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Research Paper

Application of the multivariate depth-based method for ranking the performance of judiciary

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Abstract: Evaluating the performance and ranking of the judiciary of the provinces periodically is always of interest. This problem can be done with univariate approaches; But considering many variables involved in this matter, the use of multivariate methods is more justified. A new method for this is ranking based on the depth concept. In this paper, the performance of prosecutors' offices, criminal courts and appeals courts of the provinces have been ranked over one year based on two indicators; case processing rate and the case congestion rate, employing the depth multivariate method. Evaluating the results of ranking intuitively confirms the appropriateness and rationality of this approach.

Keywords: Congestion rate; Depth notion; Performance of judiciary; Processing rate; Ranking.
Mathematics Subject Classification (2010): 62Hxx, 62Gxx, 62G30, 62F07.

1 Introduction

The performance of judicial system as the last line of defending justice has a great impact on public's trust or distrust. A judicial system is made up of different divisions such as prosecutor's office and court (criminal, civil, and appeal), all of which work towards the goal of the system. The goal is to assert the rights and to defend the public good of the nation. Therefore, evaluating the performance of this system to improve efficiency is worthwhile. Since technically speaking, the judicial process and its procedures are very complicated, appropriate data, indicators and also accurate methods should be used for visualizing the status of divisions.

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A wide range of data is collected on different aspects of the judicial system, which can be summarized and communicated using justice indicators. They are lucrative for measuring and then comparing the performance of equivalent divisions. Ranking which is a consequence of performance evaluation is done monthly and annually for the thirty-one provinces of Iran with respect to different indicators.

One of the most notable rankings is from three perspectives, prosecutor's offices, criminal courts and appeal courts. This is done based on the indicators extracted from some variables such as the number of cases entered in the judicial units of the province (input), the number of cases exists in the judicial units of the province (inventory), the number of cases withdrawn (output) and the number of remaining cases (remaining). Based on these variables, two indicators named *processing rate (PR)* and *congestion rate (CR)* are exploited which are taken into consideration for the ranking issue because they can present a clear illustration of evaluation.

The named variables and subsequent indicators are correlated, so univariate analysis should be limited for them, as they ignore the correlations and treat them individually. Instead, multivariate analysis which comprises the correlations between variables for more detailed and more accurate outcomes, is more appropriate. So, multivariate methods are more suitable options for the ranking issue.

There are a few methods of multivariate ordering and ranking in literature that are based on component-wise approaches. Barnet (1976) presented some different sub-ordering principles for multivariate random observations. These principles comprised a four-fold classification of Marginal ordering (M-ordering), Reduced ordering (R-ordering), Partial ordering (P-ordering) and Conditional ordering (C-ordering). Later, other multivariate orderings such as norm ordering (Bairamov and Gebizlioglu, 1998) and N-conditional ordering (Bairamov, 2006). (Arnold et al., 2009) reported that a general class of complete ordering of some independent identically distributed p-dimensional absolutely continuous random vectors could be supplied by considering them as the concomitants of an auxiliary random variable. Finally, (Arnold et al., 2009) discussed two new concepts of order statistics for multivariate observations. For more details see Tat and Faridrohani (2021).

A complementary approach is based on the notion of data depth. The concept of data depth is a more developed way for statistical extension of ordering to multivariate context (Zuo and Serfling, 2000). Data depth is a measure of outlyingness of a given point with respect to an underlying distribution or a multivariate data cloud, leading to a natural centre-outward ordering of multivariate sample points (Liu and Singh, 2006). Based on this ordering, the set of available observations form a data cloud and the situation of any point is measured with respect to that. Some points are located in the center of data cloud and some of the others display a tendency to the outskirt of it and maybe appear at a considerable distance from the rest of the observations.

Overall, the depth-based approach is a proper candidate for use in multivariate ranking problems due to its desirable properties and is applied to the discussed problem of this paper. For this purpose, an algorithm is written during which the measure of two indicators, processing and congestion rates, are calculated for all provinces based on the available reliable data. Then, the depth value of each province is measured with respect to the data cloud, which is the set of all provinces. Afterward, ranking is done separately for prosecutor's office, criminal and appeal courts. It should be noted here that a better rank means desirable performance in terms of two indicators simultaneously.

This paper is organized as follows. For more familiarity and clarity, a brief review of data depth is presented and the Mahalanobis depth function, the only one used here, is introduced in Section 2. In this section, the definitions and the method of ranking based on depth and then determining the orientation of each observation based on the obtained depth value are discussed. Section 3 contains information about indicators that are used to evaluate performance and also, the layout of the study. In Section 4, the results of analysis based on depth approach and ranking provinces in three divisions, prosecutor's offices, criminal courts and appeal courts are presented separately in detail. Section 5 is dedicated to the conclusion and discussion.

2 Depth notions

Depth is a useful tool in nonparametric inference for multivariate data. Data depth is a measure of deepness or outlyingness of a point relative to a data cloud or distribution F. This notion was introduced by Tukey (1975) in order to provide a center-outward ordering of multidimensional points with respect to a given data cloud or an underlying distribution. The author defined the median of a multivariate sample as the deepest point with respect to the data cloud $\mathbb{R}^p$, and introduced Tuckey depth function for measuring the values of data depth. Accordingly, different depth functions, such as Mahalanobis and projection-based were introduced.

2.1 Depth function

Let $\mathscr{F}$ be the class of distributions on the Borel sets of $\mathbb{R}^p$ and $F \in \mathscr{F}$. An associated depth function $D(\boldsymbol{x}, F)$ is defined to provide a center-outward ordering of points $\boldsymbol{x} \in \mathbb{R}^p$ relative to distribution F. Based on center-outward ordering interpretation, the set of points which globally maximize depth is the center and the points near the center have a higher depth (Zuo and Serfling, 2000). A formal definition of statistical depth function is presented by Zuo and Serfling (2000) as follow:

Let, $D(.,.) : \mathbb{R}^p * \mathscr{F} \to \mathbb{R}^1$ is bounded, nonnegative and satisfies the following properties:

• Affine invariance: $D(A\boldsymbol{x} + b, F_{A\boldsymbol{x}+b}) = D(\boldsymbol{x}, F_{\boldsymbol{x}})$, holds for any random $\boldsymbol{x}$ in $\mathbb{R}^p$ and $p \times p$ nonsingular matrix A and p-vector b.

• Maximality at center: $D(\theta, F) = sup_{\boldsymbol{x} \in \mathbb{R}^p} D(\boldsymbol{x}, F)$ holds if F is symmetric about θ in some sense.

• Monotonicity relative to the deepest point: for any F having deepest point θ, $D(\boldsymbol{x}, F) \leq D(\theta + \alpha(\boldsymbol{x} - \theta), F)$ holds for $\alpha \in [0, 1]$.

• Vanishing at infinity: if $||\boldsymbol{x}|| \to \infty$ then $D(\boldsymbol{x}, F) \to 0$.

Then $D(., F)$ is called statistical depth function.

Depth functions are necessary for measuring the depth value of a point. Various depth functions are presented and introduced in the literature. Here, only one of them, i.e. Mahalanobis depth, used in this paper, is introduced.

2.2 The Mahalanobis depth

The Mahalanobis depth of $\boldsymbol{x} \in \mathbb{R}^p$ with respect to the underlying distribution F is measured by:

$$MD(\boldsymbol{x}, F) = \frac{1}{1 + (\boldsymbol{x} - \mu_F)' \Sigma_F^{-1} (\boldsymbol{x} - \mu_F)},$$

where $(\boldsymbol{x} - \mu_F)' \Sigma_F^{-1} (\boldsymbol{x} - \mu_F)$ is a Mahalanobis distance between $\boldsymbol{x}$ and centered vector, μ_F, with respect to the dispersion matrix of distribution F, Σ_F (Liu and Singh, 1993).

A sample version of the Mahalanobis depth can be obtained by replacing the sample estimates of μ_F and Σ_F. Typically, the sample mean vector, $\bar{x}$ and the sample covariance matrix S, will be used as sample estimates. It should be noted that if μ_F agrees with the symmetric point of F, the Mahalanobis depth function is a statistical depth function.

2.3 Depth ranking

Let $\chi_n = \{\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_n\}$ be a random sample from distribution function F. The center-outward ranking of $\boldsymbol{X}_k$, $k = 1, 2, \cdots, n$ with respect to F is

$$\begin{aligned} r_{k,n} = r(\boldsymbol{X}_k; \chi_n) &= \#\{\boldsymbol{X}_j \in \chi_n : D(\boldsymbol{X}_j) \leq D(\boldsymbol{X}_k)\} \\ &= \sum_{j=1}^{n} 1_{\{D(\boldsymbol{X}_j) \leq D(\boldsymbol{X}_k) : \boldsymbol{X}_j \in \chi_n\}}, \end{aligned}$$

Obviously, when there is no tie, the largest rank, n, is attributed to the deepest point, and the smallest rank, 1, is assigned to the most outlying point.

2.4 Orientation

Depth-based ranking cannot provide enough information from the multivariate sample separately compared to the ranking which preserves the directions of data (Liu et al., 1999). Thus, accompanying depth values with orientation makes them more informative. So, the proposed method in Tat and Faridrohani (2021) can be used for orientation here. A compendious of that is as follows:

Determine the median of a set of multivariate observations, M. Translate observations to be centered at M. So, now observations are median-centered by the translation and a set of sign vectors is formed which its elements determine the sign of corresponding observation. For example, in $p = 2$, the set of signs is $\{(+, +), (+, -), (-, +), (-, -)\}$. Observations with the sign $(+, +)$ have orientation 1, $O = 1$, $(+, -)$ orientation 2, $O = 2$, $(-, +)$ orientation 3, $O = 3$ and $(-, -)$ orientation 4, $O = 4$. Therefore, in $\mathbb{R}^2$, a pair of $(D(\boldsymbol{X}_k), O_k)$ is attributed to k^{th} observation.

3 Materials and method

Various indicators are used to measure and monitor judicial performance. Some indicators are more interesting due to their nature. Multivariate methods can do performance review based on these indicators. In the following, two of the most prominent indicators are introduced and the depth-based performance evaluation method based on these two indicators is examined.

3.1 Indicators description

Justice indicators can be used to summarize and communicate large amount of critical data on various aspects of the judicial system. They are useful tools to evaluate performance, draw attention to issues, establish benchmarks and monitor progress. Indicators, together with other monitoring and evaluation mechanisms, are essential to making justice divisions more transparent and accountable. They are also necessary for providing valuable feedback to policy makers and reformers (Dandurand, 2015).

Input, inventory, output and remaining variables are the four ones that are important to evaluate the performance of the judicial system and are monitored regularly. These variables are measured continuously by the Judiciary Statistics and Information Technology Center. Based on them, two judicial indicators, Congestion Rate (CR) and Processing Rate (PR), which are in the forefront of attention are made in order to draw a clear visualization of the state of performance of provincial courts.

Processing rate indicator is the percentage ratio of the number of output cases to the number of inputs. It means,

$$Processing \quad rate = \frac{output}{input} \times 100.$$

A value higher than 100 of this index indicates the more ability of the judicial unit in dealing with cases.

The congestion rate index shows the time required to close the inputting and inventory cases during the desired period.

$$Congestion \quad rate = \frac{input + inventory}{output}.$$

The higher value of this rates indicates the need of more planning and organization for reducing the processing time. Therefore, desirable performance is defined by a higher processing rate and a lower congestion rate, and unfavorable performance is defined by a lower processing rate and a higher congestion rate.

Since correlation analysis is an essential prerequisite in multivariate analyses, the degree of association between two indicators should be checked. A proper analysis reveals a significant correlation with coefficient 0.64 between congestion and processing rates indicators. So, they are suitable for using in depth-based methods.

3.2 Scheme of the study

Opting a proper depth function from a large set of functions is a substantial step in any problem related to depth. An appropriate depth function is selected considering the nature of the problem and some features of an acceptable depth function (Tat and Faridrohani, 2021). So, during the first step, the Mahalanobis depth function was selected as one of the best ones for using here.

In the second step, ranking process of this study based on PR and CR is defined in the form of an algorithm. The algorithm is carried out in four steps and is as follows.

Algorithm 3.1.
• ***Defining data cloud*** *Let $\boldsymbol{X}_i$=(processing rate of the i^{th} province, congestion rate of the i^{th} province) for $i = 1, 2, \cdots, 31$. Then, define the data cloud of the problem as $\chi_{31} = \{\boldsymbol{X}_1, \cdots, \boldsymbol{X}_{31}\}$.*
• ***Measuring depth values and median determination*** *Compute the depth value of each point $\boldsymbol{X}_i$ with respect to the defined data cloud, χ_{31}, by using a proper depth function. Then, select the $\boldsymbol{X}_i$, for $i = 1, 2, \cdots, 31$ with the largest depth value as the median of data and named it M.*
• ***Determining orientations*** *Transfer observations to be centred at M. Now there is a median-centred data set and a corresponding set of signs is formed. How to apply orientation is presented in the Table 1.*

Table 1: Applying orientation to observations

Signs	(+,+)	(+,-)	(-,+)	(-,-)
Orientation	1	2	3	4
Inference	The better PR, the worse CR	The better PR, the better CR	The worse PR, the worse CR	The worse PR, the better CR

From point of view of two indicators simultaneously, provinces with orientation 2 have the better rank, so the better performance and provinces which has orientation 3 have the worst ranks and the worst performance compared with others.
• ***Ranking provinces*** *Now there is a set of pairs $(D(\boldsymbol{X}_i), O_i)$, $i = 1, 2, \cdots, 31$. Again, the depth values of the points in each orientation are calculated and updated with respect to the data cloud, χ_{31}. Then provinces are ranked considering the updated depth values and orientations.*

For the last step, the algorithm was written by using some built-in functions in the R package ddalpha (Pokotylo et al., 2016), as well as the codes from the author, and then was implemented in R language for statistical computing.

4 Results

In this section, the results of ranking are presented in three divisions, prosecutor's offices, criminal courts and appeal courts, seperately in details. The analyses have been based on depth approach by using two indicators, PR and CR. Some results have been viasualized for more ease of understanding and the others in some tables. The main purpose of this analysis is ranking provinces with respect to their performance by employing a proper multivariate method.

Prosecutor's office

The plots in Figure 1 represent the position of the prosecutor's office of each province in terms of PR and CR compared to others. The scatter plot in the top of Figure 1 shows the positions in two-dimensional status and two other plots in the below of Figure 1 depict the positions in one-dimensional status. The differences between multivariate analysis and univariate analysis is well shown by these three plots.

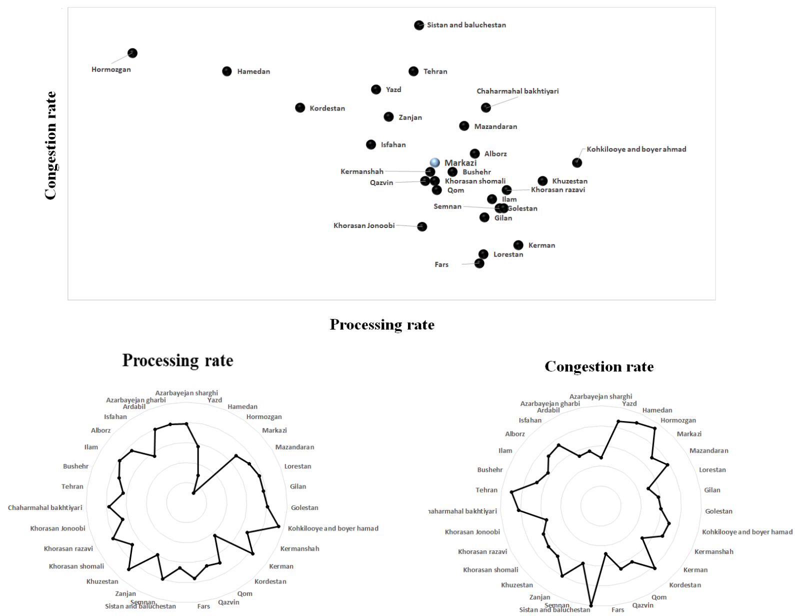

Figure 1: Bivariate and univariate position of prosecutor's office of provinces in terms of processing rate and congestion rate.

Markazi province with the largest depth value is located in the center of provinces cloud and is selected as the median. So, rank and orientation of all provinces is determined in comparison with this province. The neighboring provinces of Markazi province have analogous performance and distant provinces show a different performance from the point of view of PR or CR or both. For instance, Hormozgan, Sistan and bluchestan and Hamedan are located in the skirt of provinces cloud and demonstrate distinct performance. These distinctions make it possible to determine the best and the worst performances.

In order to obtain more accurate results, in addition to visual representation, calculation of depth values is necessary. So, the depth value and orientation, (D, O), of the prosecutor's office of each province along with the results of the ranking have been presented in the second column of Table 2. Based on the depth results and the previous content, provinces with the best performance are located in the second orientation and the worst of them are in the fourth orientation. The performance of provinces in orientations 1 and 4 is relative. Kohkilooye and boyerahmad province has the first rank, so the best performance. As Figure 1 shows, this province has performed well in both components, PR and CR. After that provinces Fars and Azarbayejan sharghi occupy the next best ranks. On the other hand, the provinces, Hormozgan, Sistan and Baluchestan and Hamedan have gained the last ranks.

Criminal court

The plots in Figure 2 represent the bivariate and univariate position of the criminal court of each province in terms of PR and CR via one scatter plot and two radar charts.

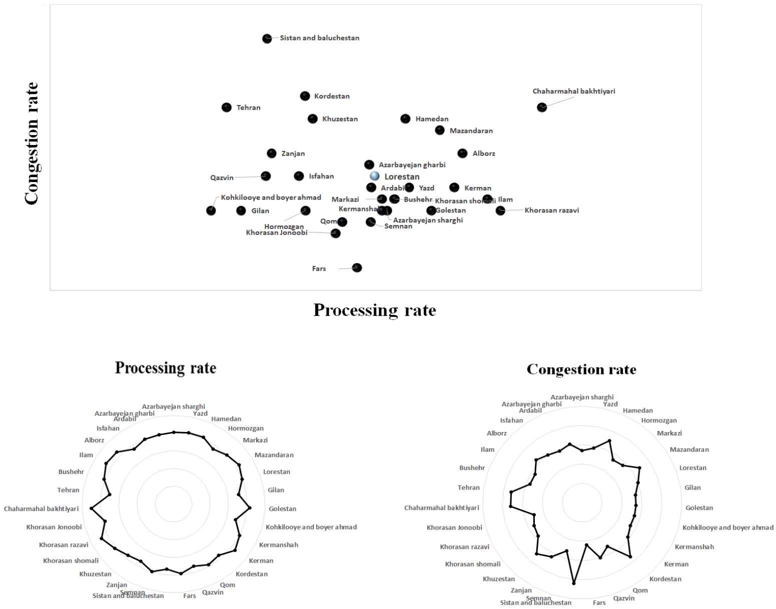

Figure 2: Bivariate and univariate position of criminal court of provinces in terms of processing rate and congestion rate.

The scatter plot in Figure 2 represents similar performance for the most provinces. There are only a few provinces that have located in the border of the data cloud and have a distinct performance. Among all of the provinces, Lorestan province have the largest depth value and is located in the center of provinces cloud. So it is selected as the median and rank and orientation of all provinces is determined in comparison with this province.

Depth value and orientation, (D, O), of criminal court of each province along with the results of the ranking have been presented in the third column of Table 2. As before, provinces with the best performance are located in the second orientation and the worst of them are in the fourth orientation. The performance of provinces in orientations 1 and 4 is relative. Based on the results, Khorasan razavi has the best performance in terms of PR and CR simultaneousely, so the best rank. After that, Ilam and Kerman provinces have gained the second and the third ranks. On the other hand, Sistan and baluchestan and Tehran provinces have obtained the largest ranks. So, their performance is less desirable in comparison with other provinces.

Appeal court

The bivariate and univariate position of the appeal court of each province in terms of PR and CR is demonstrated via the scatter plot and radar charts in Figure 3.

The scatter plot in Figure 3 demonstrates the data cloud consist of PR and CR information of provincial appeals courts. Based on this visual representation, some provinces such as Tehran, Kordestan, Mazandaran and Qom are located further away from the others that means they were different at least from point of view of one

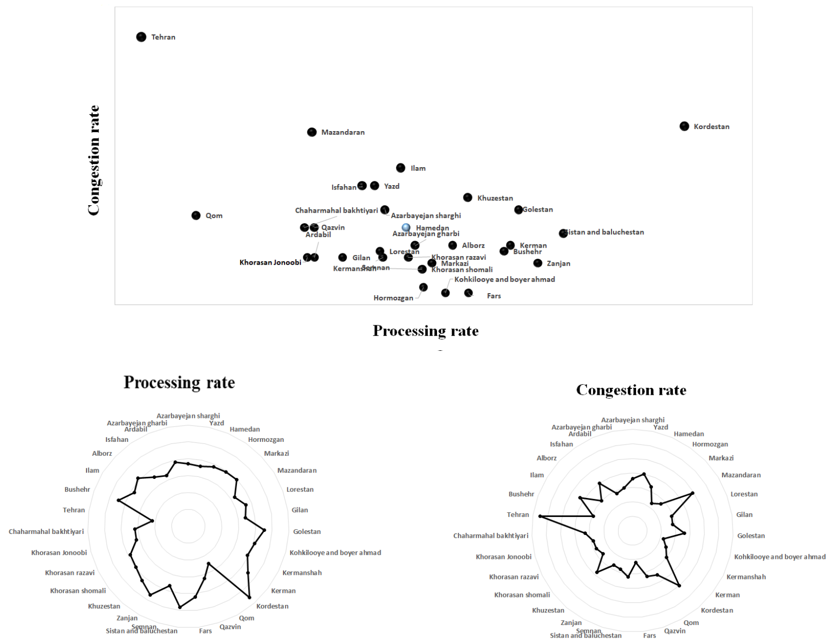

Figure 3: Bivariate and univariate position of appeal court of provinces in terms of processing rate and congestion rate.

indicators, PR and CR. This difference can be favorable or unfavorable.

The depth value, orientation, (D, O), and ranking of appeal courts related to each province have been presented in the last column of Table 2. Based on the depth results, Sistann and baluchestan, Zanjan and Fars provinces havegained the first to the third ranks. On the other hand, Tehran, Qom and Mazandaran provinces have obtained the last ranks. Other provinces have been lacated among these provinces.

5 Conclusion and Discussion

Observing the performance of provincial courts and comparing them with each other can be done using judicial indicators. These indicators are numerous and sometimes are correlated. So, multivariate approaches can be more suitable for analyzing them than univariate methods. It seems multivariate methods have not been considered so far for studying judicial indicators.

Nonparametric multivariate approach based on depth notion can be a suitable method for analyzing the judicial indicators. This method, in addition to showing the status of the provinces in relation to each other, can assign a rank to each province by simultaneously considering some indicators. The result of ranking shows the superiority of the performance. This means the better rank is the better performance in all indicators at the same time. Obviously if the problem is examined from the point

Table 2: Depth value, orientation and rank of prosecutor's office, criminal and appeal courts of provinces

	prosecutor's office		criminal court		appeal court	
province	(D, O)	rank	(D, O)	rank	(D, O)	rank
Azarbayejan sharghi	(0.294, 2)	3	(0.663, 2)	6	(0.893, 3)	23
Azarbayejan gharbi	(0.831, 2)	8	(0.931, 3)	25	(0.901, 2)	13
Ardabil	(0.690, 2)	7	(0.953, 4)	23	(0.415, 4)	17
Isfahan	(0.612, 3)	24	(0.608, 2)	5	(0.613, 3)	25
Alborz	(0.659, 1)	20	(0.354, 1)	18	(0.805, 2)	12
Ilam	(0.742, 2)	13	(0.299, 2)	2	(0.445, 3)	28
Bushehr	(0.258, 2)	16	(0.768, 2)	10	(0.528, 2)	7
Tehran	(0.246, 3)	28	(0.184, 3)	30	(0.063, 3)	31
Chaharmahal bakhtiyari	(0.251, 1)	17	(0.118, 1)	12	(0.498, 3)	27
Khorasan Jonoobi	(0.342, 4)	18	(0.394, 4)	19	(0.389, 4)	15
Khorasan razavi	(0.412, 2)	10	(0.248, 2)	1	(0.757, 2)	11
Khorasan shomali	(0.203, 2)	15	(0.765, 2)	9	(0.627, 2)	9
Khuzestan	(0.140, 2)	5	(0.371, 3)	28	(0.537, 1)	19
Zanjan	(0.607, 3)	25	(0.412, 3)	26	(0.375, 2)	2
Semnan	(0.788, 2)	12	(0.543, 4)	22	(0.690, 4)	20
Sistan and baluchestan	(0.119, 3)	30	(0.101, 3)	31	(0.304, 2)	1
Fars	(0.262, 2)	2	(0.221, 4)	14	(0.395, 2)	3
Qazvin	(0.835, 4)	21	(0.405, 3)	27	(0.549, 3)	26
Qom	(0.231, 2)	14	(0.505, 4)	21	(0.197, 3)	30
Kordestan	(0.289, 3)	26	(0.244, 3)	29	(0.067, 1)	14
Kerman	(0.379, 2)	6	(0.452, 2)	3	(0.505, 2)	6
Kermanshah	(0.957, 4)	22	(0.673, 2)	7	(0.627, 2)	8
Kohkilooye and boyerahmad	(0.192, 1)	1	(0.189, 4)	13	(0.407, 2)	4
Golestan	(0.706, 2)	11	(0.493, 2)	4	(0.396, 1)	16
Gilan	(0.974, 2)	9	(0.251, 4)	15	(0.524, 4)	18
Lorestan	(0.366, 2)	4	(0.984, Med)	24	(0.746, 4)	21
Mazandaran	(0.450, 1)	19	(0.335, 1)	16	(0.235, 3)	29
Markazi	(0.993, Med)	23	(0.814, 2)	11	(0.495, 2)	10
Hormozgan	(0.077, 3)	31	(0.477, 4)	20	(0.450, 2)	5
Hamedan	(0.149, 3)	29	(0.354, 1)	17	(0.999, Med)	22
Yazd	(0.397, 3)	27	(0.756, 2)	8	(0.629, 3)	24

of view of one-dimension, different results will be obtained. Overall, in this paper, the study of judicial performance and ranking in the three divisions, prosecutor's office, criminal court and appeal court has been done by considering two indicators, PR and CR, by employing depth-based approach. Also, to obtain more advantageous results, orientation in 4 directions has been used. The obtained results indicate that due to the multiplicity of indicators and their correlation with each other, it is more accurate to use the multivariate approach. This method is effective for the case where there are more than two indicators in the study. Also, to calculate more precise results, the orientation of observation can be extended to more than 4.

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