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Estimation and prediction from progressively censored data using proportional hazards model

Adeleh Fallah, Roshanak Zaman*
Department of Statistics, Payame Noor University, Tehran, Iran

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Abstract: In this paper, based on progressively Type-II censored samples, the maximum likelihood and Bayes estimators are derived for some lifetime parameters. In the Bayesian framework, the point estimations of unknown parameters under both symmetric and asymmetric loss functions are discussed. The Bayesian estimations have been obtained using the conjugate prior and discrete priors for the shape and scale parameters, respectively. We also provide Bayes prediction intervals for the times to failure of units censored in multiple stages in a progressively censored sample. Finally, two numerical examples are presented to illustrate the results.

Keywords: Bayes estimation; Maximum likelihood estimation; Prediction intervals; Progressively censored samples; Proportional hazard model.
Mathematics Subject Classification (2010): 62G30, 62F15, 62G32.

1 Introduction

We consider a versatile scheme of censoring called progressive Type-II censoring. Under this scheme of censoring, from a total of n units placed on a life-test, only $m(< n)$ are completely observed until failure. At the time of the first failure, R_1 of the $n-1$ surviving units are randomly withdrawn (or censored) from the life-testing experiment. At the time of the next failure, R_2 of the $n-2-R_1$ surviving units are censored, and so on. Finally, at the time of the mth failure, all the remaining $R_m = n-m-R_1-\ldots-R_{m-1}$ surviving units are censored. Note that censoring takes place here progressively in m stages. Clearly, this scheme includes as special cases the complete sample situation (when $n = m$ and $R_1 = R_2 = \ldots = R_m = 0$) and the conventional Type-II right censoring situation (when $R_1 = R_2 = \ldots = R_{m-1} = 0$ and $R_m = n-m$). The ordered failure

*Corresponding author: `rzaman@pnu.ac.ir`

times arising from such a progressively Type-II right censored sample are called progressively Type-II right censored order statistics. Let $X_{1:m:n}, \ldots, X_{m:m:n}$ be a progressively Type-II censored sample with the censoring scheme $R = (R_1, R_2, \ldots, R_m)$. To simplify the notation, we will use $x_1, x_2, \ldots, x_m$ in place of $x_{1:m:n}, x_{2:m:n}, \ldots, x_{m:m:n}$. For further details on progressively censoring, inferences and their applications, one may refer to Balakrishnan and Aggarwala (2000) and Balakrishnan and Cramer (2014).

Let us consider the continuous random variable X with the cumulative distribution function (cdf) $F(x;\theta,\sigma)$. In many situations $F(x;\theta,\sigma)$ can be written as

$$F(x;\theta,\sigma) = 1 - [\bar{F}_0(x/\sigma)]^{\theta}, \quad -\infty \le c < x < d \le \infty, \quad \theta > 0, \tag{1}$$

where $\bar{F}_0(.) = 1 - F_0(.)$, and $F_0(.)$ is an arbitrary continuous cdf with $F_0(c) = 0$ and $F_0(d) = 1$. Here, $\{F(x;\theta), \theta > 0\}$ is called a proportional hazards family with underlying distribution F_0, see, Marshall and Olkin (2007). From the model 1, the probability density function $f(x;\theta)$, the reliability function $R(t;\theta,\sigma)$ and hazard rate function $H(t;\theta,\sigma)$, (at some t) are given, respectively, by

$$f(x;\theta) = \frac{\theta}{\sigma} f_0(x/\sigma)[\bar{F}_0(x/\sigma)]^{\theta-1}, \quad -\infty \le c < x < d \le \infty, \tag{2}$$

$$R(t;\theta,\sigma) = \left(\bar{F}_0(\frac{t}{\sigma})\right)^{\theta}, \quad t > 0, \tag{3}$$

$$H(t;\theta,\sigma) = \frac{\theta}{\sigma}\frac{f_0(\frac{t}{\sigma})}{\bar{F}_0(\frac{t}{\sigma})}, \quad t > 0, \tag{4}$$

where $f_0(.) = F_0'(.)$ is the corresponding probability density function. Many authors have discussed inference under progressive Type-II censored using different lifetime distributions: see Aggarwala and Balakrishnan (1999), Cacciari and Montanari (2014), Ng (2005), Mahmoud et al. (2014), El-Sagheer (2018), Zhang and Gui (2021), Soliman (2005), and Soliman et al. (2006).

In this paper, we address the growing need for reliable estimation techniques in survival analysis, particularly in contexts where progressively Type-II censored samples are prevalent. The proportional hazards (PH) family of distributions offers remarkable flexibility in modeling hazard functions without imposing restrictive assumptions about the baseline hazard shape, making it particularly valuable for analyzing complex survival data. Unlike traditional lifetime distributions, the PH family maintains the fundamental property that hazard ratios between different groups remain constant over time, allowing for a more adaptable approach to understanding survival processes. Understanding lifetime parameters such as reliability and hazard functions is crucial for effective decision-making across various fields, as the PH family enables straightforward interpretation of covariate effects in terms of relative risk. This feature facilitates meaningful comparisons across different groups or treatment conditions, enhancing the practical applicability of our findings. Furthermore, the mathematical structure of PH distributions allows for the incorporation of time-varying covariates, enabling dynamic modeling that reflects real-world complexities. By leveraging these advantages, this study aims to contribute to the existing literature by providing robust estimation techniques that address the challenges posed by progressively censored data while offering valuable insights into survival analysis. The rest of this paper is organized as follows.

the maximum likelihood estimate (MLE) of the parameters θ and σ are discussed in Section 2. Bayes estimators relative to different loss functions are considered in Section 3. In Section 4, We also provide Bayes prediction intervals for the times to failure of units censored in multiple stages in a progressively censored sample. In Section 5, two numerical example are used to illustrate the methodologies developed in this paper.

2 Maximum likelihood estimation

Let $X_1, X_2, \ldots, X_m$ denote a progressively Type-II censored sample from the proportional hazards family (1) obtained from a sample of size n with the censoring scheme $R_1, \ldots, R_m$. The likelihood function is given (Balakrishnan and Aggarwala, 2000) by

$$L(\theta, \sigma|\boldsymbol{x}) = A \prod_{i=1}^{m} \left\{ f(x_i; \theta, \sigma)[1 - F(x_i; \theta, \sigma)]^{R_i} \right\}, \tag{5}$$

where $A = n(n-1-R_1)(n-2-R_1-R_2)\ldots(n-m+1-R_1\ldots-R_{m-1})$. It follows from (1), (2) and (5), that

$$L(\theta, \sigma \mid \boldsymbol{x}) = A\frac{\theta^m}{\sigma^m}\left[\prod_{i=1}^{m}\frac{f_0(\frac{x_i}{\sigma})}{\bar{F}_0(\frac{x_i}{\sigma})}\right]\exp\left(\theta\sum_{i=1}^{m}(R_i+1)\log\bar{F}_0(\frac{x_i}{\sigma})\right). \tag{6}$$

The log-likelihood function is

$$\begin{aligned} L = \log L(\theta, \sigma \mid \boldsymbol{x}) &= \log A + m\log\theta - m\log\sigma + \theta\sum_{i=1}^{m}(R_i+1)\log\bar{F}_0(\frac{x_i}{\sigma}) \\ &\quad + \sum_{i=1}^{m}\log\left[\frac{f_0(\frac{x_i}{\sigma})}{\bar{F}_0(\frac{x_i}{\sigma})}\right]. \end{aligned} \tag{7}$$

From (7), we obtain the likelihood equations as

$$\begin{aligned} \frac{dL}{d\sigma} &= -\frac{m}{\sigma} + \frac{\theta}{\sigma^2}\sum_{i=1}^{m}(R_i+1)\frac{x_i\ f_0(\frac{x_i}{\sigma})}{\bar{F}_0(\frac{x_i}{\sigma})} - \frac{1}{\sigma^2}\sum_{i=1}^{m}\frac{x_i f_0'(\frac{x_i}{\sigma})}{f_0(\frac{x_i}{\sigma})} - \frac{1}{\sigma^2}\sum_{i=1}^{m}\frac{x_i f_0(\frac{x_i}{\sigma})}{\bar{F}_0(\frac{x_i}{\sigma})}, \\ \frac{dL}{d\theta} &= \frac{m}{\theta} + \sum_{i=1}^{m}(R_i+1)\log\bar{F}_0(\frac{x_i}{\sigma}). \end{aligned} \tag{8}$$

The maximum likelihood estimators (MLEs) $\hat{\theta}$ and $\hat{\sigma}$ can be obtained by solving the likelihood equations. Solving $\frac{dL}{d\theta} = 0$ for θ gives, from (8)

$$\hat{\theta} = -\frac{m}{\sum_{i=1}^{m}(R_i+1)\log\bar{F}_0(\frac{x_i}{\hat{\sigma}})}, \tag{9}$$

where $\hat{\sigma}$ is the solution of

$$-m - \frac{m}{\hat{\sigma}\sum_{i=1}^{m}(R_i+1)\log\bar{F}_0(\frac{x_i}{\hat{\sigma}})}\sum_{i=1}^{m}(R_i+1)\frac{x_i\ f_0(\frac{x_i}{\hat{\sigma}})}{\bar{F}_0(\frac{x_i}{\hat{\sigma}})} - \frac{1}{\hat{\sigma}}\sum_{i=1}^{m}\frac{x_i f_0'(\frac{x_i}{\hat{\sigma}})}{f_0(\frac{x_i}{\hat{\sigma}})}$$

$$-\frac{1}{\hat{\sigma}}\sum_{i=1}^{m}\frac{x_i f_0(\frac{x_i}{\hat{\sigma}})}{\bar{F}_0(\frac{x_i}{\hat{\sigma}})} = 0. \tag{10}$$

Newton-Raphson iteration is employed to solve (10). The corresponding MLE of the reliability function $R(t)$ and the hazard function $H(t)$, are given respectively by 3 and (4) after replacing σ, and θ by their MLE $\hat{\sigma}$, and $\hat{\theta}$.
Special Case: Taking

$$\bar{F}_0(x) = \frac{1}{1+x}, \quad x > 0,$$

X has parato distribution, and using (9), the MLE of θ is

$$\hat{\theta} = \frac{m}{\sum_{i=1}^{m}(R_i+1)\log(1+x_i/\hat{\sigma})},$$

and the MLE, $\hat{\sigma}$, is the solution of

$$-m + \frac{1}{\hat{\sigma}}\sum_{i=1}^{m}\frac{x_i}{(1+x_i/\hat{\sigma})} + \frac{m}{\hat{\sigma}(\sum_{i=1}^{m}(R_i+1)\log(1+x_i/\hat{\sigma}))}\sum_{i=1}^{m}(R_i+1)\frac{x_i}{(1+x_i/\hat{\sigma})} = 0.$$

This must be solved by Newton-Raphson method in order to obtain the MLE of the scale parameter σ.

3 Bayes estimation

For a parameter θ and a decision rule $\hat{\theta}$, the most commonly used loss function is squared error loss (SEL) function $L_1(\hat{\theta},\theta) = (\hat{\theta}-\theta)^2$. The symmetric nature of this function gives equal weight to overestimation as well as underestimation, while in the estimation of parameters of life time model, overestimation may by more serious than underestimation or vice-versa. For example, in the estimation of reliability and failure rate functions, an overestimate is usually much more serious than underestimate, in this case the use of symmetric loss function may be inappropriate as has been recognized by Basu and Ebrahimi (1991). This leads us to thinking that an asymmetrical loss function may be more appropriate. One of the most popular asymmetric loss function is the linear-exponential loss function (LINEX). This loss function was introduced by Varrian (1975) and was extensively discussed by Zellner (1986).

$$L_2(\hat{\theta},\theta) \propto e^{c(\hat{\theta}-\theta)} - c(\hat{\theta}-\theta) - 1, \tag{11}$$

where c is the shape parameter of the loss function. It controls the direction and degree of symmetry. If $c > 0$, the overestimation is more serious than underestimation, and vice-versa. For c close to zero, the LINEX loss is approximately SEL and therefore almost symmetric. The posterior expectation of the LINEX loss function (11) is

$$E_\theta[L_2(\hat{\theta},\theta)] \propto e^{c\hat{\theta}}E_\theta[\exp(-c\theta)] - c(\hat{\theta} - E_\theta(\theta)) - 1, \tag{12}$$

where $E_\theta(\cdot)$ denotes the posterior expectation with respect to the posterior density of θ. The Bayes estimator of θ, denote by $\hat{\theta}_{BL}$ under the LINEX loss function is the value

$\hat{\theta}$ which minimizes (12). It is

$$\hat{\theta}_{BL} = \frac{-1}{c} \log\{E_\theta[\exp\,(-c\theta)]\},$$

provided that the expectation $E_\theta[\exp\,(-c\theta)]$ exists and is finite. Another useful asymmetric loss function is the entropy loss (EL) function

$$L(\hat{\theta}, \phi) \propto \left(\frac{\hat{\theta}}{\theta}\right) - \log\left(\frac{\hat{\theta}}{\theta}\right) - 1, \tag{13}$$

whose minimum occurs at $\hat{\theta} = \theta$. This loss is also known as Stein loss. The Bayes estimate $\hat{\theta}_{BG}$ of θ under the entropy loss (13) is

$$\hat{\theta}_{BG} = \left(E_\theta(\theta^{-1})\right)^{-1}.$$

Symmetric loss functions, such as Mean Squared Error and Mean Absolute Error, treat positive and negative errors equally, providing simplicity and general applicability but potentially leading to underfitting in asymmetric contexts. Conversely, asymmetric loss functions like LINEX assign different penalties for over- and underestimations, improving model performance by addressing specific error implications. Entropy loss offers significant benefits over LINEX loss, particularly in Bayesian methods, due to its interpretability and robustness to noisy data. It effectively aligns well with probabilistic interpretations, making it suitable for real-world scenarios with imperfect data. In contrast, LINEX loss is designed for situations where the costs of overestimations and underestimations differ, allowing practitioners to tailor error costs for enhanced predictive performance.

Under the assumption that both the parameters θ and σ are unknown, specifying a general joint prior for θ and σ leads to computational complexities for the Bayes estimates. To solve this problem and simplify the Bayesian analysis, we consider the method advocated by Soland (1969). In this method, we use a conjugate continuous-discrete joint prior distribution for the parameters θ and σ. The continuous component of this distribution is related to θ and the discrete one is related to σ. We assume that the scale parameter σ is restricted to a finite number of values $\sigma_1, \sigma_2, \ldots, \sigma_j$ with prior probabilities $\eta_1, \eta_2, \ldots, \eta_j$ respectively, where $0 \leq \eta_j \leq 1$, and $\sum_{j=1}^{k} \eta_j = 1$, i.e.

$$\pi(\sigma_j) = p_r(\sigma = \sigma_j) = \eta_j, \qquad j = 1, 2, \ldots, k.$$

Further, suppose that conditional upon $\sigma = \sigma_j$, $j = 1, 2, \ldots, k$, θ has a natural conjugate gamma prior with parameters a_j and b_j

$$\pi(\theta|\sigma_j) = \frac{b_j^{a_j} \theta^{a_j - 1} e^{-\theta b_j}}{\Gamma(a_j)}, \qquad a_j, b_j, \theta > 0. \tag{14}$$

Combining the likelihood function in (6) and (14), we obtain the conditional posterior probability density function of θ given $\sigma = \sigma_j$ as

$$\pi^*(\theta|\sigma_j; \boldsymbol{x}) = \frac{B_j^{A_j} \theta^{A_j - 1}}{\Gamma(A_j)} \exp(-\theta B_j), \quad A_j, B_j, \theta > 0, \tag{15}$$

where $A_j = m + a_j$ and $B_j = b_j - \sum_{i=1}^{m}(R_i+1)\log \bar{F}_0(\frac{x_i}{\sigma})$. The joint posterior of θ and σ_j is

$$\pi^*(\theta, \sigma_j|\boldsymbol{x}) = \frac{A\ b_j^{a_j} u_j \eta_j \theta^{A_j-1} e^{-\theta B_j}}{G\ \Gamma(a_j)\sigma_j^m},$$

and the marginal posterior probability of σ_j is

$$p_j = p_r(\sigma = \sigma_j|\boldsymbol{x}) = \frac{b_j^{a_j}\eta_j u_j \Gamma(A_j)}{\Gamma(a_j)\sigma_j^m B_j^{A_j} G}, \tag{16}$$

where

$$G = \sum_{j=1}^{k} \frac{u_j b_j^{a_j}\eta_j \Gamma(A_j)}{\sigma_j^m \Gamma(a_j) B_j^{A_j}} \quad \textit{and} \quad u_j = \prod_{i=1}^{m} \frac{f_0(\frac{x_i}{\sigma_j})}{\bar{F}_0(\frac{x_i}{\sigma_j})}. \tag{17}$$

Under a squared error loss function, the usual estimate of a parameter is the posterior mean. Thus, Bayes estimates of the parameters, the reliability function and the hazard function are obtained by using (15) and (16). The Bayes estimates $\hat{\theta}_{BS}$, and $\hat{\sigma}_{BS}$ of parameters θ, and σ are

$$\begin{aligned} \hat{\theta}_{BS} &= \int_0^\infty \sum_{j=1}^{k} p_j\ \theta\ \pi^*(\theta|\sigma_j;\boldsymbol{x})d\theta \\ &= \sum_{j=1}^{k} p_j \frac{A_j}{B_j} = \sum_{j=1}^{k} p_j \frac{(m+a_j)}{(b_j - \sum_{i=1}^{m}(R_i+1)\log \bar{F}_0(\frac{x_i}{\sigma_j}))}, \end{aligned} \tag{18}$$

$$\hat{\sigma}_{BS} = \int_0^\infty \sum_{j=1}^{k} p_j \sigma_j \pi^*(\theta|\sigma_j;\boldsymbol{x})d\theta = \sum_{j=1}^{k} p_j\sigma_j. \tag{19}$$

The Bayes estimate, $\hat{R}_{BS}(t)$, of the reliability function $R(t;\theta,\sigma)$ and $\hat{H}_{BS}(t)$, of the hazard function $H(t;\theta,\sigma)$ are

$$\begin{aligned} \hat{R}(t)_{BS} &= \sum_{j=1}^{k} p_j \int_0^\infty \left[\bar{F}_0(\frac{t}{\sigma_j})\right]^\theta \pi^*(\theta|\sigma_j;\boldsymbol{x})d\theta \\ &= \sum_{j=1}^{k} p_j \left(1 - \frac{\log \bar{F}_0(\frac{t}{\sigma_j})}{(b_j - \sum_{i=1}^{m}(R_i+1)\log \bar{F}_0(\frac{t}{\sigma_j}))}\right)^{-(m+a_j)}, \end{aligned} \tag{20}$$

$$\begin{aligned} \hat{H}(t)_{BS} &= \sum_{j=1}^{k} p_j \int_0^\infty \frac{\theta f_0(\frac{t}{\sigma_j})}{\bar{F}_0(\frac{t}{\sigma_j})\sigma_j} \pi^*(\theta|\sigma_j;\boldsymbol{x})d\theta \\ &= \sum_{j=1}^{k} p_j \frac{f_0(\frac{t}{\sigma_j})(m+a_j)}{\bar{F}_0(\frac{t}{\sigma_j})\sigma_j(b_j - \sum_{i=1}^{m}(R_i+1)\log \bar{F}_0(\frac{t}{\sigma_j}))}. \end{aligned} \tag{21}$$

respectively. Under the LINEX loss function, the Bayes estimate $\hat{\theta}_{BL}$, and $\hat{\sigma}_{BL}$ of parameters θ, and σ are

$$\hat{\theta}_{BL} = \frac{-1}{c}\log\left[\int_0^\infty \sum_{j=1}^{k} p_j \exp(-c\theta_j)\pi^*(\theta|\sigma_j;\boldsymbol{x})d\theta\right]$$

$$= \frac{-1}{c}\log\left[\sum_{j=1}^{k} p_j\left(1+\frac{c}{b_j-\sum_{i=1}^{m}(R_i+1)\log\bar{F}_0(\frac{x_i}{\sigma_j})}\right)^{-(m+a_j)}\right], \quad (22)$$

$$\hat{\sigma}_{BL} = \frac{-1}{c}\log\left[\int_0^\infty \sum_{j=1}^{k} p_j \exp(-c\sigma_j)\pi^*(\theta|\sigma_j;\boldsymbol{x})d\theta\right] \quad (23)$$

$$= \frac{-1}{c}\log\sum_{j=1}^{k} p_j e^{-c\sigma_j}.$$

The Bayes estimate, $\hat{R}_{BL}(t)$, of the reliability function $R(t;\theta,\sigma)$ is given by

$$\hat{R}(t)_{BL} = \frac{-1}{c}\log\left[\int_0^\infty \sum_{j=1}^{k} p_j \exp(-cR(t;\theta,\sigma_j))\pi^*(\theta|\sigma_j;\boldsymbol{x})d\theta\right],$$

where $R(t;\theta,\sigma_j) = \left[\bar{F}_0(\frac{t}{\sigma_j})\right]^\theta$. By using the exponential series

$$e^{-c(\bar{F}_0(t/\sigma_j))^\theta} = \sum_{s=0}^{\infty}\frac{(-c)^s}{s!}\left[\bar{F}_0(\frac{t}{\sigma_j})\right]^{s\theta} = \sum_{s=0}^{\infty}\frac{(-c)^s}{s!}e^{\theta s\log(\bar{F}_0(t/\sigma_j))},$$

and after some simplification, we obtain

$$\hat{R}(t)_{BL} = \frac{-1}{c}\log\left[\sum_{j=1}^{k}\sum_{s=0}^{\infty} p_j\frac{(-c)^s}{s!}\left(1-\frac{s\log(\bar{F}_0(\frac{t}{\sigma_j}))}{(b_j-\sum_{i=1}^{m}(R_i+1)\log\bar{F}_0(\frac{t}{\sigma_j}))}\right)^{-(m+a_j)}\right]. \quad (24)$$

Similarly, the Bayes estimate, $\hat{H}_{BL}(t)$, of the hazard function $H(t;\theta,\sigma)$ is given by

$$\hat{H}(t)_{BL} = \frac{-1}{c}\log\left[\int_0^\infty \sum_{j=1}^{k} p_j \exp(-cH(t;\theta,\sigma_j))\pi^*(\theta|\sigma_j;\boldsymbol{x})d\theta\right],$$

where $H(t;\theta,\sigma_j) = \frac{\theta f_0(t/\sigma_j)}{\sigma_j\bar{F}_0(t/\sigma_j)}$. Therefore, we obtain

$$\hat{H}(t)_{BL} = \frac{-1}{c}\log\left[\sum_{j=1}^{k} p_j\left(1+\frac{cf_0(\frac{t}{\sigma_j})}{\sigma_j\bar{F}_0(\frac{t}{\sigma_j})(b_j-\sum_{i=1}^{m}(R_i+1)\log\bar{F}_0(\frac{t}{\sigma_j}))}\right)^{-(m+a_j)}\right]. \quad (25)$$

Under the entropy loss function, the Bayes estimate $\hat{\theta}_{BG}$, and $\hat{\sigma}_{BG}$ of parameters θ, and σ are given by

$$\hat{\theta}_{BG} = \left[\sum_{j=1}^{k}\int_0^\infty \frac{p_j}{\theta}\pi^*(\theta|\sigma_j;\boldsymbol{x})d\theta\right]^{-1} = \left[\sum_{j=1}^{k}\frac{p_j(b_j-\sum_{i=1}^{m}(R_i+1)\log\bar{F}_0(\frac{x_i}{\sigma_j}))}{(m+a_j-1)}\right]^{-1} \quad (26)$$

$$\hat{\sigma}_{BG} = \left[\sum_{j=1}^{k}\int_0^\infty p_j\sigma_j^{-1}\pi(\theta|\sigma_j;\boldsymbol{x})d\theta\right]^{-1} = \left[\sum_{j=1}^{k} p_j\sigma_j^{-1}\right]^{-1}. \quad (27)$$

Similarly, The Bayes estimate, $\hat{R}_{BG}(t)$, of the reliability function $R(t;\theta,\sigma)$ and $\hat{H}_{BG}(t)$, of the hazard function $H(t;\theta,\sigma)$ are given by

$$\begin{aligned}
\hat{R}(t)_{BG} &= \left[\sum_{j=1}^{k}\int_0^\infty p_j\left(\left[\bar{F}_0(\frac{t}{\sigma_j})\right]^\theta\right)^{-1}\pi^*(\theta|\sigma_j;\boldsymbol{x})d\theta\right]^{-1} \\
&= \left[\sum_{j=1}^{k}p_j\left(1+\frac{\log\bar{F}_0(\frac{t}{\sigma_j})}{(b_j-\sum_{i=1}^m(R_i+1)\log\bar{F}_0(\frac{t}{\sigma_j}))}\right)^{-(m+a_j)}\right]^{-1}. \qquad (28)
\end{aligned}$$

$$\begin{aligned}
\hat{H}(t)_{BG} &= \left[\int_0^\infty\sum_{j=1}^{k}p_j\left[\frac{\theta f_0(t/\sigma_j)}{\sigma_j\bar{F}_0(t/\sigma_j)}\right]^{-1}\pi^*(\theta|\sigma_j;\boldsymbol{x})d\theta\right]^{-1} \\
&= \left[\sum_{j=1}^{k}p_j\frac{\left(b_j-\sum_{i=1}^m(R_i+1)\log\bar{F}_0(\frac{t}{\sigma_j})\right)\sigma_j\bar{F}_0(\frac{t}{\sigma_j})}{f_0(\frac{t}{\sigma_j})(m+a_j-1)}\right]^{-1}. \qquad (29)
\end{aligned}$$

4 Prediction

Balakrishnan and Rao (1997) considered the problem of predicting the lifetime of an item censored in the last step of the progressive censoring procedure. Here, it has to be assumed that, at the termination time $x_{m:m:n}$ of the experiment, $R_m > 1$ observations are left, see also Balakrishnan and Aggarwala (2000). This approach has been extended by Basak et al. (2006) to predict lifetimes of items progressively censored in the lifetime experiment at some stage of the censoring procedure. Based on the progressively Type-II right censored sample $\boldsymbol{X}=(X_1,\ldots,X_m)$ from the proportional hazards family (1), our interest is to find prediction interval for the life-lengths $X_{s:R_i}$ $(s=1,2,...,R_i;\ i=1,2,...,m)$ of all censored units in all m stages of censoring. Here $Y=X_{s:R_i}$ denotes the s-th order statistic out of R_i removed units at stage i $(i=1,2,\ldots,m)$. Let $\boldsymbol{x}=(x_1,\ldots,x_m)$ and $Y=y$ denote the observed value of $\boldsymbol{X}$ and the unobserved value of Y, respectively. The conditional distribution of $Y=X_{s:R_i}$ given $\boldsymbol{X}$ is just the distribution of Y given $X_i=x_i$ due to the well-known Markovian property of progressively Type-II censored ordered statistics. It follows (Balakrishnan and Aggarwala, 2000), that

$$\begin{aligned}
f(y|x_i;\theta,\sigma_j) &= s\binom{R_i}{s}f(y;\theta,\sigma_j)\left[F(y;\theta,\sigma_j)-F(x_i;\theta,\sigma_j)\right]^{s-1}\left[1-F(y;\theta,\sigma_j)\right]^{R_i-s} \\
&\quad\times\left[1-F(x_i;\theta,\sigma_j)\right]^{-R_i}, \qquad y\geq x_i. \qquad (30)
\end{aligned}$$

Substituting (1) and (2), in (30), the conditional probability density function for $y\geq x_i$, is

$$\begin{aligned}
f(y|x_i;\theta,\sigma_j) = s\binom{R_i}{s}\frac{\theta}{\sigma_j}\frac{f_0(y/\sigma_j)}{\bar{F}_0(y/\sigma_j)}\left[(\bar{F}_0(y/\sigma_j))^\theta\right]^{R_i-s+1} \\
\times\left[(\bar{F}_0(x_i/\sigma_j))^\theta-(\bar{F}_0(y/\sigma_j))^\theta\right]^{s-1}\left[(\bar{F}_0(x_i/\sigma_j))^\theta\right]^{-R_i}, y\geq x_i. \qquad (31)
\end{aligned}$$

The Bayes predictive density function of $Y = X_{s:R_i}$ given $X_i = x_i$ is given by

$$f^*(y|x_i) = \int_\theta f(y|x_i, \theta, \sigma_j) \sum_{j=1}^{k} p_j \pi^*(\theta|\sigma_j, x_i) d\theta. \quad (32)$$

Substituting (31) and (15) into (32), the Bayes predictive density function is

$$\begin{aligned} f^*(y|x_i) &= \sum_{j=1}^{k} p_j \binom{R_i}{s} \frac{s}{\sigma_j \Gamma(m+a_j)} \left[\frac{f_0(\frac{y}{\sigma_j})(b_j - \sum_{i=1}^{m}(R_i+1)\log \bar{F}_0(\frac{x_i}{\sigma_j}))^{m+a_j}}{\bar{F}_0(\frac{y}{\sigma_j})} \right] \\ &\times \int_0^\infty \theta^{m+a_j} e^{-\theta(b_j - \sum_{i=1}^{m}(R_i+1)\log \bar{F}_0(x_i/\sigma_j))} (\bar{F}_0(y/\sigma_j))^{\theta(R_i - s_i + 1)} \\ &\times (\bar{F}_0(x_i/\sigma_j))^{-\theta R_i} \left[(\bar{F}_0(x_i/\sigma_j))^\theta - (\bar{F}_0(y/\sigma_j))^\theta \right]^{s-1} d\theta. \quad (33) \end{aligned}$$

Using (33) and the binomial expansion, we have

$$\left[(\bar{F}_0(x_i/\sigma_j))^\theta - (\bar{F}_0(y/\sigma_j))^\theta \right]^{s-1} = \sum_{l=0}^{s-1} \binom{s-1}{l} (-1)^l [\bar{F}_0(y/\sigma_j)]^{\theta l} [\bar{F}_0(x_i/\sigma_j)]^{\theta(s-l-1)}.$$

The Bayes predictive density function $Y = X_{s:R_i}$ given $X_i = x_i$ is given by

$$\begin{aligned} f^*(y|x_i) &= \sum_{j=1}^{k} p_j \binom{R_i}{s} \frac{s}{\sigma_j} \left[\frac{f_0(y/\sigma_j)(m+a_j)}{\bar{F}_0(y/\sigma_j)(b_j - \sum_{i=1}^{m}(R_i+1)\log \bar{F}_0(x_i/\sigma_j))} \right] \\ &\times \sum_{l=0}^{s-1} \binom{s-1}{l} (-1)^l \left[1 - \frac{(R_i - s + l + 1)\log(\frac{\bar{F}_0(y/\sigma_j)}{\bar{F}_0(x_i/\sigma_j)})}{(b_j - \sum_{i=1}^{m}(R_i+1)\log \bar{F}_0(x_i/\sigma_j))} \right]^{-(m+a_j+1)} \\ &. \quad (34) \end{aligned}$$

Now, for constructing a Bayesian prediction interval for $Y = X_{s:R_i}$, we consider the predictive function $P(Y \le \nu|x_i)$, for some positive ν. It follows from (34), that

$$\begin{aligned} P(Y \le \nu|x_i) &= \int_{x_i}^{\nu} f^*(y|x_i) dy \\ &= \sum_{j=1}^{k} p_j s \binom{R_i}{s} \sum_{l=0}^{s-1} \binom{s-1}{l} (-1)^l \frac{1}{R_i - s + l + 1} \\ &\times \left[1 - \left(1 - \frac{(R_i - s + l + 1)\log(\frac{\bar{F}_0(\nu/\sigma_j)}{\bar{F}_0(x_i/\sigma_j)})}{(b_j - \sum_{i=1}^{m}(R_i+1)\log \bar{F}_0(x_i/\sigma_j))} \right)^{-(m+a_j)} \right]. \quad (35) \end{aligned}$$

Hence, the $100(1-\gamma)\%$ prediction interval for $Y = X_{s:R_i}$ is given by $(L(x_i), U(x_i))$, where $L(x_i)$ and $U(x_i)$ are the lower and upper prediction bounds, respectively, satisfying

$$Pr[Y \le L(x_i)|x_i] = \frac{\gamma}{2}, \quad \text{and} \quad P[Y \le U(x_i)|x_i] = 1 - \frac{\gamma}{2}. \quad (36)$$

Iterative numerical methods are required to obtain the lower and upper $100(1-\gamma)\%$ prediction bounds for Y by finding ν from (36), using (35).

5 Numerical computations

In this section, two numerical examples study are presented to illustrate all the estimation and prediction methods described in the preceding sections. We consider the parato distribution with cdf

$$F(x;\theta,\sigma) = 1 - (1 + \frac{x}{\sigma})^{-\theta}, \quad x > 0,\ \theta > 0, \sigma > 0,$$

as a special case from the model 1 with

$$\bar{F}_0(x) = \frac{1}{1+x}, \quad x > 0.$$

To implement the calculations in this section, it is first necessary to elicit the values of (σ_j, η_j) and the hyper parameters (a_j, b_j) in the conjugate prior (14) for $j = 1, 2, \ldots, k$. The hyper parameters (a_j, b_j) can be obtained based on the expected value of the reliability function $R(t)$ conditional on $\sigma = \sigma_j$, which is given using (3) and (14) by

$$\begin{aligned} E_{\theta|\sigma_j}[R(t)|\sigma_j] &= \int_0^\infty (1 + t/\sigma_j)^{-\theta} \ \frac{b_j^{a_j} \theta^{a_j - 1}}{\Gamma(a_j)} e^{(-\theta b_j)} \\ &= (1 + (\log(1 + t/\sigma_j)/b_j))^{-a_j}. \end{aligned} \tag{37}$$

Now, suppose that prior beliefs about the lifetime distribution enable one to specify two values $(R(t_1), t_1)$, $(R(t_2), t_2)$. Thus, for these two prior values $R(t = t_1)$ and $R(t = t_2)$, the values of a_j and b_j for each value σ_j, can be obtained numerically from (37) If there are no prior beliefs, the non parametric procedure

$$\tilde{R}(t_i = X_i) = \frac{m - i + 0.625}{m + 0.25}, \qquad i = 1, 2, \ldots, m,$$

can be used to estimate the reliability function $R(t)$; see Martz and Waller (1982).

Example 5.1. (Simulated data): *The progressive Type-II censored sample used here has been simulated from the Pareto distribution with $\theta = 2$ and $\sigma = 1$. The sample and the corresponding censoring scheme were summarized in Table 1.*

Table 1: Progressively censored sample in Example 1.

i	1	2	3	4	5	6	7	8	9	10
X_i	0.0009	0.0136	0.0390	0.0873	0.24667	0.3467	0.4865	0.5131	0.6658	0.6975
R_i	1	0	1	2	0	0	3	0	1	2

For this example, we have $n = 19$ and $m = 10$. The MLE of θ, and σ, using a New-Raphson method when solving (9) and (10), are obtained as $\hat{\theta} = 1.7944$ and $\hat{\sigma} = 0.9773$. Substituting $\hat{\theta}$ and $\hat{\sigma}$ into (3) and (4), we obtain MLE of the reliability function at $t = 2$ as $\hat{R(2)} = 0.1354$ and $\hat{H(2)} = 0.6027$. To obtain Bayes estimates, it is first necessary to elicit the values of (σ_j, η_j) and the hyper parameters (a_j, b_j) in the conjugate prior (14) for $j = 1, 2, \ldots, k$. Based on observations, we estimate two values of the reliability function as

$$\tilde{R}(t = 0.0873) = \frac{m - i + 0.625}{m + 0.25} = \frac{10 - 4 + 0.625}{10 + 0.25} = 0.646,$$

$$\tilde{R}(t = 0.5131) = \frac{m - i + 0.625}{m + 0.25} = \frac{10 - 8 + 0.625}{10 + 0.25} = 0.256.$$

Since $\sigma = 1$, *we approximate the prior over the interval* $(0.5, 1.4)$ *by the discrete prior with* σ *taking the* 10 *values* 0.5(0.1)1.4, *each with probability* 0.1. *The two prior values obtained in* $\tilde{R}(0.0873) = 0.646$ *and* $\tilde{R}(0.5131) = 0.256$ *are substituted into* (37), *where* a_j *and* b_j *are solved numerically for each given* $\sigma_j, j = 1, 2, \ldots, 10$, *using the Newton-Raphson method.* u_j *and* p_j *are computed using the* (17) *for each* σ_j. *Table 2 gives the values of the hyper parameters and the posterior probabilities derived for each* σ_j. *The MLEs* $(\cdot)_{ML}$, *and the Bayes estimates* $((\cdot)_{BS}, (\cdot)_{BL}, (\cdot)_{BG})$ *for the parameters* θ, σ, *the reliability function* $R(t)$, *and the failure rate function* $H(t)$ *(at* $t = 2$*) are computed using the* (18)-(29), *and are given in Table 3. From Table 3, as anticipated, we note that for c close to* 0, *Bayes estimates relative to LINEX loss are very close to the corresponding estimates under SEL function. This is one of the useful properties of working with the LINEX loss function. Using the prediction procedure described in Section 4, we computed the* 95% *prediction intervals for* $Y = X_{s:R_i}$ $(s = 1, 2, \ldots, R_i;\ i = 1, 2, \ldots, m)$. *The results are presented in Table 4.*

Table 2: Prior information, hyper parameter values and the posterior probabilities for Example 5.1.

j	1	2	3	4	5	6	7	8	9	10
σ_j	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4
η_j	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
a_j	2.8007	1.3777	1.2756	1.2149	1.1697	1.1346	1.0839	1.0649	1.0488	1.0351
b_j	1.1337	0.3651	0.2878	0.2394	0.2045	0.1782	0.1415	0.1282	0.1171	0.1077
u_j	0.0135	0.0234	0.0356	0.0497	0.0652	0.0816	0.0988	0.1163	0.1340	0.1516
p_j	0.1175	0.0965	0.0977	0.0990	0.1001	0.1010	0.0952	0.0964	0.0975	0.0985

Table 3: The MLEs and BEs of $\theta, \sigma, R(t)$, and $H(t)$ where $R(2) = 0.11$ and $H(2) = 0.66$.

	$(\cdot)_{ML}$	$(\cdot)_{BS}$	$(\cdot)_{BL}$					$(\cdot)_{BG}$
			$c=-2$	$c=-0.5$	$c=0.01$	$c=0.5$	$c=2$	
θ	1.7944	1.8829	2.3919	2.0344	1.8802	1.7608	1.5058	1.6155
σ	0.9773	0.9409	1.0229	0.9620	0.9405	0.9198	0.8595	0.8443
$R(t)$	0.1354	0.1426	0.1499	0.1444	0.1426	0.1409	0.1361	0.0929
$H(t)$	0.6027	0.6311	0.6813	0.6426	0.6309	0.6202	0.5907	0.5629

Table 4: The 95% Bayes prediction intervals for $Y = X_{s:R_i}$ in Example 5.1.

$X_{1:R_1}$	$X_{1:R_3}$	$X_{1:R_4}$	$X_{2:R_4}$	$X_{1:R_7}$
(0.0107,2.4936)	(0.0493,2.6005)	(0.0927,2.8891)	(0.1597,3.5301)	(0.4917,3.6499)
$X_{2:R_7}$	$X_{3:R_7}$	$X_{1:R_9}$	$X_{1:R_{10}}$	$X_{2:R_{10}}$
(0.5466,4.0722)	(0.7018,4.3759)	(0.6838,4.4827)	(0.7066,5.0626)	(0.8213,5.4902)

Example 5.2. (real data): *In this example we present a data analysis and illustrate application of the results in Sections 2-4 to the amount of annual rainfall (in inches) during February recorded at Los Angeles Civic Center from 1982 to 2004 (see the website of Los Angeles Almanac: www.laalmanac.com/weather/we08aa.htm). The data are ordered as follows:*

0.00 0.08 0.29 0.56 0.70 1.22 1.30 1.72 1.90 2.84 3.12 3.21
4.13 4.37 4.64 4.89 4.94 5.54 6.10 6.61 7.96 8.87 13.68

Here, we checked the validity of the Pareto distribution based on the parameters $\theta = 0.5$, *and* $\sigma = 0.67$ *using the Kolmogorov-Smirnov (K-S) test. It is observed that the K-S distance is* $K - S = 0.1811$ *with a corresponding* $p-value = 0.4354$. *This indicates that the Pareto distribution provides a good fit to the above data. We have generated a progressively censored sample using the censoring scheme* $R = (3, 0, 1, 2, 0, 0, 3, 0, 1, 3)$ *from the above data with* $n = 23$ *and* $m = 10$. *The censoring scheme and the corresponding progressively censored sample are given in Table 5.*

Table 5: Progressively censored sample in Example 5.2.

i	1	2	3	4	5	6	7	8	9	10
X_i	0.00	0.70	1.22	1.72	3.12	3.21	4.13	4.94	5.54	6.61
R_i	3	0	1	2	0	0	3	0	1	3

The MLE of θ *and* σ *using a New-Raphson method when solving* (9) *and* (10), *are obtained as* $\hat{\theta} = 0.2750$ *and* $\hat{\sigma} = 0.5790$. *Substituting* $\hat{\theta}$ *and* $\hat{\sigma}$ *into* (3) *and* (4), *we obtain MLE of the reliability function as* $\hat{R(t)} = 0.6630$ *and the failure rate function* $\hat{H(t)} = 0.1066$ *at* $t = 2$. *To obtain Bayes estimates, it is first necessary to elicit the values of* (σ_j, η_j) *and the hyper parameters* (a_j, b_j) *in the conjugate prior* (14) *for* $j = 1, 2, \ldots, N$. *These values are derived by the following steps:*

• *Step 1: Based on observations, we estimate two values of the reliability function as*

$$\tilde{R}(t = 0.70) = \frac{m - i + 0.625}{m + 0.25} = \frac{10 - 2 + 0.625}{10 + 0.25} = 0.841,$$
$$\tilde{R}(t = 4.94) = \frac{m - i + 0.625}{m + 0.25} = \frac{10 - 8 + 0.625}{10 + 0.25} = 0.256.$$

• *Step 2: Since the MLE of* σ *is* $\hat{\sigma} = 0.5790$, *we assume that* σ_j *takes the* 10 *values* 0.1(0.1)1, *each with probability* 0.1.

• *Step 3: The two prior values obtained in step 1 are substituted into* (37), *where* a_j *and* b_j *are solved numerically for each given* σ_j, $j = 1, 2, \ldots, 10$ *using the Newton-Raphson method.*

• *Step 4:* u_j *and* p_j *are computed using the* (17) *for each* σ_j.

Table 6 gives the values of the hyper parameters and the posterior probabilities derived for each σ_j. *The MLEs* $(\cdot)_{ML}$, *and the Bayes estimates*$((\cdot)_{BS}, (\cdot)_{BL}, (\cdot)_{BG})$ *of* θ, σ *and* $R(t)$, $H(t)$ *are computed and are given in Table 6. From Table 6, as anticipated, we note that for c close to* 0, *Bayes estimates relative to LINEX loss are very close to the corresponding estimates under SEL function. Using the prediction procedure described in Section 4, we computed the* 95% *prediction intervals for* $Y = X_{s:R_i}$ $(s = 1, 2, \ldots, R_i;\ i = 1, 2, \ldots, m)$. *The results are presented in Table 8.*

6 Conclusion

In this paper, the well-known proportional hazards model which includes several well-known lifetime distributions such as exponential, Pareto, Lomax, Burr type XII, and so on is considered. Based on progressively Type-II censored samples, the maximum likelihood, and Bayes estimators for some lifetime parameters (reliability, and hazard functions), as well as the parameters of the proportional hazards model, are derived.

Table 6: Prior information, hyper parameter values and the posterior probabilities for Example 5.2.

j	1	2	3	4	5	6	7	8	9	10
σ_j	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
η_j	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
a_j	0.6201	0.6584	0.6846	0.7051	0.7221	0.7366	0.7494	0.7607	0.7709	0.7801
b_j	0.4900	0.4691	0.4527	0.4388	0.4263	0.4149	0.4043	0.3944	0.3851	0.3763
u_j	6.0e-01	2.1e-01	5.7e-01	5.4e-00	2.9e-00	1.1e-00	3.3e-00	8.4e-00	1.8e-00	3.7e-00
p_j	0.0057	0.0162	0.0314	0.0509	0.0742	0.1008	0.1302	0.1621	0.1961	0.2319

Table 7: The MLEs and BEs of $\theta, \sigma, R(t)$, and $H(t)$ where $R(2) = 0.5008$ and $H(2) = 0.1872$.

	$(\cdot)_{ML}$	$(\cdot)_{BS}$	$(\cdot)_{BL}$					$(\cdot)_{BG}$
			$c=-2$	$c=-0.5$	$c=0.01$	$c=0.5$	$c=2$	
θ	0.2750	0.3282	0.3412	0.3313	0.3281	0.3251	0.3163	0.2901
σ	0.5790	0.7606	0.8017	0.7719	0.7604	0.7487	0.7092	0.6553
$R(t)$	0.6630	0.6578	0.6646	0.6595	0.6577	0.6560	0.6508	0.6466
$H(t)$	0.1066	0.1182	0.1196	0.1185	0.1182	0.1178	0.1168	0.1065

Table 8: The 95% Bayes prediction intervals for $Y = X_{s:R_i}$ in Example 5.2.

$X_{1:R_1}$	$X_{2:R_1}$	$X_{3:R_1}$	$X_{1:R_3}$
(0.0188,1.7632)	(0.2414,3.2695)	(0.4846,3.7485)	(1.2528,4.2055)
$X_{1:R_4}$	$X_{2:R_4}$	$X_{1:R_7}$	$X_{2:R_7}$
(1.8258,4.3765)	(2.3681,7.7679)	(4.2690,8.8947)	(4.4592,11.4892)
$X_{3:R_7}$	$X_{1:R_9}$	$X_{1:R_{10}}$	$X_{2:R_{10}}$
(4.8630,13.6995)	(6.0931,15.8347)	(6.8197,17.6961)	(8.3702,17.7101)
$X_{3:R_{10}}$			
(12.4075,30.9281)			

The Bayes estimators are obtained using both the symmetric (Squared Error, SE) loss function, and asymmetric (LINEX, and Entropy, E) loss functions. This was done with respect to the conjugate prior for the one shape parameter, and discrete prior for the other parameter of this model. We also provide Bayes prediction intervals for the times to failure of units censored in multiple stages in a progressively censored sample.

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