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Bi-objective optimization problem of a parallel system with random number of units

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Abstract: This paper considers a parallel system that has a random number of units. The number of units follows a power series distribution which includes several distributions such as geometric, logarithmic and zero-truncated Poisson distributions. Pareto distribution is considered for the lifetime distribution of units. The optimal parameter is obtained for the distribution of sample size so that the expected cost is minimized and the whole reliability of the system is maximized. The weighted sum method has been utilized to convert this bi-objective model into a one-objective model. Numerical calculations have been performed to evaluate the obtained results.

Keywords: Bi-objective optimization; Optimal sample size; Parallel system.
Mathematics Subject Classification (2010): 62N05, 90C29.

1 Introduction

Todays, we are dealing with systems that consist of several components and are placed together based on a predetermined structure for a specific purpose, and each component performs a task independently or dependent on other components. These systems are called coherent systems. Among the most famous and simplest coherent systems, we can mention series and parallel systems. A series system works if and only if all the components work. A parallel system works if at least one of the components works.

Reliability is one of the most important characteristics of a system because by identifying the reliability of a system, it is possible to predict the failure time of that system. It is clear that the reliability of a system depends on the structure and reliability of its

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components. Therefore, one way to increase the reliability of a system is to improve the reliability of its components. For this purpose, maintenance and repair activities should be done, which will increase the associated costs. Since cost is always one of the important criteria for decision-making, it is necessary to create a balance between these two criteria.

The problems related to systems have been studied by many researchers. For example, the most economical parallel and k-out-of-n systems were investigated by Nakagawa (1984) and Nakagawa (1985), respectively. To do this, two problems as finding the optimal number of elements and the optimal replacement time were solved by minimizing the mean cost rate. Coit and Smith (1996) studied the reliability and cost optimization problem by using a genetic algorithm in a series-parallel system with multiple choices for each subsystem. The problem of redundancy allocation for a series-parallel system by using a genetic algorithm and with the aim of maximizing the reliability of the system was investigated by Tavakkoli-Moghaddam et al. (2008). A bi-objective reliability-cost optimization problem in a series-parallel system, by considering a combination of objective functions and fuzzy membership functions, was studied by Garg et al. (2014) and Garg (2021). Two optimization problems for a parallel system that consists of dependent components were studied by Eryilmaz and Ozkut (2020). The first problem was finding the number of elements in the system that minimizes the mean cost rate of the system. The second problem was concerned with the optimal replacement time of the system. Considering series and parallel systems with their component's lifetimes follow the discrete Weibull distribution, Barmalzan (2020) obtained some ordering results for comparing these systems. Statistical evidences in lifetimes of sequential r-out-of-n systems, which were modelled by the concept of sequential order statistics, coming from homogeneous exponential populations were considered by Hashempour and Doostparast (2020). Kim and Ahn (2021) discussed one of the reliability optimization problems, i.e, locating the components of a series system. The problem investigated in this research was to minimize the cost function in order to achieve the desired reliability. Obtaining the Birnbaum component importance of the components for a consecutive-(k, k)-out-of-n:F system having two dependent subcomponents was discussed by Estabraqi and Meshkat (2021).

In all mentioned works, number of units was constant and predetermined. However, real systems are complex and large, so we might not know the exact number of units. The optimal number of units in a parallel system and replacement time based on minimizing the mean cost rate were studied by Nakagawa and Zhao (2012). In their paper, the number of units was a random variable from Poisson distribution. By minimizing the mean cost rate, Eryilmaz (2017) computed optimal number of units and replacement time for a parallel system with a random number of units that follows a power series class of distributions. Other works that deal with a system consisting of a random number of components were investigated by Al-Mutairi et al. (2011), Gupta et al. (2012), Hazra et al. (2014) and Ito et al. (2017).

In the present paper, the optimal number of units in a parallel system is the aim of paper, when the number of units is a random variable from a power series class of distributions and the failure time has a Pareto distribution with parameters α and β. As a special case, geometric, logarithmic and zero-truncated Poisson distributions with parameter θ are considered for the random sample size and a bi-objective optimization problem is solved. Since the sample size is random, the decision variable is the parameter of its distribution, i.e, θ. The objective functions in this optimization problem are the expected cost and the reliability of the system. So, the model proposed in this paper has two objective functions as maximizing the reliability of the system and minimizing the associated cost. In order to convert this bi-objective problem into a one-objective problem the weighted sum method is applied.

The rest of the present paper is organized as follows. Section 2 provides the objective functions as well as the constraints of the optimization problem. To solve this constrained bi-objective optimization problem the weighted sum method is utilized. Three cases for the random sample size are considered and the results for each case are derived. Section 3 deals with the numerical computations for illustrating the theoretical results. Finally, the paper concludes in Section 4.

2 Bi-objective optimization problem

Consider a parallel system consisting of N components in which the ith component has a lifetime T_i, $i = 1, \ldots, N$. Let $T_1, \ldots, T_N$ be independent and identically distributed continuous non-negative random variables from a Pareto distribution with parameters α and β with probability density function (pdf) and cumulative distribution function given by

$$\begin{aligned} f_{\alpha,\beta}(t) &= \frac{\alpha\beta^{\alpha}}{t^{\alpha+1}}; \quad t > \beta, \quad \alpha, \beta > 0, \\ F_{\alpha,\beta}(t) &= 1 - \left(\frac{\beta}{t}\right)^{\alpha}; \quad t > \beta, \quad \alpha, \beta > 0, \end{aligned} \tag{1}$$

respectively. This famous and continuous lifetime model was first introduced by Vilfredo Pareto in 1898. After that, Pareto distribution has been used to model population sizes, environmental extrema, and insurance claims. In the past decades, Pareto distribution also has been used for lifetime data. See, for example, Ahmadi and Doostparast (2019), Sauer et al. (2020) and Ahmadi and Almetwally (2021).

Independently, suppose that the number of components, i.e. N, is a random variable having a power series distribution with probability mass function (pmf) given by

$$P(N = n) = \frac{a(n)\theta^n}{\psi(\theta)}, \qquad n = 1, 2, \ldots, \tag{2}$$

where $\psi(\theta) = \sum_{n=1}^{\infty} a(n)\theta^n$ is positive, finite and differentiable and $a(n) \geq 0$ depends on only n. In this case, the lifetime of such a system is equal to $\max(T_1, \ldots, T_N)$. So, the survival function of such parallel system is given by (see, for example, Gupta et al. (2012))

$$\begin{aligned} R_{sys}(t) &= P\left(\max(T_1, \ldots, T_N) > t\right) \\ &= 1 - \sum_{n=1}^{\infty} \left(F_{\alpha,\beta}(t)\right)^n P(N = n) \\ &= 1 - \sum_{n=1}^{\infty} \left(F_{\alpha,\beta}(t)\right)^n \frac{a(n)\theta^n}{\psi(\theta)} \end{aligned}$$

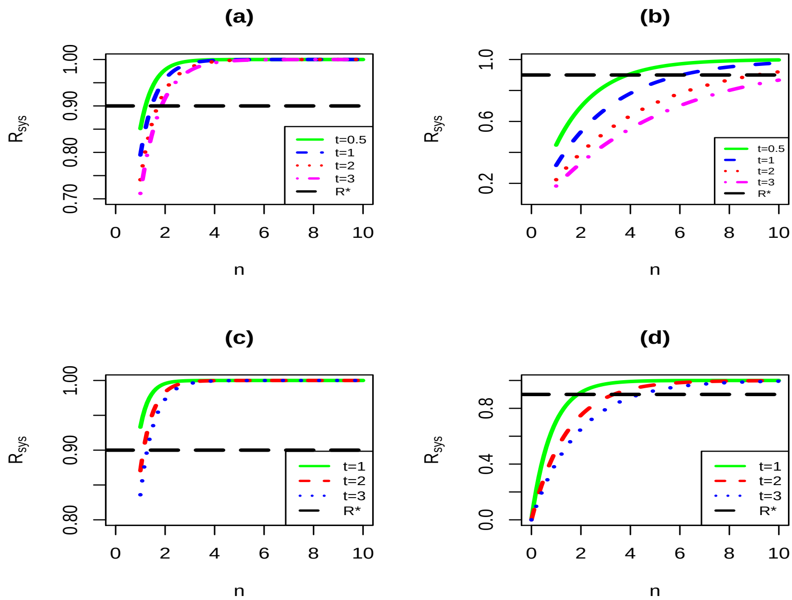

Figure 1: The values of R_{sys} for **(a):** $\alpha = 0.1$ and $\beta = 0.1$, **(b):** $\alpha = 0.5$ and $\beta = 0.1$, **(c):** $\alpha = 0.1$ and $\beta = 0.5$, **(d):** $\alpha = 0.5$ and $\beta = 0.5$, when $N = n$ is fixed and $R^* = 0.9$.

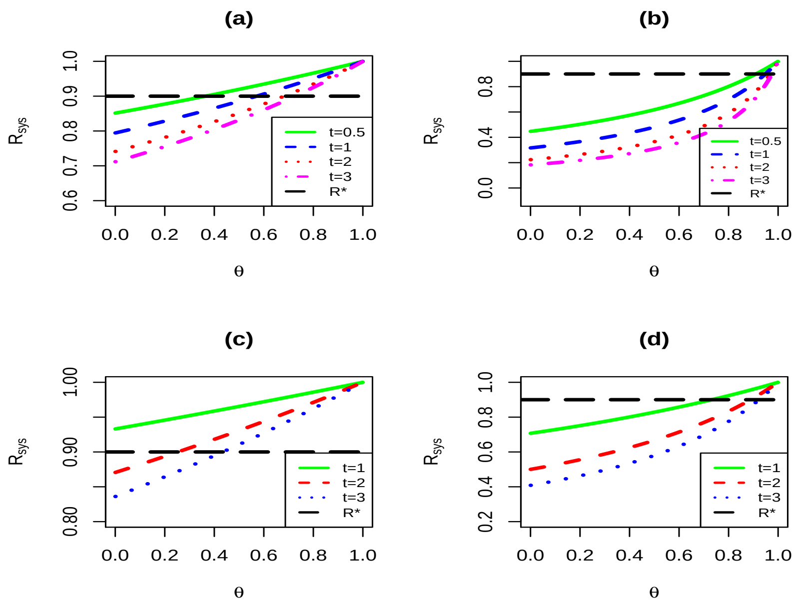

Figure 2: The values of R_{sys} for **(a):** $\alpha = 0.1$ and $\beta = 0.1$, **(b):** $\alpha = 0.5$ and $\beta = 0.1$, **(c):** $\alpha = 0.1$ and $\beta = 0.5$, **(d):** $\alpha = 0.5$ and $\beta = 0.5$, when $N \sim Ge(\theta)$ and $R^* = 0.9$.

$$
\begin{aligned}
&= 1 - \frac{\psi\big(F_{\alpha,\beta}(t)\theta\big)}{\psi(\theta)} \\
&= 1 - \frac{\psi\left[\left(1 - \frac{\beta^{\alpha}}{t^{\alpha}}\right)\theta\right]}{\psi(\theta)},
\end{aligned} \tag{3}
$$

where (1) is used for the last equation.

Another criterion considered in this paper is the expected cost of system, which plays an important role in practices. Assuming c_1 is the acquisition cost of one unit and c_2 includes all costs resulting from system failure, the expected cost associated with the system is given by

$$
C_{sys} = c_1 \mathbb{E}_{\theta}(N) + c_2 = c_1 \theta \frac{\psi'(\theta)}{\psi(\theta)} + c_2, \tag{4}
$$

since, from (2) the mean of the random variable N is

$$
\mathbb{E}_{\theta}(N) = \sum_{n=1}^{\infty} n \frac{a(n)\theta^n}{\psi(\theta)} = \theta \frac{\psi'(\theta)}{\psi(\theta)}. \tag{5}
$$

In this paper, the problem is finding the optimal value for the number of units by maximizing R_{sys} and minimizing C_{sys}, i.e.,

$$
\begin{cases} \text{maximize} & R_{sys} \\ \text{minimize} & C_{sys} \end{cases}
$$

subject to constraints

$$
C_{sys} \leq c^* \quad \text{and} \quad R_{sys} \geq R^*, \tag{6}
$$

where c^* and R^* are pre-fixed values. First notice that here N is a random variable but $\mathbb{E}_{\theta}(N)$ is always fixed that is a function of θ. So, the optimal value for θ, i.e. θ^{opt}, is the aim of paper.

Maximizing R_{sys} is equivalent to minimizing $1 - R_{sys} = \frac{\psi\left(F_{\alpha,\beta}(t)\theta\right)}{\psi(\theta)}$. So, the optimization problem changes to

$$
\text{minimize} \begin{cases} 1 - R_{sys} \\ C_{sys} \end{cases}
$$

subject to constraints in (6).

For multi-objective optimization problems determination of a single solution can be done using different methods such as utility theory, weighted sum method, and so on. To study these methods, for example, we can refer the reader to Ehrgott (2005). One of the most intuitive methods for solving multi-objective optimization problems is to optimize a weighted sum of the objective functions using any methods for single objective optimization. The general approach is to assign to each objective function f_i a weight $w_i > 0$ and minimize the objective function $\sum w_i f_i$ subject to the problem constraints. In the present paper, in order to convert this multi-objective problem into a one-objective, the weighted sum method has been used. Since the cost and reliability functions have different scales, these values are divided by the corresponding optimal

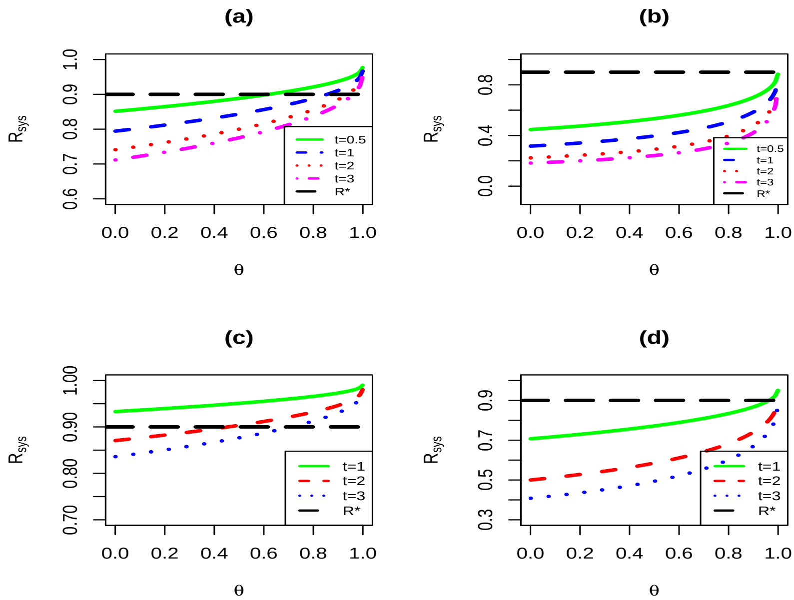

Figure 3: The values of R_{sys} for **(a):** $\alpha = 0.1$ and $\beta = 0.1$, **(b):** $\alpha = 0.5$ and $\beta = 0.1$, **(c):** $\alpha = 0.1$ and $\beta = 0.5$, **(d):** $\alpha = 0.5$ and $\beta = 0.5$, when $N \sim Log(\theta)$ and $R^* = 0.9$.

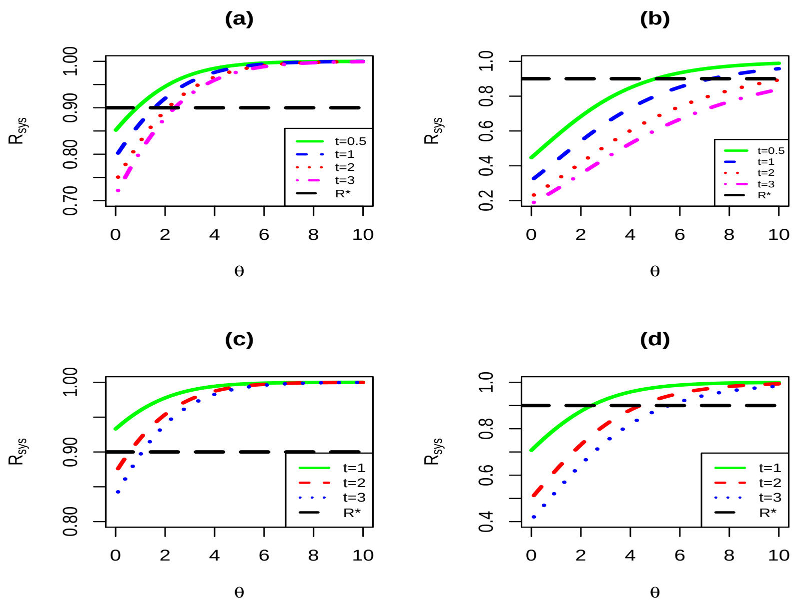

Figure 4: The values of R_{sys} for **(a):** $\alpha = 0.1$ and $\beta = 0.1$, **(b):** $\alpha = 0.5$ and $\beta = 0.1$, **(c):** $\alpha = 0.1$ and $\beta = 0.5$, **(d):** $\alpha = 0.5$ and $\beta = 0.5$, when $N \sim TP(\theta)$ and $R^* = 0.9$.

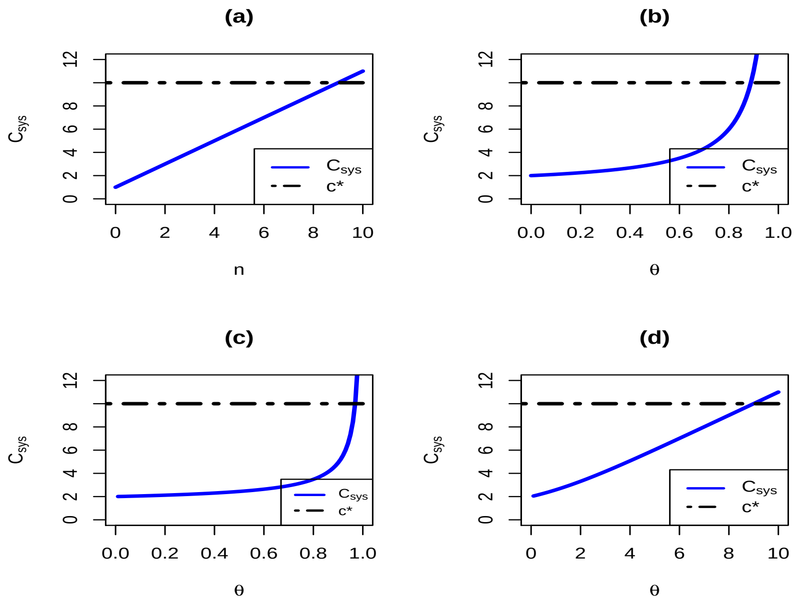

Figure 5: The values of C_{sys} for **(a):** $N = n$, **(b):** $N \sim Ge(\theta)$, **(c):** $N \sim Log(\theta)$, **(d):** $N \sim TP(\theta)$, when $c_1 = c_2 = 1$ and $c^* = 10$.

values, say R^*_{sys} and C^*_{sys}, from the one-objective problems. So, the optimal value for θ would be obtained by minimizing a weighted sum of the objective functions where all weights are positive, as

$$H = w\frac{1 - R_{sys}}{1 - R^*_{sys}} + (1 - w)\frac{C_{sys}}{C^*_{sys}}, \qquad 0 \leq w \leq 1, \tag{7}$$

subject to constraints

$$C_{sys} \leq c^* \quad \text{and} \quad 1 - R_{sys} \leq 1 - R^* = R^{**}.$$

Remark 2.1. *When $N = n$ is a fixed value,* (3) *and* (4) *reduce to*

$$\begin{aligned} R_{sys}(t) &= 1 - (F_{\alpha,\beta}(t))^n = 1 - \left(1 - \frac{\beta^\alpha}{t^\alpha}\right)^n, \\ C_{sys} &= c_1 n + c_2, \end{aligned}$$

respectively. In this case, the problem in (7) *under the constraints in* (6)*, is finding the best value for n, say n_{opt}.*

In the sequel, for an illustration, special cases of power series distribution are considered and for each case the problem is solved.

1. **Geometric distribution**: If N is a geometric random variable with parameter θ, denoted by $Ge(\theta)$, then in (2) we have $\psi(\theta) = \frac{\theta}{1-\theta}$ and $a(n) = 1$, i.e.,

$$P(N = n) = (1 - \theta)\theta^{n-1}, \qquad n = 1, 2, \ldots.$$

Table 1: The values of R_{sys} for different values of n, t, α and β, when $N = n$ is fixed.

			n									
α	β	t	1	2	3	4	5	6	7	8	9	10
0.1	0.1	0.5	0.8513	0.9779	0.9967	0.9995	0.9999	0.9999	0.9999	0.9999	1.0000	1.0000
		1	0.7943	0.9576	0.9912	0.9982	0.9996	0.9999	0.9999	0.9999	0.9999	0.9999
		2	0.7411	0.9329	0.9826	0.9955	0.9988	0.9996	0.9999	0.9999	0.9999	0.9999
		3	0.7116	0.9168	0.9760	0.9930	0.9980	0.9994	0.9998	0.9999	0.9999	0.9999
	0.5	1	0.9330	0.9955	0.9996	0.9999	0.9999	0.9999	1.0000	1.0000	1.0000	1.0000
		2	0.8705	0.9832	0.9978	0.9997	0.9999	0.9999	0.9999	0.9999	1.0000	1.0000
		3	0.8359	0.9730	0.9955	0.9992	0.9998	0.9999	0.9999	0.9999	0.9999	1.0000
	1	2	0.9330	0.9955	0.9996	0.9999	0.9999	0.9999	1.0000	1.0000	1.0000	1.0000
		3	0.8959	0.9891	0.9988	0.9998	0.9999	0.9999	0.9999	1.0000	1.0000	1.0000
	2	3	0.9602	0.9984	0.9999	0.9999	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000
0.5	0.1	0.5	0.4472	0.6944	0.8310	0.9066	0.9483	0.9714	0.9842	0.9912	0.9951	0.9973
		1	0.3162	0.5324	0.6803	0.7814	0.8505	0.8977	0.9301	0.9522	0.9673	0.9776
		2	0.2236	0.3972	0.5320	0.6366	0.7178	0.7809	0.8299	0.8679	0.8974	0.9204
		3	0.1825	0.3318	0.4538	0.5535	0.6350	0.7016	0.7561	0.8006	0.8370	0.8668
	0.5	1	0.7071	0.9142	0.9748	0.9926	0.9978	0.9993	0.9998	0.9999	0.9999	0.9999
		2	0.5000	0.7500	0.8750	0.9375	0.9687	0.9843	0.9921	0.9960	0.9980	0.9990
		3	0.4082	0.6498	0.7927	0.8773	0.9274	0.9570	0.9745	0.9849	0.9911	0.9947
	1	2	0.7071	0.9142	0.9748	0.9926	0.9978	0.9993	0.9998	0.9999	0.9999	0.9999
		3	0.5773	0.8213	0.9245	0.9680	0.9865	0.9942	0.9975	0.9989	0.9995	0.9998
	2	3	0.8164	0.9663	0.9938	0.9988	0.9997	0.9999	0.9999	0.9999	0.9999	1.0000
1	0.1	0.5	0.2000	0.3600	0.4880	0.5904	0.6723	0.7378	0.7902	0.8322	0.8657	0.8926
		1	0.2000	0.3600	0.4880	0.5904	0.6723	0.7378	0.7902	0.8322	0.8657	0.8926
		2	0.0500	0.0975	0.1426	0.1854	0.2262	0.2649	0.3016	0.3365	0.3697	0.4012
		3	0.0333	0.0655	0.0967	0.1268	0.1559	0.1840	0.2112	0.2375	0.2629	0.2875
	0.5	1	0.5000	0.7500	0.8750	0.9375	0.9687	0.9843	0.9921	0.9960	0.9980	0.9990
		2	0.2500	0.4375	0.5781	0.6835	0.7626	0.8220	0.8665	0.8998	0.9249	0.9436
		3	0.1666	0.3055	0.4212	0.5177	0.5981	0.6651	0.7209	0.7674	0.8061	0.8384
	1	2	0.5000	0.7500	0.8750	0.9375	0.9687	0.9843	0.9921	0.9960	0.9980	0.9990
		3	0.3333	0.5555	0.7037	0.8024	0.8683	0.9122	0.9414	0.9609	0.9739	0.9826
	2	3	0.6666	0.8888	0.9629	0.9876	0.9958	0.9986	0.9995	0.9998	0.9999	0.9999
2	0.1	0.5	0.0400	0.0784	0.1152	0.1506	0.1846	0.2172	0.2485	0.2786	0.3074	0.3351
		1	0.0100	0.0199	0.0297	0.0394	0.0490	0.0585	0.0679	0.0772	0.0864	0.0956
		2	0.0025	0.0049	0.0074	0.0099	0.0124	0.0149	0.0173	0.0198	0.0222	0.0247
		3	0.0011	0.0020	0.0033	0.0044	0.0055	0.0066	0.0077	0.0088	0.0099	0.0110
	0.5	1	0.2500	0.4375	0.5781	0.6835	0.7626	0.8220	0.8665	0.8998	0.9249	0.9436
		2	0.0625	0.1210	0.1760	0.2275	0.2758	0.3210	0.3634	0.4032	0.4405	0.4755
		3	0.0277	0.0547	0.0810	0.1065	0.1313	0.1555	0.1789	0.2017	0.2239	0.2455
	1	2	0.2500	0.4375	0.5781	0.6835	0.7626	0.8220	0.8665	0.8998	0.9249	0.9436
		3	0.1111	0.2098	0.2976	0.3757	0.4450	0.5067	0.5615	0.6102	0.6535	0.6920
	2	3	0.4444	0.6913	0.8285	0.9047	0.9470	0.9705	0.9836	0.9909	0.9949	0.9971

In this case, we find $\psi'(\theta) = \frac{1}{(1-\theta)^2}$, and (3), (4) and (5) can be rewritten as

$$R_{sys}(t) \quad = \quad 1 - \frac{(1-\theta)\Big(1 - \frac{\beta^\alpha}{t^\alpha}\Big)}{1 - \theta\Big(1 - \frac{\beta^\alpha}{t^\alpha}\Big)},$$

Table 2: The values of R_{sys} for different values of θ, t, α and β, when $N \sim Ge(\theta)$.

			θ								
α	β	t	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	0.1	0.5	0.8641	0.8774	0.8910	0.9051	0.9197	0.9347	0.9502	0.9662	0.9828
		1	0.8110	0.8284	0.8465	0.8655	0.8853	0.9061	0.9279	0.9507	0.9747
		2	0.7608	0.7816	0.8035	0.8267	0.8513	0.8774	0.9051	0.9347	0.9662
		3	0.7328	0.7552	0.7790	0.8044	0.8315	0.8605	0.8916	0.9250	0.9610
	0.5	1	0.9393	0.9456	0.9521	0.9587	0.9653	0.9720	0.9789	0.9858	0.9928
		2	0.8819	0.8936	0.9057	0.9180	0.9307	0.9438	0.9572	0.9711	0.9853
		3	0.8499	0.8643	0.8792	0.8946	0.9106	0.9272	0.9444	0.9622	0.9807
	1	2	0.9393	0.9456	0.9521	0.9587	0.9653	0.9720	0.9789	0.9858	0.9928
		3	0.9053	0.9149	0.9248	0.9348	0.9451	0.9556	0.9663	0.9773	0.9885
	2	3	0.9640	0.9679	0.9718	0.9757	0.9797	0.9837	0.9877	0.9917	0.9958
0.5	0.1	0.5	0.4733	0.5028	0.5361	0.5741	0.6180	0.6691	0.7294	0.8017	0.8899
		1	0.3394	0.3663	0.3978	0.4352	0.4805	0.5362	0.6065	0.6981	0.8222
		2	0.2424	0.2647	0.2915	0.3243	0.3654	0.4186	0.4898	0.5901	0.7422
		3	0.1988	0.2182	0.2418	0.2712	0.3087	0.3583	0.4267	0.5275	0.6907
	0.5	1	0.7284	0.7511	0.7752	0.8009	0.8284	0.8578	0.8894	0.9234	0.9602
		2	0.5263	0.5555	0.5882	0.6250	0.6666	0.7142	0.7692	0.8333	0.9090
		3	0.4339	0.4630	0.4963	0.5348	0.5797	0.6329	0.6969	0.7752	0.8734
	1	2	0.7284	0.7511	0.7752	0.8009	0.8284	0.8578	0.8894	0.9234	0.9602
		3	0.6028	0.6306	0.6611	0.6948	0.7320	0.7735	0.8199	0.8722	0.9317
	2	3	0.8317	0.8476	0.8640	0.8811	0.8989	0.9175	0.9368	0.9569	0.9780
1	0.1	0.5	0.2173	0.2380	0.2631	0.2941	0.3333	0.3846	0.4545	0.5555	0.7142
		1	0.1098	0.1219	0.1369	0.1562	0.1818	0.2173	0.2702	0.3571	0.5263
		2	0.0552	0.0617	0.0699	0.0806	0.0952	0.1162	0.1492	0.2083	0.3448
		3	0.0369	0.0413	0.0469	0.0543	0.0645	0.0793	0.1030	0.1470	0.2564
	0.5	1	0.5263	0.5555	0.5882	0.6250	0.6666	0.7142	0.7692	0.8333	0.9090
		2	0.2702	0.2941	0.3225	0.3571	0.4000	0.4545	0.5263	0.6250	0.7692
		3	0.1818	0.2000	0.2222	0.2500	0.2857	0.3333	0.4000	0.5000	0.6666
	1	2	0.5263	0.5555	0.5882	0.6250	0.6666	0.7142	0.7692	0.8333	0.9090
		3	0.3571	0.3846	0.4166	0.4545	0.5000	0.5555	0.6250	0.7142	0.8333
	2	3	0.6896	0.7142	0.7407	0.7692	0.8000	0.8333	0.8695	0.9090	0.9523
2	0.1	0.5	0.2173	0.2380	0.2631	0.2941	0.3333	0.3846	0.4545	0.5555	0.7142
		1	0.1098	0.1219	0.1369	0.1562	0.1818	0.2173	0.2702	0.3571	0.5263
		2	0.0552	0.0617	0.0699	0.0806	0.0952	0.1162	0.1492	0.2083	0.3448
		3	0.0369	0.0413	0.0469	0.0543	0.0645	0.0793	0.1030	0.1470	0.2564
	0.5	1	0.2702	0.2941	0.3225	0.3571	0.4000	0.4545	0.5263	0.6250	0.7692
		2	0.0689	0.0769	0.0869	0.1000	0.1176	0.1428	0.1818	0.2500	0.4000
		3	0.0307	0.0344	0.0392	0.0454	0.0540	0.0666	0.0869	0.1250	0.2222
	1	2	0.2702	0.2941	0.3225	0.3571	0.4000	0.4545	0.5263	0.6250	0.7692
		3	0.1219	0.0.1351	0.1515	0.1724	0.2000	0.2380	0.2941	0.3846	0.5555
	2	3	0.4705	0.5000	0.5333	0.5714	0.6153	0.6666	0.7272	0.8000	0.8888

$$C_{sys} = c_1 \frac{1}{1-\theta} + c_2,$$

$$\mathbb{E}_\theta(N) = \frac{1}{1-\theta}, \tag{8}$$

respectively.
2. **Logarithmic distribution**: Let N be a logarithmic random variable with parameter θ, denoted by $Log(\theta)$, i.e.,

$$P(N=n) = \frac{-\theta^n}{n\log(1-\theta)}, \qquad n = 1,2,\dots,$$

since $\psi(\theta) = -\log(1-\theta)$, $a(n) = \frac{1}{n}$ and $\psi'(\theta) = \frac{1}{1-\theta}$. So, (3), (4) and (5) become

$$\begin{aligned} R_{sys}(t) &= 1 - \frac{\log\left[1-\theta\left(1-\frac{\beta^\alpha}{t^\alpha}\right)\right]}{\log(1-\theta)}, \\ C_{sys} &= -c_1\frac{\theta}{(1-\theta)\log(1-\theta)} + c_2, \\ \mathbb{E}_\theta(N) &= \frac{-\theta}{(1-\theta)\log(1-\theta)}, \end{aligned} \tag{9}$$

respectively.
3. **Zero-truncated Poisson distribution**: Suppose N has a Poisson distribution truncated at point zero, with parameter θ, denoted by $TP(\theta)$, i.e.,

$$P(N=n) = \frac{\theta^n}{n!(e^\theta - 1)}, \qquad n = 1,2,\dots.$$

It means in (2), $\psi(\theta) = (e^\theta - 1)$, $a(n) = \frac{1}{n!}$ and $\psi'(\theta) = e^\theta$. Therefore, we have

$$\begin{aligned} R_{sys}(t) &= 1 - \frac{\exp\left[\theta\left(1-\frac{\beta^\alpha}{t^\alpha}\right)\theta\right] - 1}{e^\theta - 1}, \\ C_{sys} &= c_1\frac{\theta e^\theta}{e^\theta - 1} + c_2, \\ \mathbb{E}_\theta(N) &= \frac{\theta e^\theta}{e^\theta - 1}. \end{aligned} \tag{10}$$

3 Numerical computations

In this section, in order to evaluate the results of Section 2, numerical computations have been performed. The values of R_{sys} for different values of n, t, α and β are presented in Table 1 and Figure 1, when $N = n$ is a fixed value and $R^* = 0.9$. Moreover, Tables 2-4 and Figures 2-4 present the values of R_{sys} for different values of θ, t, α, β and different distributions for N, when $R^* = 0.9$. Also, Tables 5, **??** and 7 and Figure 5 report the values of C_{sys} for different distributions for N, when $c^* = 10$ and $c_1 = c_2 = 1$. The results and plots in this paper have been obtained using R Statistical Software v3.3.1; (R Core Team, 2021).

Table 3: The values of R_{sys} for different values of θ, t, α and β, when $N \sim Log(\theta)$.

α	β	t	θ = 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	0.1	0.5	0.8578	0.8647	0.8720	0.8799	0.8885	0.8980	0.9087	0.9213	0.9376
		1	0.8027	0.8117	0.8214	0.8319	0.8434	0.8562	0.8708	0.8883	0.9111
		2	0.7510	0.7617	0.7733	0.7860	0.8000	0.8157	0.8339	0.8550	0.8848
		3	0.7223	0.7338	0.7463	0.7601	0.7754	0.7927	0.8127	0.8370	0.8695
	0.5	1	0.9362	0.9395	0.9431	0.9468	0.9508	0.9552	0.9601	0.9657	0.9730
		2	0.8763	0.8824	0.8889	0.8959	0.9034	0.9117	0.9211	0.9320	0.9462
		3	0.8430	0.8505	0.8585	0.8671	0.8765	0.8869	0.8986	0.9125	0.9306
	1	2	0.9362	0.9395	0.9431	0.9468	0.9508	0.9552	0.9601	0.9657	0.9730
		3	0.9007	0.9057	0.9110	0.9167	0.9229	0.9296	0.9371	0.9460	0.9573
	2	3	0.9622	0.9642	0.9663	0.9686	0.9710	0.9736	0.9765	0.9799	0.9841
0.5	0.1	0.5	0.4602	0.4749	0.4916	0.5108	0.5332	0.5602	0.5935	0.6372	0.7011
		1	0.3277	0.3409	0.3563	0.3744	0.3964	0.4236	0.4590	0.5079	0.5850
		2	0.2329	0.2437	0.2565	0.2720	0.2911	0.3156	0.3487	0.3969	0.4789
		3	0.1906	0.2000	0.2112	0.2248	0.2419	0.2641	0.2947	0.3406	0.4221
	0.5	1	0.7178	0.7294	0.7421	0.7560	0.7715	0.7890	0.8094	0.8341	0.8671
		2	0.5131	0.5278	0.5443	0.5631	0.5849	0.6107	0.6421	0.6826	0.7403
		3	0.4210	0.4355	0.4520	0.4712	0.4939	0.5213	0.5557	0.6015	0.6697
	1	2	0.7178	0.7294	0.7421	0.7560	0.7715	0.7890	0.8094	0.8341	0.8671
		3	0.5901	0.6042	0.6198	0.6374	0.6575	0.6808	0.7086	0.7435	0.7921
	2	3	0.8242	0.8324	0.8412	0.8507	0.8611	0.8726	0.8858	0.9013	0.9216
1	0.1	0.5	0.2086	0.2186	0.2305	0.2450	0.2630	0.2863	0.3181	0.3652	0.4471
		1	0.1048	0.1106	0.1176	0.1263	0.1375	0.1525	0.1741	0.2090	0.2787
		2	0.0525	0.0556	0.0594	0.0641	0.0703	0.0789	0.0916	0.1132	0.1613
		3	0.0350	0.0371	0.0397	0.0430	0.0473	0.0532	0.0622	0.0777	0.1139
	0.5	1	0.5131	0.5278	0.5443	0.5631	0.5849	0.6107	0.6421	0.6826	0.7403
		2	0.2600	0.2716	0.2853	0.3017	0.3219	0.3475	0.3816	0.4306	0.5118
		3	0.1741	0.1829	0.1934	0.2062	0.2223	0.2435	0.2728	0.3173	0.3979
	1	2	0.5131	0.5278	0.5443	0.5631	0.5849	0.6107	0.6421	0.6826	0.7403
		3	0.3451	0.3587	0.3743	0.3928	0.4150	0.4425	0.4778	0.5264	0.6020
	2	3	0.6782	0.6908	0.7046	0.7198	0.7369	0.7564	0.7793	0.8072	0.8450
2	0.1	0.5	0.0420	0.0445	0.0476	0.0515	0.0565	0.0635	0.0741	0.0922	0.1335
		1	0.0105	0.0111	0.0119	0.0130	0.0143	0.0162	0.0191	0.0243	0.0374
		2	0.0026	0.0028	0.0030	0.0032	0.0036	0.0040	0.0048	0.0061	0.0096
		3	0.0011	0.0012	0.0013	0.0014	0.0016	0.0018	0.0021	0.0027	0.0043
	0.5	1	0.2600	0.2716	0.2853	0.3017	0.3219	0.3475	0.3816	0.4306	0.5118
		2	0.0656	0.0694	0.0741	0.0799	0.0874	0.0977	0.1130	0.1386	0.1938
		3	0.029	0.0310	0.0331	0.0359	0.0395	0.0445	0.0521	0.0654	0.0969
	1	2	0.2600	0.2716	0.2853	0.3017	0.3219	0.3475	0.3816	0.4306	0.5118
		3	0.1164	0.1227	0.1304	0.1398	0.1520	0.1682	0.1914	0.2284	0.3010
	2	3	0.4574	0.4721	0.4888	0.5080	0.5305	0.5574	0.5909	0.6347	0.6989

From the results of Tables 1-7 and Figures 1-5, we find the following:

- By increasing values of n and θ, values of R_{sys} and C_{sys} increase, when all other

Table 4: The values of R_{sys} for different values of θ, t, α and β, when $N \sim TP(\theta)$.

			θ								
α	β	t	1	2	3	4	5	6	7	8	9
0.1	0.1	0.5	0.9067	0.9458	0.9705	0.9848	0.9925	0.9964	0.9983	0.9992	0.9996
		1	0.8671	0.9203	0.9552	0.9761	0.9878	0.9939	0.9970	0.9985	0.9993
		2	0.8280	0.8938	0.9384	0.9661	0.9820	0.9907	0.9953	0.9976	0.9988
		3	0.8055	0.8779	0.9279	0.9595	0.9781	0.9884	0.9940	0.9969	0.9984
	0.5	1	0.9596	0.9775	0.9883	0.9942	0.9973	0.9987	0.9994	0.9997	0.9998
		2	0.9195	0.9537	0.9751	0.9873	0.9938	0.9970	0.9986	0.9993	0.9997
		3	0.8962	0.9392	0.9666	0.9826	0.9913	0.9958	0.9980	0.9990	0.9995
	1	2	0.9596	0.9775	0.9883	0.9942	0.9973	0.9987	0.9994	0.9997	0.9998
		3	0.9361	0.9637	0.9808	0.9903	0.9953	0.9978	0.9990	0.9995	0.9998
	2	3	0.9764	0.9870	0.9933	0.9967	0.9985	0.9993	0.9997	0.9998	0.9999
0.5	0.1	0.5	0.5704	0.6836	0.7772	0.8483	0.8991	0.9339	0.9571	0.9723	0.9822
		1	0.4288	0.5420	0.6448	0.7311	0.7996	0.8521	0.8915	0.9206	0.9420
		2	0.3169	0.4170	0.5143	0.6021	0.6776	0.7404	0.7916	0.8331	0.8664
		3	0.2639	0.3537	0.4438	0.5279	0.6026	0.6672	0.7220	0.7681	0.8067
	0.5	1	0.8019	0.8753	0.9262	0.9584	0.9774	0.9880	0.9938	0.9968	0.9984
		2	0.6224	0.7310	0.8175	0.8807	0.9241	0.9525	0.9706	0.9820	0.9890
		3	0.5302	0.6453	0.7431	0.8196	0.8760	0.9159	0.9434	0.9621	0.9747
	1	2	0.8019	0.8753	0.9262	0.9584	0.9774	0.9880	0.9938	0.9968	0.9984
		3	0.6938	0.7920	0.8662	0.9174	0.9506	0.9711	0.9833	0.9904	0.9945
	2	3	0.8827	0.9305	0.9615	0.9797	0.9898	0.9950	0.9976	0.9988	0.9994
1	0.1	0.5	0.2867	0.3812	0.4748	0.5609	0.6364	0.7005	0.7540	0.7983	0.8348
		1	0.1505	0.2096	0.2727	0.3358	0.3961	0.4523	0.5038	0.5508	0.5935
		2	0.0771	0.1100	0.1465	0.1846	0.2226	0.25982	0.2955	0.3297	0.3624
		3	0.0518	0.0745	0.1001	0.1271	0.1545	0.18171	0.2083	0.2341	0.2592
	0.5	1	0.6224	0.7310	0.8175	0.8807	0.9241	0.9525	0.9706	0.9820	0.9890
		2	0.3499	0.4550	0.5552	0.6439	0.7183	0.7788	0.8269	0.8649	0.8947
		3	0.2428	0.3278	0.4140	0.4956	0.5692	0.6336	0.6892	0.7366	0.7769
	1	2	0.6224	0.7310	0.8175	0.8807	0.9241	0.9525	0.9706	0.9820	0.9890
		3	0.4484	0.5627	0.6652	0.7501	0.8166	0.8668	0.9038	0.9308	0.9503
	2	3	0.7697	0.8516	0.9099	0.9478	0.9708	0.9841	0.9915	0.9955	0.9976
2	0.1	0.5	0.0620	0.0889	0.1190	0.1506	0.1824	0.2139	0.2444	0.2739	0.3023
		1	0.0157	0.0229	0.0311	0.0399	0.0491	0.0583	0.0676	0.0769	0.0860
		2	0.0039	0.0057	0.0078	0.0101	0.0125	0.0149	0.0173	0.0198	0.0222
		3	0.0017	0.0025	0.0035	0.0045	0.0055	0.0066	0.0077	0.0088	0.0099
	0.5	1	0.3499	0.4550	0.5552	0.6439	0.7183	0.7788	0.8269	0.8649	0.8947
		2	0.0958	0.1358	0.1799	0.2253	0.2702	0.3134	0.3546	0.3936	0.4302
		3	0.0433	0.0624	0.0841	0.1071	0.1305	0.1538	0.1768	0.1993	0.2212
	1	2	0.3499	0.4550	0.5552	0.6439	0.7183	0.7788	0.8269	0.8649	0.8947
		3	0.1663	0.2304	0.2983	0.3655	0.4291	0.4877	0.5410	0.5890	0.6321
	2	3	0.5676	0.6810	0.7749	0.8464	0.8976	0.9328	0.9563	0.9717	0.9818

parameters are held fixed.

- For fixed values of n, θ and β, R_{sys} is a decreasing function of α and t.

Table 5: The values of C_{sys} for different values of n, when $c_1 = c_2 = 1$ and $N = n$ is fixed.

n	1	2	3	4	5	6	7	8	9	10
C_{sys}	2	3	4	5	6	7	8	9	10	11

Table 6: The values of C_{sys} for different values of θ, when $c_1 = c_2 = 1$.

	θ								
Distribution	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Geometric	2.1111	2.2500	2.4285	2.6666	3.000	3.5000	4.3333	6.0000	11.0000
Logarithmic	2.0545	2.1203	2.2015	2.3050	2.4426	2.6370	2.9380	3.4853	4.9086

Table 7: The values of C_{sys} for different values of θ, when $c_1 = c_2 = 1$ and $N \sim TP(\theta)$.

θ	1	2	3	4	5	6	7	8	9	10
C_{sys}	2.5819	3.3130	4.1571	5.0746	6.0339	7.0149	8.0063	9.0026	10.0011	11.0004

Table 8: The values of n_{opt}, R_{sys}^{opt} and C_{sys}^{opt} for some selected values of w, t, α and β, when $N = n$, $R^* = 0.9$, $c_1 = c_2 = 1$ and $c^* = 10$.

	α	0.1						0.5		
	β	0.1			0.5			0.5		
w	t	0.5	1	2	1	2	3	1	2	3
	n_1	2	2	2	1	2	2	2	4	5
	n_2	9	9	9	9	9	9	9	9	9
	R_{sys}^*	1.000	0.999	0.999	1.000	1.000	0.999	0.999	0.998	0.991
	C_{sys}^*	3	3	3	2	3	3	3	5	6
0	n_{opt}	2	2	2	1	2	2	2	4	5
	R_{sys}^{opt}	0.978	0.958	0.933	0.933	0.983	0.973	0.914	0.938	0.927
	C_{sys}^{opt}	3	3	3	2	3	3	3	5	6
0.1	n_{opt}	9	9	9	9	9	9	9	8	7
	R_{sys}^{opt}	1.000	0.999	0.999	1.000	1.000	0.999	0.999	0.996	0.974
	C_{sys}^{opt}	10	10	10	10	10	10	10	9	8
$\{0.2, \ldots, 1\}$	n_{opt}	9	9	9	9	9	9	9	9	9
	R_{sys}^{opt}	1.000	0.999	0.999	1.000	1.000	0.999	0.999	0.998	0.991
	C_{sys}^{opt}	10	10	10	10	10	10	10	10	10

- R_{sys} increases as β increases, when all other parameters are held fixed.

It is troublesome to compute the optimal value for θ, θ^{opt}, that minimizes H with respect to $R_{sys} \geq R^*$ and $C_{sys} \leq c^*$. However, it would be easy to do this with a computer. For this purpose, first we should find the interval (θ_1, ∞) that the condition $R_{sys} \geq R^*$ is satisfied. Since R_{sys} is increasing in θ, the aforementioned inequality is equivalent to say that $\theta \geq \theta_1$, where θ_1 can be obtained by solving the equation $R_{sys} = R^*$.

Then, we should obtain the interval $(0, \theta_2)$ that the condition $C_{sys} \leq c^*$ is satisfied. Similarly, since C_{sys} is an increasing function of θ, we can write

$$C_{sys} \leq c^* \qquad \Longleftrightarrow \qquad \theta \leq \theta_2,$$

in which θ_2 is the solution of the equation $C_{sys} = c^*$.

Three cases will be happened:

- If $\theta_1 > \theta_2$, then the problem has no answer.
- If $\theta_1 = \theta_2$, then the optimal value for θ is $\theta^{opt} = \theta_1 = \theta_2$.

Table 9: The values of θ^{opt}, R_{sys}^{opt} and C_{sys}^{opt} for some selected values of w, t, α and β, when $N \sim Ge(\theta)$, $R^* = 0.9$, $c_1 = c_2 = 1$ and $c^* = 10$.

	α	0.1						0.5		
	β	0.1			0.5			0.5		
w	t	0.5	1	2	1	2	3	1	2	3
	θ_1	0.3636	0.5708	0.6819	0.0000	0.2527	0.4337	0.7317	0.8888	0.9233
	θ_2	0.8888	0.8888	0.8888	0.8888	0.8888	0.8888	0.8888	0.8888	0.8888
	R_{sys}^*	0.9809	0.9720	0.9626	0.9920	0.9837	0.9786	0.9559	0.9000	-
	C_{sys}^*	2.5715	3.3303	4.1436	2.0000	2.3382	2.7660	4.7281	10.0000	-
0	θ^{opt}	0.3636	0.5708	0.6819	0.0000	0.2527	0.4337	0.7317	0.8888	-
	R_{sys}^{opt}	0.9000	0.9000	0.9000	0.9330	0.9000	0.9000	0.9000	0.9000	-
	C_{sys}^{opt}	2.5714	3.3302	4.1442	2.0000	2.3382	2.7659	4.7276	10.0000	-
0.1	θ^{opt}	0.3636	0.5708	0.6819	0.2578	0.2818	0.4337	0.7317	0.8888	-
	R_{sys}^{opt}	0.9000	0.9000	0.9000	0.9494	0.9035	0.9000	0.9000	0.9000	-
	C_{sys}^{opt}	2.5714	3.3302	4.1442	2.3474	2.3923	2.7659	4.7276	10.0000	-
0.2	θ^{opt}	0.5561	0.6025	0.6819	0.5138	0.5376	0.5697	0.7317	0.8888	-
	R_{sys}^{opt}	0.9280	0.9067	0.9000	0.9662	0.9356	0.9221	0.9000	0.9000	-
	C_{sys}^{opt}	3.2529	3.5162	4.1442	3.0571	3.1629	3.3243	4.7276	10.0000	-
0.3	θ^{opt}	0.6670	0.7036	0.7314	0.6317	0.6525	0.6778	0.7469	0.8888	-
	R_{sys}^{opt}	0.9450	0.9287	0.9142	0.9742	0.9508	0.9405	0.9051	0.9000	-
	C_{sys}^{opt}	4.0038	4.3747	4.7237	3.7156	3.8778	4.1041	4.9514	10.0000	-
0.4	θ^{opt}	0.7361	0.7659	0.7886	0.7062	0.7242	0.7448	0.8012	0.8888	-
	R_{sys}^{opt}	0.9559	0.9428	0.9312	0.9793	0.9606	0.9523	0.9239	0.9000	-
	C_{sys}^{opt}	4.7894	5.2727	5.7309	4.4047	4.6261	4.9198	6.0306	10.0000	-
0.5	θ^{opt}	0.7863	0.8110	0.8297	0.7611	0.7765	0.7936	0.8401	0.8888	-
	R_{sys}^{opt}	0.9640	0.9533	0.9438	0.9831	0.9678	0.9610	0.9378	0.9000	-
	C_{sys}^{opt}	5.6807	6.2911	6.8730	5.1861	5.4742	5.8450	7.2542	10.0000	-
0.6	θ^{opt}	0.8267	0.8470	0.8624	0.8055	0.8186	0.8327	0.8710	0.8888	-
	R_{sys}^{opt}	0.9706	0.9619	0.9541	0.9862	0.9737	0.9682	0.9492	0.9000	-
	C_{sys}^{opt}	6.7716	7.5383	8.2710	6.1425	6.5137	6.9784	8.7527	10.0000	-
0.7	θ^{opt}	0.8619	0.8783	0.8887	0.8445	0.8553	0.8667	0.8887	0.8888	-
	R_{sys}^{opt}	0.9764	0.9694	0.9625	0.9889	0.9789	0.9745	0.9559	0.9000	-
	C_{sys}^{opt}	8.2418	9.2193	9.9881	7.4317	7.9135	8.5047	9.9897	10.0000	-
0.8	θ^{opt}	0.8887	0.8887	0.8887	0.8815	0.8887	0.8887	0.8887	0.8888	-
	R_{sys}^{opt}	0.9809	0.9719	0.9625	0.9915	0.9837	0.9786	0.9559	0.9000	-
	C_{sys}^{opt}	9.9882	9.9883	9.9881	9.4439	9.9900	9.9891	9.9897	10.0000	-
$\{0.9, 1\}$	θ^{opt}	0.8887	0.8887	0.8887	0.8887	0.8887	0.8887	0.8887	0.8888	-
	R_{sys}^{opt}	0.9809	0.9719	0.9625	0.9920	0.8887	0.9786	0.9559	0.9000	-
	C_{sys}^{opt}	9.9882	9.9883	9.9881	9.9880	9.9900	9.9891	9.9897	10.0000	-

• If $\theta_1 < \theta_2$, then we should find the optimal answer in (θ_1, θ_2), i.e., $\theta_1 \leq \theta^{opt} \leq \theta_2$. For this in (7), we set $R_{sys}^* = R_{sys}(\theta_2)$ and $C_{sys}^* = C_{sys}(\theta_1)$. Finally, we obtain the optimal value for θ, θ^{opt}, numerically. For the case that $N = n$ is a fixed value, with a similar analysis we can find the optimal value for n, n_{opt}.

Table 8 gives the optimal values for n, n_{opt}, and the resulting cost and reliability functions for some selected values of w, t, α and β, when $R^* = 0.9$, $c_1 = c_2 = 1$ and $c^* = 10$. Similarly, the optimal values of θ, θ^{opt}, for different distributions for N,

Table 10: The values of θ^{opt}, R^{opt}_{sys} and C^{opt}_{sys} for some selected values of w, t, α and β, when $N \sim Log(\theta)$, $R^* = 0.9$, $c_1 = c_2 = 1$ and $c^* = 10$.

	α	0.1						0.5		
	β	0.1			0.5			0.5		
w	t	0.5	1	2	1	2	3	1	2	3
	θ_1	0.6193	0.8556	0.9380	0.0000	0.4551	0.7102	0.9637	0.9990	0.9998
	θ_2	0.9690	0.9690	0.9690	0.9690	0.9690	0.9690	0.9690	0.9690	0.9690
	R^*_{sys}	0.9552	0.9360	0.9168	0.9806	0.9614	0.9501	0.9039	-	-
	C^*_{sys}	2.6846	4.0629	6.4451	2.0000	2.3757	2.9788	9.0101	-	-
0	θ^{opt}	0.6193	0.8556	0.9380	0.0000	0.4551	0.7102	0.9637	-	-
	R^{opt}_{sys}	0.9000	0.9000	0.9000	0.9330	0.9000	0.9000	0.9000	-	-
	C^{opt}_{sys}	2.6845	4.0625	6.4445	2.0000	2.3756	2.9787	9.0121	-	-
0.1	θ^{opt}	0.6193	0.8556	0.9380	0.0000	0.4551	0.7102	0.9637	-	-
	R^{opt}_{sys}	0.9000	0.9000	0.9000	0.9330	0.9000	0.9000	0.9000	-	-
	C^{opt}_{sys}	2.6845	4.0625	6.4445	2.0000	2.3756	2.9787	9.0121	-	-
0.2	θ^{opt}	0.6193	0.8556	0.9380	0.3179	0.4551	0.7102	0.9637	-	-
	R^{opt}_{sys}	0.9000	0.9000	0.9000	0.9437	0.9000	0.9000	0.9000	-	-
	C^{opt}_{sys}	2.6845	4.0625	6.4445	2.2182	2.3756	2.9787	9.0121	-	-
0.3	θ^{opt}	0.6275	0.8556	0.9380	0.5528	0.5939	0.7102	0.9637	-	-
	R^{opt}_{sys}	0.9008	0.9000	0.9000	0.9531	0.9112	0.9000	0.9000	-	-
	C^{opt}_{sys}	2.7058	4.0625	6.4445	2.5360	2.6229	2.9787	9.0121	-	-
0.4	θ^{opt}	0.7386	0.8556	0.9380	0.6826	0.7145	0.7578	0.9637	-	-
	R^{opt}_{sys}	0.9133	0.9000	0.9000	0.9592	0.9225	0.9063	0.9000	-	-
	C^{opt}_{sys}	3.1062	4.0625	6.4445	2.8743	2.9966	3.2065	9.0121	-	-
0.5	θ^{opt}	0.8105	0.8612	0.9380	0.7681	0.7927	0.8246	0.9637	-	-
	R^{opt}_{sys}	0.9228	0.9012	0.9000	0.9638	0.9312	0.9165	0.9000	-	-
	C^{opt}_{sys}	3.5720	4.1430	6.4445	3.2667	3.4307	3.7013	9.0121	-	-
0.6	θ^{opt}	0.8624	0.8996	0.9380	0.8305	0.8493	0.8728	0.9637	-	-
	R^{opt}_{sys}	0.9308	0.9109	0.9000	0.9677	0.9384	0.9250	0.9000	-	-
	C^{opt}_{sys}	4.1609	4.8985	6.4445	3.7613	3.9787	4.3284	9.0121	-	-
0.7	θ^{opt}	0.9029	0.9293	0.9507	0.8796	0.8935	0.9103	0.9637	-	-
	R^{opt}_{sys}	0.9382	0.9199	0.9061	0.9713	0.9451	0.9329	0.9000	-	-
	C^{opt}_{sys}	4.9883	5.9660	7.4151	4.4525	4.7466	5.2093	9.0121	-	-
0.8	θ^{opt}	0.9365	0.9540	0.9680	0.9209	0.9303	0.9414	0.9689	-	-
	R^{opt}_{sys}	0.9456	0.9290	0.9162	0.9749	0.9518	0.9408	0.9038	-	-
	C^{opt}_{sys}	6.3529	7.7389	9.8046	5.5896	6.0120	6.6666	9.9904	-	-
0.9	θ^{opt}	0.9666	0.9689	0.9689	0.9581	0.9633	0.9689	0.9689	-	-
	R^{opt}_{sys}	0.9543	0.9359	0.9168	0.9791	0.9597	0.9501	0.9038	-	-
	C^{opt}_{sys}	9.5313	9.9875	9.9854	8.2163	8.9440	9.9898	9.9904	-	-
1	θ^{opt}	0.9689	0.9689	0.9689	0.9689	0.9689	0.9689	0.9689	-	-
	R^{opt}_{sys}	0.9552	0.9359	0.9168	0.9806	0.9613	0.9501	0.9038	-	-
	C^{opt}_{sys}	9.9852	9.9875	9.9854	9.9845	9.9864	9.9898	9.9904	-	-

are presented in Tables 9-11. In Tables 9 and 10, dash $(-)$ means that there is no θ^{opt} which satisfies the conditions in (6). From Tables 8-11 we find that the values of n_{opt} (when $N = n$ is fixed), θ^{opt} (when N is a random variable), R^{opt}_{sys} and C^{opt}_{sys} are non-decreasing in w, when all other parameters are held fixed. Also, from the results of Tables 8-11 one can see that the optimal values may change by selecting different

Table 11: The values of θ^{opt}, R_{sys}^{opt} and C_{sys}^{opt} for some selected values of w, t, α and β, when $N \sim TP(\theta)$, $R^* = 0.9$, $c_1 = c_2 = 1$ and $c^* = 10$.

	α	0.1						0.5		
	β	0.1			0.5			0.5		
w	t	0.5	1	2	1	2	3	1	2	3
	θ_1	0.8609	1.5723	2.1149	0.0000	0.5632	1.0734	2.4329	4.3944	5.5564
	θ_2	8.9988	8.9988	8.9988	8.9988	8.9988	8.9988	8.9988	8.9988	8.9988
	R^*_{sys}	0.9996	0.9993	0.9988	0.9998	0.9997	0.9995	0.9983	0.9989	0.9747
	C^*_{sys}	2.4914	2.9842	3.4050	2.0000	2.3079	2.6309	3.6670	5.4493	6.5780
0	θ^{opt}	0.8609	1.5723	2.1149	0.0000	0.5632	1.0734	2.4329	4.3944	5.5564
	R_{sys}^{opt}	0.9000	0.9000	0.9000	0.9330	0.9000	0.9000	0.9000	0.9000	0.9000
	C_{sys}^{opt}	2.4914	2.9841	3.4051	2.0000	2.3079	2.6309	3.6670	5.4493	6.5779
0.1	θ^{opt}	7.0427	7.1086	7.1049	6.9762	6.9973	7.0695	7.0770	6.4953	5.9164
	R_{sys}^{opt}	0.9983	0.9972	0.9956	0.9994	0.9986	0.9981	0.9941	0.9625	0.9132
	C_{sys}^{opt}	8.0489	8.1144	8.1108	7.9827	8.0037	8.0755	8.0830	7.5051	6.9324
0.2	θ^{opt}	8.1046	8.2262	8.2866	7.9735	8.0431	8.1451	8.3067	8.1909	7.9894
	R_{sys}^{opt}	0.9992	0.9988	0.9980	0.9997	0.9994	0.9991	0.9974	0.9836	0.9620
	C_{sys}^{opt}	9.1070	9.2284	9.2887	8.9763	9.0457	9.1475	9.3087	9.1932	8.9922
0.3	θ^{opt}	8.7970	8.9550	8.9987	8.6244	8.7251	8.8466	8.9987	8.9987	8.9987
	R_{sys}^{opt}	0.9995	0.9993	0.9988	0.9998	0.9996	0.9995	0.9983	0.9890	0.9747
	C_{sys}^{opt}	9.7983	9.9562	9.9988	9.6259	9.7265	9.8478	9.9998	9.9998	9.9998
≥ 0.4	θ^{opt}	8.9987	8.9987	8.9987	8.9987	8.9987	8.9987	8.9987	8.9987	8.9987
	R_{sys}^{opt}	0.9996	0.9993	0.9988	0.9998	0.9997	0.9995	0.9983	0.9890	0.9747
	C_{sys}^{opt}	9.9998	9.9998	9.9988	9.9998	9.9998	9.9998	9.9998	9.9998	9.9998

values of w. If the cost is a more important criterion than reliability, then the w can be considered less than 0.5. On the other hand, if we consider the reliability criterion as the most important one, then we must choose w larger than 0.5, as we expected intuitively. So, the values of w can be determined by the decision-maker.

For comparing the obtained results and based on Tables 8-11, and (8), (9) and (10), the values of n^{opt} and $\mathbb{E}_{\theta^{opt}}(N)$ have been obtained in Table 12. From Table 12 one can observe that, for most cases the optimal number of units is larger under truncated Poisson distribution. We also can observe that an increasing in w leads to a smaller difference between the results of different distributions.

4 Conclusions

This paper studied the problem of optimization in a parallel system with a random number of units. The distribution of the number of units is assumed to follow a power series class of distributions that contains well-known distributions such as geometric, logarithmic and modified or truncated Poisson distributions. Optimal number of units for the system which minimizes the expected cost and maximizes the reliability of the system is computed. The two-objective optimization problem is reduced to a one-objective problem by using the weighted sum method. In this method, we consider the weight w for the reliability and the weight $1 - w$ for the cost function. If the cost is more important criterion than reliability, then w can be considered less than 0.5. On

Table 12: The values of n_{opt} and $\mathbb{E}_{\theta^{opt}}(N)$ for some selected values of w, t, α, β and different distributions for N, when $R^* = 0.9$, $c_1 = c_2 = 1$ and $c^* = 10$.

	α	0.1						0.5		
	β	0.1			0.5			0.5		
w	t	0.5	1	2	1	2	3	1	2	3
0	$N=n$	2	2	2	1	2	2	2	4	5
	$Ge(\theta)$	1.5713	2.3299	3.1436	1.0000	1.3381	1.7658	3.7271	8.9928	-
	$Log(\theta)$	1.6844	3.0618	5.4408	1.0000	1.3755	1.9786	8.0062	-	-
	$TP(\theta)$	1.4914	1.9841	2.4050	1.0000	1.3078	1.6309	2.6670	4.4493	5.5779
0.1	$N=n$	9	9	9	9	9	9	9	8	7
	$Ge(\theta)$	1.5713	2.3299	3.1436	1.3473	1.3923	1.7658	3.7271	8.9928	-
	$Log(\theta)$	1.6844	3.0618	5.4408	1.0000	1.3755	1.9786	8.0062	-	-
	$TP(\theta)$	7.0488	7.1144	7.1107	6.9827	7.0037	7.0755	7.0829	6.5051	5.9323
0.2	$N=n$	9	9	9	9	9	9	9	9	9
	$Ge(\theta)$	2.2527	2.5157	3.1436	2.0567	2.1626	2.3239	3.7271	8.9928	-
	$Log(\theta)$	1.6844	3.0618	5.4408	1.2182	1.3755	1.9786	8.0062	-	-
	$TP(\theta)$	8.1070	8.2284	8.2886	7.9762	8.0456	8.1474	8.3087	8.1931	7.9921
0.3	$N=n$	9	9	9	9	9	9	9	9	9
	$Ge(\theta)$	3.0030	3.3738	3.7230	2.7151	2.8776	3.1036	3.9510	8.9928	-
	$Log(\theta)$	1.7058	3.0618	5.4408	1.5360	1.6228	1.9786	8.0062	-	-
	$TP(\theta)$	8.7983	8.9561	8.9998	8.6259	8.7265	8.8478	8.9998	8.9998	8.9998
0.4	$N=n$	9	9	9	9	9	9	9	9	9
	$Ge(\theta)$	3.7893	4.2716	4.7303	3.4036	3.6258	3.9184	5.0301	8.9928	-
	$Log(\theta)$	2.1059	3.0618	5.4408	1.8740	1.9964	2.2065	8.0062	-	-
	$TP(\theta)$	8.9998	8.9998	8.9998	8.9998	8.9998	8.9998	8.9998	8.9998	8.9998
0.5	$N=n$	9	9	9	9	9	9	9	9	9
	$Ge(\theta)$	4.6794	5.29100	5.8719	4.1858	4.4742	4.8449	6.2539	8.9928	-
	$Log(\theta)$	2.5713	3.1420	5.4408	2.2663	2.4300	2.7008	8.0062	-	-
	$TP(\theta)$	8.9998	8.9998	8.9998	8.9998	8.9998	8.9998	8.9998	8.9998	8.9998
0.6	$N=n$	9	9	9	9	9	9	9	9	9
	$Ge(\theta)$	5.7703	6.5359	7.2674	5.1413	5.5126	5.9772	7.7519	8.9928	-
	$Log(\theta)$	3.1599	3.8981	5.4408	2.7605	2.9779	3.3276	8.0062	-	-
	$TP(\theta)$	8.9998	8.9998	8.9998	8.9998	8.9998	8.9998	8.9998	8.9998	8.9998
0.7	$N=n$	9	9	9	9	9	9	9	9	9
	$Ge(\theta)$	7.2411	8.2169	8.9847	6.4308	6.9108	7.5018	8.9847	8.9928	-
	$Log(\theta)$	3.9873	4.9613	6.4069	3.4510	3.7460	4.2086	8.0062	-	-
	$TP(\theta)$	8.9998	8.9998	8.9998	8.9998	8.9998	8.9998	8.9998	8.9998	8.9998
0.8	$N=n$	9	9	9	9	9	9	9	9	9
	$Ge(\theta)$	8.9847	8.9847	8.9847	8.4388	8.9847	8.9847	8.9847	8.9928	-
	$Log(\theta)$	5.3498	6.7354	8.7884	4.5888	5.0110	5.6625	8.9767	-	-
	$TP(\theta)$	8.9998	8.9998	8.9998	8.9998	8.9998	8.9998	8.9998	8.9998	8.9998
0.9	$N=n$	9	9	9	9	9	9	9	9	9
	$Ge(\theta)$	8.9847	8.9847	8.9847	8.9847	8.9847	8.9847	8.9847	8.9928	-
	$Log(\theta)$	8.5138	8.9767	8.9767	7.2077	7.9419	8.9767	8.9767	-	-
	$TP(\theta)$	8.9998	8.9998	8.9998	8.9998	8.9998	8.9998	8.9998	8.9998	8.9998
1	$N=n$	9	9	9	9	9	9	9	9	9
	$Ge(\theta)$	8.9847	8.9847	8.9847	8.9847	8.9847	8.9847	8.9847	8.9928	-
	$Log(\theta)$	8.9767	8.9767	8.9767	8.9767	8.9767	8.9767	8.9767	-	-
	$TP(\theta)$	8.9998	8.9998	8.9998	8.9998	8.9998	8.9998	8.9998	8.9998	8.9998

the other hand, if we consider the reliability criterion as the most important one, then we must choose w larger than 0.5, as we expected intuitively.

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