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Research Paper

Inference on the probability density and cumulative distribution functions for conducting an efficient estimation on the inverse Weibull distribution

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Abstract: Since the inverse Weibull distribution is one of the well-known lifetime distributions in statistical inference, estimating its parameters, as well as estimating its probability density function and cumulative distribution function through various statistical methods, is recommended. Throughout this study, we develop the best two-observation percentile estimation for both the probability density function and cumulative distribution function associated with the inverse Weibull distribution. To gauge its efficacy, we conduct a comparative analysis, pitting the best two-observation percentile estimation against estimations derived through maximum likelihood and percentiles, utilizing Monte Carlo simulations and scrutiny of two real datasets.

Keywords: Cumulative distribution function; Inverse Weibull distribution; Percentile estimation; Two-observational percentile estimation.
Mathematics Subject Classification (2010): 62G07, 62F10.

1 Introduction

Scholars have conducted a comparative analysis of the estimation of probability density functions (PDFs) and cumulative distribution functions (CDF) for diverse lifetime distributions. This investigation involved the utilization of various estimators, namely maximum likelihood (ML), uniformly minimum variance unbiased (UMVU), percentile (PC), least squares (LS), and weighted least squares (WLS). Numerous research papers have aimed to estimate parameters for various lifetime distributions. For instance,

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Dixit and Nooghabi (2010) focused on estimating the PDF and cumulative distribution function CDF for the Pareto distribution. In a similar vein, Jabbari Nooghabi (2010) delved into the exponentiated Pareto distribution, while Bagheri et al. (2016a) tackled the exponentiated Gumbel distribution. Alizadeh et al. (2013) addressed the generalized Rayleigh distribution, Bagheri et al. (2014) investigated the generalized Poisson-exponential distribution, and Maleki Jebely et al. (2018) concentrated their efforts on the Inverse Rayleigh distribution. Bagheri et al. (2016b) considered estimation of the PDF and the CDF of the Weibull extension model. Ghasemi Cherati et al. (2021) carried out the estimation of the PDF and CDF for the generalized inverted Weibull distribution.

Since the Rayleigh and Weibull distributions, particularly the Weibull distribution, are of significant interest to statisticians in many statistical fields, and are especially useful in data related to lifetime distributions, research on distributions derived from the Weibull distribution or various models based on the Weibull distribution is also important. Furthermore, all previous studies on this topic have focused on specific estimation methods such as maximum likelihood estimation (MLE), method of moments, percentile estimation, UMVUE, and least squares estimation, or combinations of these methods. However, none of these researches have used the method of best two-observation percentile estimation (BTPE). The innovation of this paper lies in the introduction of the BTPE method and its application.

In this investigation, the PDF and CDF of the inverse Weibull (IW) distribution are derived using the BTPE method. These results are then juxtaposed with corresponding estimations obtained through percentile (PC) and MLE procedures. If a random variable Y follows a Weibull distribution with the PDF

$$f(y;\alpha,\lambda) = \alpha\lambda^{\alpha}y^{\alpha-1}e^{-(\lambda y)^{\alpha}}, \quad y>0,$$

then the random variable $X = \frac{1}{Y}$ assumes an IW distribution with the PDF

$$f(x;\alpha,\lambda) = \alpha\lambda^{\alpha}x^{-(\alpha+1)}e^{-(\lambda x^{-1})^{\alpha}}, \quad x>0. \tag{1}$$

Here, $\alpha > 0$ and $\lambda > 0$ represent the shape and scale parameters, respectively. The obtained PDF and CDF using the BTPE method are compared with estimations derived from PC and MLE procedures.

Henceforth, the IW distribution with parameters α and λ will be denoted as $IW(\alpha,\lambda)$. If a random variable X follows $IW(\alpha,\lambda)$, then its distribution function, denoted by $F(x;\alpha,\lambda)$, is expressed as

$$F(x;\alpha,\lambda) = e^{-(\lambda x^{-1})^{\alpha}}, \quad x>0. \tag{2}$$

The IW model has been established as a suitable framework for characterizing the degradation phenomena observed in mechanical components, particularly dynamic elements of diesel engines, as illustrated in the work of Murthy et al. (2004). Additionally, the IW model has been identified in the context of the physical failure process, as outlined by Erto and Rapone (1984). The study by Erto and Rapone (1984) demonstrated the efficacy of the IW model in fitting survival data, such as the breakdown times of an insulating fluid under constant tension, as also noted by Nelson (2005). In alignment with the structure of this paper, Section 2 presents the derivation of BTPE, PCE, and

MLE. Section 3 employs Monte Carlo simulations and a real dataset to compare the performance of these estimators. The findings are presented in Section 4.

2 Approaches for determining estimates

In this section, consider a random sample $X_1, \dots, X_n$ with order statistics $Y_1, \dots, Y_n$, originating from a distribution characterized by the PDF (1) and CDF (2). The BTPE, PCE, and MLE for the PDF (1) and cumulative distribution function (2) are derived and discussed.

2.1 Estimation of the BTPE

Assume we have a random sample $X_1, \dots, X_n$ from a distribution with a CDF denoted by (2). The order statistics of this sample are $Y_1, \dots, Y_n$, and let p_i represent the percentile of Y_i then, $F(Y_i, \alpha, \lambda) = p_i$ or

$$\alpha(\log\lambda - \log Y_i) = \log(-\log p_i). \tag{3}$$

Considering two real values, p_1 and p_2 $0 < p_1 < p_2 < 1$, and utilizing relation (3), a two-observation percentile estimation of α, denoted as α^* can be calculated as follows

$$\begin{aligned}\alpha^* &= \frac{\log(-\log p_2) - \log(-\log p_1)}{\log Y_{k_1} - \log Y_{k_2}} \\ &= \frac{\log[-\log(1-p_1^*)] - \log[-\log(1-p_2^*)]}{\log Y_{k_1} - \log Y_{k_2}} = \frac{k}{\log Y_{k_1} - \log Y_{k_2}},\end{aligned}$$

where $k = \log[-\log(1-p_1^*)] - \log[-\log(1-p_2^*)]$, and for each $i = 1, 2$, let k_i be equal to $[np_i]$ if np_i is an integer; otherwise, set $k_i = [np_i] + 1$, where $[np_i]$ represents the greatest integer smaller than np_i. Additionally, define $p_1^* = 1 - p_2$ and $p_2^* = 1 - p_1$. According to Dubey (1967), the asymptotic normal distribution of α^* is characterized by a mean of α and a variance given by

$$\begin{aligned}\mathrm{Var}(\alpha^*) &= \frac{\alpha^2}{nk^2}\left[\frac{p_1^*}{(1-p_1^*)\log^2(1-p_1^*)} + \frac{p_2^*}{(1-p_2^*)\log^2(1-p_2^*)}\right. \\ &\quad \left. - \frac{2p_1^*p_2^*}{(1-p_1^*)(1-p_2^*)\log(1-p_1^*)\log(1-p_2^*)}\right].\end{aligned}$$

To minimize $\mathrm{Var}(\alpha^*)$, it is necessary to determine p_1^* and p_2^*. As suggested by Dubey (1967), optimal values are $p_1^* = 0.16730679$ and $p_2^* = 0.97366352$. Consequently, the estimation of α using the BTPE, denoted as $\hat{\alpha}_{\mathrm{BTPE}}$, is calculated as

$$\hat{\alpha}_{\mathrm{BTPE}} = \frac{-2.988881}{\log Y_{k_1} - \log Y_{k_2}},$$

where, $k_1 = [0.02633648n]$ or $k_1 = [0.8326932n] + 1$, and $k_2 = [0.8326932n]$ or $k_2 = [0.8326932n] + 1$.

In addition, for p_3 and p_4 $(0 < p_3 < p_4 < 1)$, with the help of (3), a BTPE of λ which is shown by λ^* is obtained as follows

$$\lambda^* = \exp(w_1 \log Y_{k_3} + w_2 \log Y_{k_4}),$$

where

$$w_1 = \frac{\log(-\log p_4)}{\log(-\log p_4) - \log(-\log p_3)}, \quad w_2 = \frac{-\log(-\log p_3)}{\log(-\log p_4) - \log(-\log p_3)}.$$

As per Dubey (1967)'s work in 1967, it is established that λ^* follows an asymptotic normal distribution, possessing a mean corresponding to λ and a variance of

$$\mathrm{Var}(\lambda^*) = \frac{\lambda^2}{nk^2}\left\{ r_1^* \left(\frac{k - \log k_1}{k_1}\right)\left[\left(\frac{k - \log k_1}{k_1}\right) + \frac{2\log k_1}{k_2}\right] + \frac{r_2^* \log^2 k_1}{k_2^2}\right\},$$

where $k = \log(-\log p_4) - \log(-\log p_3)$, $r_1 = 1 - p_4$, $r_2 = 1 - p_3$, and $r_i^* = \frac{r_i}{1-r_i}$, $k_i = -\log(1 - r_i)$, for $i = 1, 2$. In order to minimize the variance of λ^*, the values of r_1 and r_2 need to be determined. According to the findings of Dubey (1967), the recommended values are $r_1 = 0.39777$ and $r_2 = 0.82111$. Therefore, the BTPE of λ which is shown by $\hat{\lambda}_{\mathrm{BTPE}}$ is obtained as

$$\hat{\lambda}_{\mathrm{BTPE}} = \exp\left(\hat{w}_1 \log Y_{k_3} + \hat{w}_2 \log Y_{k_4}\right) = \exp\left(0.5556994 \log Y_{k_3} + 0.4443006 \log Y_{k_4}\right),$$

where

$$\begin{aligned}\hat{w}_1 &= \frac{\log(-\log(1-r_1))}{\log(-\log(1-r_1)) - \log(-\log(1-r_2))},\\ \hat{w}_2 &= \frac{-\log(-\log(1-r_2))}{\log(-\log(1-r_1)) - \log(-\log(1-r_2))},\end{aligned}$$

and $k_3 = [0.17889n]$ or $k_3 = [0.17889n] + 1$, $k_4 = [0.60223n]$ or $k_4 = [0.60223n] + 1$. Therefore, the BTPE of the PDF given by equation (1) and CDF denoted by equation (2) are obtained through the following relations, respectively.

$$\begin{aligned}\hat{f}_{\mathrm{BTPE}}(x;\alpha,\lambda) &= \hat{\alpha}_{\mathrm{BTPE}} \hat{\lambda}_{\mathrm{BTPE}}^{\hat{\alpha}_{\mathrm{BTPE}}} x^{-(\hat{\alpha}_{\mathrm{BTPE}}+1)} \exp\left(-\left(\hat{\lambda}_{\mathrm{BTPE}} x^{-1}\right)^{\hat{\alpha}_{\mathrm{BTPE}}}\right),\\ \hat{F}_{\mathrm{BTPE}}(x;\alpha,\lambda) &= \exp\left(-\left(\hat{\lambda}_{\mathrm{BTPE}} x^{-1}\right)^{\hat{\alpha}_{\mathrm{BTPE}}}\right).\end{aligned}$$

The Monte Carlo simulation method of the sample mean is employed to calculate the mean square error (MSE) for the optimal percentile estimations of the PDF defined in equation (1) and the CDF represented by equation (2).

2.2 PCE

Assuming that $X_1, \ldots, X_n$ constitute a random sample from a distribution characterized by the CDF denoted as (2), with corresponding order statistics $Y_1, \ldots, Y_n$, and p_i is the percentile of Y_i, then, $F(Y_i, \alpha, \lambda) = p_i$ or

$$\alpha \log \lambda - \alpha \log Y_i = \log(-\log p_i).$$

Estimators of α and λ, denoted as $\hat{\alpha}_{\text{PCE}}$ and $\hat{\lambda}_{\text{PCE}}$ respectively, are conducted through the minimization of the sum given by

$$\sum_{i=1}^{n} \left(\alpha \log \lambda - \alpha \log Y_i - \log(-\log p_i)\right)^2,$$

where $p_i = \frac{i}{n+1}$. This minimization is performed with respect to α and λ. The resulting equations are then solved using the Newton-Raphson numerical method.

$$n\alpha(\log \lambda)^2 - \log \lambda \sum_{i=1}^{n} \log(-\log p_i) + \left(\sum_{i=1}^{n} \log Y_i \log(-\log p_i) - \alpha \sum_{i=1}^{n} (\log Y_i)^2\right) = 0,$$

$$n \log \lambda - \alpha \sum_{i=1}^{n} \log Y_i - \sum_{i=1}^{n} \log(-\log p_i) = 0.$$

Upon substituting $\hat{\alpha}_{\text{PCE}}$ for α and $\hat{\lambda}_{\text{PCE}}$ for λ in equations (1) and (2), the PCE for the PDF and the CDF of the IW distribution, as well as the MSE of these estimators, are derived.

2.3 Maximum likelihood estimation

In this section, considering a random sample $X_1, \ldots, X_n$ originating from a distribution characterized by the PDF presented in (1), the MLE for the parameters α and λ, denoted as $\hat{\alpha}_{\text{MLE}}$ and $\hat{\lambda}_{\text{MLE}}$ respectively, are determined through the utilization of a set of equations and the numerical approach of Newton-Raphson.

$$\frac{n}{\alpha} + n \log \lambda \sum_{i=1}^{n} \log x_i + \sum_{i=1}^{n} \left(\frac{\lambda}{x_i}\right)^{\alpha} \log\left(\frac{\lambda}{x_i}\right) = 0,$$

$$\hat{\lambda} = \left(n \sum_{i=1}^{n} x_i^{\hat{\alpha}_{\text{MLE}}-1}\right)^{\frac{1}{\hat{\alpha}_{\text{MLE}}}}.$$

To obtain α, we use this equation

$$\alpha_i = \alpha_{i-1} - \frac{g(\alpha)}{g'(\alpha)},$$

where

$$g(\alpha) = \frac{n}{\alpha} + n \log \hat{\lambda} \sum_{i=1}^{n} \log x_i + \sum_{i=1}^{n} \left(\frac{\hat{\lambda}}{x_i}\right)^{\alpha} \log\left(\frac{\hat{\lambda}}{x_i}\right),$$

and $g'(\alpha)$ is the derivative of $g(\alpha)$ with respect to α. Substituting $\hat{\alpha}_{\text{MLE}}$ for α and $\hat{\lambda}_{\text{MLE}}$ for λ in equations (1) and (2), the MLE for the PDF and CDF of the IW distribution, as well as the MSE of these estimators, can be determined.

3 Computational analyses

Within this section, we present a Monte Carlo simulation and a numerical example to illustrate the various estimation methods detailed in the preceding sections.

3.1 Simulation Studies

In this subsection, initially, employing the first step, $X = \lambda\left[-\log U\right]^{-\frac{1}{\alpha}}$, in which U follows the Uniform distribution on interval (0,1), and for $\alpha = 1.5, 2, 4, \; = 2, 3, 4, 4.5$ and random samples are created, generating as $n = 100, 200, , 500$. In the second phase, BTPE, PCE, and MLE of parameters α and λ, as well as the BTPE, PCE, and MLE of the PDF given by equation (1) and the CDF denoted by equation (2) are computed. Moving on to the third step, the MSE for the estimations of the PDF (1) and CDF (2) is determined. These steps, from 1 to 3, are repeated 5000 times, and the average MSE along with the average estimations of parameters α and λ are calculated. The optimal estimator is identified as the one with the smallest average MSE. A comparison of the results from the simulation studies presented in Tables 1 and 2.

Table 1: Estimate the average parameters α , λ and MSE of PDF (1) of the estimation methods.

							AM		
n	$\hat{\alpha}_{\text{BTPE}}$	$\hat{\lambda}_{\text{BTPE}}$	$\hat{\alpha}_{\text{PCE}}$	$\hat{\lambda}_{\text{PCE}}$	$\hat{\alpha}_{\text{MLE}}$	$\hat{\lambda}_{\text{MLE}}$	$\hat{f}_{\text{PCE}}$	$\hat{f}_{\text{MLE}}$	$\hat{f}_{\text{BTPE}}$
				$(\alpha, \lambda) = (2, 3)$					
100	4.8367	1.0079	1.4786	2.2954	1.9731	3.6372	4.34874	0.77035	0.48280
200	4.6467	1.0102	1.4728	2.2611	1.9642	3.6005	4.46875	0.79027	0.43818
300	4.5511	1.0110	1.4551	2.2551	1.9634	3.5936	4.48392	0.78236	0.46429
400	4.6769	1.0109	1.4498	2.2295	1.9586	3.5841	4.42212	0.78222	0.46307
500	4.6228	1.0108	1.4518	2.2351	1.9594	3.5835	4.47561	0.77847	0.48618
				$(\alpha, \lambda) = (4, 1.5)$					
100	4.8612	1.1072	2.4553	0.8936	0.3301	1.1672	0.00994	0.02392	0.00931
200	4.2922	1.1492	2.4476	0.8332	0.3299	1.1652	0.01020	0.02798	0.00859
300	4.0555	1.1412	2.4444	0.7956	0.3295	1.1652	0.01051	0.02561	0.00859
400	3.8799	1.1577	2.4409	0.7693	0.3292	1.1656	0.01077	0.02709	0.00793
500	3.7940	1.1510	2.4426	0.7542	0.3293	1.1651	0.01095	0.02778	0.00795
				$(\alpha, \lambda) = (1.5, 0.25)$					
100	2.0932	1.5782	0.1531	1.6962	2.5424	0.2540	0.01412	3.47621	0.00817
200	2.0281	1.5801	0.1093	3.0855	2.5072	0.2534	0.00890	3.14124	0.00808
300	2.0289	1.5799	0.0902	6.4183	2.4900	0.2531	0.00786	2.65151	0.00745
400	2.0029	1.5769	0.0783	24.016	2.4718	0.2526	0.00866	2.53194	0.00816
500	2.0118	1.5771	0.0696	11.233	2.4811	0.2529	0.00862	2.28138	0.00798
				$(\alpha, \lambda) = (2, 3.5)$					
100	4.1289	1.0064	0.9515	2.7688	2.3723	3.8177	0.80433	1.87422	0.12703
200	4.1988	1.0113	0.9424	2.7571	2.3536	3.8049	0.70437	1.89349	0.00981
300	4.0719	1.0101	0.9372	2.7964	2.3449	3.7993	0.60702	1.84600	0.06714
400	4.1220	1.0111	0.9341	2.8412	2.3407	3.7961	0.62749	1.87497	0.95869
500	4.0517	1.0105	0.9313	2.7955	2.3429	3.7972	0.65273	1.89357	0.92011

In these Tables, since the MSE for the BTPE method is the smallest, it was concluded that the BTPE estimation method is superior to the other methods. However, for certain values of (α, λ), the MLE estimation performed better than the PCE estimation, and in some cases, the PCE estimation outperformed the MLE. These results are confirmed by Figures 1, 2 and 3.

3.2 Implementation using an actual dataset

In this section, we conduct computations and comparisons for the BTPE, PCE, and MLE for the PDF and the CDF of the IW distribution, focusing on two distinct real

Table 2: Estimate the average parameters α , λ and MSE of CDF (2) of the estimation methods.

								AM	
n	$\hat{\alpha}_{\text{BTPE}}$	$\hat{\lambda}_{\text{BTPE}}$	$\hat{\alpha}_{\text{PCE}}$	$\hat{\lambda}_{\text{PCE}}$	$\hat{\alpha}_{\text{MLE}}$	$\hat{\lambda}_{\text{MLE}}$	$\hat{F}_{\text{PCE}}$	$\hat{F}_{\text{MLE}}$	$\hat{F}_{\text{BTPE}}$
$(\alpha, \lambda) = (2, 3)$									
100	4.8367	1.0079	1.4786	2.2954	1.9731	3.6372	0.15101	0.13021	0.03613
200	4.6467	1.0102	1.4728	2.2611	1.9642	3.6005	0.15825	0.07952	0.03390
300	4.5511	1.0110	1.4551	2.2551	1.9634	3.5936	0.15983	0.08319	0.03276
400	4.6769	1.0109	1.4498	2.2295	1.9586	3.5841	0.15672	0.08829	0.03338
500	4.6228	1.0108	1.4518	2.2351	1.9594	3.5835	0.15933	0.08519	0.03356
$(\alpha, \lambda) = (4, 1.5)$									
100	4.8612	1.1072	2.4553	0.8936	0.3301	1.1672	0.05273	0.07924	0.03038
200	4.2922	1.1492	2.4476	0.8332	0.3299	1.1652	0.03054	0.05521	0.02974
300	4.0555	1.1412	2.4444	0.7956	0.3295	1.1652	0.02117	0.03104	0.01774
400	3.8799	1.1577	2.4409	0.7693	0.3292	1.1656	0.02989	0.04122	0.01507
500	3.7940	1.1510	2.4426	0.7542	0.3293	1.1651	0.02296	0.10221	0.00343
$(\alpha, \lambda) = (1.5, 0.25)$									
100	2.0932	1.5782	0.1531	1.6962	2.5424	0.2540	0.22450	0.09921	0.00272
200	2.0281	1.5801	0.1093	3.0855	2.5072	0.2534	0.22421	0.09616	0.00240
300	2.0289	1.5799	0.0902	6.4183	2.4900	0.2531	0.22577	0.09368	0.00213
400	2.0029	1.5769	0.0783	24.016	2.4718	0.2526	0.2258	0.09434	0.00215
500	2.0118	1.5771	0.0696	11.233	2.4811	0.2529	0.22661	0.09229	0.01925
$(\alpha, \lambda) = (2, 3.5)$									
100	4.1289	1.0064	0.9515	2.7688	2.3723	3.8177	0.18144	0.07928	0.01769
200	4.1988	1.0113	0.9424	2.7571	2.3536	3.8049	0.17706	0.10130	0.01735
300	4.0719	1.0101	0.9372	2.7964	2.3449	3.7993	0.18281	0.08311	0.01692
400	4.1220	1.0111	0.9341	2.8412	2.3407	3.7961	0.17867	0.08771	0.01699
500	4.0517	1.0105	0.9313	2.7955	2.3429	3.7972	0.18344	0.09134	0.01726

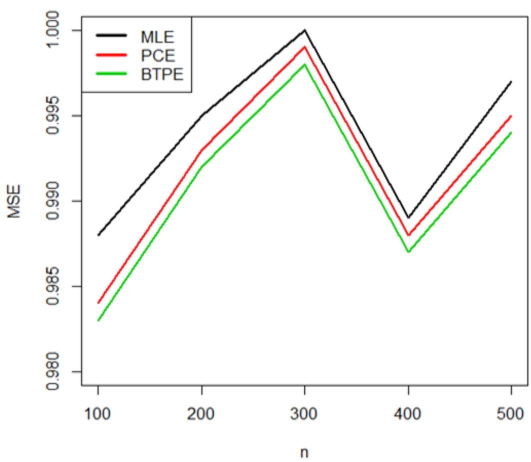

Figure 1: MSE plot with respect to n for $(\alpha, \lambda) = (2, 3)$.

datasets. The initial dataset, sourced from Lawless (2011), consists of 23 observations related to ball bearings. The second dataset represents repair times (measured in hours) for an airborne communication transceiver and was initially studied by Alven (1964). Both datasets are detailed in Table 3. We individually fit the IW distribution to each dataset, and the results are presented in Table 4, encompassing the estimated

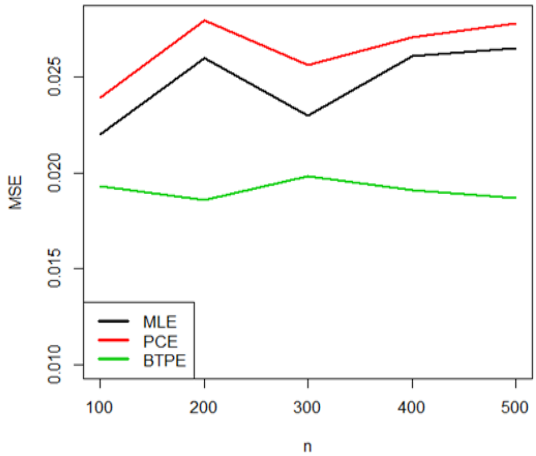

Figure 2: MSE plot with respect to n for $(\alpha, \lambda) = (4, 1.5)$.

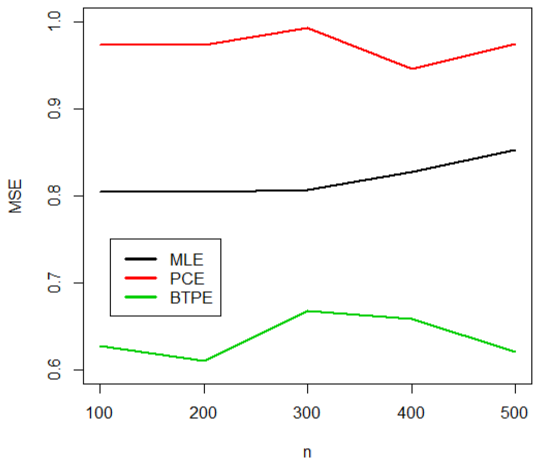

Figure 3: MSE plot with respect to n for $(\alpha, \lambda) = (2, 3.5)$.

shape and scale parameters, Kolmogorov-Smirnov (K-S) distances between the fitted and empirical distribution functions, along with corresponding p-values. The outcomes from Table 4 affirm the suitability of the IW distribution in effectively capturing the characteristics of both datasets.

The IW distribution was applied to these datasets using ML, BTP, and PC estimators, assuming that α and λ are unknown. Table 5 provides the parameter estimates for α and λ, along with the corresponding log-likelihood values. It is noteworthy that the log-likelihood value is maximized for the BTPE method.

We further assessed the estimation methods using model selection criteria, including the Akaike information criterion (AIC), Bayes information criterion (BIC), and Hannan–Quinn criterion (HQC), defined as follows: $AIC = -2\log L(\theta) + 2k$, $BIC = -2\log L(\theta) + k\log n$ and $HQC = -2\log L(\theta) + 2k\log(\log n)$, Here, $\log L(\theta)$ represents the log-likelihood, n is the number of observations, and k is the number of distribution parameters. Smaller values for these criteria indicate a better fit. Table 6

Table 3: Data Sets.

Data Set	Values
Data Set 1 (Ball Bearing)	17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.48, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40
Data Set 2 (Repair Time)	0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.7, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 7.5, 8.8, 9.0, 10.3

Table 4: Shape parameter, scale parameter, K-S, and p-values of the fitted IW models to Data sets 1 and 2.

Data Set	Shape Parameter	Scale Parameter	K-S	P-Value
1	1.834384	48.57896	0.133	0.8103
2	1.011941	1.125229	0.0815	0.9197

Table 5: Estimate of Parameters and Corresponding Log-Likelihood.

Data Set	Method	Estimate of α	Estimate of λ	Log-Likelihood
1	MLE	1.834384	48.57896	-240.9375
	BTPE	2.303755	36.54517	-115.7821
	PCE	3.786125	56.43215	-146.5857
2	MLE	1.011941	1.125229	-132.6268
	BTPE	1.034082	2.125229	-113.6210
	PCE	1.124321	2.897111	-144.4124

provides the model selection criterion values for the three different estimation methods. Notably, the BTPE estimators yield the smallest values across all three criteria. Additionally, Figures 4 and 5 present density plots (fitted PDF versus empirical PDF) and distribution plots (fitted CDF versus empirical CDF) for the three different estimation methods. The visualizations confirm that the BTPE estimators offer the most favorable fit.

Table 6: Model selection criteria for the Data sets 1 and 2.

Data Set	Method	AIC	BIC	HQC
1	MLE	483.1714	487.4424	490.3136
	BTPE	233.5642	237.8352	236.1353
	PCE	295.8750	300.1461	298.4461
2	MLE	290.8248	296.4821	294.1948
	BTPE	229.2420	234.8993	232.6120
	PCE	203.2536	308.9109	306.6236

4 Conclusion

In this study, we conducted a comprehensive comparison of three distinct estimators the ML estimator, the BTPE estimator, and the PC estimator for both the PDF and

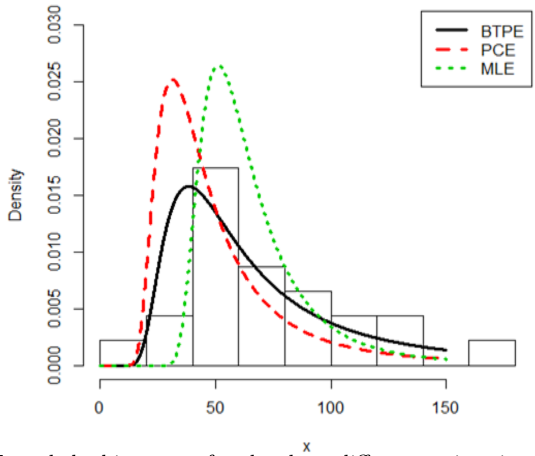

Figure 4: Fitted PDFs and the histogram for the three different estimation methods based on Data set 1.

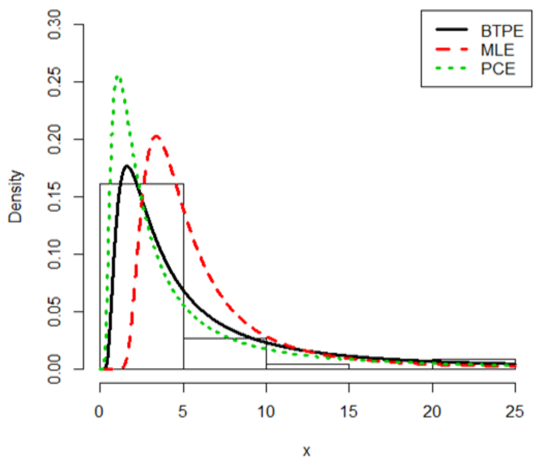

Figure 5: Fitted PDFs and the histogram for the three different estimation methods based on Data set 2.

the CDF of the IW distribution. The performance evaluations involved simulations and applications to two real datasets. The outcomes consistently demonstrate that the BTPE estimator outperforms the other estimators across various metrics, including MSE in simulation studies, log-likelihood values, density plots, and model selection criteria such as AIC, BIC, and HQC.

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