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# Approximation of stochastic delay differential equations based on Haar functions

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**Abstract:** The main objective of this article is to solve stochastic delay differential equations via Haar wavelets. We present fundamental concepts of stochastic process, Haar, block pulse functions, and their operational matrix relevant to time-delayed Haar. Analytic solutions of two examples are solved for the first time to approximate two kinds of single time-delayed stochastic differential equations with additive and multiplicative noise. This orthogonal basis function not only simplifies the problem but also speeds up the computations and lessens the computational complexity of the stochastic delay differential equations to a lower triangular system of linear algebraic equations. The equation can be solved via forward substitution, such as lower-upper decomposition method. Finally, we examine the order of convergence and error analysis of two visual samples to validate the efficiency and effectiveness of the suggested procedure.

**Keywords:** Computational methods; Ito integral; Stochastic delay equations; Volterra integral equations.
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## 1 Introduction

The entrance of random influences into differential equations (DEs) resulted in two specific equation categories, respectively, for which the sample path of the solution process is differentiable and non-differentiable. The first class is random differential equations and occurs when an ordinary differential equation (ODE) has random coefficient, an initial random value, or a fairly regular stochastic process with a differentiable

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sample trajectory. However, the second class, called stochastic differential equations (SDEs), usually happens when an irregular stochastic process is forcing Gaussian white noise. Their solution inherits the non-differentiability in stochastic integrals of sample paths from the Wiener process Klebaner (2005). Observations of stock prices in finance, diffusing particle locations in microelectronics, and several processes recorded at the moment are formulated mostly through computer simulations like stochastic differential equations or stochastic integro-differential equations.

Alternatively, there are also many processes that require time to mature, react, feed, grow, digest, and so on. The treatment of such equations not only needs the present condition information but also some data regarding the preceding situation Kuang (1993). These problems are called stochastic delay differential equations (SDDEs) and are generated form of SDEs. So a more logical model must include some of the system's past history of the system which neglecting such factor is to ignore reality. Various numerical methods have been dedicated over the past decade to the development of delay differential equations (DDE) approach with time delay Marzban and Razaghi (2006), Hosseini and Marzban (2011), Behroozifar and Yousefi (2013), Tang et al. (2017), Wang et al. (2015), Buckwar (2000), Falbo (2006), Baker et al. (2005). Nouri and Maleknejad (2016) obtained the solution of delay integral equations (DIEs) by BPFs, numerically. Hafashjani et al. (2011) applied the Legendre wavelet method for DDEs. Heydari et al. (2013) discussed a Volterra multiple-delay integral equation with a large domain via operational matrices of Chebyshev cardinal functions. Umer (2016) combined the method of steps with the RBF networks to solve DDEs. Mohammadi (2015) described a method for solving multi-dimensional stochastic *Ito* -Volterra integral equations (SVIEs) relying on the stochastic operational matrix of Haar wavelets. Recently, Kiaee et al. (2022, 2024) presented a new scheme based on theoperational matrix of integration block pulse and triangular function to solve SDDEs. Overall, SDDEs display much more complex mechanics than SDEs since time delay could cause a stable equilibrium to be unstable and cause the populations to vary. Furthermore, any template of species dynamics is at best an approximation without delays. Since analytical solutions to such equation, in many cases are not available, numerical approximation becomes a logical way of tackling this challenge. Beginning in 1991 onwards, various orthogonal functions, like Fourier series, Haar, Block pulse, Walsh functions, orthogonal polynomials, and wavelets have been used to solve these problems.

Hariharan et al. (2013) introduced the concept of wavelets, which has since been expanded into various applications. This theory allows the representation of functions through a combination of step functions and wavelets within specific interval widths. The Haar wavelet transform is an early example of a compact, orthonormal wavelet transform. Haar wavelets consist of pairs of piecewise constant functions and are the simplest mathematically among all wavelet families. One advantageous feature of Haar wavelets is the ability to analytically integrate multiple times. They are particularly effective in handling singularities, as they can be seen as intermediate boundary conditions. A short investigation of these papers can be found in Graps (1995).

In the following, we consider the integral form of single time delay SVIEs with multiplicative noise

$$\begin{cases} u(\upsilon) = u_0(\upsilon) + \int_0^{\upsilon} k_1\,(s,\upsilon)\; f(s,u(s-\tau))\,ds & \upsilon \in [0,\Upsilon)\,, \\ \qquad\qquad + \int_0^{\upsilon} k_2\,(s,\upsilon)\; g(s,u(s-\tau))\,dW(s), & \\ u(\upsilon) = \mu(\upsilon), & \upsilon \in [-\tau,0)\,, \end{cases} \tag{1}$$

where $u(\upsilon)$, $u_0(\upsilon)$, $k_1(s,\upsilon)$,$k_2(s,\upsilon)$ and $1D$ Wiener process named $W(\upsilon)$ for $s,\upsilon \in [0,\Upsilon)$ are the stochastic processes defined on the same probability space with a filtration $F_\upsilon$ satisfying the usual conditions, i.e. The filtration $(F_\upsilon)_{\upsilon\geq 0}$ is right-continuous, $F_\upsilon$, $(\upsilon \geq 0)$ and $F_0$ covering all P-null sets in $F$. $u(\upsilon)$ is unknown and the second integral is in Itô integral sense Klebaner (2005). If $g(s,u(s-\tau))$ in the 2nd integrand is equal to 1, we are studying additive noise instead of multiplicative ones.

In order to investigate the effect of Haar wavelet to approximate single time delay SVIEs, we obtain the analytic solution named method of steps Smith (2011) which used to convert DDEs to ODEs. The main way in approximate the equation, is to convert the integral form of SDDEs to an algebraic form through the use of Haar function's operational matrices of integration.

The overall paper structure would be as follows: Section 2 provides an overview of block pulse and Haar functions and a definition of delay. Section 3 describes the concept of our problem. Section 4 investigates error analysis and rate of convergence. Section 5 is devoted to numerical examples that depict the accuracy of the approach recommended. Section 6 finalizes the result of the paper.

# 2 Fundamental concepts

We study particular classical description of the stochastic calculus, orthogonal basis function like block pulse and Haar functions, and operational matrix of delay Haar function.

## 2.1 Stochastic calculus

In this section we will clarify some of the relevant information respectively Klebaner (2005).

**Definition 2.1.** *Wiener process is a real valued process with independent increments,* $W(0) = 0$ *and* $W(\upsilon + h) - W(h) \sim N(0, h)$.

**Definition 2.2.** *Let* $\{W(\upsilon), \upsilon \geq 0\}$, *be Wiener process with equation of the form,*

$$du(\upsilon) = \mu(u(\upsilon), \upsilon)d\upsilon + \sigma(u(\upsilon), \upsilon)\, dW(\upsilon). \tag{2}$$

*where* $\mu(u(\upsilon), \upsilon)$, $\sigma(u(\upsilon), \upsilon)$ *are given coefficients and* $u(\upsilon)$ *is unknown process, is called a stochastic differential equation driven by Wiener process. They are called diffusion type SDEs where coefficients depend on the whole past of process.*

**Definition 2.3.** *Strong solution of the SDE in* (2) *is called process* $u(\upsilon)$ *if for all* $\upsilon > 0$ *the integrals of coefficients exists, with the second being an Ito integral, and*

$$u(\upsilon) = u_0(\upsilon) + \int_0^{\upsilon} \mu(u(s), s)ds + \int_0^{\upsilon} \sigma(u(s), s)dW(s).$$

*$u(\upsilon)$ is strong solution of some function (functional) of $F(\upsilon,(W(\upsilon), s \leq \upsilon))$ of the given Wiener process $W(\upsilon)$.*

**Theorem 2.4.** *(Existence and Uniqueness) 1. Coefficiens are locally Lipschitz in $x$ uniformly in $\upsilon$ that is, for every $\Upsilon$ and $N$, there is a constant $L$ depending only on $\Upsilon$ and $N$, such that for all $|x|$ , $|y| \leq N$ and all $0 < \upsilon \leq \Upsilon$*

$$|\mu(x,\upsilon) - \mu(y,\upsilon)| + |\sigma(x,\upsilon) - \sigma(y,\upsilon)| \leq L|x-y|.$$

*2. Coefficiens satisfy the linear growth condition*

$$|\mu(x,\upsilon)| + |\sigma(x,\upsilon)| \leq L(1+|x|).$$

*3. $u(0)$ is independent of $(W(\upsilon), 0 \leq \upsilon \leq \Upsilon)$ and $E(u^2(0)) < \infty$. Then there exists a unique strong solution $u(\upsilon)$ of the SDE. $u(\upsilon)$ has continuous paths, moreover*

$$E\left(\sup_{0\leq\upsilon\leq\Upsilon} u^2(\upsilon)\right) < C(1+E(u^2(0))),$$

*where constant depends only on $L$ and $\Upsilon$.*

**Remark 2.5.** *TIf there is a strong solution, it is adapted to the filtration of the given Wiener process, by definition, and it is intuitively clear that it is a path function of $(W(s), s \leq \upsilon)$.*

**Theorem 2.6.** *(Yamada-Watanabe) Suppose that $\mu(\upsilon)$ satisfies the Lipshitz condition and $\sigma(\upsilon)$ satisfies a Holder condition of order $\alpha \geq 0.5$. There is a constant $L$ such that*

$$|\mu(x) - \sigma(y)| \leq L|x-y|^{\alpha}.$$

*Then there is the strong solution and is unique.*

**Lemma 2.7.** *(The Gronwall inequality) Let $\alpha$, $\beta \in [\upsilon_\theta, \Upsilon] \to R$ be integral with*

$$\theta \leq \alpha(\upsilon) \leq \beta(\upsilon) + L(\int_{\upsilon_\theta}^{\upsilon} \alpha(s)\, ds,$$

*for $\upsilon \in [\upsilon_\theta, \Upsilon]$ where $L > 0$. Then*

$$\alpha(\upsilon) \leq \beta(\upsilon) + L(\int_{\upsilon_0}^{\upsilon} e^{L(\upsilon-s)} \beta(s)\, ds), \qquad \upsilon \in [\upsilon_0, \Upsilon].$$

*For more details see Klebaner (2005).*

## 2.2 Block pulse functions

Numerous researchers have discussed BPFs and aimed to solve various problems. In this section we remind description and certain properties of BPFs Behroozifar and Yousefi (2013), Mohammadi (2015), Maleknejad et al. (2012). Consider

$$\Phi(\upsilon) = [\varphi_i(\upsilon)]^T, \qquad i = 1, \dots, m, \tag{3}$$

with $m$ component. Each $\varphi_i(\upsilon)$ is defined as

$$\varphi_i(\upsilon) = \begin{cases} 1, & (i-1)h \le \upsilon < ih, \\ 0, & o.w., \end{cases}$$

where $h = \dfrac{\Upsilon}{m}$.

**Properties.** BPFs has some properties in $\upsilon \in [0, \Upsilon)$ and for $i, j = 1, \ldots, m$ like:
1. Disjointness:

$$\varphi_i(\upsilon) . \varphi_j(\upsilon) = \begin{cases} \varphi_i(\upsilon), & i = j \\ 0, & i \neq j. \end{cases}$$

2. Orthogonality:

$$\int_0^{\Upsilon} \varphi_i(\upsilon).\varphi_j(\upsilon) ds = \begin{cases} h, & i = j, \\ 0, & i \neq j. \end{cases}$$

3. $\Phi^T(\upsilon).\Phi(\upsilon) = 1$, and

$$\Phi(\upsilon).\Phi^T(\upsilon) = \begin{bmatrix} \varphi_1(\upsilon) & 0 & 0 & \cdots & 0 \\ 0 & \varphi_2(\upsilon) & 0 & \cdots & 0 \\ 0 & 0 & \varphi_3(\upsilon) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \varphi_m(\upsilon) \end{bmatrix}_{m \times m}.$$

Every $f(\upsilon)$ real bounded and square integrable function can be expanded with $m$-term block pulse functions

$$f(\upsilon) \simeq \sum_{i=1}^{m} f_i . b_i(\upsilon) = F^{\Upsilon}.\Phi(\upsilon) = \Phi^T(\upsilon).F,$$

where $F = [f_1, f_2, \ldots, f_m]^T$ and can be calculated as

$$f_i = \frac{1}{h} \int_0^{\Upsilon} \varphi_i(\upsilon) . f(\upsilon) \, d\upsilon = \frac{1}{h} \int_{ih}^{(i+1)h} f(\upsilon) \, d\upsilon, \ i = 1, 2, \ldots, m.$$

Each $k(s, \upsilon)$ is 2D-function can be expanded as

$$k(s, \upsilon) = \Phi^T(\upsilon).K.\Phi(\upsilon),$$

where $K_{m \times m}$ is Haar coefficient matrix with $(i, j)\,th$ components calculated as

$$k_{ij} = \frac{1}{h^2} ( \int_0^{\Upsilon_1} \int_0^{\Upsilon_2} k(s, \upsilon) . b_i(\upsilon) . b_j(s) \, d\upsilon \, ds), \quad i, j = 1, 2, \ldots, m.$$

For simpilicity, we let $\Upsilon_1 = \Upsilon_2 = \Upsilon$.

## 2.3 Haar wavelets

In 1930 a physicist Paul Levy studied a kind of random signal titled Brownian motion, who used a scale-varying basis function called Haar function. He found that the study of small, complicated details is greater than the Fourier functions in the Brownian motion. Major benefits of Haar wavelets are simple, orthogonality, compact support, sparse representation and possibility of implementation of fast algorithm in matrix representation. Unfortunately, they are not continuously differentiable which constrains their usages somewhat. Haar wavelet with the lowest computational cost arises when the polynomial degree is zero in the spline wavelet.

Mathematically, Haar wavelets $h_i(\upsilon)$ includes a set of square orthogonal waves represented below Mohammadi (2015), Smith (2011):

$$h_i(\upsilon) = \begin{cases} \sqrt{2^j}, & \frac{k}{m} \leq \upsilon \leq \frac{k+0.5}{m}, \\ -\sqrt{2^j}, & \frac{k+0.5}{m} \leq \upsilon \leq \frac{k+1}{m}, \\ 0, & o.w., \end{cases}$$

each $h_i(\upsilon)$ has compact support in $\left[\frac{k}{2^j}, \frac{k+1}{2^j}\right)$ where

$$\begin{cases} M = 2^J, & j = 0, 1, \ldots, J, \quad (j \geq 0), \\ m = 2^j, & \text{Wavelet level}, \\ k = 0, 1, \ldots, m-1, & \text{transition parameter } \ 0 \leq k < 2^j, \\ m = 2M = 2 \times 2^J = 2^{J+1}, & \\ i = m + k + 1, & m, j, k \in N, \end{cases}$$

and for $i = 1$, we have

$$h_1(\upsilon) = \begin{cases} 1, & 0 \leq \upsilon < 1 \\ 0, & o.w., \end{cases}$$

by the pairwise orthonormality property, we have

$$\int_0^1 h_i(\upsilon) . h_j(\upsilon)\, d\upsilon = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Any $f(\upsilon)$ square integrable function in the interval $[0, \Upsilon)$ can be extended in terms of Haar wavelets series as $f(\upsilon) = \sum_{i=1}^{\infty} f_i . h_i(\upsilon)$, each $f_i$ can be calculated as

$$f_i = 2^j \left( \int_0^{\Upsilon} f(\upsilon) .\, h_i(\upsilon)\, d\upsilon \right) = 2^j \left( \int_{ih}^{(i+1)h} f(\upsilon)\, d\upsilon \right),$$

or can be expressed as vector form,

$$f(\upsilon) \simeq F^T . H(\upsilon) \ = H^T(\upsilon) . F, \tag{4}$$

in which $F$ and $H(\upsilon)$ are Haar coefficient and wavelets vectors respectively as

$$F \ = \ [f_1, \ldots, f_m]^T, \tag{5}$$

$$H(\upsilon) \quad = \quad [h_1(\upsilon), \ldots, h_m(\upsilon)]. \tag{6}$$

Any 2D function $k(s, \upsilon) \in L^2[0, \Upsilon) \times L^2[0, \Upsilon)$ can be expanded with regard to Haar wavelets as

$$k(s, \upsilon). \in H^T(\upsilon).K.H(\upsilon),$$

where $K_{m\times m}$ is Haar wavelets coefficients matrix with $(i, l)-th$ element can be obtained as

$$k_{il} = \int_0^\Upsilon \int_0^\Upsilon k(s, \upsilon).H_i(\upsilon).H_l(s)\,ds, \qquad i, l = 1, 2, \ldots, m.$$

## 2.4 Delay Haar function

Based on Tang et al. (2017), delay Haar function is shift of the m-set of $H(\upsilon)$ by $\tau = (k + \lambda)h$ for a non-negative integer $k = 1, \ldots, m$, $h = \frac{\Upsilon}{m}$ and $0 \le \lambda < 1$. Let

$$H(\upsilon - \tau) = [h_1(\upsilon - \tau) \quad h_2(\upsilon - \tau) \quad \cdots \quad h_i(\upsilon - \tau)]^T, \qquad i = 1, \ldots, m,$$

where $[\ldots]^T$, denotes transpose and each

$$\begin{aligned} h_i(\upsilon - \tau) = h_i(\upsilon - (k + \lambda)h) &= \begin{cases} 1, & (i-1)h \le \upsilon - (k+\lambda)h < ih, \\ 0, & o.w. \end{cases} \\ &= \begin{cases} 1, & (i+k+\lambda-1)h \le \upsilon < (i+k+\lambda)h, \\ 0, & o.w. \end{cases} \\ &= h_{i+k+\lambda}(\upsilon). \end{aligned}$$

Shift of Haar function can be declared as

$$H(\upsilon - \tau) = D.H(\upsilon), \qquad 0 \le \upsilon < \Upsilon, \ \ \upsilon > \tau. \tag{7}$$

Let

$$D = (1-\lambda)H^k + \lambda H^{k+1} = \begin{array}{c} \begin{array}{r} (k+1)\,th - column \\ \downarrow \end{array} \\ \begin{pmatrix} 0 & \cdots & 0 & 1-\lambda & \lambda & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1-\lambda & \lambda & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \ddots & \lambda \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1-\lambda \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{m\times m} \end{array},$$

in which $1 - \lambda$ in the first row is $(k + 1)$th-column component. Here we set $\lambda = 0$.

## 2.5 Relation between the BPFs and Haar wavelets

In this section, we will derive the relationship between the BPF and Haar wavelets. We set $\Upsilon = 1$ in the context of BPFs Mohammadi (2015), Maleknejad et al. (2012).

**Theorem 2.8.** *Let $H(\upsilon)$ and $\Phi(\upsilon)$ be the one dimensional Haar wavelets and BPFs vector respectively, the vector $H(\upsilon)$ can be expanded by BPFs vector $\Phi(\upsilon)$ as*

$$H(\upsilon) = Q.\,\Phi(\upsilon)\,, \tag{8}$$

*where $Q_{il}$ is $(i,l)$th component of $Q_{m\times m}$ matrix.*

$$Q_{il} = 2^{\frac{j}{2}}.h_i.\left(\frac{2l-1}{2m}\right), \qquad 1,2,\ldots,m;\; i = 2,\ldots,m. \tag{9}$$

**Remark 2.9.** *As shown in the matrix defenition $Q$ in (8), it is convenient to consider*

$$Q^{-1} = \frac{1}{m}.Q^T.$$

**Remark 2.10.** *For a m-vector $F$,*

$$H(\upsilon)\,.H^T(\upsilon)\,.F = \tilde{F}.\,H(\upsilon)\,.$$

*In which $\tilde{F}_{m\times m} = Q.\bar{F}.Q^{-1}$ where*

$$\bar{F}_{m\times m} = \begin{bmatrix} \bar{F}_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \bar{F}_m \end{bmatrix}; \qquad \bar{F}_i = diag\big(Q^T.F_i\big)_{m\times 1}.$$

**Remark 2.11.** *Let an arbitrary matrix $F_{m\times m}$, then for Haar wavelets vector $H(\upsilon)$, we have*

$$H^T(\upsilon)\,.F.\,H(\upsilon) = \hat{F}^T.\,H(\upsilon) = H^T(\upsilon).\hat{F},$$

*where $\hat{F}^T_{1\times m}$ and diagonal of any square matrix, is $m\times 1$ vector*

$$\hat{F}^T_{1\times m} = \left(diag\left(Q^T F Q\right)\right)^T.Q^{-1}.$$

## 2.6 Operational matrix of stochastic Haar wavelets integration

We derive the operational matrix for Haar Wavelets in stochastic integration. The main characteristic of operational matrices is that they reduces these stochastic delay Volterra integral equations (SDVIEs) to those of solving a linear system of algebraic equations, thus greatly simplifying the problem and speeds up the computation. We remember some relevant results for block pulse and haar functions (Marzban and Razaghi, 2006; Hosseini and Marzban, 2011; Mohammadi, 2015; Maleknejad et al., 2012).

**Lemma 2.12.** *Let $\Phi(\upsilon)$ be the BPFs vector defined in (3). Then Riemann and Ito integration of this vector can be derived as*

$$\int_0^{\upsilon} \Phi(s)\,ds \simeq P.\,\Phi(\upsilon)\,, \tag{10}$$

$$\int_0^{\upsilon} \Phi(s)\, dW(s) \simeq P_s.\Phi(\upsilon), \tag{11}$$

*where $P, P_s$ are called the deterministic and stochastic operational matrix of integration for BPFs and are given by*

$$P = \frac{h}{2}\begin{bmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 2 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}_{m\times m}, \tag{12}$$

$$P_s = \begin{bmatrix} W(\frac{h}{2}) & W(h) & W(h) & & W(h) \\ 0 & W(3\frac{h}{2}) - W(h) & W(2h) - W(h) & & W(2h) - W(h) \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & & W((2m-1)\frac{h}{2}) - W((m-1)h) \end{bmatrix}_{m\times m}. \tag{13}$$

**Theorem 2.13.** *Suppose that $H(\upsilon)$ is the Haar wavelets vector defined in* (5)*. The Riemann and Ito integral of this vector can be derived as*

$$\int_0^{\upsilon} H(s)\, ds \simeq \frac{1}{m} QPQ^T H(\upsilon) = \Lambda.H(\upsilon), \tag{14}$$

$$\int_0^{\upsilon} H(s)\, dW(s) \simeq \frac{1}{m} Q\, P_s\, {Q_s}^T H(\upsilon) = \Lambda_s.H(\upsilon), \tag{15}$$

*where $Q$ is stated in* (8)*, $\Lambda$, $\Lambda_s$ are the operational matrix of integration for deterministic and stochastic Haar wavelets derived in* (14) *and* (15) *and $P, P_S$ are the operational matrix of integration for deterministic and stochastic BPFs* (12) *and* (13)*.*

# 3 Problem statement

Both normal and human frameworks are impacted by positive and negative feedback. Such mechanisms push a system to a new state of equilibrium or back it to its primary state Kuang (1993). Consider an ordinary delay equation over the time set $R$

$$\begin{cases} u'(\upsilon) = r(\upsilon, s).u(\upsilon - \tau), & \upsilon \in [0, \Upsilon), \\ u(\upsilon) = \mu(\upsilon), & \upsilon \in [-\tau, 0), \end{cases}$$

where $r(\upsilon, s) > 0$ defines growth rate and a positive feedback system. $u(\upsilon - \tau)$ shows the trajectory of the solution in the past. We shall refer to the function $\{\mu(\upsilon);\, \upsilon \in [-\tau, 0)\}$ as an initial data and Banach space of all continuous path from $[-\tau, 0) \to R$ equipped with the supremum norm. Case $\tau = 0$ corresponds to no delay, and recovers the ODEs. SDDEs arise as a randomly perturbed ODDEs coefficients by Gaussian white noise $\xi(\upsilon) = \frac{dW(\upsilon)}{d\upsilon}$ by $b(\upsilon, s)$ noise intensity to $r(\upsilon, s) + b\,\xi_\upsilon$ or $u(\upsilon - \tau) + b\,\xi_\upsilon$ and therefore additive and multiplicative noise occur respectively.

## 3.1 Solving SDVIEs based on Haar functions with additive noise

In what follows, we consider the integral form of a linear SDDE with single time delay and obtain the Haar coefficients of $U(\upsilon)$ in the interval $\upsilon \in [0, \Upsilon]$,

$$\begin{cases} u(\upsilon) = u_0(\upsilon) + \int_0^{\upsilon} k_1(s,\upsilon).\, u(s-\tau)\, ds + \int_0^{\upsilon} k_2(s,\upsilon)\, dW(s), & \upsilon \in [0,\Upsilon), \\ u(0) = u_0, \\ u(\upsilon) = \mu(\upsilon), & -\tau \le \upsilon < 0, \end{cases} \tag{16}$$

where $u_0(\upsilon)$ is a specified constant vector, and $k_1(s,\upsilon), k_2(s,\upsilon)$ are known matrices, $\mu(\upsilon)$ is an arbitrary function known to the initial history. We approximate the $u(\upsilon), u_0(\upsilon), k_1(s,\upsilon)$ and $k_2(s,\upsilon)$ functions by Haar functions as mentioned below,

$$u(\upsilon) \simeq \overline{u}(\upsilon) \approx U^T.H(\upsilon) = H^T(\upsilon).\, U_\upsilon, \quad \tau \le \upsilon < \Upsilon, \tag{17}$$

$$u_0(\upsilon) \simeq U_0^{\Upsilon}.H(\upsilon) = H^T(\upsilon).U_0, \tag{18}$$

$$k_1(s,\upsilon) \simeq H^T(s)\ .K_1.\, H(\upsilon) = H^T(\upsilon).\, K_1^T\, .\, H(s), \tag{19}$$

$$k_2(s,\upsilon) \simeq H^T(s)\ .\, K_2\, .\, H(\upsilon) = H^T(\upsilon)\, .\, K_2^T .\, H(s), \tag{20}$$

$$u(\upsilon-\tau) = \begin{cases} \mu(\upsilon-\tau), & -\tau \le \upsilon-\tau < 0, \\ U^T.H(\upsilon-\tau) = U^T.D.\, H(\upsilon), \\ (H(\upsilon-\tau))^T U = (D.H(\upsilon))^T U = H^T(\upsilon).D^T.U, & 0 \le \upsilon-\tau < \Upsilon-\tau, \end{cases} \tag{21}$$

where the vectors $U, U_0$ and matrices $K_1, K_2$ are Haar functions coefficients of $u(\upsilon)$, $u_0(\upsilon)$, $k_1(s,\upsilon)$, $k_2(s,\upsilon)$, respectivly. With replacement of the above approximation (17)-(21) into (16), we arrived to

$$\begin{aligned} H^T(\upsilon).U \simeq\ & H^T(\upsilon).U_0 + H^T(\upsilon).\, K_1^T.\left(\int_0^{\tau} H(s).\mu(s-\tau)\ ds\right) \\ & + H^T(\upsilon)\, .\, K_1^T\left(\int_{\tau}^{\upsilon} H(s).H^T(s).D^T.U\, ds\right) \\ & + H^T(\upsilon)\, .\, K_2^T\left(\int_0^{\upsilon} H(s)\ dW(s)\right), \end{aligned}$$

also

$$H^T(\upsilon).U \simeq H^T(\upsilon).U_0 + H^T(\upsilon).\, K_1^T.I_1 + H^T(\upsilon)\, .\, K_1^T.I_2 + H^T(\upsilon)\, .\, K_2^T.I_3, \tag{22}$$

where $I_1, I_2$ and $I_3$ are defined as follows,

$$\begin{aligned} I_1 &= \int_0^{\tau} H(s).\mu(s-\tau)\ ds = \int_0^{\tau} H(s)\ ds = \left(\frac{1}{m}.Q.P.Q^T\right).H(\upsilon) = \Lambda.H(\upsilon), \\ I_2 &= \int_{\tau}^{\upsilon} H(s).H^T(s).\left(D^T.U\right) ds = \left(\int_{\tau}^{\upsilon} \tilde{Z}.H(s)\ ds\right) = \tilde{U}.\left(\int_{\tau}^{\upsilon} H(s)\ ds\right) \\ &= \tilde{U}.\Lambda.H(\upsilon), \\ I_3 &= \int_0^{\upsilon} H(s)\ dW(s) \simeq \left(\frac{1}{m}.Q.P_S.Q^T\right).H(\upsilon) = \Lambda_S.H(\upsilon). \end{aligned}$$

$\Lambda, \Lambda_S$ are defined in (14), (15), $D^T, Q, \tilde{U}$ and $\bar{U}$ are matrices. We put $I_1, I_2$ and $I_3$ in (22)

$$\begin{aligned} H^T(\upsilon).U \quad &\simeq \quad H^T(\upsilon).U_0 + H^T(\upsilon).\left(K_1^T.\Lambda\right).H(\upsilon) + H^T(\upsilon).\left(K_1^T.\tilde{U}.\Lambda\right).H(\upsilon) \\ &\quad +H^T(\upsilon).\left(K_2^T.\Lambda_S\right).H(\upsilon). \end{aligned} \tag{23}$$

In (23) we multiply both sides on $H(\upsilon)$ and write it as an algebraic equation as follows,

$$\left(U - \hat{A}_2\right) \simeq \left(U_0 + \hat{A}_1 + \hat{A}_3\right),$$

then by setting $\simeq$ with $=$ in (23), we solve algebric equation and obtain $U_\upsilon$. finally by substituting in (17), Haar function coeficients of $u_\upsilon$ are fulfilled.

## 3.2 Solving SDVIEs based on Haar functions with multiplicative noise

Consider

$$\begin{cases} u(\upsilon) = u_0(\upsilon) + \int_0^\upsilon k_1(s,\upsilon) \,.\, u(s-\tau)\,ds & \tau \in [0, \Upsilon), \\ \qquad\qquad + \int_0^\upsilon k_2(s,\upsilon).\, u(s-\tau)dW(s), & \\ u(0) = u_0, & \\ u(\upsilon) = \mu(\upsilon), & -\tau \le \upsilon < 0. \end{cases} \tag{24}$$

With substituting function's approximation (17)-(21) into (24), we get

$$\begin{aligned} H^T(\upsilon).U \quad &\simeq \quad H^T(\upsilon).U_0 + \left(\int_0^\tau H^T(\upsilon).\, K_1^T \,.\, H(s).\mu(s-\tau)\;ds\right) \\ &\quad + \left(\int_\tau^\upsilon H^T(\upsilon)\,.\,K_1^T.\,H(s).H^T(s-\tau)U\,ds\right) \\ &\quad + \left(\int_0^\tau H^T(\upsilon).\, K_2^T \,.\, H(s).\mu(s-\tau)\;dW(s)\right) \\ &\quad + \left(\int_\tau^\upsilon H^T(\upsilon)\,.\,K_2^T.\,H(s).H^T(s-\tau)U\,dW(s)\right), \\ H^T(\upsilon).U \quad &\simeq \quad H^T(\upsilon).U_0 + H^T(\upsilon).\,K_1^T.G_1 + H^T(\upsilon)\,.\,K_1^T.G_2 + H^T(\upsilon)\,.\,K_2^T.G_3 \\ &\quad +H^T(\upsilon)\,.\,K_2^T.G_4, \end{aligned} \tag{25}$$

we assume that $\{\mu(\upsilon-\tau) = c \,:\, 0 \le \upsilon < \tau, c \in R\}$ and,

$$\begin{aligned} G_1 &= \int_0^\tau \mu(s-\tau)\;H(s)\;\;ds = c\left(\int_0^\tau H(s)\;ds\right) = c.\,\Lambda.H(\upsilon), \\ G_2 &= I_2 = \int_\tau^\upsilon H(s).H^T(s-\tau).\,U\,ds = \int_\tau^\upsilon H(s).H^T(s).\left(D^T.\,U\right)ds = \tilde{U}.\Lambda.H(\upsilon), \\ G_3 &= \int_0^\tau H(s)\;.\,\mu(s-\tau)\;dW(s) = c\left(\int_0^\tau H(s)\;dW(s)\right) = c.\,\Lambda_S.H(\upsilon), \\ G_4 &= \int_\tau^\upsilon H(s).H^T(s-\tau).\,U\,dW(s) = \int_\tau^\upsilon H(s).H^T(s).\left(D^T.\,U\right)dW(s) \end{aligned}$$

$$= \quad \tilde{U}.\Lambda_S.H(\upsilon).$$

We substitute $G_1, G_2, G_3$ and $G_4$ into (25),

$$\begin{aligned} H^T(\upsilon).U \quad &\simeq \quad H^T(\upsilon)U_0 + H^T(\upsilon).\left(K_1^T c\Lambda\right).H(\upsilon) + H^T(\upsilon).\left(K_1^T.\tilde{U}.\Lambda\right).H(\upsilon) \\ &\quad + H^T(\upsilon).(K_2^T.c.\Lambda_S).H(\upsilon) + H^T(\upsilon).\left(K_2^T.\tilde{U}.\Lambda_S\right).H(\upsilon), \end{aligned}$$

By the aid of Remark 2.11, we put

$$\begin{aligned} A &= H^T(\upsilon).\left(K_1^T.\Lambda\right).H(\upsilon) = H^T(\upsilon).\hat{A}_1, \\ B &= H^T(\upsilon).\left(K_1^T.\tilde{U}.\Lambda\right).H(\upsilon) = H^T(\upsilon).\hat{A}_2, \\ C &= H^T(\upsilon).\left(K_2^T.\Lambda_S\right).H(\upsilon) = H^T(\upsilon).\hat{A}_3, \\ D &= H^T(\upsilon).\left(K_2^T.\tilde{U}.\Lambda_S\right).H(\upsilon) = H^T(\upsilon).\hat{A}_4, \end{aligned}$$

then,

$$H^T(\upsilon).U \simeq H^T(\upsilon).U_0 + H^T(\upsilon).\hat{A}_1 + H^T(\upsilon).\hat{A}_2 + H^T(\upsilon).\hat{A}_3 + H^T(\upsilon).\hat{A}_4,$$

where

$$\begin{aligned} \hat{A}_1 &= \left[\left(\left\{diag\left(\left(Q^T.K_1^T.\Lambda.Q\right)\right)\right\}\right)^T.Q^{-1}\right]^T, \\ \hat{A}_2 &= \left[\left(\left\{diag\left(\left(Q^T.K_1^T.\tilde{U}.\Lambda.Q\right)\right)\right\}\right)^T.Q^{-1}\right]^T, \\ \hat{A}_3 &= \left[\left(\left\{diag\left(\left(Q^T.K_2^T.\Lambda_s.Q\right)\right)\right\}\right)^T.Q^{-1}\right]^T, \\ \hat{A}_4 &= \left[\left(\left\{diag\left(\left(Q^T.K_2^T.\tilde{U}.\Lambda_s.Q\right)\right)\right\}\right)^T.Q^{-1}\right]^T. \end{aligned}$$

By multiplying $H(\upsilon)$ and rearranging the equation, we have

$$\left(U - \hat{A}_2 - \hat{A}_4\right) \simeq \left(U_0 + \hat{A}_1 + \hat{A}_3\right), \tag{26}$$

$U(\upsilon)$ is obtained in (26). In both additive and multiplicative noise, we get the $u(\upsilon)$ by relation (17) and the solution is completed.

# 4 Error analysis and rate of convergence

This part is allocated to the convergence rate proof. The result show a good level of accuracy of order. We utilize two theorems of 4.1 and 4.2 bellow and assert theorems from 4.3 and 4.4. It is essential that approximation of trajectories or sample paths be near enough to their Ito process when filtering or testing estimators. Among this part, the convergence and error analysis of the presented scheme for solving SDVIEs will be discussed. We utilize the first two theorems Mohammadi (2016).

**Theorem 4.1.** *Assume the square integrable function $u_0(x)$ in the interval $L^2\,[0,1)$, which is an arbitrary function with bounded first derivative, and $e_m\,(\upsilon) = u_0(\upsilon) - \overline{u}_0(x)_m$ $x \in I = [0,1)$ such that $\overline{u}_0(x)_m = \sum_{i=1}^{m} (u_0)_i.\, h_i{}^{(m)}(x)$, where $(u_0)_i$ is the Haar function series of $u_0(x)$. Then,*

$$\|e_m(\upsilon)\|_2 \leq \frac{M}{\sqrt{3}m}.$$

*That means the Haar wavelets series will be convergent.*

**Theorem 4.2.** *Suppose that $f\,(s,\upsilon) \in L^2\,([0,1)\,,[0,1))$ is a function with bounded partial derivative, $\left|\frac{\partial^2 f}{\partial s \partial \upsilon}\right| \leq M$ and $e_m\,(s,\upsilon) = f(s,\upsilon) - \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} f_{ij} h_i(s) h_j(\upsilon)$, then*

$$\|e_m(s,\upsilon)\|_2 \leq \frac{M}{3m^2}.$$

*By definition of error*

$$\|e_m(s,\upsilon)\|_2^2 = \int_0^1 \left(\sum_{i=m}^{\infty}\sum_{l=m}^{\infty} f_{il} h_i(s) h_i(\upsilon)\right)^2 d\upsilon = \sum_{i=m}^{m-1}\sum_{l=m}^{\infty} f_{il}^2.$$

**Theorem 4.3.** *Assume the square integrable function $u(\upsilon-\tau)$ in the interval $L^2\,[0,1)$, which is an arbitrary function with bounded first derivative, and $e_m\,(\upsilon-\tau) = u\,(\upsilon-\tau) - \overline{u}(\upsilon-\tau)_m$ , $(\upsilon-\tau) \in I = [0,1)$, such that $\overline{u}(\upsilon-\tau)_m = \sum_{i=1}^{m} (u)_i.\, h_i{}^{(m)}\,(\upsilon-\tau)$, where $(u)_i$ is the Haar function series of $u(\upsilon-\tau)$. Then,*

$$\|e_m(\upsilon-\tau)\|_2 \leq \frac{M}{\sqrt{3}m}.$$

*This means that the Haar wavelets series will be convergent.*

*Proof.* Let $e_m\,(\upsilon-\tau) = u\,(\upsilon-\tau) - \sum_{i=0}^{m-1} u_i.\, h_i(\upsilon-\tau) = \sum_{i=m}^{\infty} u_i.\, h_i(\upsilon-\tau)$, when,

$$\begin{aligned}
\|e_m\,(\upsilon-\tau)\|_2^2 &= \int_0^1 |e_i\,(\upsilon-\tau)|^2 d\upsilon = \int_0^1 \left(\sum_{i=m}^{\infty} u_i.h_i(\upsilon-\tau)\right)^2 d\upsilon \\
&= \int_0^1 \sum_{i=m}^{\infty} (u_i)^2.\,(h_i(\upsilon-\tau))^2 d\upsilon = \sum_{i=m}^{\infty} (u_i)^2 \left(\int_0^1 (h_i(\upsilon-\tau))^2 d\upsilon\right) \\
&= \sum_{i=m}^{\infty} (u_i)^2, \\
u_i &= \int_0^1 h_i(\upsilon-\tau)\, u(\upsilon)\, d\upsilon \\
&= \left(\int_{\frac{k}{2^j}}^{\frac{k+0.5}{2^j}} \sqrt{2^j}\;\, u(\upsilon)\, d\upsilon + \int_{\frac{k+0.5}{2^j}}^{\frac{k+1}{2^j}} (-\sqrt{2^j})\;\, u(\upsilon)\, d\upsilon\right).
\end{aligned}$$

There exist $\eta_{j1}$, $\eta_{j2}$ in $\left(\frac{k}{2^j}, \frac{k+0.5}{2^j}\right)$ and $\left(\frac{k+0.5}{2^j}, \frac{k+1}{2^j}\right)$, respectively.

$$
\begin{aligned}
u_i &= 2^{\frac{j}{2}} \left( \int_{\frac{k}{2^j}}^{\frac{k+0.5}{2^j}} u(\eta_{j1})\, d\upsilon - \int_{\frac{k+0.5}{2^j}}^{\frac{k+1}{2^j}} u(\eta_{j2})\, d\upsilon \right) \\
&= 2^{\frac{j}{2}} \left( u\left(\eta_{j1}\right) \ \frac{k+0.5-k}{2^j} \ - \ u\left(\eta_{j2}\right) \ \frac{k+1-k-0.5}{2^j} \right) \\
&= 2^{-\frac{j}{2}-1} \left(u\left(\eta_{j1}\right) \ - \ u\left(\eta_{j2}\right)\right).
\end{aligned}
$$

Furthermore there exist

$$
\exists\, \eta_j \quad \eta_{j1} < \ \eta_j < \eta_{j2} \quad ; \quad u_i = 2^{-\frac{j}{2}-1} \left(\eta_{j1} - \eta_{j2}\right) \ u'\left(\eta_j\right)
$$

this result,

$$
\begin{aligned}
\left\| e_m \left(\upsilon - \tau\right) \right\|_2^2 &= \sum_{i=m}^{\infty} \left(u_i\right)^2 = \sum_{i=m}^{\infty} \left( 2^{-\frac{j}{2}-1} \left(\eta_{j1} - \eta_{j2}\right) \ u'\left(\eta_j\right) \right)^2 \\
&= \sum_{i=m}^{\infty} 2^{-j-2} \left(\eta_{j1} - \eta_{j2}\right)^2 \ \left(u'\left(\eta_j\right)\right)^{\,2} = \sum_{i=m}^{\infty} 2^{-j-2} \left(2^{-j}\right)^2 M^{\,2} \\
&= \frac{M^2}{4} \sum_{j=J}^{\infty} \sum_{k=0}^{2^{j-1}} 2^{-3j} = \frac{M^2}{4} \sum_{j=J}^{\infty} 2^{-3j} \left( \sum_{k=0}^{2^j-1} 1 \right) \\
&= \frac{M^2}{4} \sum_{j=J}^{\infty} 2^{-2j} = \frac{M^2}{3} 2^{-2J},
\end{aligned}
$$

since $m = 2^j$, we have $\left\| e_m \left(\upsilon - \tau\right) \right\|_2 \leq \frac{M}{\sqrt{3}m}$. ☐

**Theorem 4.4.** *Let $u\left(\nu\right)$ and $\bar{u}\left(\nu\right)$ be exact and approximated solutions respectively which are calculated in* (1) *and* (17)*, $\left\| u\left(\nu - \tau\right) \right\| < C$ and $\left\| K_i \right\| < C$ , $i = 1, 2$. Then*

$$
E\left( \left\| u\left(\nu\right) - \bar{u}\left(\nu\right) \right\|^2 \right) \leq O\left(h\right) \quad \nu \in [0, 1).
$$

*Proof.* Let

$$
\begin{aligned}
u(\nu) - \bar{u}(\nu) &= u_0(\nu) - \bar{u}_0(\nu) + \int_0^{\nu} k_1\left(s, \nu\right) u(\nu - \tau) - \bar{k}_1\left(s, \nu\right) \bar{u}\left(\nu - \tau\right) \, ds \\
&\quad + \int_0^{\nu} k_2\left(s, \nu\right) u(\nu - \tau) - \bar{k}_2\left(s, \nu\right) \bar{u}\left(\nu - \tau\right) \, dW(s),
\end{aligned}
$$

based on Iso Ito property and Fubini theorem,

$$
\begin{aligned}
E \left| \int_0^{\nu} k_2\left(s, \nu\right) u(\nu - \tau) - \bar{k}_2\left(s, \nu\right) \bar{u}(\nu - \tau) dW\left(s\right)) \right|^2 &= \int_0^{\nu} E | k_2\left(s, \nu\right) \ u(\nu - \tau) \\
&\quad - \bar{k}_2\left(s, \nu\right) \bar{u}(\nu - \tau) |^2 ds,
\end{aligned}
$$

then,

$$\begin{aligned}E\|u(\nu)-\bar{u}(\nu)\|^2 \quad &\leq \quad 3\Big(E\left\|u_0(\nu)-\bar{u}_0(\nu)\right\|^2 \\ &\quad +E\left\|\int_0^{\nu} k_1\left(s,\nu\right)u(\nu-\tau)-\bar{k}_1\left(s,\nu\right)\bar{u}(\nu-\tau)ds\right\|^2 \\ &\quad +E\int_0^{\nu}\left\|k_2\left(s,\nu\right)u(\nu-\tau)-\bar{k}_2\left(s,\nu\right)\bar{u}(\nu-\tau)\right\|^2 ds\Big),\end{aligned}$$

where

$$\begin{aligned}\left\|k_i\left(s,\nu\right)\ u(\nu-\tau)-\bar{k}_i\left(s,\nu\right)\bar{u}(\nu-\tau)\right\|^2 \quad &= \quad 2\|k_i\left(s,\nu\right)\left(u(\nu-\tau)-\bar{u}(\nu-\tau)\right)\|^2 \\ &\quad +2\left\|\left(k_i\left(s,\nu\right)-\bar{k}_i\left(s,\nu\right)\right)\bar{u}(\nu-\tau)\right\|^2 \\ &\leq \quad C.\|u(\nu-\tau)-\bar{u}(\nu-\tau)\|^2 \\ &\quad +C.\left\|k_i\left(s,\nu\right)-\bar{k}_i\left(s,\nu\right)\right\|^2, \qquad i=1,2.\end{aligned}$$

Furthermore,

$$\begin{aligned}&\left\|k_i\left(s,\nu\right)-\bar{k}_i\left(s,\nu\right)\right\|^2 = O(h^2), \quad i=1,2. \\ &\|z(\nu-\tau)-\bar{z}(\nu-\tau)\|^2 = O(h^2),\end{aligned}$$

hence,

$$\begin{aligned}E\|u(\nu)-\bar{u}(\nu)\|^2 \quad &\leq \quad CO(h^2), \\ E\|u(\nu)-\bar{u}(\nu)\| \quad &= \quad O(h).\end{aligned}$$

□

# 5 Illustrative examples

In this section, we solve the integral form of two types of linear SDDEs using Haar basis function analytically and numerically for the first time. The related computations were performed using Matlab (2015$a$). Let
$u_i$: Haar function coefficient of analytic solution.
$\bar{u}_i$: Haar function coefficient of approximated method.
$e$: Absolute error computed by $\|e\|_{\infty} = \max_{1\leq i\leq m}|u_i-\bar{u}_i|$.
$n$: Number of iterations.
$\bar{u}_e$: Mean absolute error.
$S_e$: Standard deviation of error.
$UB$: Upper bound.
$LB$: Lower bound.
The numerical results of mean and standard deviation with 95 percent confidence interval for some different values of $\upsilon_i$ in the points $h=\frac{1}{m}, m=2^k$ are shown in tables. We compare the results $\bar{u}_e, S_e, UB$ and $LB$ iterations with analytical solution respectively. Time delayed equations are mostly solved analytically in a step wise procedure called the method of steps with given initial condition (Mohammadi, 2016).

**Example 5.1.** *Consider a linear stochastic Volterra integral equation with additive noise input and constant time delay*

$$\begin{cases} u(\upsilon) = u(0) + \int_0^{\upsilon} 4u(s-0.25)\,ds + 4b\int_0^{\upsilon} dW(s), & \upsilon \in [0,1)\,, \\ u(\upsilon) = 1, & \upsilon \in [-0.25, 0)\,, \\ u(0) = 1, \end{cases}$$

*where $u(\upsilon)$ is an unknown stochastic processes on the probability space $(\Omega, F, P)$, and $W(\upsilon)$ is a Wiener process. The analytical solution is as bellow:*

$$u(\upsilon) = \begin{cases} 1 + 4bW(\upsilon), & 0 \le \upsilon < \frac{1}{4}, \\ 1 + 4(\upsilon - \frac{1}{4}) + 4b\left(4\left(\int_{\frac{1}{4}}^{\upsilon} W(s-\frac{1}{4})ds\right) + W(\upsilon)\right), & \frac{1}{4} \le \upsilon < \frac{2}{4}, \\ 1 + 4\left(\upsilon - \frac{1}{4}\right) + 8(\upsilon - \frac{1}{2})^2 & \frac{2}{4} \le \upsilon < \frac{3}{4}, \\ \quad +4b\left[16(\upsilon - \frac{1}{2}) + \left(\int_{\frac{1}{4}}^{\upsilon-\frac{1}{4}} W(s-\frac{1}{4})ds\right)\right. & \\ \quad \left. +4\left(\int_{\frac{1}{2}}^{\upsilon} W(u-\frac{1}{4})du\right) + 4\left(\int_{\frac{1}{4}}^{\frac{1}{2}} W(s-\frac{1}{4})ds\right) + W(\upsilon)\right], & \\ 1 + 4\left(\upsilon - \frac{1}{4}\right) + 8\left(\upsilon - \frac{1}{2}\right)^2 + \frac{32}{3}(\upsilon - \frac{3}{4})^3 & \frac{3}{4} \le \upsilon < 1, \\ \quad +4b\left[\begin{array}{l} \frac{4^3}{2}(\upsilon - \frac{3}{4})\left(\int_{\frac{1}{4}}^{\upsilon-\frac{1}{2}} W(s-\frac{1}{4})ds\right) \\ 4\left(\int_{\frac{3}{4}}^{\upsilon} W(\upsilon-\frac{1}{4})d\upsilon\right) + 8(1 + 2(\upsilon - \frac{3}{4}))\left(\int_{\frac{1}{4}}^{\frac{2}{4}} W(s-\frac{1}{4})ds\right) \\ +4^2(\upsilon - \frac{3}{4})\left(\int_{\frac{1}{2}}^{\upsilon-\frac{1}{4}} W(u-\frac{1}{4})du\right) + W(\upsilon) \\ +4\left(\int_{\frac{1}{2}}^{\frac{3}{4}} W(u-\frac{1}{4})du\right) \end{array}\right]. & \end{cases}$$

Table 1: Approximation of SDDE via Haar functions with (additive noise), $k = 1$, $\sigma = 0.99$, $b = 0.95$, $m = 4$.

| $\upsilon_i$ | $\bar{u}_e$ | $S_e$ | $LB$ | $UB$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0.25 | -0.0010967 | 0.03535 | -0.00103 | -0.0011633 |
| 0.5 | -0.00043463 | 0.01401 | -0.00040821 | -0.00046104 |
| 0.75 | 0.00070197 | 0.022627 | 0.0006593 | 0.00074463 |
| 1 | 0.0039387 | 0.12696 | 0.0036993 | 0.0041781 |

**Example 5.2.** *Consider a linear stochastic delay Volterra integral equation with multiplicative noise input and intensity of $b$ as follows*

$$\begin{cases} u(\upsilon) = u(0) + \int_0^{\upsilon} 4, u(s-0.25)\,ds + \int_0^{\upsilon} b\,u(s-0.25)\,dW(s), & \upsilon \in [0,1)\,, \\ u(\upsilon) = 0, & \upsilon \in [-0.25, 0)\,, \\ u\,(0) = 1. \end{cases}$$

*The analytical solution is obtained as follows:*

$$u(\upsilon)=\begin{cases}1, & 0\le \upsilon<\frac{1}{4},\\ 1+4(\upsilon-\frac{1}{4})+b.\left(W(\upsilon)-W(\frac{1}{4})\right), & \frac{1}{4}\le \upsilon<\frac{2}{4},\\ 1+4\left(\upsilon-\frac{1}{4}\right)+8(\upsilon-\frac{1}{2})^2+4b\left[\left(\int_{\frac{1}{2}}^{\upsilon}W(s-\frac{1}{4})ds\right)\right. & \frac{2}{4}\le t<\frac{3}{4},\\ \quad -\left(\int_{\frac{1}{2}}^{\upsilon}W(s)\,ds\right)+\upsilon.\left.\left(W(\upsilon)-W(\frac{1}{4})\right)\right] & \\ \quad +b.\left[b.\left(\int_{\frac{1}{2}}^{\upsilon}W(s-\frac{1}{4})dW(s)\right)-(1+b.W(\frac{1}{4}))W(\upsilon)\right. & \\ \quad \left.+W(\frac{1}{4})(1+b.W(\frac{1}{2}))\right], & \\ 1+4\left(\upsilon-\frac{1}{4}\right)+8\left(\upsilon-\frac{1}{2}\right)^2+\frac{32}{3}(\upsilon-\frac{3}{4})^3 & \frac{3}{4}\le \upsilon<1,\\ \quad +4b.\left[\left(3+b.W(\frac{1}{4})\right)\left(\int_{\frac{3}{4}}^{\upsilon}W(u)du\right)\right. & \\ \quad -\left(b\left(W(\upsilon)-W(\frac{3}{4})\right)+4(\upsilon-\frac{3}{4})\right)\left(\int_{\frac{1}{2}}^{\upsilon-\frac{1}{4}}W(s)ds\right) & \\ \quad \times\left(\int_{\frac{1}{2}}^{\frac{3}{4}}W(s)\,ds\right)+\left(3+b.W(\frac{1}{4})\right)\left(\int_{\frac{3}{4}}^{\upsilon}W(u)\,du\right)-\left(\int_{\frac{1}{2}}^{\frac{3}{4}}W(s)\,ds\right) & \\ \quad -(2+b.W(\frac{1}{4}))\left(\int_{\frac{3}{4}}^{\upsilon}W(u-\frac{1}{4})du\right)+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}W(s-\frac{1}{4})ds\right) & \\ \quad +2\upsilon^2\left(W(\upsilon)-W(\frac{1}{4})\right)+\left(b.\left(W(\upsilon)-W(\frac{3}{4})\right)\right. & \\ \quad \left.+4(\upsilon-\frac{3}{4})\right)\left(\int_{\frac{1}{2}}^{\upsilon-\frac{1}{4}}W(s-\frac{1}{4})ds\right)+b.\left(\int_{\frac{3}{4}}^{\upsilon}uW(u-\frac{1}{4})dW(u)\right) & \\ \quad -\frac{7}{8}W(\frac{1}{4})+\upsilon\left(\left(2+bW(\frac{1}{2})\right)W(\frac{1}{4})-\left(2+bW(\frac{1}{4})\right)W(\upsilon)\right) & \\ \quad \left.-\left(b\left(W(\upsilon)-W(\frac{3}{4})\right)+4(\upsilon-\frac{3}{4})\right)\left(\int_{\frac{1}{2}}^{\upsilon-\frac{1}{4}}W(s)ds\right)\right] & \\ \quad +b^2.\left(\left(b.(\left(W(\upsilon)-W(\frac{3}{4})\right)+4(\upsilon-\frac{3}{4})\right)\left(\int_{\frac{1}{2}}^{\upsilon-\frac{1}{4}}W(s-\frac{1}{4})dW(s)\right)\right. & \\ \quad +\left(\int_{\frac{1}{2}}^{\frac{3}{4}}W(s-\frac{1}{4})dW(s)\right)-(2+b.W(\frac{1}{4}))\left(\int_{\frac{3}{4}}^{\upsilon}W(u-\frac{1}{4})dW(u)\right) & \\ \quad +\left(2W(\frac{1}{4})+b.W(\frac{1}{4}).W(\frac{1}{2})+\frac{7}{2}b\right)W(\upsilon) & \\ \quad \left.-W(\frac{1}{2}).W(\frac{1}{4}).\left(2+b.W(\frac{3}{4})\right)\right). & \end{cases}$$

Table 2: Approximation of SDDEs via Haar functions (multiplicative noise), $k=1$, $\sigma=0.99$, $b=0.95$, $m=4$.

| $\upsilon_i$ | $\bar{u}_e$ | $S_e$ | $LB$ | $UB$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0.25 | 0.00088243 | 0.028444 | 0.0008288 | 0.00093607 |
| 0.5 | -0.0020305 | 0.065452 | -0.0019071 | -0.002154 |
| 0.75 | 0.0010416 | 0.033574 | 0.00097827 | 0.0011049 |
| 1 | 0.00025052 | 0.0080751 | 0.00023529 | 0.00026574 |

# 6 Discussion and conclusions

A numerical method based on the Haar wavelets and their stochastic operational matrix was proposed for solving the integral form of linear SDDEs with constant time delay. The main characteristic of orthogonal basis functions like Haar wavelets is to reduce the computational burden of SDDE to a linear triangular lower algebraic equation which simplifies and speeds up the computations as well. The algorithm is simple and

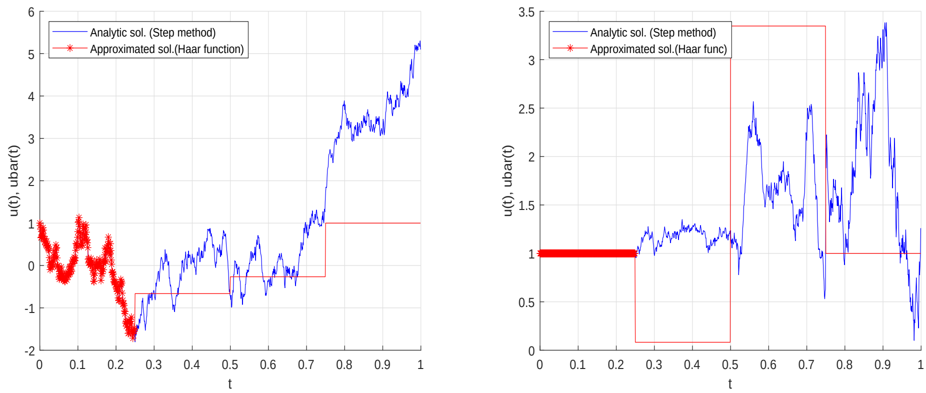


Figure 1: SDHDEs, additive noise (left) and multiplicative noise (right) with $k = 1, m = 4$ .

clear to use and can be implemented easily. The convergence and error analysis were investigated.

Samples illustrate the profitability and accuracy of the method with a $O(h)$ rate of convergence. Moreover, the running times of the algorithm were in a reasonable range and the mean absolute errors quantitatively confirmed that the method is convergent. As a result,the implementation of the method was quite general, without limitation. Therefore, it can be used for numerically solving an extensive variety of linear SDDEs.

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