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# Ruin related quantities in a class of state-space compound binomial models

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**Abstract:** The main focus of this paper is to extend the analysis of some ruin related problems to a class of state-space compound binomial risk models for a sequence of independent and identically distributed random variables of interclaim times when the claim occurrences are homogeneous. First, we obtain the mass function of a defective renewal sequence of random $\{F_n\}_{n\geq 0}$-stopping times, using the compound binomial of aggregate claim amount together the net profit condition, and compute the infinite time ruin probability with Markov property of risk process. Moreover, we derive the distribution of the time to ruin among many random variables associated with ruin using the convolution of claim amount and Lagrange's implicit function theorem. Lastly, the theoretical results are illustrated with numerical computations.

**Keywords:** Compound binomial risk model; Homogenous claim occurrences; Ruin probability; Time to ruin.
**Mathematics Subject Classification (2010):** 62E15, 91B30.

## 1 Introduction

Let us first start with the description of the compound binomial risk model. Assume that the premium income for each period is one and the total claim amount up to time $n \in N = \{0, 1, 2, \dots\}$ is defined by a binomial process $\{N(n); n \in N\}$ with

$$N(n) = I_1 + I_2 + \cdots + I_n = \sum_{i=1}^{n} I_i, \qquad n \in N^+ = \{1, 2, 3, \dots\},$$

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with $N(0) = 0$, where $\{I_i, i = 1, 2, \dots\}$ are homogeneous, independent and identically distributed (i.i.d.) Bernoulli random variables representing the claim occurrences with mean $p \in (0, 1)$. That is, in any time period there is at most one claim; the probability of having a claim is $p$ and the probability of no claim is $1 - p$. The occurrence of the claims in different time periods are assumed to be independent events and they are homogeneous in the compound binomial risk model. Then $\{N(n); n \in N\}$ is called a binomial process with probability mass function (p.m.f.)

$$P\big(N(n) = k\big) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0, 1, \dots, n.$$

The claim amounts $X_1, X_2, \dots$ are mutually independent, identically distributed, positive, integer–valued, random variables with a finite expectation $\mu_X = E(X)$, common p.m.f. $p(x) = P(X = x), x = 1, 2, \dots$, cumulative distribution function (c.d.f.) $p(y) = \sum_{x=1}^{y} p(x)$, probability generating function $G(z) = \sum_{x=1}^{\infty} z^x p(x)$ and $r$th moment denoted by $E(X^r) = \sum_{x=1}^{\infty} x^r p(x)$.

Premiums are payable at the rate of one per period. Then, the surplus of an insurance company at time $n$ with a constant premium rate is described as

$$R(n) = u + n - \sum_{i=1}^{N(n)} X_i, \qquad n = 1, 2, \dots, \tag{1}$$

where $R(0) = u \in N = \{0, 1, \dots\}$, $N(n)$ and $\sum_{i=1}^{N(n)} X_i$ are the initial surplus, the number of claims up to time $n$ and the total claim amount over $n$ periods, respectively. We assume that $\{N(n); n \in N\}$ and $\{X_i; i = 1, 2, \dots, N(n)\}$ are independent. We further assume that the net profit condition $p\mu_X < 1$ holds. Under the net profit condition the ruin is not certain to occur eventually. This model is so-called compound binomial model and includes information about the premium income rate and the initial capital necessary to meet the expected claims costs and the ruin probability function depends on the some quantities. The risk process (1) is a stationary homogeneous Markov chain, (to see the proof, one can refer to Bazyari (2023a), where the proof is given for a perturbed renewal risk process). From (1), since $\{N(n)\}$ and $\{X_i\}$ are independent, then

$$\begin{aligned} E\big(R(n)\big) &= u + n - E(X)E\big(N(n)\big) \\ &= u + n - np\mu_X \\ &= u + n\big(1 - p\mu_X\big), \end{aligned}$$

and under the net profit condition $p\mu_X < 1$, we have $\lim_{n\to\infty} E\big(R(n)\big) = \infty$.

The risk model in (1) can be written as

$$R(n) = u + n - \sum_{i=1}^{n} Y_i, \qquad n = 1, 2, \dots, \tag{2}$$

where $Y_i = I_i X_i$ is the claim amount in period $i$, with probability function

$$k(x) = \begin{cases} 1 - p, & \text{if } x = 0, \\ pp(x), & \text{if } x = 1, 2, 3, \dots, \end{cases} \tag{3}$$

with probability generating function $\hat{k}(x) = \sum_{x=0}^{\infty} z^x k(x) = 1 - p + pG(z)$ and c.d.f. $K(x) = \sum_{y=0}^{x} k(y)$. Let $T$ be a random variable which represents the random time to ruin. That is, $T = \{\min n \geq 1 : R(n) \leq 0 | R(0) = u\}$. In this case

$$\psi(u) = P\big(T < \infty | R(0) = u\big),$$

denotes the infinite time ruin probability. The survival time ruin probability given by $\phi(u) = 1 - \psi(u)$. In this case, for $n = 1, 2, \dots$, the finite time ruin probability is denoted by

$$\psi(u, n) = P\big(T \leq n | R(0) = u\big). \tag{4}$$

As a complement of (4), the finite time survival probability is denoted by

$$\phi(u, n) = 1 - \psi(u, n) = P\big(T > n | R(0) = u\big) = P_u\big(T > n\big).$$

Define the sequence of random variables $\{\mathcal{V}_k\}_{k\geq 1}$ by

$$T_1 = T, \qquad \mathcal{V}_1 = \inf\{n > T_1 : R(n) = 0\},$$

and for $k = 2, 3, \dots,$

$$T_k = \inf\{n > \mathcal{V}_{k-1} : R(n) < 0\}, \qquad and \qquad \mathcal{V}_k = \inf\{n > T_k : R(n) = 0\}.$$

If the insurance company never stops its activities whenever it is in the red or not, $T_k$ can be viewed as the $k^{th}$ ruin time and $\{\mathcal{V}_k\}$ and $k^{th}$ recovery time. From the definition $\{\mathcal{V}_k\}_{k\geq 1}$, it is clear that $R(\mathcal{V}_k) = 0$ for each $k$, therefore, $R(\mathcal{V}_k - 1) = -1$. Let $F_n = \sigma\big(R(k), k \leq n\big)$ for $n \geq 0$. Then for $k \geq 1$, $T_k$ and $\mathcal{V}_k$ are all $\{F_n\}_{n\geq 0}$-stopping times. For $n = 0, 1, \dots$, we define a counting process $N^*(n) = \sup\{k > 0; \mathcal{V}_k \leq n\}$. It is easy to see that $N^*(n)$ is a renewal counting process, since the risk process $R(n)$ is a Markov process and $R(\mathcal{V}_k) = 0$ for all $k > 0$.

Suppose that $\{w_k\}_{k\geq 1}$ denotes the sequence of inter-occurrence times of the renewal counting process $N^*(n)$ with $w_1 = \mathcal{V}_1$ and $w_k = \mathcal{V}_k - \mathcal{V}_{k-1}$ for $k = 2, 3, \dots$. We can verify that $\{w_k\}_{k\geq 1}$ is a sequence of independent variables and $\{w_k\}_{k\geq 2}$ is a sequence of i.i.d. random variables. Denote the common distribution of $w_k, k \geq 2$ by $W_1$ and that of $w_1$ by $W_1^u$. Then both of them are defective with $W_1(0) = W_1^u(0) = 0$, $W_1(\infty) < 1$ and $W_1^u(\infty) < 1$ for $u \geq 0$. For $u < 0$ distribution of $W_1^u$ is proper.

Among the different types of risk models, the classical binomial risk model, first proposed by Gerber (1988), is a discrete time which assume that the premium income is constant over time and the claim amounts form a sequence of i.i.d. random variables.

Shiu (1989) derived several formulas for the probability of eventual ruin in a discrete time model when the total claim amount process is assumed to be binomial. Willmot (1993) computed some explicit formulas for finite time ruin probabilities in the discrete time and state-space compound binomial model using the technique of generating functions. Dickson (1994) suggested that the compound binomial model is useful in approximating the classical continuous-time compound Poisson mode. Cheng et al. (2000) computed the discounted probability of ruin compound binomial risk model. Cossette et al. (2003) and Cossette et al. (2004) extended the compound binomial model to the so-called compound Markov binomial model which introduces time dependence in the aggregate claim amount increments governed by a Markov process.

Yuen and Guo (2006) gave results including the expected discounted penalty function, ruin probability and deficit at ruin in a compound Markov binomial model. Xiao and Guo (2007) btained the recursive formula of the joint distribution of the surplus immediately prior to ruin and deficit at ruin in the compound binomial risk model with the time-correlated claims. Moreover, Lefèvre and Loisel (2008) investigated the finite time ruin probabilities for some classical risk models, including the compound binomial model.

Li et al. (2009) presented a review of results for discrete-time risk models, including the compound binomial risk model and some of its extensions. Leipus and Šiaulys (2011) found the asymptotics of finite horizon ruin probability in the compound discrete time risk model for a subclass of heavy-tailed claim sizes and claim numbers. Bao and Liu (2012) computed the recursive equations for both the infinite time ruin probability, the joint distribution of the surplus one period prior to ruin and the deficit at ruin under the compound binomial risk model with delayed claims and random income.

In addition to ruin-related problems, total dividends payable before ruin is another focus point in risk theory. There are many papers in the literature studying risk models with deterministic dividend strategies (constant, linear, etc.). Among them, Wu and Li (2012) studied expected total dividends until ruin in a discrete time risk model with delayed claims and a constant dividend barrier. Li (2008) analyzed the moments of the present value of the dividends in the compound binomial model under a constant dividend barrier and stochastic interest rates. Eryilmaz (2012) studied the distribution of the total number of claims and its conditional Distributions. Tuncel and Tank (2014) computed the distribution of total number of claims with nonhomogeneous claim occurrences. Wat et al. (2018) obtained explicit results for the discount-free Gerber–Shiu function for a compound binomial risk model in the presence of delayed claims and a randomized dividend strategy.

The key difference between the discrete time risk model and compound binomial risk model comes from their distributions of the inter-occurrence time. Yang et al. (2012) computed the ruin probability in a dependent discrete time risk model with insurance and financial risks. Sun and Wei (2014) studied an insurance risk model in which the insurer makes both risk free, and risky investments. Yang and Konstantinides (2015) derived the precise estimates for ruin probabilities in a discrete time insurance risk model with dependent financial and insurance risks. Dickson and Qazvini (2017) studied the compound Markov binomial model and derived the numerical algorithms that provide approximations to the infinite time ruin probability. Bazyari (2022) computed the ruin probabilities in a generalized dual binomial risk model using Markov property. Bazyari (2023b) obtained the ruin probabilities in the discrete time insurance risk process with capital injections and analyzed the effect of capital injections when claim amounts follow a heavy-tailed distribution.

The aim of the present paper is to derive the probability generating function of time to ruin according to the joint probability function of a class of state-space compound binomial risk model. Moreover, the ruin probabilities and the distribution of the time to ruin are computed.

The remainder of this paper organized as follows. In Section 2, we give the expressions for some related problems of the ruin probabilities. In Section 3, for a class of state-space compound binomial risk model with homogenous claim occurrences, we obtain the mass function of a defective renewal sequence of random $\{F_n\}_{n\geq 0}$-stopping times using the Markov property of risk process together the net profit condition. Then the infinite time ruin probability is computed. Section 4 deals with the distribution of the time to ruin among many random variables associated with ruin using the convolution of claim amount. The numerical examples to illustrate the application of risk model are presented in Section 5. Discussion and conclusions are given in Section 6.

# 2 Expressions for some ruin related problems

In this section, for convenience of later use, we give the notations for the distribution of time to ruin and the probability generating function of time to ruin and obtain an expression for the probability generating function of time to ruin according to the joint probability function of the surplus prior to ruin, the deficit at ruin and the time to ruin.

For $u \in N$ and $n = 1, 2, \dots,$ the distribution of time to ruin and probability generating function of time to ruin are defined by $\tau(u, n) = P\big(T = n|R(0) = n\big)$ and $G_T(u) = E\big\{\gamma^T I(T < \infty)|R(0) = u\big\}$ respectively, where $0 < \gamma \leq 1$. It is clear that

$$G_T(0) = \sum_{n=1}^{\infty} \tau(0, n)\gamma^n. \tag{5}$$

On the other hand, from Li and Garrido (2002) the function $G_T(u)$ can be expressed by $G_T(u) = \sum_{n=1}^{\infty} \big(1 - G_T(0)\big)\big(1 - D^{*n}(u)\big)$, where $D(u) = \sum_{y=1}^{u} d(y|0)$, is the defective cumulative distribution function of the deficit at ruin with no initial reserve.

Suppose that $f_{XYT}(x, y, t|u)$, $x, u \in N$ , $y, t \in N^+$ be the joint probability function of the surplus prior to ruin, $x$, the deficit at ruin $y$, and the time to ruin $t$, starting with an initial surplus of $u$, then

$$G_T(u) = \sum_{x=0}^{\infty}\sum_{y=1}^{\infty}\sum_{t=1}^{\infty} \gamma^t f_{XYT}(x, y, t|u). \tag{6}$$

Furthermore, for $u \in N$, $\psi(u) = \sum_{n=1}^{\infty} \tau(u, n)$, and

$$\psi(u, n) = P\big(T \leq n|R(0) = u\big) = \sum_{j=1}^{n} \tau(u, j).$$

Therefore, with computing the probability generating function of time to ruin we can obtain the ruin probabilities.

# 3 Mass function of the defective renewal

In this section, we derive the mass function of the defective renewal of the counting process when the claim occurrences are homogeneous, using the compound binomial of aggregate claim amount, and compute the infinite time ruin probability of risk process in (2) with Markov property.

Let $W^u$ is the renewal function of the renewal process $N^*(n)$ with the initial value $u$, i.e.

$$\begin{aligned} W^u(n) &= \sum_{k=1}^{\infty} P\big(\mathcal{V}_k \le n | R(0) = u\big) \\ &= \sum_{k=1}^{n} P\big(\mathcal{V}_k \le n | R(0) = u\big) \\ &= \sum_{k=1}^{n} W_1^u * W_1^{*(k-1)}, \end{aligned}$$

with the mass function

$$W^u(n) = \sum_{k=1}^{\infty} P\big(\mathcal{V}_k = n | R(0) = u\big) = \sum_{k=1}^{n} P\big(\mathcal{V}_k = n | R(0) = u\big), \tag{7}$$

for $n = 1, 2, \ldots$. It is well known that the distribution law of the aggregate claim amount $S(n) = \sum_{i=1}^{N(n)} X_i$ is compound binomial, i.e.,

$$f_{S(n)}(j) = P(S(n) = j) = \sum_{k=1}^{n \wedge j} \binom{n}{k} p^k (1-p)^{n-k} H_j^{*k}, \tag{8}$$

where $H_j^{*k} = P\big(\sum_{b=1}^{k} X_b = j\big)$ and $n \wedge j = \min\{n, j\}$.

**Lemma 3.1.** *For the mass function $W^u(n)$, we have $W^u(0) = 0$ and for $n = 1, 2, \ldots,$*

$$W^u(n) = \begin{cases} 0, & \text{if } u + n < 0, \\ (1-p)^n, & \text{if } u + n = 0, \\ \sum_{k=1}^{(n-1)\wedge(u+n)} \binom{n-1}{k} p^k (1-p)^{n-k} H_{u+n}^{*k}, & \text{if } u + n > 0, \end{cases}$$

*Proof.* Clearly, for $n = 0$, the mass function $W^u(0)$ is equal to zero and for $u + n < 0$, $W^u(n) = 0$. Also, if $u + n = 0$, there is no claim until the time $n - 1$. From (7), we have

$$\begin{aligned} W^u(n) &= \sum_{k=1}^{n} P\big(\mathcal{V}_k = n | R(0) = u\big) \\ &= P\big(R(n-1) = -1, N(n) - N(n-1) = 0 | R(0) = u\big) \\ &= (1-p) P\big(u + n - 1 - S(n-1) = -1 | R(0) = u\big) \\ &= (1-p) f_{S(n-1)}(u + n), \end{aligned} \tag{9}$$

obviously, for $u + n = 0$, $w^u(n) = (1-p)^n$ and for $u + n > 0$, from (8) and (9) we get

$$\begin{aligned} W^u(n) &= (1-p) \sum_{k=1}^{(n-1)\wedge(u+n)} \binom{n-1}{k} p^k (1-p)^{n-k-1} H_{u+n}^{*k} \\ &= \sum_{k=1}^{(n-1)\wedge(u+n)} \binom{n-1}{k} p^k (1-p)^{n-k} H_{u+n}^{*k}. \end{aligned}$$

Thus, the proof is completed. □

### 3.1 Infinite time ruin probability

One central problem in the theory of ruin is to find $\psi(u)$. Now, we are ready to obtain the infinite time ruin probability according to the mass function of the defective renewal.

**Theorem 3.2.** *Suppose that the net profit condition $pE(X) < 1$ holds. Then for $u \geq 0$ the infinite time ruin probability given by*

$$\psi(u) = \frac{1 - pE(X)}{1 - p} W^u(\infty),$$

*where* $W^u(\infty) = \sum_{n=1}^{\infty} w^u(n) = \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-k} H_{u+n}^{*k}$.

*Proof.* With the Markov property of the risk process, we have

$$\begin{aligned} \psi(u) &= \sum_{k=1}^{\infty} P\big(\mathcal{V}_k \leq \infty, \mathcal{V}_{k+1} = \infty | R(0) = u\big) \\ &= \sum_{k=1}^{\infty} P\big(\mathcal{V}_k \leq \infty | R(0) = u\big) P\big(\mathcal{V}_1 = \infty | R(0) = 0\big) \\ &= \phi(0) W^u(\infty), \end{aligned}$$

where from Shiu (1989), $\phi(0) = \frac{1-pE(X)}{1-p}$, and this completes the proof. □

# 4 Distribution of time to ruin with homogenous claim occurrences

In this section, we present two separately formulas for the distribution of time to ruin with homogenous claim occurrences given the discrete nature of the insurance risk model.

**Theorem 4.1.** ***(Lagrange's implicit function theorem).*** *If $t$ is determined by the following implicit function*

$$t = c + zg(t), \tag{10}$$

*where $t$, $c$ and $z$ are scalars, and $g(y)$ is a scalar function developable as a power series in $t = c$ for $|t - c|$ sufficiently small. Suppose that $L(t)$ is any function of $t$, then the Lagrange's formula is given by*

$$L(t) = L(c) + \sum_{n=1}^{\infty} \frac{z^n}{n!} \frac{d^{n-1}}{dc^{n-1}} \big(g^n(c) \frac{d}{dc} L(c)\big).$$

*In particular, if $L(t) = t$, one obtains the equality $t = c + \sum_{n=1}^{\infty} \frac{z^n}{n!} \frac{d^{n-1}}{dc^{n-1}} \big(g^n(c)\big)$, which inverts* (10) *giving as a power series in $z$.*

*Proof.* See p. 15 of Goulden and Jackson (1983). □

Using Lemma 1 in Li and Garrido (2002), there exists the scalar $\nu \in (0,1)$, such that $\nu = \gamma\hat{k}(\nu)$. Therefore, using Theorem 4.1, we have

$$L(\nu) = L(0) + \sum_{n=1}^{\infty} \frac{z^n}{n!} \frac{d^{n-1}}{dc^{n-1}} \big(\frac{d}{dc} L(c) \sum_{z=0}^{\infty} c^z k^{*n}(z)\big)_{c=0},$$

where $k^{*n}$ is the $n^{th}$ fold convolution of claim amount $Y_1$ with itself, and is computed by

$$k^{*n}(z) = \frac{1}{k(0)} \sum_{j=1}^{z} \big[(n+1)\frac{j}{z} - 1\big] k(j) k^{*n}(z-j),$$

where $k^{*n}(0) = \big[k(0)\big]^n$ (To proof the equation $k^{*n}(z)$ see Dickson and Willmot (2005)).

In particular, $\nu$ may be computed explicitly by

$$\begin{aligned} \nu &= \sum_{n=1}^{\infty} \frac{z^n}{n!} \frac{d^{n-1}}{dc^{n-1}} \big(c^z k^{*n}(z)\big)_{c=0} \\ &= \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_{z=n-1}^{\infty} z^{(n-1)} c^{z-(n-1)} k^{*n}(z)|_{c=0} \\ &= \sum_{n=1}^{\infty} \frac{z^n}{n!} k^{*n}(n-1), \end{aligned}$$

where $z^{(n)} = z(z-1)(z-2)\dots(z-n+1)$ is the $n^{th}$ factorial power of $z$.

**Lemma 4.2.** *For any function $h(t)$, $t \in N^+$, such that*

$$\hat{h}(\nu) = \sum_{t=1}^{\infty} \nu^t h(t) = \sum_{n=1}^{\infty} \gamma^n g(n) = \hat{g}(\gamma),$$

*with $\nu = \nu(\gamma)$ as defined by Lemma 1 in Li and Garrido (2002), then for $n$, $n = 1, 2, \dots,$ we have*

$$g(n) = \frac{1}{n} \sum_{t=1}^{\infty} t h(t) k^{*n}(n-t).$$

*Proof.* See Li and Sendova (2013) to prove this Lemma. □

**Lemma 4.3.** *For any $n = 1, 2, \dots,$ in the case of no initial surplus, the distribution of time to ruin $\tau(0, n)$ is derived by*

$$\tau(0,n) = \frac{1}{1-p} \big(\frac{1}{n} \sum_{t=1}^{n} t k^{*n}(n-t) - \frac{1}{n+1} \sum_{t=1}^{n+1} t k^{*(n+1)}(n+1-t)\big).$$

*Proof.* See Li and Sendova (2013) to prove this Lemma. □

**Lemma 4.4.** *For any $n = 1, 2, \dots,$ in the case of $u > 0$, the distribution of time to ruin $\tau(u, n)$ is derived by*

$$\tau(u,n) = P\big(T = n | R(0) = u\big) = \frac{1}{n} \sum_{z=1}^{n} z M(u,z) k^{*(n-z)}, \qquad (11)$$

*where*

$$
\begin{aligned}
M(u,z) &= \frac{pE(Z)}{1-p}b_1(z)+\sum_{n=1}^{u}\big(\frac{pE(Z)}{1-p}\sum_{z=1}^{n}b_1(z-t)\theta_n(u,t)-\theta_n(u,z)\big),\\
\theta_n(u,t) &= (\frac{p}{1-p})^n\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}\sum_{z=0}^{u-n}\binom{z+n-1}{n-1}\\
&\quad\times P^{*k}(u-n+k-z)P^{*(n-k)}(t+n-k+z),
\end{aligned}
$$

*for* $u\geq n$ *and* $t=1,2,\dots,$ *where* $P^{*k}(u-n+k-z)=\sum_{y=x+n}^{u}p^{*l}(y-n+k-z)$.

*Proof.* Using the implementation of the last equation on page 6 given in Li and Garrido (2002). With the penalty function being identically equal to 1,

$$
\begin{aligned}
G_T(0) &= E\{\gamma^T I(T<\infty)|R(0)=u\}\\
&= \frac{\nu}{k(0)}\sum_{x=0}^{\infty}\sum_{t=1}^{\infty}\nu^x k(x+t+1)\\
&= \frac{\nu}{1-p}\sum_{t=2}^{\infty}\sum_{x=0}^{t-2}\nu^x k(t). \qquad (12)
\end{aligned}
$$

The relation (12) implies that the probability function of the deficit at ruin with no initial reserve is

$$d(y|0)=\sum_{x=0}^{\infty}\sum_{t=1}^{\infty}\gamma^t f_{XYT}(x,y,t|0)=\frac{1}{k(0)}\sum_{x=0}^{\infty}\nu^{x+1}k(x+y+1).$$

Moreover, the identity (6) can be written as

$$
\begin{aligned}
G_T(u) &= G_T(0)+\sum_{n=1}^{\infty}G_T(0)D^{*n}(u)-\sum_{n=1}^{\infty}D^{*n}(u)\\
&= G_T(0)+\sum_{n=1}^{u}G_T(0)D^{*n}(u)-\sum_{n=1}^{u}D^{*n}(u), \qquad (13)
\end{aligned}
$$

where $D^{*n}(u)=0$ for $n>u$ and $G_T(0)$ obtains from (12). To compute $D^{*n}(u)$, consider $D^{*n}(u)=\sum_{y=n}^{u}d^{*n}(y|0)$ and we note that $d^{*n}(r)=\sum_{y=n}^{u}r^y d^{*n}(y|0)=\hat{d}^n(r)$ with $\hat{d}(r)=\frac{p}{1-p}\frac{(r/\nu)\hat{p}(\nu)-\hat{p}(r)}{1-r/\nu}$. Then

$$d^{*n}(r)=\big(\frac{p}{(1-p)(1-r/\nu)}\big)^n=\sum_{l=0}^{n}(-1)^l\binom{n}{l}\hat{p}^l(r)\big(\frac{r}{\nu}\big)^{n-1}\hat{p}^{n-l}(r).$$

Now, let the function $w_n(y,m)$, $y\geq n$, exists such that $d^{*n}(y|0)=\sum_{m=0}^{\infty}\nu^m w_n(y,m)$. Then $d^{*n}(r)=\sum_{m=0}^{\infty}\nu^m\sum_{y=n}^{\infty}r^y w_n(y,m)$.

Note that, if $\nu=0$, then $d^{*n}(r)=0$. Moreover, we have $D^{*n}(u)=\sum_{m=0}^{\infty}\nu^m\theta_n(u,m)$, where $\theta_n(u,m)=\sum_{y=n}^{\infty}w_n(y,m)$. Therefore,

$$D^{*n}(u) = (\frac{p}{1-p})^n\sum_{k=0}^{n}(-1)^k\binom{n}{l}(\frac{\hat{p}(\nu)}{\nu})^{n-l}\sum_{x=0}^{u-n}\binom{x+n-1}{n-1}P^{*l}(u-n+l-x)\nu^{-x}$$

$$+(\frac{p}{1-p})^n(\frac{\hat{p}(\nu)}{\nu})^n\sum_{x=0}^{u-n}\binom{x+n-1}{n-1}\nu^{-x}$$

$$= (\frac{p}{1-p})^n\sum_{k=0}^{n}(-1)^k\binom{n}{l}(\frac{\hat{p}(\nu)}{\nu})^{n-l}\sum_{x=0}^{u-n}\binom{x+n-1}{n-1}P^{*l}(u-n+l-x)\nu^{-x},$$

since for $y \geq 0$, $p^{*0}(y) = 1$. Also, after a change of variables we observe that for $0 \leq l \leq n$,

$$\hat{p}^{n-l}(\nu)\nu^{u-n+l-x} = \sum_{m=u-x}^{\infty}\hat{p}^{*(n-l)}(m)(m-u+n-l+x)\nu^m.$$

Thus

$$\begin{aligned} D^{*n}(u) &= (\frac{p}{1-p})^n\sum_{l=0}^{n}(-1)^k\binom{n}{l}\sum_{m=n}^{u}[\sum_{x=u-m}^{u-n}\binom{x+n-1}{n-1}P^{*l}(u-n+l-x)p^{*(n-l)} \\ &\quad\times(m-u+n-x)\nu^m] + (\frac{p}{1-p})^n\sum_{l=0}^{n}(-1)^k\binom{n}{l} \\ &\quad\times\sum_{m=u+1}^{\infty}[\sum_{x=0}^{u-n}\binom{x+n-1}{n-1}P^{*n}(u-n+l-x)\nu^{-x} \\ &\quad\times p^{*(n-l)}(m-u+n-x)\nu^m]\nu^m \\ &\quad+(\frac{-p}{1-p})^n\sum_{m=n}^{u}\binom{x-m+n-1}{n-1}P^{*n}(m)\nu^m. \end{aligned} \tag{14}$$

Using identity (14), we get

$$\begin{aligned} \theta_n(u,y) &= (\frac{p}{1-p})^n\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}\sum_{z=0}^{u-n}\binom{z+n-1}{n-1} \\ &\quad\times P^{*n}(u-n+k-z)p^{*(n-k)}(y+n-k+z). \end{aligned}$$

Finally, combining the equations (12) and (13), we have

$$G_T(0) = \sum_{y=1}^{\infty}\nu^y M(u,z) = \sum_{n=1}^{\infty}\gamma^n\tau(u,n),$$

where $M(u,z)$ defined in Lemma 4.4. Finally, using Lemma 4.2 the equation (11) will be proved. □

# 5 Numerical illustrations

In the following, we will proceed to numerically illustrate the usefulness of the related problems on the obtained ruin probability in this manuscript. For that reason, we consider three different p.m.f. for claim amounts and different values of the initial reserve

$u$ and the time $n$. Furthermore, the finite time ruin probability and the distribution for time to ruin have been computed for the fixed values of $u$ and $n$. Then, to assess the performance of the ruin probabilities and the distributions for time to ruin computed in this paper, these values have been calculated using Monte Carlo simulation.

**Example 5.1.** *Assume that the claim amounts are distributed with common p.m.f.* $p(x) = (1-\theta)^{x-1}\theta I(x \geq 1)$, $\theta \in (0,1)$, *so that by* (3), $k(0) = 1 - p$ *and* $k(x) = pp(x)$, $x = 1, 2, \ldots$. *Then for* $x \geq n$, *we have*

$$p^{*n}(x) = \binom{x-1}{x-n}(1-\theta)^n \theta^{x-n},$$

*and for* $n = 1, 2, \ldots,$ *we have*

$$k^{*n}(x) = \begin{cases} (1-p)^n, & \text{if } x = 0, \\ \sum_{n=1}^{x} \binom{n}{k}\binom{x-1}{x-k} p^x (1-p)^{n-k}(1-\theta)^k \theta^{x-k}, & \text{if } x \in N^+. \end{cases}$$

*Let* $p = 0.3$ *and* $\theta = 0.5$, *so that* $pE(X) = 0.6 < 1$. *We compute the mass function* $w^u(n)$ *for* $u = 0, 5, 10, 15, 20, 25$ *and* $n = 1, 2, \ldots, 10$. *We display the numerical results in Table 1.*

Table 1: The mass function $w^u(n)$.

| | $u$ | | | | | |
|---|---|---|---|---|---|---|
| $n$ | 0 | 5 | 10 | 15 | 20 | 25 |
| 1 | 0.03518 | 0.06702 | 0.09627 | 0.34150 | 0.53022 | 0.64710 |
| 2 | 0.05272 | 0.08215 | 0.15253 | 0.37802 | 0.55131 | 0.68302 |
| 3 | 0.07045 | 0.09843 | 0.18038 | 0.39224 | 0.57602 | 0.71155 |
| 4 | 0.08913 | 0.13029 | 0.21545 | 0.41055 | 0.59325 | 0.74224 |
| 5 | 0.16506 | 0.17648 | 0.24700 | 0.42061 | 0.60873 | 0.75046 |
| 6 | 0.19850 | 0.18551 | 0.28183 | 0.45412 | 0.63113 | 0.78287 |
| 7 | 0.22603 | 0.21080 | 0.31934 | 0.48306 | 0.65002 | 0.80338 |
| 8 | 0.25048 | 0.24320 | 0.33408 | 0.50072 | 0.68351 | 0.83019 |
| 9 | 0.29072 | 0.27592 | 0.36241 | 0.52921 | 0.71616 | 0.86553 |
| 10 | 0.33081 | 0.30315 | 0.39605 | 0.54800 | 0.73454 | 0.88245 |

*The infinite time ruin probabilities are obtained and the results are reported in Table 2. From this Table, it can be seen that, the infinite time ruin probability decrease as the initial reserve increases.*

Table 2: The infinite time ruin probabilities.

| $u$ | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 5 | 10 | 15 | 20 | 25 | 30 |
| 0.52852 | 0.47401 | 0.35226 | 0.30719 | 0.24630 | 0.21083 | 0.18914 |

*Using Lemma 4.3 and formula* (11) *the distribution of time to ruin is computed for* $u = 0, 5, 10, 15, 20, 25$ *and* $n = 1, 2, \ldots, 10$. *We display the numerical results in Table 3.*

*The finite time ruin probabilities are computed and numerical results are presented in Table 4. From this Table, it can be seen that the finite time ruin decreases as the initial reserve increases.*

Table 3: The distribution of time to ruin $\tau(u,n)$.

| | $u$ | | | | | |
|---|---|---|---|---|---|---|
| $n$ | 0 | 5 | 10 | 15 | 20 | 25 |
| 1 | 0.20316 | 0.17250 | 0.14724 | 0.07132 | 0.01165 | 0.00752 |
| 2 | 0.17980 | 0.13925 | 0.10370 | 0.05542 | 0.00831 | 0.00351 |
| 3 | 0.12351 | 0.09542 | 0.07650 | 0.05106 | 0.00723 | 0.00085 |
| 4 | 0.08392 | 0.06181 | 0.05023 | 0.04685 | 0.00484 | 0.00026 |
| 5 | 0.05514 | 0.04870 | 0.04215 | 0.04175 | 0.00217 | 0.00008 |
| 6 | 0.04920 | 0.04312 | 0.03902 | 0.03702 | 0.000951 | 0.00006 |
| 7 | 0.04361 | 0.03706 | 0.03264 | 0.03114 | 0.000528 | 0.00005 |
| 8 | 0.03783 | 0.03127 | 0.02756 | 0.02651 | 0.000400 | 0.00004 |
| 9 | 0.02972 | 0.02552 | 0.01937 | 0.02260 | 0.000242 | 0.00003 |
| 10 | 0.02439 | 0.01842 | 0.01403 | 0.01992 | 0.000113 | 0.00002 |

Table 4: The finite time ruin probability $\psi(u,n)$.

| | $u$ | | | | | |
|---|---|---|---|---|---|---|
| $n$ | 0 | 5 | 10 | 15 | 20 | 25 |
| 1 | 0.20316 | 0.17250 | 0.14724 | 0.07132 | 0.01165 | 0.00752 |
| 2 | 0.38296 | 0.31175 | 0.25094 | 0.12674 | 0.01996 | 0.01103 |
| 3 | 0.50647 | 0.40717 | 0.32744 | 0.17780 | 0.02719 | 0.01188 |
| 4 | 0.59039 | 0.46898 | 0.37767 | 0.22465 | 0.03203 | 0.01214 |
| 5 | 0.64553 | 0.51768 | 0.41982 | 0.26640 | 0.03420 | 0.01222 |
| 6 | 0.69473 | 0.56080 | 0.45884 | 0.30342 | 0.035151 | 0.01228 |
| 7 | 0.73834 | 0.59786 | 0.49148 | 0.33456 | 0.035679 | 0.01233 |
| 8 | 0.77617 | 0.62913 | 0.51904 | 0.36107 | 0.036079 | 0.01237 |
| 9 | 0.80589 | 0.65465 | 0.53841 | 0.38367 | 0.036321 | 0.01240 |
| 10 | 0.83028 | 0.67307 | 0.55244 | 0.40359 | 0.036434 | 0.01242 |

*We now describe the Monte Carlo simulation process employed to generate estimates of finite time ruin probabilities and the distributions for time to ruin resulting from those that experienced ruin. We simulated a large number of trajectories or sample paths of the surplus process by first initializing a surplus at time 0 of u, and then generating claims and receiving premiums. It was possible that some of these sample paths will never lead to ruin and therefore necessary to terminate the process at some finite time.*

*In any process that has started, if* $T < n$ *then we say that process has led to ruin, otherwise ruin has not occurred in that process. We assume that the total number of simulated routes for each individual risk model is equal to* $M$ *and for each of the individual risk model with initial capital u,* $I_m(.)$ *is an indicator function of the ruin probability for the mth path and* $T$ *is the time to ruin. In this case, the estimated finite time ruin probability is given by* $\hat{\psi}(u,n) = \frac{1}{M}\sum_{m=1}^{M} I_m(T < n)$.

*It is clear that* $\hat{\psi}(u,n)$ *is an unbiased estimator for the finite time ruin probability. Moreover, we assume a total of 1000 static policies, that is, there will always be this much exposure in the insurance portfolio for each time period. We simulate* $10^4$ *trajectories of the risk process and we calculate how many times on average they fall below zero in order to get values of* $\hat{\psi}(u,n)$. *We present the numerical results in Table 5.*

*From the Tables 4 and 5, it can be seen that, the exact and simulated estimate for the finite time ruin probability decrease as the initial reserve increases. In addition, comparison of the exact values and simulated estimates for the finite time ruin proba-*

Table 5: The simulation of finite time ruin probability $\hat{\psi}(u, n)$.

| | $u$ | | | | | |
|---|---|---|---|---|---|---|
| $n$ | 0 | 5 | 10 | 15 | 20 | 25 |
| 1 | 0.20857 | 0.16941 | 0.15301 | 0.06605 | 0.01726 | 0.00718 |
| 2 | 0.37340 | 0.30829 | 0.24157 | 0.11902 | 0.02261 | 0.01154 |
| 3 | 0.51227 | 0.41603 | 0.31025 | 0.18286 | 0.02960 | 0.01186 |
| 4 | 0.58344 | 0.47900 | 0.36017 | 0.25870 | 0.03355 | 0.01230 |
| 5 | 0.62907 | 0.52263 | 0.40036 | 0.27385 | 0.03512 | 0.01297 |
| 6 | 0.69855 | 0.54470 | 0.46718 | 0.28337 | 0.03680 | 0.01299 |
| 7 | 0.74522 | 0.59230 | 0.50388 | 0.30477 | 0.03824 | 0.01305 |
| 8 | 0.78260 | 0.64800 | 0.53746 | 0.35951 | 0.03918 | 0.01340 |
| 9 | 0.82133 | 0.67991 | 0.55844 | 0.37760 | 0.03974 | 0.01389 |
| 10 | 0.83470 | 0.68251 | 0.57261 | 0.40115 | 0.04031 | 0.01395 |

*bilities for $n = 1, 5, 10$ are depicted in Figures 1, 2 and 3. From these Figures, it can be seen that the exact value and simulated estimate for the finite time ruin probability are close and almost coincide.*

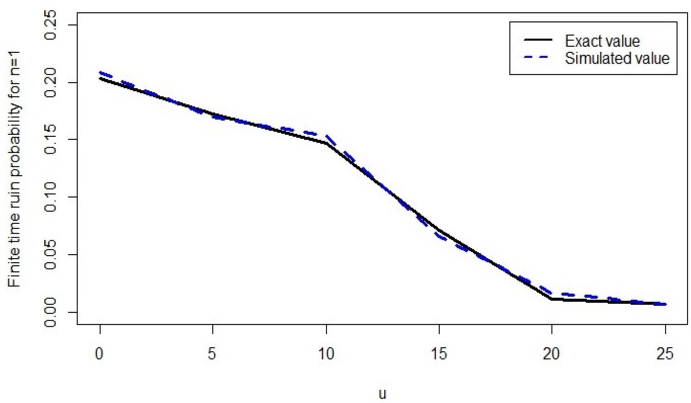


Figure 1: The values of ruin probabilities $\psi(x, n)$ and $\hat{\psi}(x, n)$ for $n = 1$.

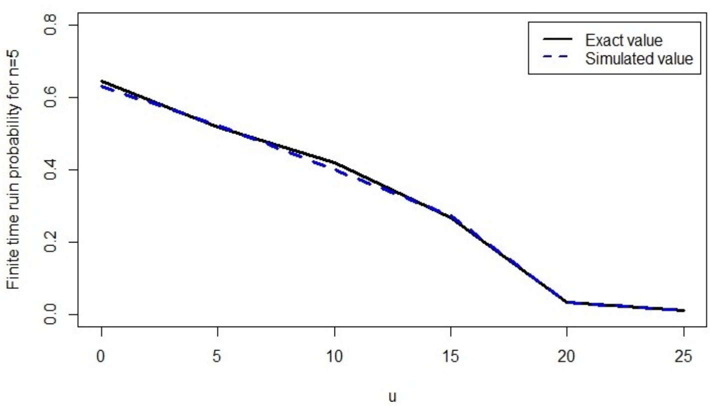


Figure 2: The values of ruin probabilities $\psi(x, n)$ and $\hat{\psi}(x, n)$ for $n = 5$.

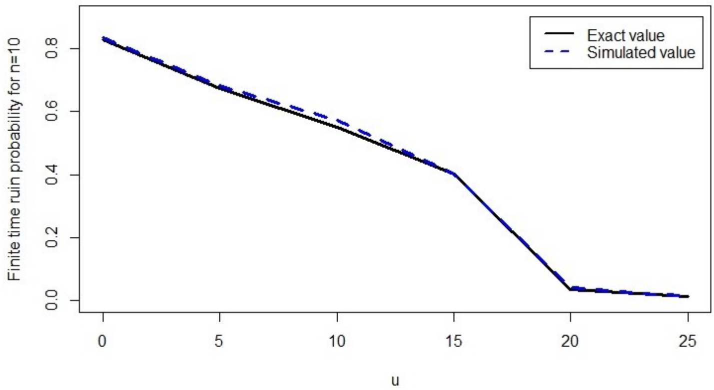


Figure 3: The values of ruin probabilities $\psi(x,n)$ and $\hat{\psi}(x,n)$ for $n = 10$.

**Example 5.2.** *In this example, we assume that the claim amounts have a negative binomial* $B(2,\theta)$ *with common p.m.f.* $p(x) = x(1-\theta)^2\theta^{x-1}I(x \geq 1)$, $\theta \in (0,1)$, *so that* $k(0) = 1-p$ *and* $k(x) = pp(x)$, $x = 1,2,\ldots$. *Let* $p = 0.15$ *and* $\theta = 0.6$, *and* $pE(X) = 0.06 < 1$. *As similar to example 1, we compute the mass function* $w^u(n)$ *for* $u = 0,5,10,15,20,25$ *and* $n = 1,2,\ldots,10$ *and display the numerical results in Table 6.*

Table 6: The mass function $w^u(n)$.

| | $u$ | | | | | |
|---|---|---|---|---|---|---|
| $n$ | 0 | 5 | 10 | 15 | 20 | 25 |
| 1 | 0.12659 | 0.15034 | 0.22075 | 0.29041 | 0.36366 | 0.44518 |
| 2 | 0.18074 | 0.20652 | 0.25330 | 0.31225 | 0.39001 | 0.46202 |
| 3 | 0.22855 | 0.23811 | 0.27810 | 0.34801 | 0.41722 | 0.47819 |
| 4 | 0.24421 | 0.26400 | 0.28923 | 0.36720 | 0.43600 | 0.49746 |
| 5 | 0.25005 | 0.29112 | 0.31275 | 0.37344 | 0.45802 | 0.53015 |
| 6 | 0.27711 | 0.31298 | 0.34811 | 0.39201 | 0.47313 | 0.55248 |
| 7 | 0.28630 | 0.33760 | 0.36654 | 0.41945 | 0.49107 | 0.57811 |
| 8 | 0.29102 | 0.35144 | 0.37992 | 0.43330 | 0.51259 | 0.59003 |
| 9 | 0.32709 | 0.36901 | 0.39015 | 0.44715 | 0.52763 | 0.62255 |
| 10 | 0.34581 | 0.39045 | 0.41307 | 0.45902 | 0.54112 | 0.64120 |

*Moreover, we compute the infinite time ruin probabilities and the results are reported in Table 7. The numerical results for the distribution of time to ruin are derived and we display the results in Table 8.*

Table 7: The infinite time ruin probabilities.

| $u$ | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 5 | 10 | 15 | 20 | 25 | 30 |
| 0.68712 | 0.59745 | 0.51093 | 0.44860 | 0.40835 | 0.34801 | 0.28123 |

*Moreover, the finite time ruin probabilities are computed and the numerical results are presented in Table 9. From this Table, it can be seen that the finite time ruin decreases as the initial reserve increases.*

Table 8: The distribution of time to ruin $\tau(u,n)$.

| | $u$ | | | | | |
|---|---|---|---|---|---|---|
| $n$ | 0 | 5 | 10 | 15 | 20 | 25 |
| 1 | 0.12894 | 0.10294 | 0.07581 | 0.03140 | 0.00551 | 0.00083 |
| 2 | 0.10679 | 0.07323 | 0.06925 | 0.01588 | 0.00169 | 0.00064 |
| 3 | 0.09011 | 0.06180 | 0.06290 | 0.00924 | 0.00087 | 0.00051 |
| 4 | 0.07455 | 0.05929 | 0.05724 | 0.00781 | 0.00046 | 0.00014 |
| 5 | 0.07152 | 0.05302 | 0.04970 | 0.00616 | 0.00032 | 0.00007 |
| 6 | 0.06880 | 0.04861 | 0.04253 | 0.00527 | 0.00021 | 0.00006 |
| 7 | 0.06272 | 0.04075 | 0.03765 | 0.00435 | 0.00015 | 0.00005 |
| 8 | 0.05863 | 0.03582 | 0.03109 | 0.00308 | 0.00008 | 0.00003 |
| 9 | 0.05375 | 0.03310 | 0.02833 | 0.00216 | 0.00005 | 0.00002 |
| 10 | 0.05072 | 0.02994 | 0.02521 | 0.00184 | 0.00002 | 0.00001 |

Table 9: The finite time ruin probability $\psi(u,n)$.

| | $u$ | | | | | |
|---|---|---|---|---|---|---|
| $n$ | 0 | 5 | 10 | 15 | 20 | 25 |
| 1 | 0.12894 | 0.10294 | 0.07581 | 0.03140 | 0.00551 | 0.00083 |
| 2 | 0.23573 | 0.17617 | 0.14506 | 0.04728 | 0.00720 | 0.00147 |
| 3 | 0.32584 | 0.23797 | 0.20796 | 0.05652 | 0.00807 | 0.00198 |
| 4 | 0.40039 | 0.29726 | 0.26520 | 0.06433 | 0.00853 | 0.00212 |
| 5 | 0.47191 | 0.35028 | 0.31490 | 0.07049 | 0.00885 | 0.00219 |
| 6 | 0.54071 | 0.39889 | 0.35743 | 0.07576 | 0.00906 | 0.00225 |
| 7 | 0.60343 | 0.43964 | 0.39508 | 0.08011 | 0.00921 | 0.00230 |
| 8 | 0.66206 | 0.47546 | 0.42617 | 0.08319 | 0.00929 | 0.00233 |
| 9 | 0.71581 | 0.50856 | 0.45450 | 0.08535 | 0.00934 | 0.00238 |
| 10 | 0.76653 | 0.55928 | 0.47971 | 0.08819 | 0.00936 | 0.00276 |

*As similar to example 1, we assume a total of 1000 static policies, that is, there will always be this much exposure in the insurance portfolio for each time period. We simulate* $10^4$ *trajectories of the risk process and we calculate how many times on average they fall below zero in order to get values of* $\hat{\psi}(u,n)$*. We present the numerical results in Table 10. In addition, comparison of the exact values and simulated estimates for the finite time ruin probabilities for* $n = 1, 5, 10$ *are depicted in Figures 4, 5 and 6. From these Figures, it can be seen that the exact value and simulated estimate for the finite time ruin probability are close and almost coincide.*

Table 10: The simulation of finite time ruin probability $\hat{\psi}(u,n)$.

| | $u$ | | | | | |
|---|---|---|---|---|---|---|
| $n$ | 0 | 5 | 10 | 15 | 20 | 25 |
| 1 | 0.13051 | 0.10847 | 0.07165 | 0.03568 | 0.00607 | 0.00077 |
| 2 | 0.22805 | 0.16210 | 0.14005 | 0.05133 | 0.00679 | 0.00150 |
| 3 | 0.33406 | 0.24289 | 0.19670 | 0.05935 | 0.00764 | 0.00203 |
| 4 | 0.39772 | 0.28130 | 0.25811 | 0.06633 | 0.00791 | 0.00228 |
| 5 | 0.45709 | 0.34863 | 0.32404 | 0.06362 | 0.00830 | 0.00247 |
| 6 | 0.55300 | 0.37925 | 0.35047 | 0.07369 | 0.00885 | 0.00266 |
| 7 | 0.59138 | 0.42550 | 0.38701 | 0.07882 | 0.00904 | 0.00285 |
| 8 | 0.65871 | 0.46902 | 0.43770 | 0.08204 | 0.00935 | 0.00299 |
| 9 | 0.72355 | 0.51222 | 0.44031 | 0.08611 | 0.00965 | 0.00314 |
| 10 | 0.76802 | 0.54240 | 0.48887 | 0.09655 | 0.00982 | 0.00378 |

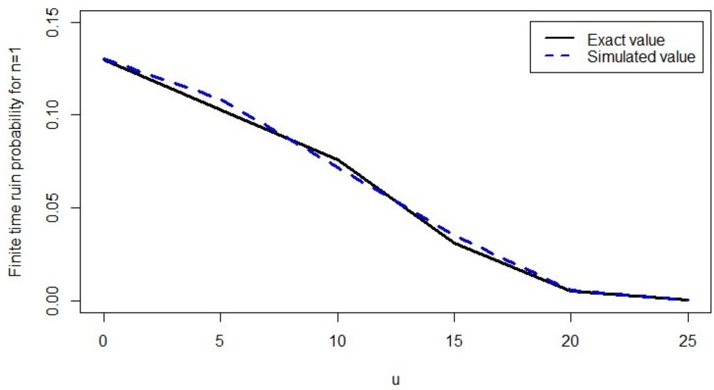


Figure 4: Exact and simulated values of ruin probabilities for $n = 1$.

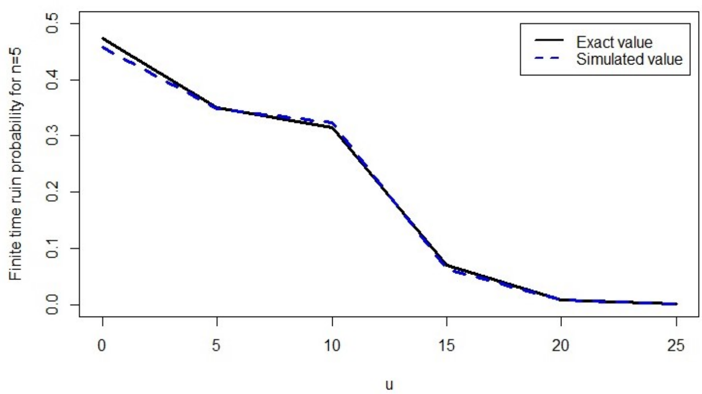


Figure 5: Exact and simulated values of ruin probabilities for $n = 5$.

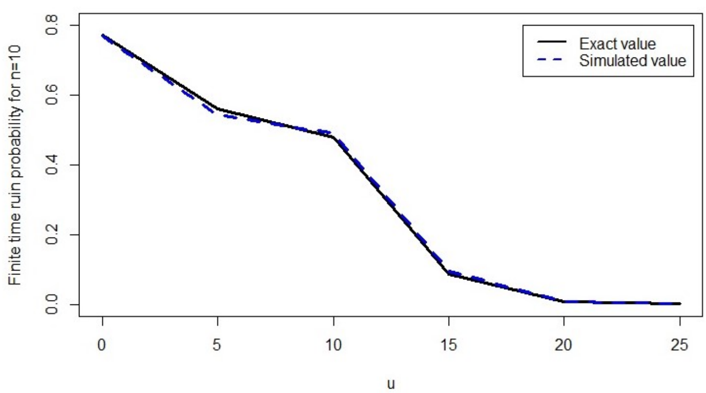


Figure 6: Exact and simulated values of ruin probabilities for $n = 10$.

**Example 5.3.** ***(England and Wales data).*** *This example concerns the number of drinking days in a reference week reported by respondents in a retrospective interview in England and Wales in 1989. The data have been compiled from Tables 6.1 and 6.9 of Goddard (1991), which gives figures for a survey in 1989 and the survey combined, and from Tables 3.3 and 2.13 of Goddard and Ikin (1988), which gives data for the earlier survey. Self-reported abstainers have been excluded. They showed that the transformed Inverse Gaussian-binomial distribution is*

$$p(x) = \binom{N}{x} \sum_{j=0}^{x} \binom{x}{j} (-1)^{x-j} \exp\big(\frac{\gamma}{\mu}\big[1 - \big\{1 + \frac{2\mu^2(N-j)}{\gamma}\big\}^{\frac{1}{2}}\big]\big),$$

*for $x = 0, 1, \ldots, N$, where $N = 1323$, and the maximum likelihood estimators of parameters are given by $\hat{\mu} = 0.79$ and $\hat{\gamma} = 0.38$, which provide the only acceptable fit to the data. As similar to the previous examples, we compute the mass function $W^u(n)$ and the numerical results display in Table 11.*

Table 11: The mass function $w^u(n)$.

| | | | $u$ | | | |
|---|---|---|---|---|---|---|
| $n$ | 0 | 5 | 10 | 15 | 20 | 25 |
| 1 | 0.32180 | 0.44338 | 0.50452 | 0.56615 | 0.67120 | 0.71026 |
| 2 | 0.42162 | 0.50297 | 0.56310 | 0.61258 | 0.68041 | 0.71318 |
| 3 | 0.48708 | 0.54904 | 0.59302 | 0.67360 | 0.72053 | 0.74002 |
| 4 | 0.52330 | 0.59609 | 0.64811 | 0.70210 | 0.75415 | 0.78190 |
| 5 | 0.53774 | 0.63002 | 0.69560 | 0.74312 | 0.78914 | 0.82305 |
| 6 | 0.57605 | 0.67330 | 0.71941 | 0.77065 | 0.80233 | 0.83170 |
| 7 | 0.60219 | 0.69235 | 0.75330 | 0.78911 | 0.82053 | 0.86122 |
| 8 | 0.62251 | 0.71622 | 0.78100 | 0.79044 | 0.83177 | 0.86873 |
| 9 | 0.65309 | 0.73507 | 0.80095 | 0.81663 | 0.83062 | 0.87551 |
| 10 | 0.67402 | 0.74913 | 0.81544 | 0.83170 | 0.84650 | 0.89114 |

*The numerical results for the distribution of time to ruin is derived and the results are reported in Table 12. Moreover, the finite time ruin probabilities are computed and the results are presented in Table 13. From this Table, it can be seen that the finite time ruin decreases as the initial reserve increases.*

Table 12: The distribution of time to ruin $\tau(u, n)$.

| | | | $u$ | | | |
|---|---|---|---|---|---|---|
| $n$ | 0 | 5 | 10 | 15 | 20 | 25 |
| 1 | 0.25118 | 0.23564 | 0.21500 | 0.18162 | 0.12394 | 0.08841 |
| 2 | 0.20320 | 0.20017 | 0.19361 | 0.17255 | 0.11505 | 0.07235 |
| 3 | 0.18741 | 0.17315 | 0.16664 | 0.14824 | 0.11480 | 0.07322 |
| 4 | 0.16057 | 0.14180 | 0.12802 | 0.11307 | 0.07032 | 0.02904 |
| 5 | 0.08305 | 0.05215 | 0.04010 | 0.01251 | 0.02884 | 0.00050 |
| 6 | 0.05120 | 0.01367 | 0.00722 | 0.00233 | 0.00109 | 0.00032 |
| 7 | 0.02748 | 0.01126 | 0.00814 | 0.00457 | 0.00102 | 0.00015 |

# 6 Discussion and conclusions

Ruin theory has always been a vital part of actuarial mathematics. In this work, we computed and investigated some related problems on the ruin probabilities in a class of

Table 13: The finite time ruin probability $\psi(u, n)$.

| | $u$ | | | | | |
|---|---|---|---|---|---|---|
| $n$ | 0 | 5 | 10 | 15 | 20 | 25 |
| 1 | 0.25118 | 0.23564 | 0.21500 | 0.18162 | 0.12394 | 0.08841 |
| 2 | 0.47438 | 0.43581 | 0.40861 | 0.35417 | 0.23899 | 0.16076 |
| 3 | 0.66179 | 0.60896 | 0.57525 | 0.50241 | 0.35379 | 0.23398 |
| 4 | 0.82236 | 0.75076 | 0.70327 | 0.61548 | 0.42411 | 0.26302 |
| 5 | 0.90541 | 0.80291 | 0.74337 | 0.62799 | 0.45295 | 0.26352 |
| 6 | 0.95661 | 0.81658 | 0.75059 | 0.63032 | 0.45404 | 0.26384 |
| 7 | 0.98409 | 0.82784 | 0.75873 | 0.63489 | 0.45506 | 0.26399 |
| 8 | 0.99279 | 0.83414 | 0.76081 | 0.63581 | 0.45539 | 0.26412 |
| 9 | 0.99704 | 0.83669 | 0.76178 | 0.63645 | 0.45554 | 0.26422 |
| 10 | 0.99835 | 0.83753 | 0.76220 | 0.63660 | 0.45554 | 0.26422 |

state-space compound binomial risk model for homogeneous claim occurrences. Lemma 3.1 gives the mass function of a defective renewal sequence of random $\{F_n\}_{n\geq 0}$-stopping times. Theorem 1 derives the infinite time ruin probability with Markov property of risk process. Lemmas 4.3 and 4.4 compute the distribution of the time to ruin using the convolution of claim amount in the case of no initial surplus and the case $u > 0$, respectively. Finally, to show the application of results, three examples presented and the implementation of Monte Carlo simulation method performed to obtain the ruin probabilities and the distribution of the time to ruin. The results show that the exact value and simulated estimate for the finite time ruin probability are close and almost coincide. This paper will serve as a detailed reference for the study of compound binomial risk models with homogeneous claim occurrences.

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