Journal of Statistical Modelling: Theory and Applications
Vol. 4, No. 2, 2023, pp. 83-95
Yazd University Press 2023

JSMTA

Research Paper

A Bayesian shrinkage approach under symmetric and asymmetric loss functions for the Rayleigh distribution with the progressively type-II censoring schemes

Mojtaba Delavari[1], Zahra Khodadadi[1*], Karim Zare[1], Einolah Deiri[2]
[1]Department of Statistics, Marvdasht Branch, Islamic Azad University, Marvdasht, Iran
[2]Department of Statistics, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran

Received: May 20, 2024/ Revised: August 03, 2024/ Accepted: August 14, 2024

Abstract: The main goal of this paper is to achieve the Bayesian shrinkage estimators for the scale parameter of the Rayleigh distribution with progressively type-II censoring data. The best linear estimators are also presented under the squared error and linear exponential loss functions. Furthermore, the relative efficiency of the proposed Bayesian shrinkage estimators is calculated for the best linear estimators. Finally, through a numerical analysis, the relative efficiency of the Bayesian shrinkage estimators is compared with the best linear estimators.

Keywords: Bayesian shrinkage estimators, Linear exponential loss function, Rayleigh distribution, Squared error loss function.
Mathematics Subject Classification (2010): 62E15, 62F10, 62F15.

1 Introduction

The failure time and the mean lifetime can be investigated and compared when they can be represented or described by a continuous random variable. Otherwise, they are nothing except the measure and value of the failure times. In recent years to deal with these problems, the Rayleigh distribution is widely used as a model to investigate the lifetime of the productions. Also, the Rayleigh distribution has been extensively considered in practical issues, real experiments, and survival analysis. Nowadays, the

*Corresponding author:`zhrkhodadadi@gmail.com`

Rayleigh distribution is one of the most common statistical models that is applied for reliability studies of lifetime data and quality control and it is broadly utilized in the different branches of sciences, such as medical and engineering.

To determine the appropriate statistical model for the data and relevant analysis, it is necessary to consider all the distributions that can be employed as a distribution of the lifetime data and choose the most convenient distribution. In choosing the appropriate distribution, the principal statistical inferences regarding the parameter of the applied model are estimating and performing the hypothesis test of the parameters, and the Rayleigh distribution plays an important role in this regard. The importance of the Rayleigh distribution is due to its application in modeling the failure time, the distribution function of lifetime data, and the behavior of the hazard rate function which can be strictly increasing with different choices of the model's parameter. The Rayleigh distribution is more flexible to modeling the different data sets, due to the behavior of its hazard rate function. See Polovko (1968) for more applications of the Rayleigh distribution.

The probability density function (PDF) and cumulative distribution function (CDF) of the Rayleigh distribution are represented as

$$\begin{aligned} f(x;\nu,\lambda) &= \frac{x}{\lambda}e^{\frac{-x^2}{2\lambda}}, \\ F_X(x;\nu,\lambda) &= 1-e^{\frac{-x^2}{2\lambda}}, \qquad x>0,\lambda>0, \end{aligned} \tag{1}$$

where λ is called the scale parameter.

In nature, it is impossible to investigate all the existing data of a variable or the measured value for a variable may be out of the measurable bands. Therefore, the data must be censored. These data are called censored data. The censored random variables (censored samples) studied by Viveros and Balakrishnan (1994); Bandyopadhyay and Chattopadhyay (1995); Yuen and Tse (1996); Aggarwala and Balakrishnan (1998); Balakrishnan et al. (2001); Guilbaud (2001); Balakrishnan and Sandhu (1995, 1996); Fernandez (2004); Dey et al. (2015); Jia et al. (2018); Dey et al. (2023) and Asar and Arabi Belaghi (2023). Among different censoring schemes, progressively type-II censoring is one of the most reputed schemes in censoring fields; See Dey et al. (2016); Basiri and Beigi (2021); Ren and Gui (2021); Tolba et al. (2023) and Kumar et al. (2023) as new works in the field of using this scheme in the Rayleigh distribution family. The aim of this paper is to present an optimized estimation of the scale parameter of the Rayleigh distribution under progressively censored data.

In some application situations, the researcher has some prior information from the unknown parameter λ as a guess λ_0. The shrinkage estimator of the unknown parameter λ is used when the guess value λ_0 for λ is available. The shrinkage estimators have been discussed by many authors and one can refer to Lehmann and Casella (1998); Prakash and Singh (2008, 2009) and Singh et al. (2007).

Estimating parameters under the squared error loss function (SELF) is not always appropriate, since the SELF assigns equal weights to positive and negative errors, whereas overestimating is more serious and worse than underestimating. Here, overestimation error is defined when the loss function is positive and underestimation error is defined when the loss function is negative. Therefore, it is more suitable to estimate the parameters under an asymmetric loss function. Zellner (1986) introduced an asymmetric loss function named as the linear exponential (LINEX) loss function, which is defined as

$$L(\Delta) = e^{a\Delta} - a\Delta - 1, \qquad a \neq 0, \quad \Delta = \frac{\hat{\lambda} - \lambda}{\lambda}, \tag{2}$$

where the sign and the absolute value of the shape parameter Δ indicate the direction and degree of non-symmetric, respectively. The positive (negative) value of a is used when the overestimation is more (less) serious than the underestimation. The loss function (2) is an approximation of the SELF and if $|a|$ be near to zero, is almost symmetric.

Recently, some authors have discussed the Bayesian estimation methods under the LLF, among them we cite Marrelec et al. (2003); Ahmadi et al. (2005) and Singh et al. (2007). However, the scale parameter λ is more desirable under this asymmetric loss function, and the estimator can still be improved.

By considering the uniformly minimum variance unbiased (UMVU) estimator λ and using the prior information λ_0, Thompson (1968) introduced a new estimator called the shrinkage estimator as

$$\hat{\lambda}_S = K\hat{\lambda} + (1-K)\lambda_0, \qquad 0 \leq K \leq 1, \tag{3}$$

where $\hat{\lambda}$ is the preliminary UMVU or unbiased maximum likelihood (UML) estimator and K is the shrinkage coefficient, which can be defined as the researcher's opinion.

The values of K near to 1 and near to zero, indicate the tendency of the estimator to the sample and the guess value, respectively. While the guess value of the parameter is near to the real value, the shrinkage estimator shows a more appropriate behavior than classic estimators such as the ML estimator, which means that the shrinkage estimator has less risk. Thompson (1968), by using the proposed method, provided a linear minimum variance unbiased estimator with the shrinkage coefficient K. In application of the BS method in the Rayleigh distribution, Dey et al. (2015) studied the BS estimator under the general entropy loss function for progressively type-II censored data.

Here, we consider the experiment is under a progressively type-II censoring scheme. We propose the Rayleigh distribution for modeling the censored data and present the BL and BS estimates of the scale parameter under two popular loss functions, as a highlight of this paper. Moreover, the comparison of the BS estimates with the BL estimates through a numerical analysis is another point of this work.

As stated above, the structure of the paper is organized as follows. In Section 2, we derive the BL estimates of the parameter of Rayleigh distribution with the progressively type-II censoring data under SELF and LLF. In Section 3, under theses conditions, the BS estimates of the parameter are also obtained. In order to evaluate the efficiency of the estimators, the risk function of the proposed estimators under both loss functions LLF and SELF are computed. In Section 4, we calculate the relative efficiency and then present a numerical analysis to compare the proposed BS estimators with respect to the BL estimators.

2 The best linear estimator

Consider $X_1, X_2, \ldots, X_n$ as a random sample of the Rayleigh distribution with size n. Suppose $\boldsymbol{X} = (X_{(1)}, X_{(2)}, \ldots, X_{(m)})$ is the sample derived from the progressively type-II censoring scheme $\boldsymbol{R} = (r_1, r_2, \ldots, r_m)$, where r_i's are pre-assigned numbers of elimination of units from the test. According to Balakrishnan and Aggarwala (2000), the likelihood function under the sample progressively type-II censoring is defined as

$$\begin{aligned} L(\lambda|\boldsymbol{X}) &= A\prod_{i=1}^{m} f(X_{(i)})[1-F(X_{(i)})]^{r_i}, \\ &= A\lambda^{-m}\prod_{i=1}^{m} X_{(i)}^2 \exp\left(-\frac{1}{2\lambda}\sum_{i=1}^{m}(1+r_i)X_{(i)}^2\right) \\ &\propto A\lambda^{-m}\exp\left(-\frac{1}{2\lambda}T\right), \end{aligned} \tag{4}$$

where $A = n(n-r_1-1)\ldots(n-\sum_{i=1}^{m-1} r_i - m + 1)$ and $T = \sum_{i=1}^{m}(1+r_i)X_{(i)}^2$. Based on the Factorization Theorem, T is the complete sufficient statistic that has $\lambda_{\chi^2_{(2m)}}$ distribution (Dey et al., 2015). To optimize the equation (4), we take the derivative of the log-likelihood function with respect to λ and then equal it with zero. The result of solving the equation will be the ML estimator of the λ as

$$\hat{\lambda} = \hat{\lambda}_{MLE} = \hat{\lambda}_{UMVUE} = \frac{T}{2m} = \frac{1}{2m}\sum_{i=1}^{m}(1+r_i)X_{(i)}^2 \sim \frac{\lambda}{2m}\chi^2_{(2m)}. \tag{5}$$

If $c\hat{\lambda}$ is a linear estimator for λ, where c is a positive constant, then the risk function under the SELF is equal to

$$R_S(c\hat{\lambda}) = E(c\hat{\lambda}-\lambda)^2 = \left[c^2\left(m^{-1}+1\right) - 2c + 1\right]\lambda^2.$$

The minimum risk is obtained, if the following conditions are satisfied

$$\begin{cases} \frac{\partial R_S(c\hat{\lambda})}{\partial c} = 2c\left(\frac{1}{m}+1\right) - 2 = 0, \\ \frac{\partial^2 R_S(c\hat{\lambda})}{\partial c^2} = \frac{1}{m} + 1 > 0. \end{cases}$$

This conclude that

$$c = c_S = \frac{m}{m+1}, \quad R_S(c_S\hat{\lambda}) = \frac{\lambda^2}{m+1}. \tag{6}$$

Similarly, the risk function of $c\hat{\lambda}$ under the LINEX loss function is computed as

$$\begin{aligned} R_L(c\hat{\lambda}) &= AE(e^{a\Delta} - a\Delta - 1) \\ &= E\left(e^{a\left((c\hat{\lambda}-\lambda)/\lambda\right)} - a(c\hat{\lambda}-\lambda)/\lambda - 1\right) \\ &= E\left(e^{a\left(\frac{c\frac{T}{2m}-\lambda}{\lambda}\right)}\right) - aE\left(\frac{c\frac{T}{2m}-\lambda}{\lambda}\right) - 1 \end{aligned}$$

$$\begin{aligned} &= e^{a}E\left(e^{\frac{ac}{2m}\chi^2_{(2m)}}\right) - aE\left(\frac{c}{2m}\chi^2_{(2m)} - 1\right) - 1 \\ &= e^{a}\left(1-\frac{ac}{m}\right)^{-m} - a(c-1) - 1. \end{aligned}$$

The minimum risk exists if $\frac{\partial R_L(c\hat{\lambda})}{\partial c} = 0$ and $\frac{\partial^2 R_L(c\hat{\lambda})}{\partial c^2} > 0$ are satisfied. So,

$$\begin{cases} \frac{\partial R_L(c\hat{\lambda})}{\partial c} = \frac{a}{m} m e^{-a}\left(1-\frac{ac}{m}\right)^{-(m+1)} - a = 0, \\ \frac{\partial^2 R_L(c\hat{\lambda})}{\partial c^2} = a^2\frac{m+1}{m} e^{-a}\left(1-\frac{ac}{m}\right)^{-(m+2)} > 0. \end{cases}$$

Therefore,

$$c_L = \frac{m}{a}(1 - e^{-a/(m+1)}), \qquad R_L(c_L\hat{\lambda}) = e^{\frac{-a}{m+1}}(a+1) - 1. \tag{7}$$

3 The Bayesian shrinkage estimator

In contrary to the classic statistic that researchers estimate the unknown parameter with the sample information, in the Bayesian approach the unknown parameter is estimated based on the combination of the sample information and prior distribution. Prakash and Singh (2009) investigated the BS estimation of the scale parameter of the Weibull distribution based on the censored data under the LINEX loss function. Dey et al. (2015) achieved the Bayesian shrinkage estimation of the scale parameter for the Rayleigh distribution based on the censored data under the general entropy loss function.

Whenever the prior information of the parameter exists, the conjugate prior distribution of λ is inverse-Gamma with the following PDF

$$g_1(\lambda) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\lambda^{-\alpha-1}e^{-\frac{\beta}{\lambda}}, \qquad \alpha > 0, \beta > 0. \tag{8}$$

Another class of the prior distributions is represented as

$$g_2(\lambda) = \lambda^{-d}e^{-\frac{cd}{\lambda}}, \qquad d > 0, c > 0.$$

In the absence of the prior information of the parameter λ, the uniform or improper distributions can be applied as the prior distribution.

The posterior density of the parameter λ under the prior density function $g_1(\lambda)$ is defined as

$$\begin{aligned} \pi_1(\lambda|\boldsymbol{X}) &\propto g_1(\lambda)L(\lambda|\boldsymbol{X}) \propto \lambda^{-(m+\alpha+1)}\exp\left(-\frac{\frac{T}{2}+\beta}{\lambda}\right) \\ &\propto \frac{\left(\frac{T}{2}+\beta\right)^{\alpha+m}}{\Gamma(\alpha+m)}\lambda^{-(m+\alpha+1)}\exp\left(-\frac{\frac{T}{2}+\beta}{\lambda}\right), \end{aligned} \tag{9}$$

which is the inverse-Gamma distribution with the parameters $(\alpha + m)$ and $(\frac{T}{2} + \beta)$. Therefore, the Bayes estimator of the parameter under the SELF is obtained as

$$\hat{\lambda}_{B1} = E_P(\lambda|\boldsymbol{X}) = \frac{\frac{T}{2}+\beta}{\alpha+m} = \left(\frac{T}{2}+\beta\right)\varphi_1, \qquad \varphi_1 = (\alpha+m)^{-1}, \tag{10}$$

where the subscript P in expectation indicates that expectation was obtained under the posterior density. Due to the presence of the unknown parameter in the estimator, in practice the computation of the estimator is unfeasible. Hence, the posterior distribution $g_1(\lambda)$ must be estimated under the point prior information $E(\hat{\lambda}_{B1}) = \lambda_0$, where the point λ_0 is a guess value of λ. Consequently, the value of β based on the point prior information would be as

$$\beta_1 = \left(\frac{1}{\varphi_1} - m\right)\lambda_0.$$

By substituting this value in (10), the Bayesian shrinkage estimator of λ under the SELF is computed as

$$\hat{\lambda}_{BS1} = \left(\frac{T}{2} + \beta_1\right)\varphi_1 = m\varphi_1\hat{\lambda} + \left(\frac{1}{\varphi_1} - m\right)\lambda_0\varphi_1 = K_1\hat{\lambda} + (1 - K_1)\lambda_0, \tag{11}$$

where the shrinkage coefficient is $K_1 = m\varphi_1$. Since the obtained estimator is similar to the shrinkage estimator defined in (3), hence $\hat{\lambda}_{BS1}$ is named as the Bayesian shrinkage estimator.

Now, the Bayes estimator of the parameter λ under the LLF is achieved through minimizing

$$E_P\left\{e^{a\left(\frac{\hat{\lambda}_{BS1}}{\lambda} - 1\right)} - a\left(\frac{\hat{\lambda}_{BS1}}{\lambda} - 1\right) - 1|\boldsymbol{X}\right\}$$

with respect to $\hat{\lambda}_{B2}$. This leads the following equation

$$E_P\left(\frac{1}{\lambda}e^{\frac{a\hat{\lambda}_{B2}}{\lambda}}|\boldsymbol{X}\right) = e^a E_P\left(\frac{1}{\lambda}|\boldsymbol{X}\right).$$

By using the posterior density in (9), we can rewrite the above equation as

$$\int_0^\infty \frac{1}{\lambda}e^{\frac{a\hat{\lambda}_{B2}}{\lambda}}\frac{\left(\frac{T}{2} + \beta\right)^{\alpha+m}}{\Gamma(\alpha+m)}\lambda^{-(m+\alpha+1)}e^{-\frac{\frac{T}{2}+\beta}{\lambda}}d\lambda = e^a\int_0^\infty \frac{1}{\lambda}\frac{\left(\frac{T}{2} + \beta\right)^{\alpha+m}}{\Gamma(\alpha+m)} \times \lambda^{-(m+\alpha+1)}e^{-\frac{\frac{T}{2}+\beta}{\lambda}}d\lambda,$$

So,

$$\frac{\left(\frac{T}{2} + \beta\right)^{\alpha+m}}{\Gamma(\alpha+m)}\frac{\Gamma(\alpha+m+1)}{\left(\frac{T}{2} + \beta - a\hat{\lambda}_{B2}\right)^{\alpha+m+1}} = e^a\frac{\left(\frac{T}{2} + \beta\right)^{\alpha+m}}{\Gamma(\alpha+m)}\frac{\Gamma(\alpha+m+1)}{\left(\frac{T}{2} + \beta\right)^{\alpha+m+1}}.$$

Therefore

$$\frac{T}{2} + \beta - a\hat{\lambda}_{B2} = \left(\frac{T}{2} + \beta\right)e^{-\frac{-a}{m+\alpha+1}}$$

and hence, the Bayes estimator of the parameter λ under the LLF would be as

$$\hat{\lambda}_{B2} = \left(\frac{T}{2} + \beta\right)\varphi_2, \qquad \varphi_2 = \frac{\left(1 - e^{-\frac{-a}{m+\alpha+1}}\right)}{a}. \tag{12}$$

Similarly, the posterior distribution $g_1(\lambda)$ must be estimated under the point prior information $E(\hat{\lambda}_{B2}) = \lambda_0$ that concludes

$$(m\lambda_0 + \beta)\varphi_2 = \lambda_0.$$

Therefore, the value of based on the point prior information would be as

$$\beta_2 = \lambda_0 \left(\frac{1}{\varphi_2} - m \right).$$

Subsequently, the BS estimator of the parameter λ under the LLF is obtained as

$$\hat{\lambda}_{BS2} = \left(\frac{T}{2} + \beta_2 \right) \varphi_2 = \left(m\hat{\lambda} + \lambda_0 \left(\frac{1}{\varphi_2} - m \right) \right) \varphi_2 = K_2\hat{\lambda} + (1 - K_2)\lambda_0. \quad (13)$$

where the shrinkage coefficient is $K_2 = m\varphi_2$.

The risk functions of the Bayes estimator under the SELF is achieved as follows

$$\begin{aligned} R_S(\hat{\lambda}_{B1}) &= E(\hat{\lambda}_{B1} - \lambda)^2 = E\left(\varphi_1 \left(\frac{T}{2} + \beta \right) - \lambda \right)^2 \\ &= \frac{1}{4}\varphi_1^2 E(T^2) + \varphi_1^2\beta^2 + \varphi_1^2 E(T)\beta - \lambda\varphi_1 E(T) - 2\varphi_1\lambda\beta + \lambda^2 \\ &= m\varphi_1^2\lambda^2 + (\lambda(m\varphi_1 - 1) + \beta\varphi_1)^2. \end{aligned} \quad (14)$$

Similarly, the risk of the Bayes estimator under the LINEX loss function is demonstrated by

$$\begin{aligned} R_L(\hat{\lambda}_{B2}) &= E(e^{a\Delta} - a\Delta - 1) = E\left(e^{a\left(\frac{\hat{\lambda}_{B2}-\lambda}{\lambda}\right)} - a\left(\frac{\hat{\lambda}_{B2} - \lambda}{\lambda} \right) - 1 \right) \\ &= e^{\frac{a\beta\varphi_2}{\lambda} - a}(1 - a\varphi_2)^{-m} - a\left(\frac{\varphi_2(m\lambda + \beta) - \lambda}{\lambda} \right) - 1. \end{aligned} \quad (15)$$

Now by considering $\hat{\lambda}_{BSi} = K_i\hat{\lambda} + (1 - K_i)\hat{\lambda}_0, i = 1, 2$, the risk functions of the Bayesian shrinkage estimator under the SELF and LLF is respectively given by

$$\begin{aligned} R_S(\hat{\lambda}_{BS1}) &= E(\hat{\lambda}_{BS1} - \lambda)^2 = E[(K_1\hat{\lambda} + (1 - K_1)\lambda_0 - \lambda)^2] \\ &= \lambda^2 \left(\frac{K_1^2}{m} + (1 - K_1)^2(1 - \delta)^2 \right), \end{aligned} \quad (16)$$

$$\begin{aligned} R_L(\hat{\lambda}_{BS2}) &= E\left[\exp\left(a\left(\frac{K_2\hat{\lambda} + (1 - K_2)\lambda_0}{\lambda} - 1 \right) \right) - a\left(\frac{K_2\hat{\lambda} + (1 - K_2)\lambda_0}{\lambda} - 1 \right) - 1 \right] \\ &= \exp(a((1 - K_2)\delta - 1)) M_{\chi^2_{(2m)}}\left(\frac{aK_2}{2m} \right) - a[K_2 + (1 - K_2)\delta - 1] - 1 \\ &= e^{a((1-K_2)\delta - 1)} \left(1 - \frac{aK_2}{2m} \right)^{-m} + a[(1 - K_2)(1 - \delta)] - 1, \end{aligned} \quad (17)$$

where $\delta = \frac{\lambda_0}{\lambda}$.

Moreover, the posterior density of λ based on the prior distribution $g_2(\lambda)$ is illustrated as

$$\begin{aligned}\pi_2(\lambda|\boldsymbol{X}) &\propto g_2(\lambda)L(\lambda|\boldsymbol{X}) \propto \left[e^{-d}e^{-\frac{cd}{\lambda}}\right]\left[\lambda^{-m}e^{-\frac{T}{2\lambda}}\right] \\ &\propto \frac{\left(\frac{T}{2}+cd\right)^{r+d-1}}{\Gamma(r+d-1)}e^{-\frac{\frac{T}{2}+cd}{\lambda}}\lambda^{-(m+d)}. \end{aligned} \tag{18}$$

Therefore, λ has the inverse-Gamma posterior density function with the parameters $m+d-1$ and $\frac{T}{2}+cd$ which has the same form as the posterior density in (9). The only difference is in the positions of α and β which are substituted by $d-1$ and cd, respectively. If we replace the following equation in (17), then all the discussed results are satisfied

$$d = (\alpha+1), \qquad c = \frac{\beta}{\alpha+1}. \tag{19}$$

4 Numerical analysis

In this section, we use the relative efficiency (RE) to compare the estimators, which is defined as follows. The relative efficiency of the BS estimators $\hat{\lambda}_{BSi}, i=1,2$ with respect best linear unbiased estimator under the SELF and LLF are determined as

$$\begin{aligned} RE_S(\hat{\lambda}_{BS1}, c_S\hat{\lambda}) &= \frac{R_S(c_S\hat{\lambda})}{R_S(\hat{\lambda}_{BS1})} = \frac{\frac{\lambda^2}{m+1}}{K_1^2\frac{\lambda^2}{m}+(1-K_1)^2(\lambda_0-\lambda)^2} \\ &= \frac{1}{(m+1)\left(\frac{K_1^2}{m}+(1-K_1)^2(1-\delta)^2\right)}, \\ RE_L(\hat{\lambda}_{BS1}, c_L\hat{\lambda}) &= \frac{R_L(c_L\hat{\lambda})}{R_L(\hat{\lambda}_{BSi})} \\ &= \frac{e^{\frac{-a}{m+1}}(a+1)-1}{e^{a((1-K_2)\delta-1)}\left(1-\frac{aK_2}{m}\right)^{-m}+a((1-K_2)(1-\delta))-1}. \end{aligned}$$

As can be seen, the relative efficiency under SELF ($RE_S(\hat{\lambda}_{BS1}, c_S\hat{\lambda})$) is a function of the parameters r, α and δ whereas the relative efficiency under the LLF ($RE_L(\hat{\lambda}_{BS2}, c_L\hat{\lambda})$) also depends on a.

In Figure 1, the relative efficiency of the BS estimator with respect to the BL estimator under SELF based on the value of $0 \leq \delta \leq 3$ is represented for different values of $m = 5, 10, 20, 50$ and $\alpha = 0.25, 0.5, 1, 2, 4, 10$. In all plots, as expected, the maximum value of the relative efficiency is attained at point $\delta = 1$.

Based on Figure 1, the efficiency of the BS estimator $\hat{\lambda}_{BS1}$ under SELF is always less than the BL estimator $c\hat{\lambda}$ for all values δ when α is very close to zero. For a small value of m, there is a remarkable difference between the efficiency of the two estimators and with increasing m, their efficiencies are close together.

Generally, with increasing α, the BS estimator is more efficient when δ is close to one. In this case, the range of the points around $\delta = 1$ that the BS estimator has better performance, decreases. Moreover, for small values of m and α, the efficiency of

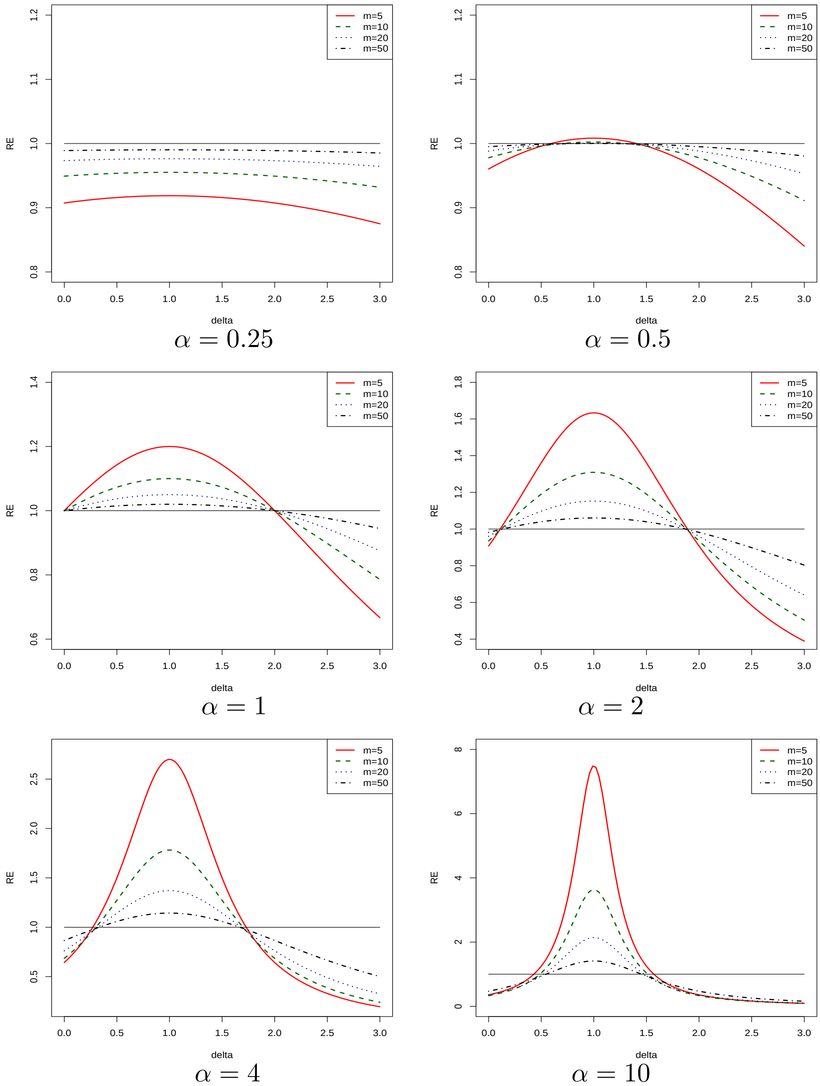


Figure 1: the relative efficiency of the BS estimator with respect to BL estimator under SELF based on the value of $0 \leq \delta \leq 3$ for different values of $m = 5, 10, 20, 50$ and $\alpha = 0.25, 0.5, 1, 2, 4, 10$.

the BS estimator strongly decreases when δ is far from one. Consequently, this shows that with the suitable choice of α, when the guess value λ_0 is near to the real value of the parameter λ, the BS estimator under SELF is more efficient, especially for small values of m.

Figures 2 and 3 present the relative efficiency of the BS estimator with respect to the BL estimator under LLF based on the value of δ for different values of $m = 5, 10$ and $\alpha = 0.5, 1, 2, 4$. In all plots of Figures 2 and 3, we observe the relative efficiency under LLF strongly depends on a values, such that for its small values, the relative

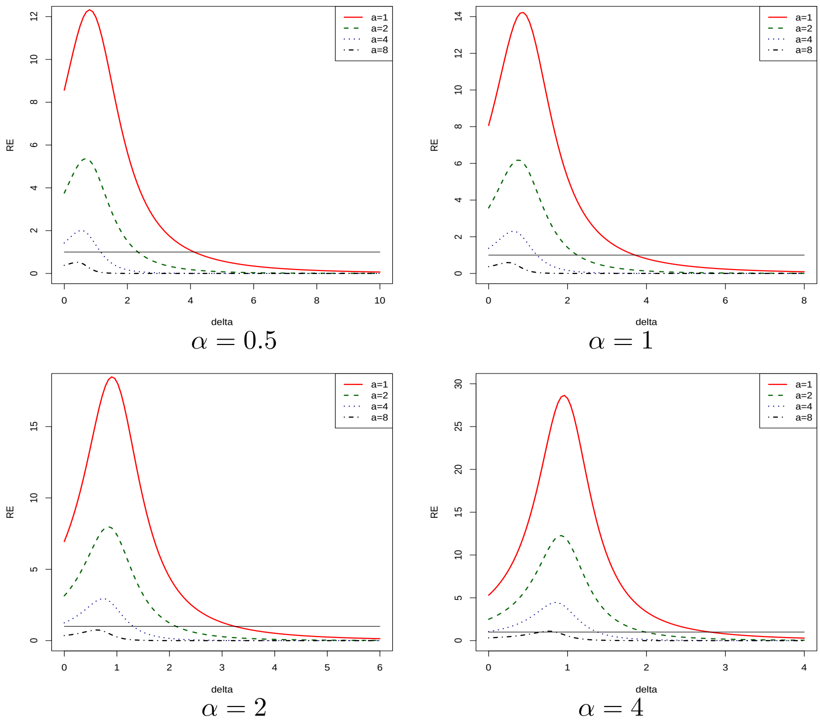


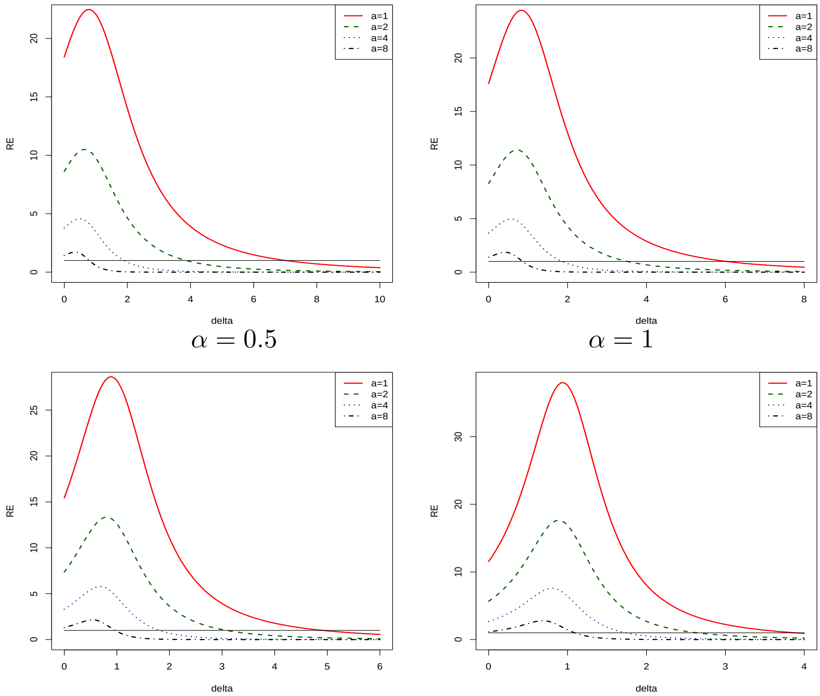
Figure 2: the relative efficiency of the BS estimator with respect to BL estimator under LLF based on the value of δ for different values of $\alpha = 0.5, 1, 2, 4$ and $m = 5$.

efficiency have large values for the points around $\delta = 1$ that indicates the BS estimator perform better than the BL estimator.

Generally, when m increases and a decreases, the relative efficiency under LLF increases such that for the points around $\delta = 1$, its values is much greater than one. Moreover, for small values of m and large values of a, the efficiency of the BS estimator is worse, especially when δ is far from one. With increasing α, the BS estimator has better performance when δ is close to one. In this case, the range of the points around $\delta = 1$ that the BS estimator is more efficient, decreases. Consequently, with the suitable choices of a and α, when the guess value λ_0 is near to the real value of the parameter λ, the BS estimator under LLF is more efficient.

Discussion and Conclusions

Here, our main goal was to introduce the BS estimation method under SELF and LLF for the scale parameter of Rayleigh distribution with progressively type-II censored data. First, we presented the ML estimator for the parameter and derived the BL estimator based on the ML estimator under SELF and LLF. Then, we presented the Bayesian method and derived the Bayesian estimator under the two loss functions. Then, we proposed the BS method and obtained the BS estimators based on Bayesian



$\alpha = 0.5$ $\alpha = 1$

$\alpha = 2$ $\alpha = 4$

Figure 3: the relative efficiency of the BS estimator with respect to BL estimator under LLF based on the value of δ for different values of $\alpha = 0.5, 1, 2, 4$ and $m = 10$.

estimators under the two loss functions. Also, we derived the risk function for all estimators. Finally, we presented the relative efficiency of the BS estimators concerning the BL estimators and used a numerical analysis to compare them. Our findings about BL and BS estimation methods verified the previous results in this field. The numerical results showed the BS estimator performs better than the BL estimator under SELF if the guess value λ_0 is near to the real value of the parameter λ, especially for small values of m. The amount of α also affected the efficiency of the BS estimator. The results indicated values of $1 \leq \lambda \leq 2$ are suitable choices, and using the BS estimator has a more applicable advantage concerning the BL estimator in this case. Under LLF, the relative efficiency depended on a, α and m. If the guess value λ_0 is near to the real value of the parameter λ, the suitable choices of a and α lead to better performance of the BS estimator under LLF, even if the amount of m is too small.

References

Aggarwala, R. and Balakrishnan, N. (1998). Some properties of progressive censored order statistics from arbitrary and uniform distributions with applications to inference and simulation. *Journal of statistical planning and inference*, **70**(1):35–49.

Ahmadi, J., Doostparast, M. and Parsian, A. (2005). Estimation and prediction in a two-parameter Exponential distribution based on k-record values under LINEX loss function. *Communications in Statistics–Theory and Methods*, **34**(4):795–805.

Asar, Y. and Arabi Belaghi, R. (2023). Estimation in Weibull distribution under progressively type-I hybrid censored data. *REVSTAT-Statistical Journal*, **20**(5):563–586.

Balakrishnan, N. and Aggarwala, R. (2000). *Progressive Censoring: Theory, Methods and Applications*. Boston: Birkhäuser.

Balakrishnan, N., Cramer, E., kamps, U. and sckenk, N. (2001). Progressive type-II censored order statistics from exponential distributions. *Statistics*, **35**(4):537–556.

Balakrishnan, N. and Sandhu, R.A. (1995). A simple simulational algorithm for generating progressive type-II censored samples. *The American Statistician*, **49**(2):229–230.

Balakrishnan, N. and Sandhu, R.A. (1996). Best linear unbiased and maximum likelihood estimation for exponential distribution under general progressive type-II censored samples. *Sankhyā: The Indian Journal of Statistics*, **Series B**:1–9.

Bandyopadhyay, U. and Chattopadhyay, G. (1995). Progressive censoring under inverse sampling for nonparametric two-sample problems. *Sequential Analysis*, **14**(1):1–28.

Basiri, E. and Beigi, S. (2021). Bayesian two-sample prediction problem for the Rayleigh distribution under progressively type-II censoring with random removals. *Caspian Journal of Mathematical Sciences*, **10**(2):156–175.

Dey, S., Dey, T. and Maiti, S.S. (2015). Bayes shrinkage estimation of the parameter of Rayleigh distribution for progressive type-II censored data. *Austrian Journal of Statistics*, **44**(4):3–15.

Dey T., Dey S., and Kundu D. (2016). On progressively type-II censored two-parameter Rayleigh distribution. *Communications in Statistics-Simulation and Computation*, **45**(2):438–455.

Dey, S., Singh, S., Tripathi, Y.M. and Asgharzadeh, A. (2023). Estimation and prediction for a progressively censored generalized inverted exponential distribution. *Statistical Methodology*, **32**:185–202.

Fernandez, A.J. (2004). On estimating exponential parameters with general type II progressive censoring. *Journal of Statistical Planning and Inference*, **121**(1):135–147.

Guilbaud, O. (2001). Exact non-parametric confidence intervals for quantiles with progressive type-II censoring. *Scandinavian Journal of Statistics*, **28**(4):699–713.

Jia, X., Nadarajah, S. and Guo B. (2018). Exact inference on Weibull parameters with multiply Type-I censored data. *IEEE Transactions on Reliability*, **67**(2):432–445.

Kumar, D., Nassa,r M. and Dey, S. (2023). Progressive type-II censored data and associated inference with application based on Li–Li Rayleigh distribution. *Annals of Data Science*, **10**(1):43–71.

Lehmann, E.L. and Casella, G. (1998). *Theory of Point Estimation.* 2nd Edition, New York: Springer.

Marrelec, G., Benali, H., Ciuciu, P., Pélégrini-Issac, M. and Poline J.B. (2003). Robust Bayesian estimation of the hemodynamic response function in event-related BOLD fMRI using basic physiological information. *Human Brain Mapping*, **19**(1):1–17.

Polovko, A.M. (1968). *Fundamentals of Reliability Theory.* San Diego: Academic Press.

Prakash, G. and Singh D.C. (2008). Shrinkage estimation in exponential type-II censored data under LINEX loss. *Journal of the Korean Statistical Society*, **37**:53–61.

Prakash, G. and Singh, D.C. (2009). A Bayesian shrinkage approach in Weibull type-II censored data using prior point information. *REVSTAT-Statistical Journal*, **7**(2):171–187.

Ren, J. and Gui, W. (2021). Inference and optimal censoring scheme for progressively Type-II censored competing risks model for generalized Rayleigh distribution. *Computational Statistics*, **36**(1):479–513.

Singh, D.C., Prakash, G. and Singh, P. (2007). Shrinkage testimators for the shape parameter of Pareto distribution using LINEX loss function. *Communication in Statics Theory and Methods*, **4**(36):741–753.

Thompson, J.R. (1968). Some shrunken techniques for estimating the mean. *Journal of the American Statistical Association*, **63**(321):113–122.

Tolba, A.H., Abushal, T.A. and Ramadan, D.A. (2023). Statistical inference with joint progressive censoring for two populations using power Rayleigh lifetime distribution. *Scientific Reports*, **13**(1):3832.

Viveros, R. and Balakrishnan, N. (1994). Interval estimation of life characteristics from progressively censored samples. *Technometrics*, **36**(1):84–91.

Yuen, H.K. and Tse, S.K. (1996). Parameters estimation for Weibull distributed lifetimes under progressive censoring with random removals. *Journal of Statistical Computation and Simulation*, **55**(1-2):57–71.

Zellner, A. (1986). Bayesian estimation and prediction using asymmetric loss functions. *Journal of the American Statistical Association*, **81**(394):446–451.