and it is defined over non-negative real numbers. This distribution has variety of parameters including three (Type-I) and four (Type-II) parameter Dagum models and can be applied in several areas of applications such as financial, wealth, reliability theory, actuarial, survival studies, meteorological

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and many more. One of the important characteristics of the Dagum model is that its hazard rate could be monotonically decreasing, bathtub and upside-down bathtub. Some discussions of the Dagum model can be found by many authors such as Dagum (1977), Dagum (2006), Kleiber and Kotz (2003), Kleiber (2008), Shahzad and Asghar (2013), among others. Many researchers have applied the Dagum model in different areas of applications including Domma and Condino (2013) who generalized the Dagum model to Beta-Dagum distribution and applied it using glass fiber and income data. Other generalized distributions to be mentioned include Log-Dagum distribution by Domma (2004), Exponentiated Kumaraswamy-Dagum by Huang and Oluyede (2014), Weighted-Dagum by Oluyede and Ye (2014), Transmuted Dagum by Elbatal and Aryal (2015), Gamma-Dagum by Rodrigues and Silva (2015), Dagum-Poisson by Oluyede et al. (2016), Weibull-Dagum by Tahir et al. (2018), Odd Log-Logistic Dagum by Domma et al. (2018), and Topp-Leone Dagum distributions by Rasheed (2020). The cumulative distribution function (cdf) and probability density function (pdf) of the Dagum distribution are given as

$$M(x;t,\delta,\lambda) = \left(1 + tx^{-\lambda}\right)^{-\delta}, \qquad x > 0, \tag{1}$$

$$m(x;t,\delta,\lambda) = t\delta\lambda x^{-\lambda-1}\left(1 + tx^{-\lambda}\right)^{-\delta-1}, \qquad x > 0, \tag{2}$$

where $\lambda, \delta > 0$ are shape parameters, and $t > 0$ is a scale parameter.

In this study, the four-parameter Maxwell-Dagum (M-D) distribution is developed based on the following motivations and novelty:

i. To obtain a distribution with different pdf shapes such as left-skewed, right-skewed and symmetric.

ii. To derive some structural properties such as stress-strength, Rényi and q-entropies.

iii. To estimate the parameters of M-D distribution by using different methods of estimation such as maximum likelihood estimation (MLE), maximum product of spacings (MPS), least squares estimation (LSE) and weighted least squares estimation (WLSE) methods.

iv. To evaluate the performance of the M-D distribution from the basis of quantile function by using a simulation study, and lastly.

v. To illustrate the flexibility and potentiality of the M-D distribution against its competing distributions and compare the performance among them.

The study is organized as follows: Section 2 presents the cdf, pdf, survival, hazard and quantile functions of Maxwell-Dagum distribution developed from the Maxwell generalized family of distributions. Some properties and parameter estimates using different methods of estimation are given in Sections 3 and 4. A simulation study is carried out from the basis of quantile function in Section 5. An application to real data sets are provided in section 6, the conclusion of the results and further study are given in Section 7.

2 Methodology

The Maxwell-Dagum distribution and method used to derive this distribution are provided in this section.

2.1 The Maxwell generalized class of distributions

The Maxwell generalized class of distributions was proposed by Ishaq and abiodun (2020) from the logit of Maxwell random variable. Its cdf and pdf are given as

$$F(x;a,\psi)=\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{2a^2}\left(\frac{M(x,\psi)}{1-M(x,\psi)}\right)^2\right),\quad x\in\Re, \tag{3}$$

$$f(x;a,\psi)=\frac{2m(x,\psi)}{a^3\sqrt{2\pi}\,(1-M(x,\psi))^2}\left(\frac{M(x,\psi)}{1-M(x,\psi)}\right)^2 e^{-\dfrac{1}{2a^2}\left(\frac{M(x,\psi)}{1-M(x,\psi)}\right)^2}, \tag{4}$$

where a is a scale parameter, $M(x,\psi)$ and $m(x,\psi)$ are cdf and pdf of the Dagum distribution. Abdullahi et al. (2021) extends Exponential distribution to propose Maxwell-Exponential distribution by applying Maxwell generalized class of distributions. Some properties and real life applications of the proposed distribution are given in Abdullahi et al. (2021). The Mukherjee-Islam distribution was generalized by Ishaq et al. (2021) to study Maxwell-Mukherjee Islam distribution. The validity test and parameter estimation of the Maxwell-Mukherjee Islam distribution was obtained by using Bayesian approach, refer to Ishaq et al. (2021) for more details and application to real life data set.

2.2 The Maxwell-Dagum distribution

The cdf and pdf in (3) and (4) were considered by Ishaq and abiodun (2020) in studying Maxwell-Dagum distribution. Its cdf is obtained by using the Dagum baseline cdf and pdf given in (1) and (2) as

$$F(x;a,t,\delta,\lambda)=F(x)=\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{2a^2}\left(\frac{\left(1+tx^{-\lambda}\right)^{-\delta}}{1-\left(1+tx^{-\lambda}\right)^{-\delta}}\right)^2\right). \tag{5}$$

The corresponding pdf is derived by inserting (1) and (2) into (4) as

$$\begin{aligned}f(x;a,t,\delta,\lambda)=&\frac{2t\lambda\delta x^{-\lambda-1}\left(1+tx^{-\lambda}\right)^{-\delta-1}}{a^3\sqrt{2\pi}\left(1-\left(1+tx^{-\lambda}\right)^{-\delta}\right)^2}\left(\frac{\left(1+tx^{-\lambda}\right)^{-\delta}}{1-\left(1+tx^{-\lambda}\right)^{-\delta}}\right)^2\\&\times\ \exp\left(-\frac{1}{2a^2}\left(\frac{\left(1+tx^{-\lambda}\right)^{-\delta}}{1-\left(1+tx^{-\lambda}\right)^{-\delta}}\right)^2\right),\quad x>0,\end{aligned} \tag{6}$$

where $\lambda,\delta>0$ are shape parameters and $a,t>0$ are scale parameters. Plots of the pdf of M-D distribution are presented in Figure 1. As provided in Figure 1, the pdf of the M-D distribution can be symmetric, right-skewed and left-skewed.

The survival, hazard and quantile functions of the M-D distribution are

$$S(x)=1-\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{2a^2}\left(\frac{\left(1+tx^{-\lambda}\right)^{-\delta}}{1-\left(1+tx^{-\lambda}\right)^{-\delta}}\right)^2\right),$$

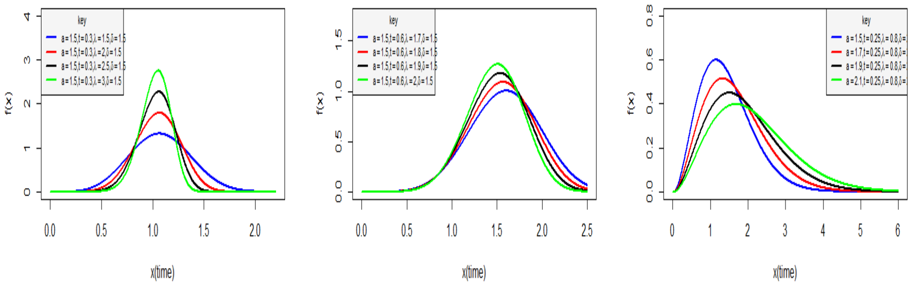

Figure 1: The plot of pdf for the Maxwell-Dagum distribution for some parameter values.

$$h(x) = \frac{2t\lambda\delta x^{-\lambda-1}\left(1+tx^{-\lambda}\right)^{-3\delta-1}\exp\left(-\frac{1}{2a^2}\left(\frac{(1+tx^{-\lambda})^{-\delta}}{1-(1+tx^{-\lambda})^{-\delta}}\right)^2\right)}{a^3\sqrt{2\pi}\left(1-(1+tx^{-\lambda})^{-\delta}\right)^4\left(1-\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{2a^2}\left(\frac{(1+tx^{-\lambda})^{-\delta}}{1-(1+tx^{-\lambda})^{-\delta}}\right)^2\right)\right)},$$

$$q_u = \left\{-\theta\left(1-\left[\frac{\left[2\alpha^2\gamma^{-1}\left(\frac{3}{2},u\Gamma\left(\frac{3}{2}\right)\right)\right]^{\frac{1}{2}}}{1+\left[2\alpha^2\gamma^{-1}\left(\frac{3}{2},u\Gamma\left(\frac{3}{2}\right)\right)\right]^{\frac{1}{2}}}\right]^{-1/\delta}\right)^{-1}\right\}^{1/\lambda}, \tag{7}$$

where u is a uniform random variable defined on interval $(0,1)$. The hazard plots of the M-D distribution are presented in Figure 2. The hazard function of the M-D distribution can be constant, decreasing and increasing as given in Figure 2.

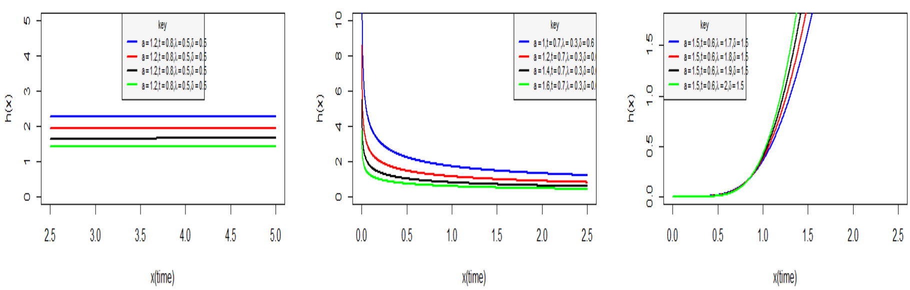

Figure 2: Hazard plots of the Maxwell-Dagum distribution for some parameter values.

3 Properties of the Maxwell-Dagum distribution

In this section, some properties of the Maxwell-Dagum distribution are presented including stress-strength, Rényi and q-entropies, and order statistics.

3.1 Stress-strength model of the Maxwell-Dagum distribution

The stress-strength model can be used for reliability estimation especially in the fields of engineering and related areas to describe the strength failure and system collapse. Suppose X_1 and X_2 are two independent random variables with pdfs $f(x_1)$ and $f(x_2)$, respectively. The stress-strength (denoted by $\boldsymbol{R}$) is a measure of the system when its subjected to random stress X_2 and has strength of X_1. The reliability of the system $\boldsymbol{R}$ can be defined as

$$\begin{aligned}\boldsymbol{R} = P(X_2 < X_1) &= \int_{-\infty}^{\infty} f(x_1; a_1, \psi) dx_1 \int_{-\infty}^{x_1} f(x_2; a_2, \psi) dx_2 \\ &= \int_{-\infty}^{\infty} f(x_1; a_1, \psi) F(x_1; a_2, \psi) dx_1, \end{aligned} \tag{8}$$

where $F(x_1; a_2, \psi)$ is as given in (3); and $f(x_1; a_1, \psi)$ and $f(x_2; a_2, \psi)$ are defined in (4). These functions can be presented as a mixture representation given as

$$F(x_1; a_2, \psi) = \sum_{l=0}^{\infty} \Omega_w Q(x_1, \psi)^w, \tag{9}$$

$$f(x_1; a_1, \psi) = \sum_{l=0}^{\infty} \Omega_{l+1}(l+1) q(x_1, \psi) Q(x_1, \psi)^l. \tag{10}$$

By taking the product of (9) and (10) gives

$$f(x_1; a_1, \psi) F(x_1; a_2, \psi) = \sum_{l,w=0}^{\infty} \Omega_{l+1} \Omega_w (l+1) m(x_1, \psi) M(x_1, \psi)^{l+w}, \tag{11}$$

where

$$\begin{aligned}\Omega_w &= \sum_{f,g,n=0}^{\infty} \frac{(-1)^{f+n+w} \Gamma(3+2f+g)}{f! g! 2^f \alpha_2^{3+2f} (3+2f) \Gamma(3+2f)} \binom{3+2f+g}{n} \binom{n}{w} \sqrt{\frac{2}{\pi}}, \\ \Omega_{l+1} &= \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+k+l} \Gamma(4+2i+j) \binom{2+2i+j}{k} \binom{k}{l}}{(l+1) 2^i \alpha_1^{3+2i} i! j! \Gamma(4+2i)} \sqrt{\frac{2}{\pi}}.\end{aligned}$$

Using generalized binomial expansion, (11) can be written as

$$\begin{aligned} f(x_1; a_1, \psi) F(x_1; a_2, \psi) &= \sum_{l,w,p,s=0}^{\infty} \Omega_{l+1} \Omega_w (l+1)(-1)^{p+s} \binom{l+w}{k} \\ &\quad \times \binom{p}{s} m(x_1, \psi) M(x_1, \psi)^s \\ &= \sum_{l,w,p,s=0}^{\infty} \frac{\Omega_{l+1} \Omega_w (l+1)(-1)^{p+s} \binom{l+w}{k} \binom{p}{s}}{(s+1)} \\ &\quad \times (s+1) m(x_1, \psi) M(x_1, \psi)^s \end{aligned}$$

$$=\sum_{s=0}^{\infty}\omega_{s+1}z_{s+1}(x), \tag{12}$$

where $z_{s+1}=(s+1)m(x_1,\psi)M(x_1,\psi)^s$ and

$$\omega_{s+1}=\frac{\sum_{l,w,p=0}^{\infty}\Omega_{l+1}\Omega_w(l+1)(-1)^{p+s}\binom{l+w}{k}\binom{p}{s}}{(s+1)}.$$

Substituting (12) into (8) becomes

$$\boldsymbol{R}=\sum_{s=0}^{\infty}\omega_{s+1}\int_{-\infty}^{\infty}z_{s+1}(x)dx_1.$$

Therefore, the stress-strength model of M-D distribution is derived by inserting (1) for $\delta=\delta_2$ and (2) when $\delta=\delta_1$ into (8) as

$$\begin{aligned}\boldsymbol{R}&=\sum_{s=0}^{\infty}\omega_{s+1}(s+1)\lambda t\delta_1\int_0^{\infty}x_1^{-\lambda-1}\left(1+tx_1^{-\lambda}\right)^{-\delta_1-1}\left(\left(1+tx_1^{-\lambda}\right)^{-\delta_2}\right)^s dx_1\\&=\sum_{s=0}^{\infty}\omega_{s+1}(s+1)\lambda t\delta_1\int_0^{\infty}x_1^{-\lambda-1}\left(1+tx_1^{-\lambda}\right)^{-\delta_1-s\delta_2-1}dx_1.\end{aligned} \tag{13}$$

Let $A=tx_1^{-\lambda}$ then (13) gives

$$\boldsymbol{R}=\sum_{s=0}^{\infty}\omega_{s+1}(s+1)\delta_1\int_0^{\infty}(1+A)^{-\delta_1-s\delta_2-1}dA. \tag{14}$$

We can also let $1+A=\frac{1}{B}$ then (14) gives the stress-strength model of the M-D distribution given by

$$\boldsymbol{R}=\sum_{s=0}^{\infty}\omega_{s+1}(s+1)\delta_1\int_0^1 B^{\delta_1+s\delta_2-1}dB=\sum_{s=0}^{\infty}\omega_{s+1}(s+1)\left(\frac{\delta_1}{\delta_1+s\delta_2}\right).$$

3.2 Rényi entropy

The Rényi entropy of the random variable X is defined as

$$R_\varrho(x)=\frac{1}{1-\varrho}\left[\int_{-\infty}^{\infty}f(x)^\varrho dx\right],\qquad \varrho>0,\varrho\neq 1;x\in\Re, \tag{15}$$

where $f(x)$ is the pdf defined in (6). The term $f(x)^\varrho$ in (15) can be presented as

$$f(x)^\varrho=\sum_{l=0}^{\infty}\varphi_l m(x;\psi)^\varrho M(x,\psi)^l,$$

where $\varphi_l=\sum_{i,j,k=0}^{\infty}A_{i,j}(-1)^{k+l}\binom{2\varrho+2i+j}{k}\binom{k}{l}$ and

$$A_{i,j}=\left(\frac{2}{\alpha^3\sqrt{2\pi}}\right)^\varrho\frac{(-\varrho)^i}{i!(2\alpha^2)^i}\frac{\Gamma(4\varrho+2i+j)}{j!\Gamma(4\varrho+2i)}.$$

The integral part in (15) can be simplified as

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)^{\varrho}dx &= (t\delta\lambda)^{\varrho}\int_{0}^{\infty}\left(x^{-\lambda-1}\left(1+tx^{-\lambda}\right)^{-\delta-1}\right)^{\varrho}\left(\left(1+tx^{-\lambda}\right)^{-\delta}\right)^{l}dx\\ &= (t\delta\lambda)^{\varrho}\int_{0}^{\infty} x^{-\varrho(\lambda+1)}\left(1+tx^{-\lambda}\right)^{-\varrho(\delta+1)-\varrho l}dx. \end{aligned} \tag{16}$$

It becomes

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)^{\varrho}dx &= \frac{(t\delta\lambda)^{\varrho}}{t\lambda}\int_{0}^{\infty}\left(\left(\frac{t}{A}\right)^{1/\lambda}\right)^{-\varrho(\lambda+1)+\lambda+1}(1+A)^{-\varrho(\delta+1)-\varrho l}dA\\ &= \frac{(t\delta\lambda)^{\varrho}}{t\lambda}t^{\frac{-\varrho(\lambda+1)+\lambda+1}{\lambda}}\int_{0}^{\infty}\left(A^{1/\lambda}\right)^{\varrho(\lambda+1)-\lambda-1}(1+A)^{-\varrho(\delta+1)-\varrho l}dA\\ &= \frac{(t\delta\lambda)^{\varrho}}{\lambda}t^{\frac{-\varrho(\lambda+1)+1}{\lambda}}\beta\left(\frac{\varrho(\lambda+1)-1}{\lambda},\quad 1-\varrho(\delta+1)-\varrho l\right). \end{aligned} \tag{17}$$

Therefore, the Rényi entropy of Maxwell-Dagum distribution is obtained by inserting (17) into (15) as

$$R_{\varrho}(x) = \frac{1}{1-\varrho}\left[\frac{(t\delta\lambda)^{\varrho}}{\lambda t^{\frac{\varrho(\lambda+1)-1}{\lambda}}}\sum_{l=0}^{\infty}\varphi_l\left\{\beta\left(\frac{\varrho(\lambda+1)-1}{\lambda},\quad 1-\varrho(\delta+1)-\varrho l\right)\right\}\right].$$

3.3 q-entropy

The q-entropy of a random variable X with pdf $f(x)$ in (6) is given by

$$Q_{\upsilon}(x) = \frac{1}{\upsilon-1}\left[1-\int_{-\infty}^{\infty} f(x)^{\upsilon}dx\right],\qquad \upsilon\neq 1; x\in\Re. \tag{18}$$

The integral part of (18) has been defined in (17). By substituting (17) into (18) for $\varrho=\upsilon$ gives the q-entropy of Maxwell-Dagum distribution

$$Q_{\upsilon}(x) = \frac{1}{\upsilon-1}\left[1-\sum_{l=0}^{\infty}\varphi_l\left\{\frac{(t\delta\lambda)^{\upsilon}}{\lambda t^{\frac{\upsilon(\lambda+1)-1}{\lambda}}}\beta\left(\frac{\upsilon(\lambda+1)-1}{\lambda},\quad 1-\upsilon(\delta+1)-\upsilon l\right)\right\}\right].$$

3.4 Order statistics

Let $X_1,\ldots,X_n$ be independent and identically distributed random variables drawn with sample size n from the cdf and pdf given in (5) and (6), respectively. Then, the τth order statistic of those variables denoted $f_{\tau,n}(x)$ is given as

$$\begin{aligned} f_{\tau,n}(x) &= \frac{n!f(x)}{(\tau-1)!(n-\tau)!}F(x)^{\tau-1}\left[1-F(x)\right]^{n-\tau}\\ &= \frac{n!f(x)}{(\tau-1)!(n-\tau)!}\sum_{l=0}^{n-\tau}(-1)^l\binom{n-\tau}{l}F(x)^{\tau+l-1}. \end{aligned}$$

By (5) and (6), the order statistic of the Maxwell-Dagum distribution is given as

$$f_{\tau,n}(x)=\frac{x^{-\lambda-1}\left(1+tx^{-\lambda}\right)^{-3\delta-1}}{\left(1-\left(1+tx^{-\lambda}\right)^{-\delta}\right)^{4}}\exp\left(-\frac{1}{2a^2}\left(\frac{\left(1+tx^{-\lambda}\right)^{-\delta}}{1-\left(1+tx^{-\lambda}\right)^{-\delta}}\right)^2\right)$$
$$\times\sum_{l=0}^{n-\tau}\Omega_l\left[\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{2a^2}\left(\frac{\left(1+tx^{-\lambda}\right)^{-\delta}}{1-\left(1+tx^{-\lambda}\right)^{-\delta}}\right)^2\right)\right]^{\tau+l-1},$$

where $\Omega_l=\frac{2t\lambda\delta n!(-1)^l\binom{n-\tau}{l}}{a^3\sqrt{2\pi}(\tau-1)!(n-\tau)!}$.

4 Parameter estimation

This section provides the methods used to estimate the unknown parameters of Maxwell-Dagum distribution, which include MLE, MPS, LSE and WLSE.

4.1 Maximum likelihood estimation method

Let $x_1, x_2, \ldots, x_n$ denotes the random sample of size n from M-D distribution with parameters a, t, λ and δ and let $\varphi=(a,t,\delta,\lambda)^T$ be the $p\times 1$ vector parameter. The MLE of the parameter φ is determined by using the likelihood function as

$$L=\left(\frac{2t\lambda\delta}{a^3\sqrt{2\pi}}\right)^n\prod_{i=1}^{n}\frac{x_i^{-\lambda-1}A_i^{-\delta-1}}{\left(1-A_i^{-\delta}\right)^2}\left(\frac{A_i^{-\delta}}{1-A_i^{-\delta}}\right)^2\exp\left(-\frac{1}{2a^2}\left(\frac{A_i^{-\delta}}{1-A_i^{-\delta}}\right)^2\right),\quad (19)$$

where $A_i=1+tx_i^{-\lambda}$. The log-likelihood function of (19) is given by

$$\ell=n\log(2)+n\log(t)+n\log(\lambda)+n\log(\delta)-\frac{n}{2}\log(2\pi)-3n\log(a)-(\lambda+1)$$
$$\times\sum_{i=1}^{n}\log(x_i)-(\delta+1)\sum_{i=1}^{n}\log(A_i)-2\sum_{i=1}^{n}\log\left(1-A_i^{-\delta}\right)$$
$$+2\sum_{i=1}^{n}\log\left(\frac{A_i^{-\delta}}{1-A_i^{-\delta}}\right)-\frac{1}{2a^2}\sum_{i=1}^{n}\left(\frac{A_i^{-\delta}}{1-A_i^{-\delta}}\right)^2.\quad (20)$$

The estimates of the parameter φ is determined by differentiating (20) partially with respect to parameters a, t, δ and λ as given respectively in the following

$$\frac{\partial\ell}{\partial a}=-\frac{3n}{a}+\frac{1}{a^3}\sum_{i=1}^{n}\left(\frac{A_i^{-\delta}}{1-A_i^{-\delta}}\right)^2,$$
$$\frac{\partial\ell}{\partial t}=\frac{n}{t}-(\delta+1)\sum_{i=1}^{n}\left(\frac{x_i^{-\lambda}}{A_i}\right)-2\delta\sum_{i=1}^{n}\left(\frac{x_i^{-\lambda}A_i^{-\delta-1}}{1-A_i^{-\delta}}\right)-2\delta\sum_{i=1}^{n}\left(\frac{x_i^{-\lambda}}{A_i\left(1-A_i^{-\delta}\right)}\right)$$
$$+\frac{\delta}{a^2}\sum_{i=1}^{n}\left(\frac{x_i^{-\lambda}A_i^{-2\delta-1}}{\left(1-A_i^{-\delta}\right)^3}\right),$$

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} \log(x_i) + t(\delta+1) \sum_{i=1}^{n} \left(\frac{x_i^{-\lambda} \ln x_i}{A_i} \right) + 2t\delta \sum_{i=1}^{n} \left(\frac{A_i^{-\delta-1} x_i^{-\lambda} \ln x_i}{1 - A_i^{-\delta}} \right)$$
$$+2t\delta \sum_{i=1}^{n} \left(\frac{x_i^{-\lambda} \ln x_i}{A_i \left(1 - A_i^{-\delta}\right)} \right) - \frac{t\delta}{a^2} \sum_{i=1}^{n} \left(\frac{A_i^{-2\delta-1} x_i^{-\lambda} \ln x_i}{(1 - A_i)^{-\delta})^3} \right),$$
$$\frac{\partial \ell}{\partial \delta} = \frac{n}{\delta} - \sum_{i=1}^{n} \log A_i - 2 \sum_{i=1}^{n} \left(\frac{A_i^{-\delta} \ln A_i}{1 - A_i^{-\delta}} \right) - 2 \sum_{i=1}^{n} \left(\frac{\ln A_i}{1 - A_i^{-\delta}} \right) + \frac{1}{a^2} \sum_{i=1}^{n} \left(\frac{A_i^{-2\delta} \ln A_i}{\left(1 - A_i^{-\delta}\right)^3} \right).$$

The MLEs of the parameter φ is derived by setting these equations to zero. As observed, theses equations are non-linear and they cannot be derived analytically, therefore statistical software such as Matlab, R-package, and so on, could be employed to obtain the estimates of the parameters. The second derivatives of log-likelihood function are presented in Appendix.

4.2 Maximum product of spacings method

The maximum product of spacings method is an alternative to MLE used in estimating the parameters of continuous probability distributions. Suppose $F(x_{(i)}|a,\lambda,\delta,t)$ and $F(x_{(i-1)}|a,\lambda,\delta,t)$ for $i = 1, 2, \ldots, n+1$ are the cdfs of M-D distribution. Now, we can let

$$D_i(a,\lambda,\delta,t) = F(x_{(i)}|a,\lambda,\delta,t) - F(x_{(i-1)}|a,\lambda,\delta,t),$$

be the uniform spacings of a random sample generated from the M-D distribution. Then we can note that $F(x_{(0)}|a,\lambda,\delta,t) = 0$ and $F(x_{(n+1)}|a,\lambda,\delta,t) = 1$, this implies $\sum_{i=1}^{n+1} D_i(a,\lambda,\delta,t) = 1$. Therefore, the estimates of the parameters $\left(\hat{a},\hat{\lambda},\hat{\delta},\hat{t}\right)$ using MPS method are obtained by maximizing the logarithm of geometric mean (GM) of the spacings as

$$C\left(a,\lambda,\delta,t\right) = \log\left(GM\left(a,\lambda,\delta,t\right)\right), \tag{21}$$

where $GM\left(a,\lambda,\delta,t\right) = \left[\prod_{i=1}^{n+1} D_i(a,\lambda,\delta,t)\right]^{\frac{1}{n+1}}$.

The estimates $\hat{a}$, $\hat{\lambda}$, $\hat{\delta}$ and $\hat{t}$ of the parameters of M-D distribution are obtained by differentiating (21) partially with respect to parameters a, λ, δ and t, and setting the result to zero as

$$\frac{\partial C\left(a,\lambda,\delta,t\right)}{\partial_{a,\lambda,\delta,t}} = \frac{1}{n+1} \sum_{i=1}^{n+1} \left(\frac{1}{D_i(a,\lambda,\delta,t)} \right) \frac{\partial D_i\left(a,\lambda,\delta,t\right)}{\partial_{a,\lambda,\delta,t}} = 0.$$

4.3 Least squares estimation method

The LSE method is another technique used for estimating the unknown parameters of probability distribution. Let $X_{(i)}$, $i = 1, 2, \ldots$ denotes the order statistics drawn from the random sample of size n with the cdf as $F(x_i|a,\lambda,\delta,t)$. The estimates of parameters of M-D distribution using LSE method are obtained by minimizing as

$$L\left(a,\lambda,\delta,t\right) = \sum_{i=1}^{n} \left[F(x_{(i)}|a,\lambda,\delta,t) - \frac{i}{n+1} \right]^2, \tag{22}$$

with respect to parameters a, λ, δ and t.

Therefore, the estimates $\hat{a}$, $\hat{\lambda}$, $\hat{\delta}$ and $\hat{t}$ can be obtained by differentiating (22) partially with respect to parameters as a, λ, δ and t, and setting the result to zero as

$$\frac{\partial L\left(a, \lambda, \delta, t\right)}{\partial_{a,\lambda,\delta,t}} = \sum_{i=1}^{n} \left[F(x_{(i)}|a, \lambda, \delta, t) - \frac{i}{n+1}\right] \frac{\partial F(x_{(i)}|a, \lambda, \delta, t)}{\partial_{a,\lambda,\delta,t}} = 0.$$

4.4 Weighted least squares estimation method

The WLSE method is modified version of LSE, and the estimation using WLSE followed similar procedure to LSE method by multiplying the LSE by a wieight. The estimation is done by minimizing

$$W\left(a, \lambda, \delta, t\right) = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{(i)}|a, \lambda, \delta, t) - \frac{i}{n+1}\right]^2, \tag{23}$$

with respect to parameters a, λ, δ and t.

Therefore, the estimates of the parameters of M-D distribution are obtained by differentiating (23) partially with respect to parameters as a, λ, δ and t, and setting result to zero as

$$\frac{\partial W\left(a, \lambda, \delta, t\right)}{\partial_{a,\lambda,\delta,t}} = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{(i)}|a, \lambda, \delta, t) - \frac{i}{n+1}\right] \frac{\partial F(x_{(i)}|a, \lambda, \delta, t)}{\partial_{a,\lambda,\delta,t}} = 0. \tag{24}$$

5 Simulation study

We provide a simulation study by using MPS, LSE and WLSE methods in order to evaluate the performance of the parameters of M-D distribution and compare the efficiency among them. The simulation study is conducted from the basis of quantile function given in (7) for sample sizes $n = 10, 20, 50, 250, 500$ and 1000; and parameter values $a = 1.5$, $\lambda = 1$, $\delta = 1.5$ and $t = 2$. The simulation was repeated 1,000 times in which the mean, bias, variance and mean square error (MSE) were obtained. This simulation can be done through the following procedures:

i. Set a random sample of size n from the M-D distribution with the stated parameter values.

ii. Compute the parameters, say $\hat{a}$, $\hat{\lambda}$, $\hat{\delta}$ and $\hat{t}$ by using MPS, LSE and WLSE.

iii. Repeat steps 1 and 2, 1,000 times, and lastly.

iv. Evaluate the mean, bias, variance and MSE.

Repeat for $n = 10, 20, 50, 250, 500$ and 1000. The mean, bias, variance and MSE are obtained. The results for simulation 1 of the Maxwell-Dagum distribution for sample sizes 10, 20, 50, 250, 500,1000 and parameter values $a = 1.5$, $\lambda = 1$, $\delta = 1.5$ and $t = 2$ using MPS, LSE and WLSE methods are given in Tables 1, 2 and 3, respectively.

It observed from Table 1 that the mean of each estimates get closer to the true parameter values as the sample size increases. The variance and MSE of each estimates decreases with increase in sample size. Similarly, it seen from Table 2 that the mean of

Table 1: Simulation results of the Maxwell-Dagum distribution using MPS method of estimation for $a = 1.5$, $\lambda = 1$, $\delta = 1.5$ and $t = 2$.

sample size	parameter	mean	bias	variance	MSE
$n = 10$	a	1.6768	0.1768	0.1656	0.1969
	λ	0.9280	-0.0720	0.0314	0.0366
	δ	1.3041	-0.1959	0.3265	0.3649
	t	2.1099	0.1099	0.1731	0.1851
$n = 20$	a	1.5881	0.0881	0.1306	0.1384
	λ	0.9515	-0.0485	0.0161	0.0185
	δ	1.3657	-0.1343	0.1885	0.2065
	t	2.0888	0.0888	0.1089	0.1168
$n = 50$	a	1.5542	0.0542	0.0802	0.0831
	λ	0.9703	-0.0297	0.0087	0.0096
	δ	1.3891	-0.1109	0.0861	0.0984
	t	2.0747	0.0747	0.0666	0.0722
$n = 250$	a	1.5247	0.0247	0.0292	0.0298
	λ	0.9914	-0.0086	0.0026	0.0026
	δ	1.4424	-0.0576	0.0265	0.0298
	t	2.0515	0.0515	0.0277	0.0304
$n = 500$	a	1.5237	0.0237	0.0214	0.0220
	λ	0.9958	-0.0042	0.0014	0.0014
	δ	1.4568	-0.0432	0.0179	0.0197
	t	2.0391	0.0391	0.0177	0.0192
$n = 1000$	a	1.5275	0.0275	0.0143	0.0151
	λ	0.9995	-0.0005	0.0007	0.0007
	δ	1.4599	-0.0401	0.0129	0.0145
	t	2.0394	0.0394	0.0114	0.0130

Table 2: Simulation results of the Maxwell-Dagum distribution using LSE method of estimation for $a = 1.5$, $\lambda = 1$, $\delta = 1.5$ and $t = 2$.

sample size	parameter	mean	bias	variance	MSE
$n = 10$	a	1.5616	0.0616	0.0854	0.0891
	λ	1.0063	0.0063	0.0206	0.0207
	δ	1.4936	-0.0064	0.1590	0.1590
	t	2.0822	0.0822	0.0899	0.0966
$n = 20$	a	1.5690	0.0690	0.0592	0.0640
	λ	1.0000	0.0000	0.0118	0.0118
	δ	1.4537	-0.0463	0.0991	0.1013
	t	2.0664	0.0664	0.0604	0.0648
$n = 50$	a	1.5457	0.0457	0.0380	0.0401
	λ	1.0014	0.0014	0.0070	0.0070
	δ	1.4535	-0.0465	0.0509	0.0531
	t	2.0625	0.0625	0.0376	0.0415
$n = 250$	a	1.5289	0.0289	0.0124	0.0132
	λ	1.0047	0.0047	0.0022	0.0022
	δ	1.4799	-0.0201	0.0106	0.0110
	t	2.0250	0.0250	0.0114	0.0120
$n = 500$	a	1.5192	0.0192	0.0066	0.0070
	λ	1.0050	0.0050	0.0013	0.0013
	δ	1.4882	-0.0118	0.0050	0.0051
	t	2.0211	0.0211	0.0064	0.0069
$n = 1000$	a	1.5155	0.0155	0.0037	0.0039
	λ	1.0039	0.0039	0.0007	0.0007
	δ	1.4909	-0.0091	0.0024	0.0025
	t	2.0149	0.0149	0.0040	0.0043

Table 3: Simulation results of the Maxwell-Dagum distribution using WLSE method of estimation for $a = 1.5$, $\lambda = 1$, $\delta = 1.5$ and $t = 2$.

sample size	parameter	mean	bias	variance	MSE
$n = 10$	a	1.5744	0.0744	0.0850	0.0906
	λ	0.9625	-0.0375	0.0202	0.0216
	δ	1.3977	-0.1023	0.1564	0.1668
	t	2.0549	0.0549	0.1015	0.1045
$n = 20$	a	1.5682	0.0682	0.0667	0.0714
	λ	0.9904	-0.0096	0.0133	0.0133
	δ	1.4367	-0.0633	0.1070	0.1110
	t	2.0662	0.0662	0.0716	0.0760
$n = 50$	a	1.5387	0.0387	0.0433	0.0448
	λ	1.0025	0.0025	0.0076	0.0076
	δ	1.4622	-0.0378	0.0545	0.0559
	t	2.0688	0.0688	0.0506	0.0553
$n = 250$	a	1.5304	0.0304	0.0155	0.0165
	λ	1.0054	0.0054	0.0022	0.0022
	δ	1.4844	-0.0156	0.0112	0.0114
	t	2.0213	0.0213	0.0140	0.0145
$n = 500$	a	1.5173	0.0173	0.0078	0.0081
	λ	1.0041	0.0041	0.0012	0.0013
	δ	1.4897	-0.0103	0.0048	0.0049
	t	2.0181	0.0181	0.0071	0.0074
$n = 1000$	a	1.5138	0.0138	0.0041	0.0042
	λ	1.0036	0.0036	0.0007	0.0007
	δ	1.4908	-0.0092	0.0024	0.0025
	t	2.0162	0.0162	0.0044	0.0047

Table 4: Summary of the estimates of the parameters of Maxwell-Dagum distribution using different methods of estimation for various parameter values.

parameter	sample size (n)	MPS		LSE		WSLE	
		mean	MSE	mean	MSE	mean	MSE
$a = 1.5$	10	1.6768	0.1969	1.5616	0.0891	1.5744	0.0906
	20	1.5881	0.1384	1.5690	0.0640	1.5682	0.0714
	50	1.5542	0.0831	1.5457	0.0401	1.5387	0.0448
	250	1.5247	0.0298	1.5289	0.0132	1.5304	0.0165
	500	1.5237	0.0220	1.5192	0.0070	1.5173	0.0081
	1000	1.5275	0.0151	1.5155	0.0039	1.5138	0.0042
$\lambda = 1$	10	0.9280	0.0366	1.0063	0.0207	0.9625	0.0216
	20	0.9515	0.0185	1.0000	0.0118	0.9904	0.0133
	50	0.9703	0.0096	1.0014	0.0070	1.0025	0.0076
	250	0.9914	0.0026	1.0047	0.0022	1.0054	0.0022
	500	0.9958	0.0014	1.0050	0.0013	1.0041	0.0013
	1000	0.9995	0.0007	1.0039	0.0007	1.0036	0.0007
$\delta = 1.5$	10	1.3041	0.3649	1.4936	0.1590	1.3977	0.1668
	20	1.3657	0.2065	1.4537	0.1013	1.4367	0.1110
	50	1.3891	0.0984	1.4535	0.0531	1.4622	0.0559
	250	1.4424	0.0298	1.4799	0.0110	1.4844	0.0114
	500	1.4568	0.0197	1.4882	0.0051	1.4897	0.0049
	1000	1.4599	0.0145	1.4909	0.0025	1.4908	0.0025
$t = 2$	10	2.1099	0.1851	2.0822	0.0966	2.0549	0.1045
	20	2.0888	0.1168	2.0664	0.0648	2.0662	0.0760
	50	2.0747	0.0722	2.0625	0.0415	2.0688	0.0553
	250	2.0515	0.0304	2.0250	0.0120	2.0213	0.0145
	500	2.0391	0.0192	2.0211	0.0069	2.0181	0.0074
	1000	2.0394	0.0130	2.0149	0.0043	2.0162	0.0047

Table 5: Simulation results of the Maxwell-Dagum distribution using MPS method of estimation for $a = 2.5$, $\lambda = 1$, $\delta = 1.5$ and $t = 2$.

sample size	parameter	mean	bias	variance	MSE
$n = 10$	a	2.6694	0.1694	0.2624	0.2911
	λ	0.9265	-0.0735	0.0288	0.0342
	δ	1.3074	-0.1926	0.4005	0.4376
	t	2.1349	0.1349	0.2428	0.2610
$n = 20$	a	2.6487	0.1487	0.1867	0.2088
	λ	0.9514	-0.0486	0.0158	0.0182
	δ	1.3282	-0.1718	0.2144	0.2439
	t	2.1033	0.1033	0.1763	0.1870
$n = 50$	a	2.6251	0.1251	0.1302	0.1458
	λ	0.9703	-0.0301	0.0076	0.0085
	δ	1.3805	-0.1195	0.1022	0.1165
	t	2.0317	0.0317	0.1079	0.1089
$n = 250$	a	2.5492	0.0492	0.0304	0.0328
	λ	0.9894	-0.0106	0.0022	0.0023
	δ	1.4595	-0.0405	0.0183	0.0200
	t	1.9878	-0.0122	0.0304	0.0305
$n = 500$	a	2.5326	0.0326	0.0196	0.0207
	λ	0.9939	-0.0061	0.0011	0.0011
	δ	1.4748	-0.0252	0.0089	0.0095
	t	1.9911	-0.0089	0.0145	0.0155
$n = 1000$	a	2.5206	0.0206	0.0113	0.0117
	λ	0.9966	-0.0034	0.0006	0.0006
	δ	1.4861	-0.0139	0.0038	0.0040
	t	1.9930	-0.0070	0.0077	0.0078

Table 6: Simulation results of the Maxwell-Dagum distribution using LSE method of estimation for $a = 2.5$, $\lambda = 1$, $\delta = 1.5$ and $t = 2$.

sample size	parameter	mean	bias	variance	MSE
$n = 10$	a	2.6076	0.1076	0.1402	0.1518
	λ	1.0086	0.0086	0.0141	0.0142
	δ	1.5079	0.0079	0.1657	0.1657
	t	2.0591	0.0591	0.1004	0.1039
$n = 20$	a	2.5913	0.0913	0.1089	0.1172
	λ	1.0015	0.0015	0.0089	0.0089
	δ	1.4757	-0.0243	0.0976	0.0982
	t	2.0505	0.0505	0.0836	0.0862
$n = 50$	a	2.5780	0.0780	0.0768	0.0829
	λ	1.0030	0.0030	0.0054	0.0054
	δ	1.4701	-0.0299	0.0461	0.0470
	t	2.0374	0.0374	0.0569	0.0583
$n = 250$	a	2.5606	0.0606	0.0256	0.0293
	λ	1.0042	0.0042	0.0019	0.0020
	δ	1.4838	-0.0162	0.0110	0.0113
	t	2.0113	0.0113	0.0207	0.0208
$n = 500$	a	2.5532	0.0532	0.0177	0.0205
	λ	1.0045	0.0045	0.0012	0.0012
	δ	1.4862	-0.0138	0.0057	0.0059
	t	2.0114	0.0114	0.0148	0.0149
$n = 1000$	a	2.5374	0.0374	0.0124	0.0138
	λ	1.0039	0.0039	0.0007	0.0007
	δ	1.4922	-0.0078	0.0028	0.0028
	t	2.0084	0.0084	0.0075	0.0076

Table 7: Simulation results of the Maxwell-Dagum distribution using WLSE method of estimation for $a = 2.5$, $\lambda = 1$, $\delta = 1.5$ and $t = 2$.

sample size	parameter	mean	bias	variance	MSE
$n = 10$	a	2.6198	0.1198	0.1535	0.1679
	λ	0.9706	-0.0294	0.0157	0.0166
	δ	1.4005	-0.0995	0.1596	0.1695
	t	2.0413	0.0413	0.1148	0.1165
$n = 20$	a	2.5899	0.0899	0.1379	0.1460
	λ	0.9931	-0.0069	0.0106	0.0107
	δ	1.4525	-0.0475	0.1069	0.1091
	t	2.0558	0.0558	0.0993	0.1024
$n = 50$	a	2.5889	0.0889	0.0863	0.0942
	λ	1.0053	0.0053	0.0060	0.0060
	δ	1.4767	-0.0233	0.0499	0.0505
	t	2.0348	0.0348	0.0565	0.0587
$n = 250$	a	2.5669	0.0669	0.0275	0.0320
	λ	1.0063	0.0063	0.0020	0.0020
	δ	1.4899	-0.0101	0.0091	0.0092
	t	2.0078	0.0078	0.0204	0.0204
$n = 500$	a	2.5630	0.0630	0.0169	0.0209
	λ	1.0052	0.0052	0.0011	0.0011
	δ	1.4829	-0.0171	0.0048	0.0051
	t	2.0111	0.0111	0.0121	0.0122
$n = 1000$	a	2.5433	0.0433	0.0100	0.0118
	λ	1.0038	0.0038	0.0006	0.0006
	δ	1.4866	-0.0134	0.0020	0.0022
	t	2.0108	0.0108	0.0067	0.0068

each estimate approaches true parameter values, and their corresponding bias, variance and MSE decrease with increase in sample size. It is also observed in Table 3 that the mean of each estimate decreases as the sample size increases. Also, the bias, variance and MSE decrease as the sample size increases.

The summary of the results of simulation 1 using different methods of estimation for the parameters of Maxwell-Dagum distribution setting $a = 1.5$, $\lambda = 1$, $\delta = 1.5$ and $t = 2$ are provided in Table 4.

As observed from the Table 4, for sample size 10, the MSE of each estimate using LSE method produces smaller values followed by WSLE and finally MPS methods. As sample size increases from 20, 50, 250, 500 and 1000, the MSE of each estimate for LSE method still provides the smallest MSE values followed by WSLE and MPS methods. This indicated that the estimation using LSE method provided a better estimates than WSLE and MPS irrespective of sample sizes. We concluded by saying that the LSE method could be a better choice in estimating the parameters of Maxwell-Dagum distribution using simulation study.

For the second simulation, the estimate using parameter a was increased to $a = 2.5$, whereas λ, δ and t remained unchanged. The results of each estimate using MPS, LSE and WLSE methods are provided in Tables 5, 6 and 7 respectively for sample sizes 10, 20, 50, 250, 500 and 1000.

It is observed from Tables 5, 6 and 7 that the mean of each estimate approaches true parameter values irrespective of sample sizes n. Similarly, the variance and MSE of each estimate decreases as sample size increases.

Summary of the simulation results by using different methods of estimation setting $a = 2.5$, $\lambda = 1$, $\delta = 1.5$ and $t = 2$ are provided in Table 8.

Table 8: Summary of the estimates of the parameters of Maxwell-Dagum distribution using different methods of estimation for various parameter values.

parameter	sample size (n)	MPS		LSE		WSLE	
		mean	MSE	mean	MSE	mean	MSE
$a = 2.5$	10	2.6694	0.2911	2.6076	0.1518	2.6198	0.1679
	20	2.6487	0.2088	2.5913	0.1172	2.5899	0.1460
	50	2.6251	0.1458	2.5780	0.0829	2.5889	0.0942
	250	2.5492	0.0328	2.5606	0.0293	2.5669	0.0320
	500	2.5326	0.0207	2.5532	0.0205	2.5630	0.0209
	1000	2.5206	0.0117	2.5374	0.0138	2.5433	0.0118
$\lambda = 1$	10	0.9265	0.0342	1.0086	0.0142	0.9706	0.0166
	20	0.9514	0.0182	1.0015	0.0089	0.9931	0.0107
	50	0.9699	0.0085	1.0030	0.0054	1.0053	0.0060
	250	0.9894	0.0023	1.0042	0.0020	1.0063	0.0020
	500	0.9939	0.0011	1.0045	0.0012	1.0052	0.0011
	1000	0.9966	0.0006	1.0039	0.0007	1.0038	0.0006
$\delta = 1.5$	10	1.3074	0.4376	1.5079	0.1657	1.4005	0.1695
	20	1.3282	0.2439	1.4757	0.0982	1.4525	0.1091
	50	1.3805	0.1165	1.4701	0.0470	1.4767	0.0505
	250	1.4595	0.0200	1.4838	0.0113	1.4899	0.0092
	500	1.4748	0.0095	1.4862	0.0059	1.4829	0.0051
	1000	1.4861	0.0040	1.4922	0.0028	1.4866	0.0022
$t = 2$	10	2.1349	0.2610	2.0591	0.1039	2.0413	0.1165
	20	2.1033	0.1870	2.0505	0.0862	2.0558	0.1024
	50	2.0317	0.1089	2.0374	0.0583	2.0348	0.0587
	250	1.9878	0.0305	2.0113	0.0208	2.0078	0.0204
	500	1.9911	0.0155	2.0114	0.0149	2.0111	0.0122
	1000	1.9930	0.0078	2.0084	0.0076	2.0108	0.0068

The mean and MSE of each estimates using MPS, LSE and WLSE methods are provided in Table 8. For sample sizes 10, 20, 50 and 250, the MSE of each estimate by using LSE method produced smaller MSE values followed by WSLE and then MPS methods. At sample sizes 500 and 1000, the MSE of each parameter using WLSE method provided smaller MSE values followed by LSE and MPS methods. This implies that as sample size goes higher, the estimation using WLSE method could be a better choice in estimating the parameters of Maxwell-Dagum distribution using simulation study.

6 Applications

Real life datasets are used in this section to assess the superiority of the Maxwell-Dagum distribution.

6.1 Data sets

We analyze two data sets in comparing the flexibility of the Maxwell-Dagum distribution against its other competing distributions. The first data set has been studied by Almetwally et al. (2021) and the data represents the mortality rate of COVID-19 belonging to Canada of about 36 days from 10 April, 2020 to 15 May, 2020. The second data was reported in Alqallaf et al. (2015) and the data represents the waiting time of bank customers before service is being rendered and it's measured in minutes. The first and second data sets are presented in Table 9.

Table 9: Data sets

First Data Set.							
3.1091	3.3825	3.1444	3.2135	2.4946	3.5146	4.9274	3.3769
6.8686	3.0914	4.9378	3.1091	3.2823	3.8594	4.0480	4.1685
3.6426	3.2110	2.8636	3.2218	2.9078	3.6346	2.7957	4.2781
4.2202	1.5157	2.6029	3.3592	2.8349	3.1348	2.5261	1.5806
2.7704	2.1901	2.4141	1.9048				
Second Data Set.							
0.8	0.8	1.3	1.5	1.8	1.9	1.9	2.1
2.6	2.7	2.9	3.1	3.2	3.3	3.5	3.6
4.0	4.1	4.2	4.2	4.3	4.3	4.4	4.4
4.6	4.7	4.7	4.8	4.9	4.9	5	5.3
5.5	5.7	5.7	6.1	6.2	6.2	6.2	6.3
6.7	6.9	7.1	7.1	7.1	7.1	7.4	7.6
7.7	8	8.2	8.6	8.6	8.6	8.8	8.8
8.9	8.9	9.5	9.6	9.7	9.8	10.7	10.9
11	11	11.1	11.2	11.2	11.5	11.9	12.4
12.5	12.9	13	13.1	13.3	13.6	13.7	13.9
14.1	15.4	15.4	17.3	17.3	18.1	18.2	18.4
18.9	19	19.9	20.6	21.3	21.4	21.9	23.0
27	31.6	33.1	38.5				

6.2 Competing distributions

The competing distributions considered in this research include Beta-Dagum (B-D) distribution by Domma and Condino (2013), Gamma-Dagum (G-D) distribution by Rodrigues and Silva (2015), Weibull-Dagum (W-D) distribution by Tahir et al. (2018), Extended-Dagum (E-D) distribution by Gomes-Silva et al. (2017), Exponentiated Generalized Exponential-Dagum (EGE-D) distribution by Nasiru et al. (2019), and Topp Leone-Dagum (T-D) distribution by Rasheed (2020).

The pdfs plots for the proposed Maxwell-Dagum distribution and the competing distributions using first and second data sets are provided in Figure 3.

6.3 Information criteria

The information criteria considered in this study include the use of Akaike Information Criterion (AIC), Corrected Akaike Information Criterion (CAIC), and Bayesian Information Criterion (BIC). The model with a minimum value of these criteria is considered the best that fits the data sets. The estimate of each parameter (estimate) and its corresponding standard error (SE), AIC, CAIC, and BIC values for the proposed distribution and other competing distributions by using first and second data sets are presented in Tables 10 and 11.

The estimate, SE, AIC, CAIC, and BIC values for the Maxwell-Dagum distribution and other competing distributions by using first and second data sets are presented in Tables 10 and 11. It is observed from these tables that the Maxwell-Dagum distribution

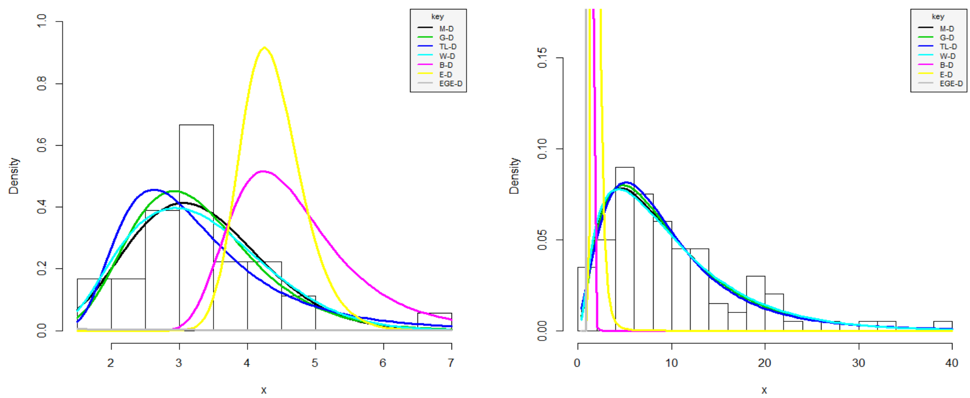

Figure 3: Fitted pdfs for the proposed and competing distributions using first data set (left) and second data set (right).

has a minimum value of AIC, CAIC, and BIC against its competing distributions. This implies that the Maxwell-Dagum distribution could be chosen as the best distribution fitted for both data sets.

7 Conclusion

In this study, we proposed and derived some important properties of the Maxwell-Dagum distribution including stress-strength, Rényi and q-entropies, and order statistics. A simulation study was conducted from the basis of quantile function by using different methods of estimation including MPS, LSE and WSLE methods. Using simulation, the LSE method could be chosen as the best method in estimating the parameters of the Maxwell-Dagum distribution, and for the second simulation, the WLSE could be a better choice. An application to two real life data sets relating to the mortality rate belonging to Canada and the waiting time of bank customers was used to assess the flexibility and potentiality of the Maxwell-Dagum distribution. It found that the Maxwell-Dagum distribution could be preferred as the best distribution used in modelling both data sets.

This research can further be extended as follows:

i. The research considered a compound Maxwell-Dagum distribution as the statistical distribution developed by the same authors in Ishaq and abiodun (2020) and obtained the estimates of its parameters by using different methods of estimation including MLE, MPS, LSE, and WLSE. Further research will focus on developing a parametric regression model from the compound distribution, deriving some of its important features, and obtaining the estimate of the parameters.

ii. The data sets used in this study were complete samples data. Further research should consider incomplete (censored) data sets and apply them in the proposed parametric regression model.

Table 10: The estimate, SE, AIC, CAIC and BIC values for the proposed and competing distributions using first data set.

Model	estimate	SE	AIC	CAIC	BIC
M-D	$a = 0.0976$	0.1557	105.4012	106.6915	111.7352
	$t = 0.1349$	0.0567			
	$\lambda = 0.6493$	0.3008			
	$\delta = 33.1759$	6.7873			
W-D	$b = 1.1833$	0.4382	106.1207	107.411	112.4548
	$t = 0.1264$	0.1080			
	$\lambda = 2.0293$	0.6608			
	$\delta = 71.9413$	20.2296			
T-D	$c = 0.5606$	0.3738	108.7253	110.0157	115.0594
	$t = 0.0979$	0.0706			
	$\lambda = 2.5864$	0.3246			
	$\delta = 337.9011$	12.2972			
G-D	$d = 20.8128$	1.4445	106.1799	107.4702	112.514
	$t = 7.7367$	13.4275			
	$\lambda = 1.2322$	0.4458			
	$\delta = 19.3742$	15.1316			
B-D	$\theta = 8.3027$	2.0350	106.2319	108.2319	114.1495
	$\phi = 2.1335$	19.1927			
	$t = 0.5641$	1.0149			
	$\lambda = 3.9661$	21.8523			
	$\delta = 147.4713$	0.3548			
E-D	$\gamma = 14.3839$	3.6720	106.2567	108.2567	114.1743
	$\tau = 3.3007$	2.0012			
	$t = 0.5243$	0.0961			
	$\lambda = 2.6039$	0.8670			
	$\delta = 192.6840$	4.1978			
EGE-D	$\nu = 0.1632$	0.0159	447.1638	450.0604	456.6649
	$\sigma = 5.3390$	1.1029			
	$\mu = 5.7547$	0.2130			
	$t = 4.8971$	0.8541			
	$\lambda = 0.0309$	0.0015			
	$\delta = 5.3330$	0.1793			

iii. Subsequent research should develop many probability distributions by proposing a Maxwell-Dagum generalized family of distributions. The properties and applications to real data sets will also be considered in the study.

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Table 11: The estimate, SE, AIC, CAIC and BIC values for the proposed and competing distributions using second data set.

Model	estimate	SE	AIC	CAIC	BIC
M-D	$a = 0.6143$	1.4989	641.9026	642.3237	652.3233
	$t = 0.1163$	0.1560			
	$\lambda = 0.4253$	0.2177			
	$\delta = 15.4595$	5.1714			
W-D	$b = 1.7342$	0.7400	641.9126	642.3337	652.3333
	$t = 0.2057$	0.1315			
	$\lambda = 0.5997$	0.2348			
	$\delta = 14.1112$	5.7188			
T-D	$c = 0.1307$	0.2713	643.1858	643.6069	653.6065
	$t = 95.3318$	8.3673			
	$\lambda = 2.0708$	0.2196			
	$\delta = 7.2292$	15.3440			
G-D	$d = 7.6308$	4.5926	642.269	642.6901	652.6897
	$t = 131.4359$	5.5952			
	$\lambda = 1.2701$	0.2413			
	$\delta = 3.1382$	1.3985			
B-D	$\theta = 0.6084$	1.2539	644.1000	644.7383	657.1259
	$\phi = 2.1390$	3.0461			
	$t = 0.5841$	0.2416			
	$\lambda = 9.3835$	20.2932			
	$\delta = 31.9279$	1.1943			
E-D	$\gamma = 0.6038$	1.0626	644.1352	644.7735	657.1611
	$\tau = 1.9882$	2.1556			
	$t = 0.5809$	0.2191			
	$\lambda = 9.7680$	17.8489			
	$\delta = 30.5689$	1.2124			
EGE-D	$\nu = 6.7634$	6.0289	646.1250	647.0282	661.7560
	$\sigma = 11.7648$	10.6343			
	$\mu = 0.2994$	0.1503			
	$t = 1.4451$	0.4103			
	$\lambda = 17.6660$	9.3523			
	$\delta = 13.4061$	16.2329			

Appendix

$$\frac{\partial^2 \ell}{\partial a^2} = \frac{3n}{a^2} - \frac{3}{a^4}\sum_{i=1}^{n} B_i^2,$$

$$\frac{\partial^2 \ell}{\partial t \partial a} = -\frac{2\delta}{a^3}\sum_{i=1}^{n}\left(\frac{x_i^{-\lambda} B_i^2}{A_i\left(1 - A_i^{-\delta}\right)}\right),$$

$$\frac{\partial^2 \ell}{\partial \lambda \partial a} = \frac{2t\delta}{a^3}\sum_{i=1}^{n}\left(\frac{B_i^2 E_i}{A_i}\right),$$

$$\frac{\partial^2 \ell}{\partial \delta \partial a} = -\frac{2}{a^3}\sum_{i=1}^{n}\left(\frac{B_i^2 \ln A_i}{1 - A_i^{-\delta}}\right),$$

$$\frac{\partial^2 \ell}{\partial a \partial t} = -\frac{2\delta}{a^3}\sum_{i=1}^{n}\left(\frac{x_i^{-\lambda} B_i^2}{A_i\left(1 - A_i^{-\delta}\right)}\right),$$

$$
\begin{aligned}
\frac{\partial^2 \ell}{\partial \delta^2} =& -\frac{n}{\delta^2} + 4\sum_{i=1}^{n}\left(\frac{B_i\left(\ln A_i\right)^2}{1-A_i^{-\delta}}\right) - \frac{1}{a^2}\sum_{i=1}^{n}\left(\frac{B_i^2\left(\ln A_i\right)^2\left(2+A_i^{-\delta}\right)}{\left(1-A_i^{-\delta}\right)^2}\right), \\
\frac{\partial^2 \ell}{\partial a \partial \delta} =& -\frac{2}{a^3}\sum_{i=1}^{n}\left(\frac{B_i^2 \ln A_i}{1-A_i^{-\delta}}\right), \\
\frac{\partial^2 \ell}{\partial a \partial \lambda} =& \frac{2t\delta}{a^3}\sum_{i=1}^{n}\left(\frac{B_i^2 E_i}{A_i}\right), \\
\frac{\partial^2 \ell}{\partial t^2} =& \frac{-n}{t^2} + (\delta+1)\sum_{i=1}^{n}\left(\frac{x_i^{-2\lambda}}{A_i}\right) + 2\delta\sum_{i=1}^{n}\left(x_i^{-2\lambda}B_i\left(\frac{\left(1-A_i^{-\delta}\right)(\delta+1)+\delta A_i^{-\delta}}{A_i^2\left(1-A_i^{-\delta}\right)}\right)\right) \\
& +2\delta\sum_{i=1}^{n}\left(x_i^{-2\lambda}\left(\frac{A_i^{-\delta}(\delta-1)+1}{A_i^2\left(1-A_i^{-\delta}\right)^2}\right)\right) - \frac{\delta}{a^2}\sum_{i=1}^{n}\left(x_i^{-2\lambda}B_i^2\right) \\
& \times\left(\frac{\left(1-A_i^{-\delta}\right)(2\delta+1)+3\delta A_i^{-\delta}}{A_i^2\left(1-A_i^{-\delta}\right)^2}\right), \\
\frac{\partial^2 \ell}{\partial \lambda \partial t} =& 2\delta\sum_{i=1}^{n}\left(B_i E_i\right)\left(\frac{\left(1-A_i^{-\delta}\right)\left[1-t(\delta+1)x_i^{-\lambda}A_i^{-1}\right]+t\delta x_i^{-\lambda}A_i^{-\delta-1}}{A_i}\right) \\
& +2\delta\sum_{i=1}^{n}\left(E_i\left(\frac{A_i\left(1-A_i^{-\delta}\right)-tx_i^{-\lambda}\left[A_i^{-\delta}(\delta-1)+1\right]}{A_i^2\left(1-A_i^{-\delta}\right)}\right)\right) \\
& +(\delta+1)\sum_{i=1}^{n}\left(F_i\left(\frac{A_i-tx_i^{-\lambda}}{A_i}\right)\right) \\
& -\frac{\delta}{a^2}\sum_{i=1}^{n}\left(B_i^2 E_i\left(\frac{\left(1-A_i^{-\delta}\right)\left[1-t(2\delta+1)x_i^{-\lambda}A_i^{-1}\right]-3t\delta x_i^{-\lambda}A_i^{-\delta-1}}{A_i\left(1-A_i^{-\delta}\right)}\right)\right), \\
\frac{\partial^2 \ell}{\partial \delta \partial t} =& -\sum_{i=1}^{n}\left(\frac{x_i^{-\lambda}}{A_i}\right) - 2\sum_{i=1}^{n}\left(x_i^{-\lambda}B_i\left(\frac{\left(1-A_i^{-\delta}\right)\left(1-\delta\ln A_i\right)-\delta A_i^{-\delta}\ln A_i}{A_i\left(1-A_i^{-\delta}\right)}\right)\right) \\
& -2\sum_{i=1}^{n}\left(\frac{x_i^{-\lambda}\left(1-A_i^{-\delta}\left(1+\delta\ln A_i\right)\right)}{A_i\left(1-A_i^{-\delta}\right)^2}\right) + \frac{1}{a^2}\sum_{i=1}^{n}\left(x_i^{-\lambda}B_i^2\right) \\
& \times\left(\frac{\left(1-A_i^{-\delta}\right)\left(1-2\delta\ln A_i\right)-3\delta A_i^{-\delta}\ln A_i}{A_i\left(1-A_i^{-\delta}\right)^4}\right), \\
\frac{\partial \ell}{\partial t \partial \lambda} =& (\delta+1)\sum_{i=1}^{n}F_i\left(\frac{A_i-tx_i^{-\lambda}}{A_i}\right) + 2\delta\sum_{i=1}^{n}\left(B_i E_i\right) \\
& \times\left(\frac{\left(1-A_i^{-\delta}\right)\left[1-t\left(1+\delta\right)x_i^{-\lambda}A_i^{-1}-t\delta x_i^{-\lambda}A_i^{-\delta-1}\right]}{A_i}\right) \\
& +2\delta\sum_{i=1}^{n}E_i\left(\frac{A_i\left(1-A_i^{-\delta}\right)-tx_i^{-\lambda}\left[A_i^{-\delta}\left(\delta-1\right)+1\right]}{A_i^2\left(1-A_i^{-\delta}\right)}\right)
\end{aligned}
$$

$$-\frac{\delta}{a^2}\sum_{i=1}^{n}\left(\frac{B_i^2E_i\left(\left(1-A_i^{-\delta}\right)\left[1-t\left(2\delta+1\right)x_i^{-\lambda}A_i^{-1}\right]-3t\delta x_i^{-\lambda}A_i^{-\delta-1}\right)}{A_i\left(1-A_i^{-\delta}\right)}\right),$$

$$\frac{\partial^2\ell}{\partial\lambda^2}=-\frac{n}{\lambda^2}-t(\delta+1)\sum_{i=1}^{n}\left(\frac{F_i\ln x_i}{A_i}\right)-2t\delta\sum_{i=1}^{n}\left(\frac{B_iE_i\ln x_i}{A_i}\right)$$
$$\times\left(\left(1-A_i^{-\delta}\right)\left[\left(1+\delta\right)A_i^{-1}tx_i^{-\lambda}+1\right]-t\delta x_i^{-\lambda}A_i^{-\delta-1}\right)$$
$$-2t\delta\sum_{i=1}^{n}\left(E_i\ln x_i\left(\frac{A_i\left(1-A_i^{-\delta}\right)-tx_i^{-\lambda}\left[A_i^{-\delta}\left(\delta-1\right)+1\right]}{A_i^2\left(1-A_i^{-\delta}\right)}\right)\right)$$
$$+\frac{t\delta}{a^2}\sum_{i=1}^{n}\left(B_i^2E_i\left(\frac{\left(1-A_i^{-\delta}\right)\left(2\ln A_i+\ln x_i\right)-3t\delta x_i^{-\lambda}\ln x_iA_i^{-\delta-1}}{A_i\left(1-A_i^{-\delta}\right)}\right)\right),$$

$$\frac{\partial^2\ell}{\partial\delta\partial\lambda}=t\sum_{i=1}^{n}\left(F_i\right)-2t\sum_{i=1}^{n}\left(\frac{B_iE_i}{A_i}\left(\left(1-A_i^{-\delta}\right)\left(\delta\ln A_i-1\right)+\delta A_i^{-\delta}\ln A_i\right)\right)$$
$$-2t\sum_{i=1}^{n}\left(\frac{E_i}{A_i\left(1-A_i\right)}\left(A_i^{-\delta}\left(\delta\ln A_i+1\right)-1\right)\right)+\frac{t}{a^2}\sum_{i=1}^{n}\left(B_i^2E_i\right)$$
$$\times\left(\frac{\left(1-A_i^{-\delta}\right)\left(2\delta\ln A_i-1\right)+3\delta A_i^{-\delta}\ln A_i}{A_i\left(1-A_i^{-\delta}\right)}\right),$$

$$\frac{\partial^2\ell}{\partial t\partial\delta}=-\sum_{i=1}^{n}\left(\frac{x_i^{-\lambda}}{A_i}\right)-2\sum_{i=1}^{n}\left(\frac{x_i^{-\lambda}B_i\left[\left(1-A_i^{-\delta}\right)\left(1-\delta\ln A_i\right)-\delta A_i^{-\delta}\ln A_i\right]}{A_i\left(1-A_i^{-\delta}\right)}\right)$$
$$-2\sum_{i=1}^{n}\left(\frac{x_i^{-\lambda}\left[1-A_i^{-\delta}\left(1+\delta\ln A_i\right)\right]}{A_i\left(1-A_i^{-\delta}\right)^2}\right)$$
$$+\frac{1}{a^2}\sum_{i=1}^{n}\left(\frac{x_i^{-\lambda}B_i^2\left[\left(1-A_i^{-\delta}\right)\left(1-2\delta\ln A_i\right)-3\delta A_i^{-\delta}\ln A_i\right]}{A_i\left(1-A_i^{-\delta}\right)^4}\right),$$

$$\frac{\partial^2\ell}{\partial\lambda\partial\delta}=-\sum_{i=1}^{n}F_i-2t\sum_{i=1}^{n}\left(\frac{B_iE_i\left[\left(1-A_i^{-\delta}\right)\left(\delta\ln A_i-1\right)+\delta A_i^{-\delta}\ln A_i\right]}{A_i}\right)$$
$$-2t\sum_{i=1}^{n}\left(\frac{E_i\left[A_i^{-\delta}\left(\delta\ln A_i+1\right)-1\right]}{A_i\left(1-A_i^{-\delta}\right)}\right)$$
$$+\frac{t}{a^2}\sum_{i=1}^{n}\left(\frac{tB_i^2E_i\left[\left(1-A_i^{-\delta}\right)\left(2\delta\ln A_i-1\right)+3\delta A_i^{-\delta}\ln A_i\right]}{A_i\left(1-A_i^{-\delta}\right)}\right),$$

where $B_i=\frac{A(x_i)^{-\delta}}{1-A(x_i)^{-\delta}}$, $E_i=\frac{X_i^{-\lambda}\ln x_i}{1-A(x_i)^{-\delta}}$ and $F_i=\frac{X_i^{-\lambda}\ln x_i}{A(x_i)^{-\delta}}$.

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