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Ruin probabilities in a discrete-time risk process with homogeneous markov chain

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Abstract: The present paper considers a discrete-time risk model with a homogeneous, irreducible, and aperiodic Markov chain. The general distribution of total claim amounts is influenced by the environmental Markov chain and in the i-th period the individual claim sizes are conditionally independent. We obtain the recursive formulae for infinite time ruin probability using the technique of ordinary generating functions. In addition, we give some restrictions which under those the ruin will not happen. In the last part, we present some numerical illustrations for the results and give the practical problem through a fully developed case study in the domain of social insurance.

Keywords: Discrete-time risk model; Homogeneous Markov chain; Ruin probability; Stationary distribution; Transition probability matrix.
Mathematics Subject Classification (2010): 62P05, 65C40.

1 Introduction

The theory of ruin has been the central interest for many researchers. The main objective of ruin theory is to obtain exact formulas or approximations of ruin probabilities in various kinds insurance risk models. The discrete-time risk process is a very popular model to describe the surplus process of an insurance portfolio as it includes the compound binomial risk and the compound Markov binomial model. Under the discrete-time setting, Yang (1999) computed the upper bounds for ruin probabilities by using martingale inequalities. Dufresne (1988) proposed the recursive algorithm to compute the ruin probabilities by using the stationary distributions of the bonus-malus

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system. Cai (2002) computed the ruin probabilities in the discrete-time risk models under rates of interest. Wagner (2002), derived recursive formulae for ruin probabilities in a two-state Markov chain risk model.

Cossette et al. (2003) presented the properties of the Markov Bernoulli and Markov binomial models and obtained the ruin probabilities. Cossette et al. (2004) pursued the analysis of the compound Markov binomial model by showing that the conditional infinite time ruin probability is a compound geometric tail and proposed an alternative algorithm to compute the ruin probabilities. Chen and Su (2006) obtained a precise asymptotic estimate for the finite time ruin probability in a discrete-time risk model, when the risk model contains the heavy-tailed distribution. Trufin and Loisel (2013) determined the behavior of the infinite time ruin probability for large initial capital in the case of light-tailed claim amounts in a discrete-time ruin model. Wu et al. (2015) derived the explicit formulae for infinite time ruin probabilities when there is a certain type of correlation between premiums and claim amounts. Liu et al. (2018a) computed the finite time ruin probability of a discrete-time risk model with GARCH discounted factors and dependent risks when the common distribution of claim amounts is heavy-tailed distribution. Liu et al. (2018b) derived a recursive formula for the Gerber-Shiu function of a discrete-time Markov additive process. Alfa (2020) applied a discrete-time Markov chain to modelled the Age of Information (AoI) and matrix-geometric method is used to obtain the probability mass function of the AoI. Santana and Rincon (2020) derived the approximations of ruin probability in a discrete-time risk model and provided several examples along with the numerical evidence of the accuracy of the approximations. Pachon et al. (2021) developed a discrete-time risk model and showed the relationships and differences with respect to the continuous time case. Nakade and Karim (2022) considered a discrete-time Markov process with a bounded continuous state space and obtained the equilibrium equations on steady-state probability. Bazyari (2022a) considered a generalized dual Binomial risk model where the periodic premium is one and derived a recursive expression for the finite time ruin probability. Bazyari (2023) computed the ruin probabilities in the discrete-time insurance risk process with capital injections and reinsurance.

The present paper focuses on computing the infinite time ruin probability in a discrete-time risk model for a general class of occurrence claim distributions with a Markovian environment process, which includes many well-known models as special cases. The obtained results for the infinite time ruin probability are based on the conditional joint distribution of claim amounts and Markovian chain.

The rest of this paper is structured as follows. In Section 2, we give the models and assumptions of our study and present the definition of the ordinary generating function of the sequence. In Section 3, we derive the infinite time ruin probability in the Markovian environment. For illustration purposes, only one-state and two-state of discrete-time risk model are examined. In addition, we compute the ruin probabilities for the compound binomial risk model and the compound Markov binomial risk model. Section 4 presents some numerical examples with practical application of results. Discussion and conclusions are given in Section 5.

2 Reviewing some insurance models and assumptions

Discrete-time risk models are of interest to us, because we can use them to approximate risk models in continuous time. The underlying model for our study is the discrete-time risk model which we now describe. Let $R(t)$ be the surplus process of an insurance company at time t, $t = 1, 2, 3, \ldots$. Then

$$R(t) = u + t - \sum_{i=1}^{t} X_i, \tag{1}$$

where $u = R(0) \geq 0$ is the insurer's initial capital and t is the total premium income up to time t ssuming that the insurer's premium income per unit time is 1. The insurer's aggregate claim size in the ith time interval is denoted by X_i and $\{X_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed (i.i.d.) random variables with $E(X_1) < 1$, probability density function $g(x)$ and distribution function $G(x)$.

Let M_n, $n \in N$, be a homogeneous, irreducible and aperiodic Markov chain with finite state space $S = \{1, 2, \ldots, s\}$, $1 \leq s < \infty$. The matrix of one-step transition probability is given by $\boldsymbol{P} = (p_{ij})_{i,j \in M_n}$, where $p_{ij} = P(M_n = j | M_{n-1} = i)$, i.e. the transition probability matrix $\boldsymbol{P}$ is the matrix consisting of the one-step transition probabilities p_{ij}. Moreover, the unique stationary distribution of matrix $\boldsymbol{P}$ is given by $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_s)$. Suppose that the distribution of claim sizes is influenced by the Markov chain M_n, $n \in N$. For any $i, j \in S$ and $z \in \mathbb{N}$ define $f_{ij}(z) = P(X_t = z, M_t = j | M_{t-1} = i)$ which describes the conditional joint distribution of claim amounts and Markovian chain given the previous state of Markovian chain, where we assume throughout the paper that for any $j \in S$ and $i \neq 1$, $f_{ij}(0) = 0$ and $\sum_{j \in S} f_{1j}(0) > 0$. This notation plays an important role in computing the ruin probabilities in the following Sections. Assume further that for all $i, j \in S$, $\theta_{ij} = \sum_{w=0}^{\infty} w f_{ij}(w) < \infty$ and for any $i \in S$ define $\theta_i = \sum_{j=1}^{s} \theta_{ij}$.

Remark 2.1. *It is well-known that the positive safety loading condition is a key role in ruin theory. To make sure that ruin is not certain, we assume that the positive safety loading condition holds, that is,* $\sum_{i=1}^{s} \pi_i \theta_i < 1$.

Moreover, for the risk model (1), we denote the time to ruin for initial capital u by T_u and define this quantity as $T_u = \min\big\{t \geq 1 : R(t) < 0 \mid R(0) = u\big\}$, with $T_u = \infty$ if for $t = 1, 2, 3, \ldots$, $R(t) \geq 0$. For the initial capital u and and the initial environment state $i \in S$, the infinite time ruin probability is

$$\psi_i(u) = P\big(T_u < \infty \mid R(0) = u, M_0 = i\big).$$

2.1 The compound binomial risk model

The compound binomial risk model, which is of special importance in actuarial studies, was first presented by Gerber (1988). Afterwards, the model was investigated by Shiu (1989), Willmot (1993) and Dickson (194). Some further extensions and properties of the model have been studied by Zhang et al. (2008), Eryilmaz (2014) and Bazyari (2022b). The compound binomial risk model is a discrete-time version of the classical risk model. The corresponding surplus process of an insurance company can be defined as

$$R(t) = u + t - \sum_{i=1}^{t} I_i Y_i, \tag{2}$$

where the claim amounts Y_i, $i = 1, 2, \dots, t$, are i.i.d. random variables with common probability function $h(y) = P(Y = y)$, $y = 1, 2, \dots$, and I_i are i.i.d. Bernoulli random variables representing the claim occurrences with mean $p \in (0, 1)$. That is, in any time period there is at most one claim; the probability of having a claim is p and the probability of no claim is $1 - p$. The occurrence of the claims in different time periods are assumed to be independent events and they are homogeneous in the compound binomial risk model.

2.2 The compound Markov binomial risk model

The compound Markov binomial model, an extension to Gerber's compound binomial model, was first proposed by Cossette et al. (2003) as a discrete-time risk model which introduces time dependence in the claim occurrence process. For convenience, a short presentation of the compound Markov binomial model follows (see Cossette et al. (2003) for more details). An extension of the compound binomial model is the compound Markov binomial model, in which I_i, $i = 1, 2, \dots, t$, in (2) is a two-state Markovian process with a transition probability matrix $\boldsymbol{P} = (p_{ij})_{i,j\in\{1,2\}}$, where $p_{ij} = P\big(I_{k+1} = j - 1 | I_k = i - 1\big)$ for $i, j \in \{1, 2\}$ and $k = 1, 2, \dots$.

2.3 Generating function of a sequence

In mathematics, a generating function is a way of encoding an infinite sequence of numbers by treating them as the coefficients of a formal power series. This series is called the generating function of the sequence. Unlike an ordinary series, the formal power series is not required to converge. In fact, the generating function is not actually regarded as a function, and the "variable" remains an indeterminate (one can refer to Knuth (1997) for more information on generating functions). The ordinary generating function of the sequence $\big\{a_n, n = 0, 1, \dots\big\}$, is defined by

$$a_n^*(x) = \sum_{n=0}^{\infty} a_n x^n.$$

For the sequence $\{b_n, n = 0, 1, \dots\}$ with generating function $b_n^*(x)$, we have

$$a_n^*(x) b_n^*(x) = \sum_{n=0}^{\infty} (a * b)_n x^n = \sum_{n=0}^{\infty} \sum_{w=0}^{n} a_w b_{n-w} x^n. \tag{3}$$

Moreover, let $\psi_i^*(x)$, $\phi_i^*(x)$ and $f_{ij}^*(x)$ denote the generating functions of $\psi_i(w)$, $\phi_i(w)$ and $f_{ij}(w)$, respectively.

3 Infinite time ruin probability

In studying the nature of the insurance risk associated with a portfolio of business, it is often of interest to assess how the portfolio may be expected to perform over an extended period of time. In addition, it is very important to know about the status of parameters in an insurance company. In this Section, we assume that the environmental

Markov chain has only two states and derive some recursive formulae for the infinite time ruin probability using technique of ordinary generating functions in the risk model (1). Conditioning on the occurrence of claims at the end of the first time period, clearly we have the following recursive formula for the infinite time ruin probability

$$\psi_i(u) = 1 - \sum_{j=1}^{2}\sum_{w=0}^{u+1} f_{ij}(w)\big(1-\psi_j(u+1-w)\big), \qquad i=1,2. \tag{4}$$

The remaining of this Section aims to derive some expressions of infinite time ruin probability for calculation purposes. By multiplying both sides of (4) by x^{u+1} and summing over u from 0 to ∞, we obtain

$$x\big(1-\psi_i^*(x)\big) = \sum_{j=1}^{2} f_{ij}^*(x)\big(1-\psi_j^*(x)\big) - \sum_{j=1}^{2} f_{ij}(0)\big(1-\psi_j(0)\big), \qquad i=1,2.$$

Then we have the following equations:

$$\begin{cases} (f_{11}^*(x)-x)\big(1-\psi_1^*(x)\big) + f_{12}^*(x)\big(1-\psi_2^*(x)\big) = \sum_{j=1}^{2} f_{1j}(0)\big(1-\psi_j(0)\big), \\ f_{21}^*(x)\big(1-\psi_1^*(x)\big) + (f_{22}^*(x)-x)\big(1-\psi_2^*(x)\big) = \sum_{j=1}^{2} f_{2j}(0)\big(1-\psi_j(0)\big). \end{cases} \tag{5}$$

It follows from (5) that

$$\begin{aligned}\big[(f_{11}^*(x)-x)(f_{22}^*(x)-x) - f_{21}^*(x)f_{12}^*(x)\big]\big(1-\psi_1^*(x)\big) &= (f_{22}^*(x)-x) \\ \times\Big(\sum_{j=1}^{2} f_{1j}(0)\big(1-\psi_j(0)\big)\Big) &- f_{12}^*(x)\Big(\sum_{j=1}^{2} f_{2j}(0)\big(1-\psi_j(0)\big)\Big). \end{aligned}\tag{6}$$

For notational convenience, we define $\hat{f}_{ii}(1) = f_{ii}(1)-1$ and $\hat{f}_{ii}(w) = f_{ii}(w)$, for $i=1,2$ and $k=2,3,\ldots$. Moreover, for any $w=1,2,\ldots$, consider the following notations:

$$\begin{aligned} v(w) &= \sum_{n=0}^{w}\big[\hat{f}_{11}(n)\hat{f}_{22}(w-n) - f_{21}(n)f_{12}(w-n)\big], \qquad (7)\\ f_w^{*1} &= \sum_{n=0}^{w}\big(1-\psi_1(n)\big)v(w-n), \\ r_w^{*1} &= \Big(\sum_{j=1}^{2} f_{1j}(0)\big(1-\psi_j(0)\big)\Big)f_{22}^*(w) - \Big(\sum_{j=1}^{2} f_{2j}(0)\big(1-\psi_j(0)\big)\Big)f_{12}^*(w), \\ r_w^{*2} &= \Big(\sum_{j=1}^{2} f_{2j}(0)\big(1-\psi_j(0)\big)\Big)f_{11}^*(w) - \Big(\sum_{j=1}^{2} f_{1j}(0)\big(1-\psi_j(0)\big)\Big)f_{21}^*(w). \end{aligned}$$

Note that r_w^{*1} can be rewritten as

$$r_w^{*1} = \zeta_w^{*1}\big(1-\psi_1(0)\big) + \gamma_w^{*1}\big(1-\psi_2(0)\big), \tag{8}$$

where $\zeta_w^{*1} = f_{11}(0)\hat{f}_{22}(w) - f_{21}(0)f_{12}(w)$ and $\gamma_w^{*1} = f_{12}(0)\hat{f}_{22}(w) - f_{22}(0)f_{12}(w)$. Let $\tilde{f}^{*1}(x)$, $\tilde{v}(x)$ and $\tilde{r}^{*1}(x)$ denote the generating functions of f_w^{*1}, $v(w)$ and r_w^{*1}, respectively. Applying the generating function's property (3) to (6) yields

$$\tilde{f}^{*1}(x) = \tilde{v}(x)\big(1-\tilde{\psi}_1(x)\big) = \tilde{r}^{*1}(x). \tag{9}$$

Then comparing the coefficients of x^w, $w = 1, 2, 3, \ldots$, in both sides of the equation (9) gives $f_w^{*1} = r_w^{*1}$ and $f_w^{*2} = r_w^{*2}$, these are

$$\sum_{n=0}^{w} \big(1 - \psi_1(n)\big) v(w-n) = r_w^{*1}, \tag{10}$$

$$\sum_{n=0}^{w} \big(1 - \psi_2(n)\big) v(w-n) = r_w^{*2},$$

respectively. Therefore, for $w = 1, 2, \ldots$, the infinite time ruin probabilities $\psi_1(w)$ and $\psi_2(w)$ can be obtained from the equations

$$\psi_1(w) = \begin{cases} 1 - \frac{1}{v(0)}\left[r_w^{*1} - \sum_{n=0}^{w-1}\big(1-\psi_1(n)\big)v(w-n)\right], & if\ v(0) \neq 0, \\ 1 - \frac{1}{v(1)}\left[r_{w+1}^{*1} - \sum_{n=0}^{w-1}\big(1-\psi_1(n)\big)v(w+1-n)\right], & if\ v(0) = 0, v(1) \neq 0, \end{cases} \tag{11}$$

$$\psi_2(w) = \begin{cases} 1 - \frac{1}{v(0)}\left[r_w^{*2} - \sum_{n=0}^{w-1}\big(1-\psi_2(n)\big)v(w-n)\right], & if\ v(0) \neq 0, \\ 1 - \frac{1}{v(1)}\left[r_{w+1}^{*2} - \sum_{n=0}^{w-1}\big(1-\psi_2(n)\big)v(w+1-n)\right], & if\ v(0) = 0, v(1) \neq 0. \end{cases} \tag{12}$$

In the following Theorem, we give some restrictions which under those the ruin will not be happened.

Theorem 3.1. *If both $v(0) = 0$ and $v(1) = 0$, then $\pi_1\theta_1 + \pi_2\theta_2 \geq 1$, that is, the positive safety loading condition does not hold.*

Proof. It is easy to see that the unique stationary of matrix $\boldsymbol{P}$ is given by

$$\boldsymbol{\pi} = (\pi_1, \pi_2) = \Big(\frac{p_{21}}{p_{21}+p_{12}}, \frac{p_{12}}{p_{21}+p_{12}}\Big).$$

On the other hand, from equality (7), we have

$$v(1) = f_{11}(0)\big(f_{22}(1) - 1\big) + f_{22}(0)\big(f_{11}(1) - 1\big) - f_{21}(0)f_{12}(1) - f_{21}(1)f_{12}(0) \leq 0.$$

By the assumption of Theorem 3.1 if $v(1) = 0$, then

$$f_{11}(0) = f_{22}(0) = f_{21}(0)f_{12}(1) = f_{21}(1)f_{12}(0) = 0. \tag{13}$$

Moreover, $f_{21}(0)f_{12}(0) = 0$, since $v(0) = 0$. Hence there are only two situations:
Situation (I): The equations $f_{12}(0) = 0$ and $f_{21}(0) \neq 0$ hold. From (13) we see $f_{12}(1) = 0$. Then

$$\begin{aligned} \theta_1 &= \sum_{w=0}^{\infty} w\big(f_{11}(w) + f_{12}(w)\big) \\ &= \sum_{w=1}^{\infty} w f_{11}(w) + \sum_{w=2}^{\infty} w f_{12}(w) \\ &\geq p_{11} + 2p_{12} = 1 + p_{12}, \\ \theta_2 &= \sum_{w=0}^{\infty} w\big(f_{21}(w) + f_{22}(w)\big) \geq p_{21} + p_{22} - f_{21}(0) = 1 - f_{21}(0), \end{aligned}$$

since $p_{11}+p_{12}=1$ and $p_{21}+p_{22}=1$. Therefore

$$\pi_1\theta_1+\pi_2\theta_2=\frac{p_{21}\theta_1}{p_{21}+p_{12}}+\frac{p_{12}\theta_2}{p_{21}+p_{12}}\geq\frac{p_{21}+p_{12}+p_{12}\big(p_{21}-f_{21}(0)\big)}{p_{21}+p_{12}}\geq 1.$$

Situation (II): The equations $f_{21}(0)=0$ and $f_{12}(0)\neq 0$ hold. By similar arguments, we have

$$\begin{aligned} f_{21}(1) &= 0, \theta_1\geq 1-f_{12}(0), \theta_2\geq 1+p_{21}, \\ \pi_1\theta_1+\pi_2\theta_2 &= \frac{p_{21}\theta_1}{p_{21}+p_{12}}+\frac{p_{12}\theta_2}{p_{21}+p_{12}}\geq\frac{p_{21}+p_{12}+p_{21}\big(p_{12}-f_{12}(0)\big)}{p_{21}+p_{12}}\geq 1, \end{aligned}$$

and this completes the proof. □

3.1 Infinite time ruin probabilities for $u=0$

For computing the whole infinite time ruin probability the remaining part is to determine the values of ruin probabilities $\psi_1(0)$ and $\psi_2(0)$. To compute these ruin probabilities we present an equality which is associated with their relations. For notational convenience, for $i=1,2$ and $u\geq 1$, we define

$$\begin{aligned} &\psi_i(0)=1-L_i(0), \qquad L_i(u)=\psi_i(u-1)-\psi_i(u), \\ &B_i(0)=r_0^{*i}, \qquad B_i(u)=r_u^{*i}-r_{u-1}^{*i}. \end{aligned}$$

It follows from (9) and (10) that

$$\begin{aligned} v(0)L_i(u+1) &= B_i(u+1)-\sum_{n=0}^{u}\big(1-\psi_i(n)\big)v(u+1-n)-\sum_{n=0}^{u-1}\big(1-\psi_i(n)\big)v(u-n) \\ &= B_i(u+1)-L_i(0)v(u+1)-\sum_{n=1}^{u}\big(1-\psi_i(n)\big)v(u+1-n) \\ &\quad+\sum_{n=0}^{u-1}\big(1-\psi_i(n)\big)v(u-n) \\ &= B_i(u+1)-L_i(0)v(u+1)-\sum_{n=1}^{u}\big(1-\psi_i(n)\big)v(u+1-n), \qquad (14) \end{aligned}$$

for $u=1,2,\ldots$. The equation (14) can be written in the following general form:

$$\sum_{q=0}^{u+1}v(q)L_i(u+1-q)=B_i(u+1). \qquad (15)$$

Let $\tilde{v}(x)$, $\tilde{L}_i(x)$ and $\tilde{B}_i(x)$ denote the generating functions of $v(u)$, $L_i(u)$ and $B_i(u)$, respectively. From equation (15) we have

$$\tilde{v}(x)\tilde{L}_i(x)-v(0)L_i(0)=\tilde{B}_i(x)-B_i(0),$$

which implies that for $i = 1, 2$

$$\tilde{v}(x)\tilde{L}_i(x) = \tilde{B}_i(x). \tag{16}$$

By derivation from equation (15), we have

$$\tilde{B}'_i(x) = \tilde{v}'(x)\tilde{L}_i(x) + \tilde{v}(x)\tilde{L}'_i(x). \tag{17}$$

On the other hand, since $\tilde{L}_i(1) = \sum_{u=0}^{\infty} L_i(u) = \lim_{n\to\infty}\big(1-\psi_i(n)\big) = 1$ and $\tilde{B}_i(1) = \lim_{n\to\infty} B_i(n) = 0$, therefore, from (16) we result that $\tilde{v}(1) = 0$ which gives the equality $\tilde{B}'_i(1) = \tilde{v}'(1)$ due to (17). On the other hand, we have

$$\tilde{B}'_1(1) = \sum_{w=1}^{\infty} w\big(r_w^{*1} - r_{w-1}^{*1}\big) = \sum_{w=1}^{\infty}\sum_{l=1}^{w}\big(r_w^{*1} - r_{w-1}^{*1}\big) = -\sum_{l=0}^{\infty} r_l^{*1},$$

which from (8), we get

$$\begin{aligned}
\tilde{B}'_1(1) &= -\big(1-\psi_1(0)\big)\sum_{l=0}^{\infty}\zeta_l^{*1} - \big(1-\psi_2(0)\big)\sum_{l=0}^{\infty}\gamma_l^{*1}\\
&= \big(1-\psi_1(0)\big)\big(f_{11}(0)p_{21} + f_{21}(0)p_{12}\big)\\
&\quad +\big(1-\psi_2(0)\big)\big(f_{12}(0)p_{21} + f_{22}(0)p_{12}\big). \qquad (18)\\
\tilde{v}'(1) &= \sum_{w=0}^{\infty} wv(w) = \sum_{w=0}^{\infty}\sum_{n=0}^{w} w\big[\hat{f}_{11}(n)\hat{f}_{22}(w-n) - f_{21}(n)f_{12}(w-n)\big]\\
&= \sum_{n=0}^{\infty}\big[\hat{f}_{11}(n)\sum_{w=n}^{\infty}(w-n)\hat{f}_{22}(w-n) + n\hat{f}_{11}(n)\sum_{w=n}^{\infty}\hat{f}_{22}(w-n)\big]\\
&\quad -\sum_{n=0}^{\infty}\big[f_{21}(n)\sum_{w=n}^{\infty}(w-n)f_{12}(w-n) + nf_{21}(n)\sum_{w=n}^{\infty}f_{12}(w-n)\big]\\
&= p_{12}(1-\theta_2) + p_{21}(1-\theta_1). \qquad (19)
\end{aligned}$$

Since the expressions (18) and (19) are equal, then

$$\begin{aligned}
\big(1-\psi_1(0)\big)\big(f_{11}(0)p_{21} + f_{21}(0)p_{12}\big) + \big(1-\psi_2(0)\big)\big(f_{12}(0)p_{21} + f_{22}(0)p_{12}\big)&\\
= p_{12}(1-\theta_2) + p_{21}(1-\theta_1),& \qquad (20)
\end{aligned}$$

This equality is a relationship between $\psi_1(0)$ and $\psi_2(0)$ which is based on the elements of transition probability matrix. To solve this equality, we need to determine the value of $v(0)$. To do it, we consider the following three cases of $v(0)$.

Case I) If $v(0) = 0$, it follows from (9) that

$$v(1)\big(\psi_1(0)\big) = r^{*1}(1) = \zeta_w^{*1}\big(1-\psi_1(0)\big) + \gamma_w^{*1}\big(1-\psi_2(0)\big),$$

which we result that

$$D_1\psi_1(0) + D_2\psi_2(0) = 0,$$

where $D_1 = \zeta_w^{*1} - v(1) = f_{12}(0)f_{21}(1) + f_{22}(0)\big(1 - (f_{11}(1)\big)$ and $D_2 = \gamma_w^{*1} - v(1) = f_{12}(0)\big(f_{22}(1) - 1\big) + f_{22}(0)f_{12}(1)$. In addition, $D_1 = D_2$ if and only if $f_{12}(0) = f_{22}(0) = 0$. Then

$$\begin{aligned}
&\sum_{j=1}^{2} f_{1j}(0)\big(1-\psi_j(0)\big) = f_{11}(0)\big(1-\psi_1(0)\big),\\
&\sum_{j=1}^{2} f_{2j}(0)\big(1-\psi_j(0)\big) = f_{21}(0)\big(1-\psi_1(0)\big),\\
&v(1)\big(1-\psi_2(0)\big) = r^{*2}(1) = \big(-f_{11}(0)f_{21}(1) + f_{21}(0)\big(f_{11}(1)-1\big)\big)\big(1-\psi_1(0)\big). \quad (21)
\end{aligned}$$

Case II) Suppose that $v(0) > 0$. Let

$$\begin{aligned}
N_1(x) &= \big(\hat{f}_{11}(x) - x\big)\big(\hat{f}_{22}(x) - x\big)\hat{f}_{21}(x)\hat{f}_{12}(x),\\
N_2(x) &= \big(\hat{f}_{22}(x) - x\big)\sum_{j=1}^{2} f_{1j}(0)\big(1-\psi_j(0)\big) - \hat{f}_{12}(x)\sum_{j=1}^{2} f_{2j}(0)\big(1-\psi_j(0)\big).
\end{aligned}$$

By (6), the equality $N_1(x)\big(1-\psi_1^*(x)\big) = N_2(x)$ holds. On the other hand, since $N_2(0) = v(0)\big(1-\psi_1(0)\big)$ and $N_2(1) = (p_{22}-1)\sum_{j=1}^{2} f_{1j}(0)\big(1-\psi_j(0)\big) - p_{12}\sum_{j=1}^{2} f_{2j}(0)\big(1-\psi_j(0)\big) < 0$, then there exists a δ, $0 < \delta < 1$, such that $N_2(\delta) = 0$, and therefore the equality

$$\begin{aligned}
\big(f_{11}(0)\big(\hat{f}_{22}(\delta) - \delta\big) - f_{21}(0)\hat{f}_{12}(\delta)\big)\big(1-\psi_1(0)\big) = \big(f_{22}(0)\hat{f}_{12}(\delta) - f_{12}(0)\big(\hat{f}_{22}(\delta) - \delta\big)\big)\\
\times\big(1-\psi_2(0)\big), \quad (22)
\end{aligned}$$

holds. Noting that for any $s \in (0,1)$, $\big(1-\psi_1^*(x)\big) > 0$ and we see that δ is a solution to the equation $N_1(x) = 0$.

Case III) If $v(0) < 0$, then $N_1(0) = v(0) < 0$. On the other hand, consider that

$$\begin{aligned}
N_1(-1) &= \big(1+\hat{f}_{11}(-1)\big)\big(1+\hat{f}_{22}(-1)\big) - \hat{f}_{21}(-1)\hat{f}_{12}(-1)\\
&> \big(1-\hat{f}_{11}(1)\big)\big(1-\hat{f}_{22}(1)\big) - \hat{f}_{21}(1)\hat{f}_{12}(1)\\
&= (1-p_{11})(1-p_{22}) - p_{21}p_{12} = 0,
\end{aligned}$$

and it shows that there exists a $\delta \in (-1,0)$, such that $N_1(\delta) = 0$, which in turn implies that $N_2(\delta) = 0$, and therefore (22) also holds in this case.

3.2 Infinite time ruin probability for two risk models

Here we compute the infinite time ruin probability for the compound binomial risk model and compound Markov binomial risk model given in Subsections 2.1 and 2.2. For $i = 1, 2$, assume that

$$f_{i1}(w) = \begin{cases} p, & if \quad w = 0,\\ 0, & if \quad w > 0,\end{cases}$$

$$f_{i2}(w) \quad = \quad \begin{cases} 0, & if \quad w = 0, \\ (1-p)v(w), & if \quad w > 0. \end{cases}$$

Then the risk model reduces to the first model, i.e., the compound binomial risk model. We suppose that the equality $\psi(u) = \psi_1(u) = \psi_2(u)$ holds. Then $v(0) = 0$, and from (11) or (12), we have

$$\psi(u) = \frac{\big(1 - \psi(u-1) - (1-p)\sum_{n=0}^{u-1}\big(1-\psi(u)\big)v(u-n)\big)}{p},$$

where $u = 1, 2, \ldots$. Using (20), we obtain $\psi(0) = 1 - \frac{1-(1-p)\sum_{w=1}^{\infty} wv(w)}{p}$, and this result is also given in Shiu (1989) and Willmot (1993).
For $i = 1, 2$, in the compound Markov binomial risk model assume that

$$f_{i1}(w) \quad = \quad \begin{cases} p_{i1}, & if \quad w = 0, \\ 0, & if \quad w > 0, \end{cases}$$

$$f_{i2}(w) \quad = \quad \begin{cases} 0, & if \quad w = 0, \\ p_{i2}v(w), & if \quad w > 0. \end{cases}$$

In this case, the risk model reduces to the second model, i.e., the compound Markov binomial risk model and $v(0) = 0$. From (11) or (12), we have

$$\begin{aligned} \psi_2(u) \quad &= \quad 1 - \frac{1}{v(1)}\big[\big(1 - \psi_2(u-1)\big) \\ &+ \sum_{n=0}^{u-1}\big(1-\psi_2(n)\big)\big[(p_{11}-p_{21})v(u+1-n) - p_{22}v(u-n)\big]\big], \end{aligned}$$

for $u = 1, 2, \ldots$. By relation (21), we obtain

$$\psi_2(0) = \frac{(p_{11}-p_{21})\big(1-v(1)\big) + p_{21}\psi_1(0)}{p_{11} - (p_{11}-p_{21})v(1)},$$

and this result is the same as relation (7) given in Cossette et al. (2004).

4 Numerical illustrations

In this Section, to better illustrate the above points, we shall study several examples in details. These examples contain different types of conditional joint distribution of claim amounts and Markovian chain. In examples 4.1 and 4.2, we give the discrete distribution of claim amounts for $w = 0, 1, 2, \ldots$ and compute the values of matrix of one-step transition probability and infinite time ruin probabilities. In example 4.3, we give the marginal distribution of each claim amounts and the conditional joint distribution of claim amounts is defined as the coefficients of marginal distribution and compute the infinite time ruin probabilities. In examples 4.4 and 4.5, we present the practical problem by a fully developed case study in the domain of social insurance.

Example 4.1. *The conditional joint distribution of claim amounts and Markovian chain for $w = 0, 1, 2, \ldots$ is given in Table 1.*

Table 1: The distribution of claim amounts.

w	$f_{11}(w)$	$f_{12}(w)$	$f_{21}(w)$	$f_{22}(w)$
0	$\frac{5}{8}$	0	0	0
1	$\frac{1}{8}$	$\frac{1}{8}$	0	$\frac{1}{6}$
2	$\frac{1}{8}$	0	$\frac{1}{2}$	$\frac{1}{6}$
3	0	0	$\frac{1}{6}$	0
≥ 4	0	0	0	0

With some calculations, we get

$$p_{12} = \frac{1}{8}, \quad p_{21} = \frac{2}{3}, \quad \theta_1 = \frac{1}{2}, \quad \theta_2 = 2,$$
$$v(0) = 0, \quad v(1) = -\frac{25}{48}, v(2) = \frac{5}{6}, \quad v(3) = -\frac{5}{16},$$

and $v(w) = 0$ *for* $w \geq 4$. *From* (20) *and* (21), *we have* $\psi_1(0) = \frac{1}{2}$ *and* $\psi_2(0) = 1$. *Moreover,*

$$r_2^{*1} = \zeta_2^{*1}\big(1 - \psi_1(0)\big) = \frac{5}{48} \times \frac{1}{2} = \frac{5}{96},$$

and $r_w^{*1} = 0$, *for* $w \geq 3$. *Therefore,* $\psi(1) = 1 - \frac{1}{v(1)}\big(r_2^{*1} - v(2)\big(1 - \psi(0)\big)\big) = \frac{7}{10}$. *Using* (10), *for* $w \geq 2$ *we get*

$$v(1)\psi_1(w) = \big(v(1) + v(3)\big)\psi_1(w-1) - v(3)\psi_1(w-2).$$

With some calculations, we obtain the recursive formula

$$\psi_1(w) - \psi_1(w-1) = \frac{v(3)}{v(1)}\big(\psi_1(w-1) - \psi_1(w-2)\big). \tag{23}$$

Using formula (23) , *we get*

$$\psi_1(w) - \psi_1(w-1) = \big(\frac{v(3)}{v(1)}\big)^2\big(\psi_1(w-2) - \psi_1(w-3)\big),$$

and therefore, for $w \geq 1$,

$$\psi_1(w) - \psi_1(w-1) = \big(\frac{v(3)}{v(1)}\big)^{w-1}\big(\psi_1(1) - \psi_1(0)\big) = \frac{1}{5}\big(\frac{3}{5}\big)^{w-1},$$

which yields the infinite time ruin probability for $w \geq 1$ *and* $i = 1$, *is given by* $\psi_1(w) = 1 - \frac{1}{2}\big(\frac{3}{5}\big)^w$. *By similar method, the infinite time ruin probability for* $w \geq 1$ *and* $i = 2$, *is given by* $\psi_2(w) = 1 - \frac{7}{10}\big(\frac{3}{5}\big)^{w-1}$.

Example 4.2. *The conditional joint distribution of claim amounts and Markovian chain for* $w = 0, 1, 2, \ldots$ *is given in Table 2.*

Table 2: The distribution of claim amounts.

w	$f_{11}(w)$	$f_{12}(w)$	$f_{21}(w)$	$f_{22}(w)$
0	$\frac{3}{8}$	$\frac{1}{8}$	0	$\frac{1}{2}$
1	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	0
2	0	$\frac{1}{8}$	0	$\frac{1}{6}$
3	$\frac{1}{8}$	0	$\frac{1}{12}$	0
≥ 4	0	0	0	0

With some calculations, we obtain

$$p_{11}=\frac{5}{8},\quad p_{12}=\frac{3}{8},\quad p_{21}=\frac{1}{3},\quad p_{22}=\frac{2}{3},$$

$$\theta_1=\theta_{11}+\theta_{12}=\frac{1}{2}+\frac{3}{8}=\frac{7}{8},\quad \theta_2=\theta_{21}+\theta_{22}=\frac{1}{2}+\frac{1}{3}=\frac{5}{6}.$$

Using (20)*, we derive the equation*

$$\frac{1}{2}\psi_1(0)+\frac{11}{12}\psi_2(0)=1. \tag{24}$$

Since $v(0)=\frac{3}{16}>0$*, therefore we have the Case (II). On the other hand,*

$$\hat{f}_{11}(x)=\frac{1}{8}(x^3+x+3),\quad \hat{f}_{12}(x)=\frac{1}{8}(x^2+x+3),$$
$$\hat{f}_{21}(x)=\frac{1}{12}(x^3+3x),\quad \hat{f}_{22}(x)=\frac{1}{6}(x^2+3),$$

and $N_1(x)=\frac{1-x}{96}\big(-x^4+12x^3+24x^2-63x+18\big)$.

Using R program, $\delta=0.3356$*, which is the solution to the equation* $N_1(x)=0$ *on the interval* $(0,1)$*. Using* (22) *and* (24)*, we obtain* $\psi_1(0)=0.7088$ *and* $\psi_2(0)=0.7043$*. We use* (11) *and* (12)*, to compute the infinite time ruin probabilities* $\psi_1(u)$ *and* $\psi_2(u)$ *for some initial capitals. These values are presented in Table 3. The graph of the ruin probabilities* $\psi_1(u)$ *and* $\psi_2(u)$ *are shown in Figure 1.*

Table 3: Infinite time ruin probabilities $\psi_1(u)$ and $\psi_2(u)$.

u	$\psi_1(u)$	$\psi_2(u)$
0	0.7088	0.7043
1	0.5689	0.5541
2	0.4311	0.4229
3	0.3346	0.3275
4	0.2573	0.2521
5	0.1987	0.1946
6	0.1531	0.1502
7	0.1181	0.1157
8	0.0911	0.0892
9	0.0702	0.0688
10	0.0542	0.0531
15	0.0147	0.0145
20	0.0040	0.0039
25	0.0014	0.0009

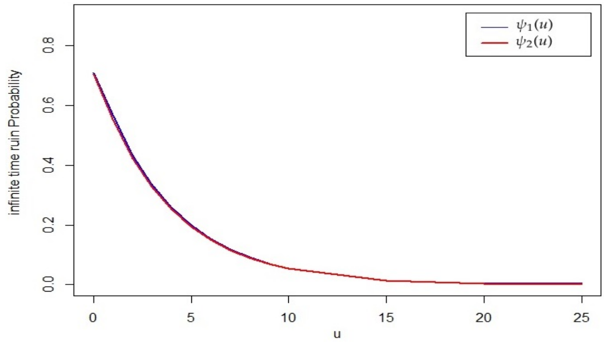

Figure 1: Infinite time ruin probabilities $\psi_1(u)$ and $\psi_2(u)$.

Example 4.3. *Let the function $f_{ij}(w)$ is given by $f_{ij}(w) = p_{ij} f_j(w)$ with the matrix of one-step transition probability $\boldsymbol{P} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$ and for $w \in \{1, 2, \dots\}$ we have*

$$f_1(w) = \big(\frac{1}{2}\big)^{w+1} \qquad and \qquad f_2(w) = \frac{2}{3}\big(\frac{1}{2}\big)^{w}.$$

With some calculations, we get $v(0) = -\frac{5}{36} < 0$, therefore we have the Case (III). Also, $\theta_1 = \theta_{11} + \theta_{12} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$ and $\theta_2 = \theta_{21} + \theta_{22} = \frac{3}{4} + \frac{1}{8} = \frac{7}{8}$. Using (20), we derive the equation

$$\frac{3}{8}\psi_1(0) + \frac{4}{9}\psi_2(0) = \frac{35}{72}. \tag{25}$$

On the other hand

$$\hat{f}_{11}(x) = \frac{1}{3(2-x)}, \quad \hat{f}_{12}(x) = \frac{4}{3(3-x)},$$
$$\hat{f}_{21}(x) = \frac{3}{4(2-x)}, \quad \hat{f}_{22}(x) = \frac{1}{2(3-x)},$$

and $N_1(x) = \frac{\big(6x^3 - 24x^2 + 17x + 5\big)(x-1)}{6(2-x)(3-x)}$.

Again, using R program, $\delta = -0.2212$, which is the solution to the equation $N_1(x) = 0$ on the interval $(-1, 0)$. Using (22) and (25), we obtain $\psi_1(0) = 0.5797$ and $\psi_2(0) = 0.6046$. We use (11) and (12), to compute the infinite time ruin probabilities $\psi_1(u)$ and $\psi_2(u)$ for some initial capitals. These values are presented in Table 4. The graph of the infinite time ruin probabilities $\psi_1(u)$ and $\psi_2(u)$ are shown in Figure 2.

Markov chains appear in many practical problems in such fields as operations research, business, social sciences and etc. To give an idea of this potential, we will present two more examples follow by a fully developed case study in the domain of social insurance.

Example 4.4. *(A transportation problem, Anton and Kolman (1978)). Let us consider a taxicab company of a city V, subdivided into two sectors V_1 and V_2. A taxicab picks up a passenger in any sector and drops her or him off in any sector. We can view*

Table 4: Infinite time ruin probabilities $\psi_1(u)$ and $\psi_2(u)$.

u	$\psi_1(u)$	$\psi_2(u)$
0	0.5797	0.6046
1	0.4493	0.4714
2	0.3496	0.3677
3	0.2725	0.2869
4	0.2125	0.2238
5	0.1658	0.1747
6	0.1294	0.1363
7	0.1009	0.1064
8	0.0788	0.0830
9	0.0615	0.0648
10	0.0479	0.0506
15	0.0139	0.0146

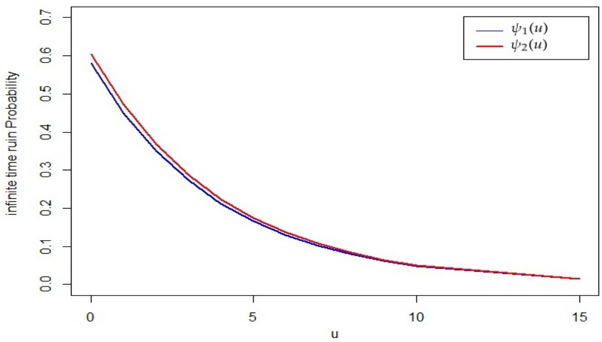

Figure 2: Infinite time ruin probabilities $\psi_1(u)$ and $\psi_2(u)$.

a taxicab as a physical system S which can be in one of three states: the sectors V_1 or V_2. The observation of taxicabs leads to the construction of a Markov chain with three states. This Markov chain has the transition probability matrix $\boldsymbol{P} = \begin{pmatrix} \frac{3}{7} & \frac{4}{7} \\ \frac{4}{5} & \frac{1}{5} \end{pmatrix}$. *This matrix is regular, hence irreducible and aperiodic since all its elements are strictly positive. Let the function $f_{ij}(w)$ is given by $f_{ij}(w) = p_{ij} f_j(w)$ and for $w \in \{1, 2, \dots\}$ we have*

$$f_1(w) = \Big(\frac{1}{3}\Big)^w \qquad and \qquad f_2(w) = \frac{1}{2}\Big(\frac{1}{4}\Big)^{w+1}.$$

With some calculations, we get $v(0) = -\frac{6}{17} < 0$, therefore we have the Case (III). Also, $\theta_1 = \frac{1}{2}$ and $\theta_2 = \frac{3}{5}$. Using (20)*, we get*

$$\frac{2}{9}\psi_1(0) + \frac{3}{5}\psi_2(0) = \frac{42}{67}.$$

From (22) *and* (25)*, we obtain $\psi_1(0) = 0.6994$ and $\psi_2(0) = 0.7028$. We use* (11) *and* (12)*, to compute the infinite time ruin probabilities $\psi_1(u)$, $\psi_2(u)$ and $\psi_3(u)$ for some initial capitals. These values are presented in Table 5.*

Example 4.5. *(A management problem in an insurance company, Anton and Kolman (1978)). A car insurance company classifies its customers in two groups:*

G_0: Those having no accidents during the year,

Table 5: Infinite time ruin probabilities $\psi_1(u)$ and $\psi_2(u)$.

u	$\psi_1(u)$	$\psi_2(u)$
1	0.6839	0.6846
2	0.6504	0.6410
3	0.6200	0.6105
4	0.6027	0.5922
5	0.5740	0.5608
10	0.4331	0.4291
20	0.1981	0.2006
30	0.0579	0.0541
50	0.0096	0.0085
80	0.0025	0.0030
100	0.0007	0.0008

G_1: Those having one or more than one accident during the year.

The statistics department of the company observes that the annual transition between the two groups can be represented by a Markov chain with state space $\{G_0, G_1\}$ and transition matrix $\boldsymbol{P} = \begin{pmatrix} 0.85 & 0.15 \\ 0 & 1 \end{pmatrix}$. Let the function $f_{ij}(w)$ is given by $f_{ij}(w) = p_{ij} f_j(w)$ and for $w \in \{1, 2, \dots\}$ we have

$$f_1(w) = \left(\frac{1}{2}\right)^{w-1} \qquad \textit{and} \qquad f_2(w) = \frac{1}{3}\left(\frac{1}{2}\right)^{w}.$$

With some calculations, we get $v(0) = \frac{3}{17} > 0$, therefore we have the Case (II). Also, $\theta_1 = \frac{2}{7}$ and $\theta_2 = \frac{1}{4}$. Using (20), we derive

$$\frac{3}{4}\psi_1(0) + \frac{5}{6}\psi_2(0) = \frac{75}{93}.$$

From (22) and (25), we obtain $\psi_1(0) = 0.7542$ and $\psi_2(0) = 0.8015$. We use (11) and (12), to compute the ruin probabilities $\psi_1(u)$ and $\psi_2(u)$ for some initial capitals. These values are presented in Table 6.

Table 6: Infinite time ruin probabilities $\psi_1(u)$ and $\psi_2(u)$.

u	$\psi_1(u)$	$\psi_2(u)$
1	0.7306	0.7816
2	0.7254	0.7722
3	0.7106	0.7529
4	0.7011	0.7536
5	0.6903	0.7485
10	0.5761	0.6335
20	0.3550	0.4072
30	0.2489	0.2956
50	0.0512	0.0944
80	0.0095	0.0070
100	0.0022	0.0074

5 Discussion and conclusions

We have provided a general formula for the infinite time ruin probability in the discrete-time risk model when general distribution of the total claim amounts is influenced by the environmental Markov chain. The ruin probability is expressed as the conditional joint distribution of claim amounts and Markovian chain. Furthermore, we proved that when $v(0) = v(1) = 0$, the inequality $\pi_1\theta_1 + \pi_2\theta_2 \geq 1$ holds and presented some numerical illustrations. As the practical results, we considered the examples 4.4 and 4.5 associated with "A transportation problem" and "A management problem in an insurance company", respectively, and computed the infinite time ruin probabilities. The obtained results in this discrete-time risk model can be a useful policy for the managers of insurance company.

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