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*Research Paper*

# Classical and Bayesian estimation of the reliability function for the inverse Lindley distribution based on lower record statistics

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**Abstract:** The reliability function, or the survival function at a specified time $t$, denotes the proportion of products that remain operational beyond time $t$ and continue to function. This interpretation underscores the pivotal role of the survival function and its estimation in understanding lifetime phenomena. This paper explores the estimation of the survival function for the inverse Lindley distribution based on lower records. The estimation techniques encompass maximum likelihood and bootstrap methods. Furthermore, Bayesian approaches employing Metropolis-Hastings and importance sampling algorithms are employed. In addition to deriving approximate confidence intervals using the delta method and percentile bootstrap intervals for the survival function, Chen and Shao's shortest width credible intervals are also determined. A comprehensive simulation study is presented to assess the effectiveness of both point and interval estimators. Finally, an application of the results is given to a real data set.

**Keywords:** Delta method; Importance sampling technique; Inverse Lindley distribution; Metropolis-Hastings algorithm; Survival function.
**Mathematics Subject Classification (2010):** 62F10, 62F15, 65C05.

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# 1 Introduction

In science of statistics, the record value or record statistic is the lowest or largest value that can be obtained from a sequence of random variables, which are usually independent and identically distributed. Let $\{X_1, X_2, \ldots\}$ be a sequance of independent and identically distributed random variables with the cumulative distribution function (CDF) $F(x;\theta)$ and probability density function (PDF) $f(x;\theta)$, where $\theta$ is an unknown parameter or parameter vector. Then, $X_j$ is a lower record extracted from this sequence if its value is less than all previous observations, in other words, $X_j$ is a lower record value if for all $i < j$, we have $X_j < X_i$. In this paper, for $m \geq 1$, the sequence $\{X_{L(m)}, m \geq 1\}$ denotes the sequance of lower record values, where

$$L(1) = 1, \quad L(m) = \min\{j | j > L(m-1), X_j < X_{L(m-1)}\},$$

and the sequnce $\{L(m), m \geq 1\}$ denotes the sequence of record times.

The joint PDF of $X_{L(1)}, \ldots, X_{L(m)}$ is given by (Arnold et al., 1998)

$$f_{X_{L(1)},\ldots,X_{L(m)}}(x_1, \ldots, x_m) = f(x_m) \prod_{i=1}^{m-1} \frac{f(x_i)}{F(x_i)}, \quad -\infty < x_m < \cdots < x_1 < \infty.$$

Besides, the marginal PDF of $m$-th lower record for $m \geq 1$ is given by

$$f_{X_{L(m)}}(x) = \frac{1}{\Gamma(m)} \big[ -\log F(x) \big]^{m-1} f(x), \quad -\infty < x < \infty.$$

Record data are important in many life areas such as industry, mining, meteorology, seismology, economics, medicine and lifetime tests. The theory of record values and its distributional properties have been studied extensively in the literature and many authors have accomplished valuable researches in this field, see Arnold et al. (1998) and the references therein for more details regarding record values.

The reliability can be interpreted as the capability of a system in doing an action under specific practical and environmental conditions in a certain period of time. Let the continuous random variable $T$, be the lifetime of a unit with PDF $f(t;\theta)$ and CDF $F(t;\theta)$. Then, the reliability or survival function for $t > 0$ is defined as $S(t) = P(T \geq t)$. In fact, the survival function can be interpreted as the percentage of products that survive (remain sound) and are active after a specified time $t$. The problem of estimation of a survival function have absorbed the atrraction of many statisticians due to its great importance. Especially, the problem of survival function estimation based on record data has been developed by Soliman and Al-Aboud (2008) and MirMostafaee et al. (2016) for the Rayleigh and Topp-Leone distributions, respectively.

The inverse Lindley distribution (ILD) has been recently introduced by Sharma et al. (2015) as a lifetime model. Ley $Y$ follow a Lindley distribution with parameter $\theta$ and $X = \frac{1}{Y}$. Then, $X$ is said to possess the inverse Lindley distribution with parameter $\theta$ and we write $X \sim ILD(\theta)$. The PDF and CDF of $X$ are given, respectively, by

$$\begin{aligned} f(x;\theta) &= \frac{\theta^2}{\theta+1}\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}, \quad x > 0, \quad \theta > 0, \qquad (1) \\ F(x;\theta) &= \left(1 + \frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}, \quad x > 0, \quad \theta > 0, \end{aligned}$$

The PDF in (1) can be written as the combination of the PDF of the inverse exponential distribution with parameter $\theta$ and the PDF of the inverse gamma with parameters 2 and $\theta$. The survival function of the inverse Lindley distribution with parameter $\theta$ is given by

$$S_t \equiv S(t) = 1 - \left(1 + \frac{\theta}{(\theta+1)t}\right)\mathrm{e}^{-\frac{\theta}{t}}, \qquad t > 0.$$

In the ensuing sections, we discuss the maximum likelihood (ML) estimation of the survival function at point $t$. We also obtain approximate confidence intervals for $S(t)$ with the help of the approximate properties of ML estimators and the delta method. Then, we work on the bootstrap estimation of $S(t)$. The Bayesian estimation of the survival function is also considered by using importance sampling and the Metropolis-Hastings algorithm. A simulation study is presented to evaluate the proposed estimators in the paper. We also present an application of the outcomes of the paper to a real data set example. Finally, we state several remarks.

# 2 ML estimation

Let $X_{L(1)}, \ldots, X_{L(m)}$ be the first $m$ lower record values from $ILD(\theta)$ and $\boldsymbol{x} = (x_1, \ldots, x_m)$ be the corresponding observed set of $(X_{L(1)}, \ldots, X_{L(m)})$. Then, the likelihood function of $\theta$ given the first $m$ lower records is given by

$$\begin{aligned} L^*(\theta \mid \boldsymbol{x}) &= \frac{\theta^2}{1+\theta}\left(\frac{1+x_m}{x_m^3}\right)\mathrm{e}^{-\frac{\theta}{x_m}}\prod_{i=1}^{m-1}\frac{\frac{\theta^2}{1+\theta}(\frac{1+x_i}{x_i^3})\mathrm{e}^{-\frac{\theta}{x_i}}}{(1+\frac{\theta}{(1+\theta)x_i})\mathrm{e}^{-\frac{\theta}{x_i}}} \\ &= \frac{\theta^{2m}}{1+\theta}\cdot\frac{\mathrm{e}^{-\frac{\theta}{x_m}}}{x_m}\prod_{i=1}^{m}\frac{(1+x_i)}{x_i^2}\prod_{i=1}^{m-1}\left(\theta\left(1+x_i\right)+x_i\right)^{-1}. \end{aligned} \tag{2}$$

The log-likelihood function is then given by

$$\ell(\theta) = \log L^*(\theta|\boldsymbol{x}) = 2m\log\theta - \log(1+\theta) - \frac{\theta}{x_m} - \sum_{i=1}^{m-1}\log\left(\theta\left(1+x_i\right)+x_i\right) + A(\boldsymbol{x}), \tag{3}$$

where $A(\boldsymbol{x}) = \sum_{i=1}^m \log(1+x_i) - \log x_m - 2\sum_{i=1}^m \log x_i$. Therefore, the ML estimate of $\theta$ will be obtained through maximizing (3) with respect to $\theta$. Upon taking the derivative of (3) with respect to $\theta$ and equating the result with zero, we have

$$\frac{\partial\ell(\theta)}{\partial\theta} = \frac{2m}{\theta} - \frac{1}{1+\theta} - \frac{1}{x_m} - \sum_{i=1}^{m-1}\frac{1+x_i}{\theta(1+x_i)+x_i} = 0.$$

Let $\hat{\theta}$ denote the ML estimator of $\theta$. Then, using the invariance property of ML estimators, the ML estimator of $S_t$ is obtained to be

$$\hat{S}_t = 1 - \left(1 + \frac{\hat{\theta}}{(\hat{\theta}+1)t}\right)\mathrm{e}^{-\frac{\hat{\theta}}{t}}, \qquad t > 0.$$

Following Kumar et al. (2024), and Etemad Golestani et al. (2024), we can show that under some regularity conditions stated in Lehmann and Casella (1998), the approximate distribution of $\hat{\theta}$ is normal with the mean $\theta$ and variance $[I_{X_{L(1)},\ldots,X_{L(m)}}(\theta)]^{-1}$, where $I_{X_{L(1)},\ldots,X_{L(m)}}(\theta)$ is the expected Fisher information of the record sample about $\theta$, namely

$$(\hat{\theta}-\theta) \overset{AppD}{\sim} N(0,[I_{X_{L(1)},\ldots,X_{L(m)}}(\theta)]^{-1}),$$

where $\overset{AppD}{\sim}$ denotes "approximately distributed as", and

$$I_{X_{L(1)},\ldots,X_{L(m)}}(\theta) = -E\left(\frac{\partial^2 \log f_{X_{L(1)},\ldots,X_{L(m)}}(\boldsymbol{X})}{\partial\theta^2}\right),$$

in which $\boldsymbol{X} = (X_{L(1)},\ldots,X_{L(m)})$, provided that the above expectation exists.

Now, we obtain the approximate distribution of $S_t$ with the help of delta method. See Shao (2003), Corollary 1.1. on page 61, for more details. Upon taking the derivative of $S_t$ with respect to $\theta$, we get

$$\eta(\theta) = \frac{\partial S_t}{\partial\theta} = \frac{\theta(t\theta+2t+\theta+1)}{t^2(1+\theta)^2}\,\mathrm{e}^{-\frac{\theta}{t}}.$$

So $(\hat{S}_t - S_t)$ approximately has a $N\big(0,(\eta(\theta))^2[I_{X_{L(1)},\ldots,X_{L(m)}}(\theta)]^{-1}\big)$ distribution, See- Saini (2024) and TanıŞ et al. (2023) for similar approaches.

From (3), we obtain

$$\frac{\partial^2\ell(\theta)}{\partial\theta^2} = \frac{-2m}{\theta^2} + \frac{1}{(1+\theta)^2} + \sum_{i=1}^{m-1}\left(\frac{1+x_i}{\theta\,(1+x_i)+x_i}\right)^2.$$

Thus, an estimator for $I_{X_{L(1)},\ldots,X_{L(m)}}(\theta)$ is given by

$$\hat{I}_{X_{L(1)},\ldots,X_{L(m)}}(\theta) = \frac{2m}{\hat{\theta}^2} - \frac{1}{(1+\hat{\theta})^2} - \sum_{i=1}^{m-1}\left(\frac{1+X_{L(i)}}{\hat{\theta}\left(1+X_{L(i)}\right)+X_{L(i)}}\right)^2.$$

Therefore, an approximate estimator for the variance of $\hat{S}_t$ becomes

$$\hat{V}ar(\hat{S}_t) = (\eta(\hat{\theta}))^2[\hat{I}_{X_{L(1)},\ldots,X_{L(m)}}(\theta)]^{-1}.$$

A two-sided equi-tailed approximate $(1-\gamma)\%$ confidence interval for $S_t$ based on lower records is given by

$$\hat{S}_t \pm z_{\gamma/2}\sqrt{\hat{V}ar(\hat{S}_t)}, \tag{4}$$

where $z_\eta$ is the $\eta$-th upper quantile of the standard normal distribution.

The lower and upper bounds of the confidence interval (4) may not belong to $(0,1)$, thus we apply the delta method once more and use the tranformation $g(S_t) = \log\left(\frac{S_t}{1-S_t}\right)$. The approximate distribution of $g(\hat{S}_t)$ is given by

$$\big(g(\hat{S}_t)-g(S_t)\big) \overset{AppD}{\sim} N\big(0,\frac{Var(\hat{S}_t)}{S_t^2(1-S_t)^2}\big).$$

Thus, an approximate estimator for the variance of $g(\hat{S}_t)$ is obtained to be

$$\hat{V}ar(g(\hat{S}_t)) = \frac{(\eta(\hat{\theta}))^2}{\hat{S}_t^{\,2}(1-\hat{S}_t)^2 \hat{I}_{X_{L(1)},\ldots,X_{L(m)}}(\theta)}.$$

So a two-sided equi-tailed approximate $(1-\gamma)\%$ confidence interval for $g(S_t)$ is given by

$$g(\hat{S}_t) \pm z_{\gamma/2}\sqrt{\hat{V}ar(g(\hat{S}_t))} \equiv (L_g, U_g).$$

Finally, an approximate $(1-\gamma)\%$ confidence interval for $S_t$ based on lower records is given by

$$\left(\frac{\mathrm{e}^{L_g}}{1+\mathrm{e}^{L_g}}, \frac{\mathrm{e}^{U_g}}{1+\mathrm{e}^{U_g}}\right).$$

# 3 Parametric bootstrap estimation

In this section, we propose the bootstrap estimation of $S_t$, see Efron and Tibshirani (1993) for the details regarding bootstrap estimating. We use the following algorithm.
Step 1. Calculate the ML estimate of $\theta$ based on the observed lower records, denoted by $\hat{\theta}$.
Step 2. Generate a bootstrap sample from $ILD(\hat{\theta})$, denoted by $X^*_{L(1)}, \ldots, X^*_{L(m)}$.
Step 3. Calculate the ML estimate of $S_t$ based on the bootstrap sample, denoted by $\hat{S}^*_{t1}$.
Step 4. Repeat Steps 2 and 3, $B$ times to obtain the bootstrap sample $\{\hat{S}^*_{t1}, \ldots, \hat{S}^*_{tB}\}$. Then, sort the bootstrap sample in order of magnitude. The sorted bootstrap sample is denoted by $\{\hat{S}^*_{t(1)}, \ldots, \hat{S}^*_{t(B)}\}$.

A sensible estimate for $S_t$ based on the generated bootstrap sample is given by $\hat{S}^{Boot}_t = \frac{1}{B}\sum_{i=1}^{B} \hat{S}^*_{t(i)}$. Besides, the $(1-\gamma)\%$ percentile bootstrap confidence interval for $S_t$ is given by

$$\left(\hat{S}^*_{t(\tau_1)}, \hat{S}^*_{t(\tau_2)}\right),$$

where $\tau_1 = (B+1)\gamma/2$ and $\tau_2 = (B+1)(1-\gamma/2)$.

# 4 Bayesian estimation

In this section, we focus on employing Bayesian techniques to estimate the survival function $S_t$ within the framework of the lower record values from the ILD. Within the context of Bayesian estimation, the parameter is treated as a random variable with a prior distribution denoted by $\pi(\theta)$. Let us assume $\theta$ follows a gamma prior distribution with the PDF given by

$$\pi(\theta) = \frac{b^a}{\Gamma(a)}\theta^{a-1}\mathrm{e}^{-b\theta}, \qquad a>0, b>0, \theta>0. \tag{5}$$

Let $\boldsymbol{X}$ represent the informative sample, and $L\big(S_t, \delta(\boldsymbol{X})\big)$ denote the loss function, where $\delta(\boldsymbol{X})$ is an estimator of $S_t$. The Bayes estimator of $S_t$ is derived by minimizing the posterior risk $E[L(S_t, \delta(\boldsymbol{X}))|\boldsymbol{X}]$ with respect to $\delta$.

In the existing literature, one commonly employed loss function is the squared error (SE) function. The Bayes estimator of $S_t$ under the squared error loss function is denoted as $\delta_{SE}(\boldsymbol{X}) = E(S_t|\boldsymbol{X})$, provided that this expectation exists and is finite.

However, in practical scenarios, the implications of overestimation and underestimation may not be symmetric, rendering the use of symmetric loss functions inappropriate. To address this, Varian (1975) proposed an asymmetric loss function known as the linear-exponential (LE or linex) loss function. The Bayes estimator of $S_t$ under this loss function is expressed as

$$\delta_{LE}(\boldsymbol{X}) = \frac{-1}{c} \log E(\mathrm{e}^{-cS_t}|\boldsymbol{X}), \quad c \neq 0,$$

provided that the above expectation exists and is finite.

Another asymmetric loss function, proposed by Calabria and Pulcini (1994), is the general entropy ($GE$) loss function. The Bayes estimator of $S_t$ under the GE loss function is given by

$$\delta_{GE}(\boldsymbol{X}) = [E(S_t^{-p}|\boldsymbol{X})]^{-\frac{1}{p}}, \quad p \neq 0,$$

provided that the above expectation exists and is finite.

From the likelihood function in (2) and the prior distribution in (5), the posterior pdf can be obtained as follows

$$\begin{aligned} \pi(\theta|\boldsymbol{x}) &= \frac{L(\theta|\boldsymbol{x})\pi(\theta)}{\int_0^\infty L(\theta|\boldsymbol{x})\pi(\theta)\mathrm{d}\theta} \\ &\propto \theta^{2m+a-1}\mathrm{e}^{-\theta(\frac{1}{x_m}+b)}\Big\{(1+\theta)\prod_{i=1}^{m-1}\big(\theta(1+x_i)+x_i\big)\Big\}^{-1}. \end{aligned} \tag{6}$$

The posterior distribution lacks a closed form, necessitating the use of approximate Bayes methods to estimate $S_t$. To this end, Metropolis-Hastings (MH) and importance sampling (IS) methods have been utilized.

## 4.1 Metropolis-Hastings method

In this subsection, we use the Metropolis-Hastings (MH) algorithm to approximate the Bayes estimates of the survival function of the ILD based on lower record values. Using a similar approach proposed by Dey and Pradhan (2014), we write the MH algorithm steps as follows:

**Algorithm 4.1.**

*Step 1: Set an initial value $\theta^{(0)}$, we propose to consider the ML estimate of $\theta$ as the initial value.*

*Step 2: For $j = 1, \ldots, N'$, repeat the following steps:*
*Set $\theta = \theta^{(j-1)}$.*
*Generate a new candidate parameter value $\delta$ from $N\Big(\log(\theta), \frac{\hat{I}^{-1}_{X_{L(1)},\ldots,X_{L(m)}}(\theta)\big|_{\theta=\theta^{(0)}}}{[\theta^{(0)}]^2}\Big)$, where $\hat{I}^{-1}_{X_{L(1)},\ldots,X_{L(m)}}(\theta)$ is the inverse of the observed Fisher information. Set $\theta' = \exp(\delta)$.*

*Calculate* $P = \min\left\{1, \frac{\pi(\theta'|\boldsymbol{x})q(\theta|\theta')}{\pi(\theta|\boldsymbol{x})q(\theta'|\theta)}\right\}$, *where* $q(x|b)$ *is the pdf of the log-normal distribution with parameters* $\log(b)$ *and* $\frac{\hat{I}^{-1}_{X_{L(1)},\ldots,X_{L(m)}}(\theta)\big|_{\theta=\theta^{(0)}}}{[\theta^{(0)}]^2}$.

*Update* $\theta^{(j)} = \theta'$ *with probability* $P$*, otherwise set* $\theta^{(j)} = \theta$.

*Step 3: Discard the first* $k$ *generated data, where* $k$ *is the burn-in period.*

*Step 4: Compute* $\{S_{t,l}, l = 1, \ldots M\}$ *based on* $\{\theta_l,\ l = 1, \ldots, M\}$*, where* $M = N' - k$.

The approximate Bayes estimates of $S_t$ under the SE, LE, and GE loss functions are given by $\tilde{S_t}^{SE} = \frac{1}{M}\sum_{l=1}^{M} S_{t,l}$, $\tilde{S_t}^{LE} = -\frac{1}{c}\log\left(\frac{1}{M}\sum_{l=1}^{M} e^{-cS_{t,l}}\right)$, and $\tilde{S_t}^{GE} = \left(\frac{1}{M}\sum_{l=1}^{M} S_{t,l}^{-p}\right)^{-\frac{1}{p}}$, respectively.

In order to obtain the Chen and Shao's shortest width credible intervals (CSSW CrI) for $S_t$, based on generated samples using the MH technique, we shall use a similar procedure described in Chen and Shao (1999) (we do not write the details for the sake of brevity).

## 4.2 Importance sampling method

To implement the importance sampling (IS) procedure, we rewrite the posterior pdf (6) as follows:

$$\pi(\theta|\boldsymbol{x}) = C(\boldsymbol{x})\text{gamma}\big(\theta; 2m + a, \frac{1}{x_m} + b\big)h(\theta),$$

where $\text{gamma}(\theta; 2m + a, \frac{1}{x_m} + b)$ is the pdf of the gamma distribution with shape and rate parameters $2m + a$ and $\frac{1}{x_m} + b$, respectively, $C(\boldsymbol{x}) = \frac{m(\boldsymbol{x})\Gamma(2m+a)}{\left(x_m^{-1}+b\right)^{2m+a}}$, $m(\boldsymbol{x})$ is the normalizing constant of the posterior distribution, and $h(\theta) = \left\{(1+\theta)\prod_{i=1}^{m-1}\left(\theta(1+x_i)+x_i\right)\right\}^{-1}$. Now, consider the following algorithm.

**Algorithm 4.2.**

*Step 1: Generate* $\theta$ *from the gamma distribution with shape and rate parameters respectively as* $2m + a$ *and* $\frac{1}{x_m} + b$.

*Step 2: Repeat Step 1,* $N$ *times to obtain the importance sample* $\theta_1, \theta_2, \ldots, \theta_N$.

*Step 3: Compute* $S_t$ *for each* $\theta$ *in Step 2, to obtain* $S_{t,i}$ *for* $i = 1, \ldots N$.

The approximate Bayes estimate of $S_t$ under the $SE$ loss function is given by $\hat{S_t}^{SE} = \sum_{i=1}^{N} S_{t,i}w_i$, where $w_i = \frac{h(\theta_i)}{\sum_{i=1}^{N} h(\theta_i)}$. Additionally, the approximate Bayes estimates of $S_t$ under the $LE$ and $GE$ loss functions are $\hat{S_t}^{LE} = -\frac{1}{c}\log\left(\sum_{i=1}^{N} \mathrm{e}_i^{-cS_{t,i}}w_i\right)$ and $\hat{S_t}^{GE} = \left(\sum_{i=1}^{N} {S_{t,i}}^{-p}w_i\right)^{-\frac{1}{p}}$, respectively.

We can also find the CSSW CrI for $S_t$ based on the set $\{S_{t,1}, \ldots, S_{t,N}\}$ generated by an IS method. In this regard, we shall use a similar procedure described in Chen and Shao (1999) (we do not write the details for the sake of brevity).

# 5 A simulation study and conclusions

In this section, we present a simulation study to assess both point and interval estimators. The simulation consists of $M = 1000$ iterations, and $B = 99$ replications for the bootstrap estimation. We consider parameter values $\theta = 0.5, 1, 2$ and numbers of lower records $m = 3, 4, 5$. Additionally, we evaluate different values of $t$, denoted as $t_1, t_2$, and $t_3$, where $t_i$ represents the $i$-th quartile of the distribution. Thus, $S(t_1) = 0.75, S(t_2) = 0.5$ and $S(t_3) = 0.25$.

For Bayesian estimation, two gamma priors are applied: Prior 1 with parameters $(a_1, b_1) = (0.2, 1.5)$ and Prior 2 with parameters $(a_2, b_2) = (3, 1)$. Furthermore, we consider $c = -0.5, 0.5$ for the $LE$ loss function, and $p = -0.5, 0.5$ for the $GE$ loss function. The simulation study results are based on $N = 1000$ iterations.

In this simulation, point estimates are obtained using the ML, bootstrap and (approximate) Bayesian methods, while the 95% interval estimates are derived using approximate, percentile bootstrap and Bayesian methods. The classical point estimators are evaluated based on two criteria: (i) the estimated mean squared error (EMSE) and (ii) the estimated bias (EB). Bayesian point estimators are assessed using two criteria: (i) the estimated risk and (ii) the estimated bias (EB). Moreover, the 95% interval estimators are evaluated based on two criteria: (i) the average length (AL) and (ii) coverage probability (CP).

The numerical results of the simulation for the point estimation are given in Tables 1, 3, and 4 and the results for the interval estimation are presented in Tables 2, and 5. In Tables 3 and 4, BS denotes the Bayes estimator under the SE loss function, BLc1 and BLc2 denote the Bayes estimators under the LE loss function with $c = -0.5$ and $c = +0.5$, respectively, and BGEp1 and BGEp2 denote the Bayes estimators under the GE loss function with $p = -0.5$ and $p = +0.5$, respectively. We note the estimated risk of each Bayes estimator is calculated according to its own loss function. From Table 1, we see that the point bootstrap estimators outperform the ML estimators in the sense of EMSE for $t_1$ and $t_2$, while the ML estimators are superior to the point bootstrap estimators for $t_3$ in terms of EMSE. Besides, we observe that the ML estimators have less EBs than the point bootstrap estimators. The EMSE criterion is decreasing with respect to the number of records, $m$.

From Table 2, we see that for $t_2$ and $t_3$, the bootstrap confidence intervals have larger ALs in comparison with the approximate confidence intervals (except for the one case). For $t_1$, the approximate intervals have larger ALs than the bootstrap confidence intervals. The approximate confidence intervals have larger CPs than the bootstrap ones. The AL criterion is decreasing with respect to the number of records, $m$. From Table 3, we see that prior selection played a significant role in the performance of Bayesian methods. Prior 1 tended to underestimate $S_t$, while Prior 2 tended to overestimate it in the most cases. Moreover, the estimated biases are decreasing with respect to $\theta$ except for the one case. From Table 4, we observe that the MH method generally outperformed the IS method in terms of estimated risk for Prior 1 in the most cases, while the opposite was observed for Prior 2. Table 5 reveals that the CSSW CrIs based on the MH method provide smaller ALs than the ones based on the IS method for $t_1$ in the most cases, whereas the reverse is true for $t_2$ and $t_3$. Prior 2 leads to not large CPs when $\theta$ equals 0.5. In conclusion, the choice of method, prior, and parameters significantly influenced the performance of the simulation results. Both classical and

Bayesian methods showed strengths and weaknesses across different scenarios, highlighting the importance of careful method selection in statistical analysis

Table 1: EMSEs and EBs of the ML and bootstrap estimators.

| $\theta$ | $m$ | | ML | | | Bootstrap | | |
|---|---|---|---|---|---|---|---|---|
| | | | $t_1$ | $t_2$ | $t_3$ | $t_1$ | $t_2$ | $t_3$ |
| 0.5 | 3 | EMSE | 0.0317 | 0.0459 | 0.0352 | 0.0234 | 0.0408 | 0.0428 |
| | | EB | 0.0041 | 0.0484 | 0.0623 | 0.0112 | 0.0843 | 0.1160 |
| | 4 | EMSE | 0.0273 | 0.0369 | 0.0248 | 0.0211 | 0.0335 | 0.0303 |
| | | EB | −0.0005 | 0.0350 | 0.0450 | 0.0054 | 0.0665 | 0.0894 |
| | 5 | EMSE | 0.0218 | 0.0302 | 0.0194 | 0.0174 | 0.0282 | 0.0238 |
| | | EB | 0.0050 | 0.0342 | 0.0399 | 0.0092 | 0.0617 | 0.0776 |
| 1 | 3 | EMSE | 0.0315 | 0.0476 | 0.0389 | 0.0237 | 0.0444 | 0.0478 |
| | | EB | 0.0185 | 0.0653 | 0.0753 | 0.0258 | 0.1022 | 0.1299 |
| | 4 | EMSE | 0.0265 | 0.0386 | 0.0271 | 0.0207 | 0.0361 | 0.0334 |
| | | EB | 0.0104 | 0.0481 | 0.0547 | 0.0167 | 0.0797 | 0.0985 |
| | 5 | EMSE | 0.0216 | 0.0278 | 0.0167 | 0.0175 | 0.0265 | 0.0209 |
| | | EB | 0.0009 | 0.0279 | 0.0330 | 0.0060 | 0.0564 | 0.0699 |
| 2 | 3 | EMSE | 0.0292 | 0.0451 | 0.0338 | 0.0222 | 0.0427 | 0.0436 |
| | | EB | 0.0171 | 0.0616 | 0.0682 | 0.0276 | 0.1018 | 0.1247 |
| | 4 | EMSE | 0.0234 | 0.0336 | 0.0220 | 0.0185 | 0.0326 | 0.0284 |
| | | EB | 0.0139 | 0.0471 | 0.0491 | 0.0214 | 0.0804 | 0.0930 |
| | 5 | EMSE | 0.0217 | 0.0292 | 0.0174 | 0.0175 | 0.0280 | 0.0218 |
| | | EB | 0.0037 | 0.0323 | 0.0359 | 0.0111 | 0.0617 | 0.0721 |

Table 2: ALs and CPs for the 95% approximate and percentile bootstrap confidence intervals.

| $\theta$ | $m$ | | Approximate | | | Bootstrap | | |
|---|---|---|---|---|---|---|---|---|
| | | | $t_1$ | $t_2$ | $t_3$ | $t_1$ | $t_2$ | $t_3$ |
| 0.5 | 3 | AL | 0.7034 | 0.6833 | 0.5582 | 0.5690 | 0.6844 | 0.6513 |
| | | CP | 0.980 | 0.987 | 0.964 | 0.913 | 0.913 | 0.913 |
| | 4 | AL | 0.6356 | 0.6300 | 0.5018 | 0.5455 | 0.6482 | 0.5807 |
| | | CP | 0.977 | 0.983 | 0.966 | 0.939 | 0.939 | 0.939 |
| | 5 | AL | 0.5894 | 0.5944 | 0.4663 | 0.5202 | 0.6158 | 0.5309 |
| | | CP | 0.985 | 0.982 | 0.964 | 0.943 | 0.943 | 0.943 |
| 1 | 3 | AL | 0.6990 | 0.6765 | 0.5485 | 0.5441 | 0.6682 | 0.6459 |
| | | CP | 0.971 | 0.983 | 0.943 | 0.897 | 0.897 | 0.897 |
| | 4 | AL | 0.6247 | 0.6184 | 0.4885 | 0.5280 | 0.6344 | 0.5695 |
| | | CP | 0.979 | 0.985 | 0.953 | 0.907 | 0.907 | 0.907 |
| | 5 | AL | 0.5822 | 0.5802 | 0.4423 | 0.5202 | 0.6101 | 0.5115 |
| | | CP | 0.972 | 0.975 | 0.958 | 0.944 | 0.944 | 0.944 |
| 2 | 3 | AL | 0.6934 | 0.6629 | 0.5260 | 0.5393 | 0.6652 | 0.6380 |
| | | CP | 0.989 | 0.994 | 0.961 | 0.881 | 0.881 | 0.881 |
| | 4 | AL | 0.6164 | 0.6057 | 0.4669 | 0.5180 | 0.6260 | 0.5545 |
| | | CP | 0.990 | 0.989 | 0.960 | 0.912 | 0.912 | 0.912 |
| | 5 | AL | 0.5656 | 0.5598 | 0.4201 | 0.5016 | 0.5902 | 0.4938 |
| | | CP | 0.978 | 0.972 | 0.955 | 0.922 | 0.922 | 0.922 |

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Table 3: EBs of the Bayes estimators.

| | | | | BS | | | BLc1 | | | BLc2 | | | BGEp1 | | | BGEp2 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\theta$ | $m$ | | | $t_1$ | $t_2$ | $t_3$ | $t_1$ | $t_2$ | $t_3$ | $t_1$ | $t_2$ | $t_3$ | $t_1$ | $t_2$ | $t_3$ | $t_1$ | $t_2$ | $t_3$ |
| 0.5 | 3 | IS | Prior 1 | -0.088 | -0.034 | 0.009 | -0.080 | -0.025 | 0.014 | -0.097 | -0.042 | 0.004 | -0.105 | -0.054 | -0.009 | -0.147 | -0.104 | -0.049 |
| | | | Prior 2 | 0.095 | 0.179 | 0.187 | 0.099 | 0.184 | 0.192 | 0.092 | 0.173 | 0.182 | 0.090 | 0.168 | 0.173 | 0.076 | 0.143 | 0.143 |
| | | MH | Prior 1 | -0.027 | 0.031 | 0.058 | -0.019 | 0.040 | 0.064 | -0.036 | 0.022 | 0.052 | -0.042 | 0.010 | 0.038 | -0.080 | -0.040 | -0.007 |
| | | | Prior 2 | 0.137 | 0.240 | 0.250 | 0.140 | 0.245 | 0.255 | 0.134 | 0.235 | 0.244 | 0.133 | 0.231 | 0.237 | 0.124 | 0.211 | 0.209 |
| | 4 | IS | Prior 1 | -0.077 | -0.030 | 0.007 | -0.070 | -0.023 | 0.011 | -0.084 | -0.036 | 0.003 | -0.090 | -0.046 | -0.008 | -0.121 | -0.085 | -0.039 |
| | | | Prior 2 | 0.080 | 0.149 | 0.151 | 0.083 | 0.155 | 0.156 | 0.076 | 0.144 | 0.147 | 0.074 | 0.139 | 0.139 | 0.062 | 0.117 | 0.113 |
| | | MH | Prior 1 | -0.021 | 0.030 | 0.052 | -0.015 | 0.037 | 0.056 | -0.028 | 0.022 | 0.047 | -0.033 | 0.013 | 0.036 | -0.060 | -0.025 | 0.001 |
| | | | Prior 2 | 0.119 | 0.205 | 0.204 | 0.122 | 0.209 | 0.209 | 0.116 | 0.200 | 0.200 | 0.115 | 0.196 | 0.193 | 0.106 | 0.177 | 0.168 |
| | 5 | IS | Prior 1 | -0.060 | -0.020 | 0.010 | -0.054 | -0.014 | 0.013 | -0.067 | -0.026 | 0.006 | -0.071 | -0.034 | -0.003 | -0.095 | -0.065 | -0.029 |
| | | | Prior 2 | 0.077 | 0.137 | 0.134 | 0.080 | 0.142 | 0.138 | 0.073 | 0.132 | 0.130 | 0.072 | 0.128 | 0.124 | 0.061 | 0.109 | 0.101 |
| | | MH | Prior 1 | -0.012 | 0.032 | 0.048 | -0.007 | 0.039 | 0.052 | -0.018 | 0.026 | 0.045 | -0.022 | 0.018 | 0.035 | -0.044 | -0.013 | 0.006 |
| | | | Prior 2 | 0.112 | 0.186 | 0.179 | 0.114 | 0.191 | 0.183 | 0.109 | 0.182 | 0.175 | 0.108 | 0.178 | 0.169 | 0.099 | 0.161 | 0.147 |
| 1 | 3 | IS | Prior 1 | -0.130 | -0.080 | -0.029 | -0.121 | -0.072 | -0.026 | -0.139 | -0.087 | -0.033 | -0.148 | -0.100 | -0.045 | -0.192 | -0.147 | -0.079 |
| | | | Prior 2 | 0.079 | 0.150 | 0.149 | 0.083 | 0.155 | 0.153 | 0.075 | 0.144 | 0.144 | 0.073 | 0.139 | 0.136 | 0.059 | 0.114 | 0.108 |
| | | MH | Prior 1 | -0.084 | -0.037 | 0.000 | -0.075 | -0.028 | 0.005 | -0.094 | -0.045 | -0.004 | -0.102 | -0.058 | -0.018 | -0.146 | -0.110 | -0.057 |
| | | | Prior 2 | 0.120 | 0.206 | 0.201 | 0.123 | 0.211 | 0.206 | 0.117 | 0.201 | 0.196 | 0.116 | 0.197 | 0.189 | 0.105 | 0.176 | 0.162 |
| | 4 | IS | Prior 1 | -0.110 | -0.067 | -0.023 | -0.103 | -0.060 | -0.020 | -0.117 | -0.073 | -0.026 | -0.124 | -0.083 | -0.036 | -0.156 | -0.120 | -0.063 |
| | | | Prior 2 | 0.066 | 0.125 | 0.121 | 0.069 | 0.130 | 0.125 | 0.062 | 0.120 | 0.117 | 0.060 | 0.115 | 0.111 | 0.047 | 0.093 | 0.088 |
| | | MH | Prior 1 | -0.064 | -0.022 | 0.007 | -0.057 | -0.015 | 0.011 | -0.071 | -0.029 | 0.004 | -0.077 | -0.039 | -0.007 | -0.108 | -0.773 | -0.038 |
| | | | Prior 2 | 0.104 | 0.176 | 0.166 | 0.107 | 0.180 | 0.170 | 0.101 | 0.171 | 0.162 | 0.099 | 0.167 | 0.156 | 0.090 | 0.148 | 0.133 |
| | 5 | IS | Prior 1 | -0.101 | -0.064 | -0.025 | -0.095 | -0.059 | -0.023 | -0.107 | -0.070 | -0.027 | -0.112 | -0.078 | -0.035 | -0.138 | -0.107 | -0.058 |
| | | | Prior 2 | 0.052 | 0.100 | 0.095 | 0.056 | 0.105 | 0.098 | 0.049 | 0.096 | 0.092 | 0.047 | 0.091 | 0.086 | 0.035 | 0.072 | 0.066 |
| | | MH | Prior 1 | -0.058 | -0.022 | 0.004 | -0.052 | -0.016 | 0.007 | -0.065 | -0.028 | 0.001 | -0.069 | -0.036 | -0.008 | -0.094 | -0.068 | -0.033 |
| | | | Prior 2 | 0.089 | 0.148 | 0.135 | 0.092 | 0.152 | 0.138 | 0.086 | 0.143 | 0.132 | 0.085 | 0.140 | 0.126 | 0.075 | 0.122 | 0.105 |
| 2 | 3 | IS | Prior 1 | -0.229 | -0.173 | -0.091 | -0.220 | -0.167 | -0.089 | -0.237 | -0.179 | -0.093 | -0.248 | -0.192 | -0.103 | -0.296 | -0.234 | -0.128 |
| | | | Prior 2 | 0.025 | 0.069 | 0.068 | 0.030 | 0.075 | 0.072 | 0.020 | 0.063 | 0.065 | 0.018 | 0.057 | 0.057 | 0.000 | 0.031 | 0.033 |
| | | MH | Prior 1 | -0.212 | -0.159 | -0.083 | -0.202 | -0.152 | -0.081 | -0.221 | -0.166 | -0.086 | -0.233 | -0.180 | -0.097 | -0.284 | -0.227 | -0.125 |
| | | | Prior 2 | 0.064 | 0.113 | 0.102 | 0.068 | 0.119 | 0.106 | 0.059 | 0.107 | 0.098 | 0.057 | 0.102 | 0.091 | 0.043 | 0.077 | 0.065 |
| | 4 | IS | Prior 1 | -0.188 | -0.144 | -0.076 | -0.181 | -0.138 | -0.074 | -0.195 | -0.149 | -0.077 | -0.203 | -0.158 | -0.085 | -0.238 | -0.192 | -0.106 |
| | | | Prior 2 | 0.022 | 0.060 | 0.059 | 0.026 | 0.065 | 0.062 | 0.018 | 0.055 | 0.056 | 0.016 | 0.050 | 0.050 | 0.002 | 0.029 | 0.030 |
| | | MH | Prior 1 | -0.163 | -0.122 | -0.063 | -0.155 | -0.116 | -0.061 | -0.171 | -0.128 | -0.065 | -0.178 | -0.139 | -0.074 | -0.215 | -0.175 | -0.098 |
| | | | Prior 2 | 0.059 | 0.103 | 0.091 | 0.062 | 0.108 | 0.094 | 0.055 | 0.097 | 0.088 | 0.053 | 0.093 | 0.081 | 0.041 | 0.072 | 0.060 |
| | 5 | IS | Prior 1 | -0.166 | -0.127 | -0.067 | -0.160 | -0.123 | -0.065 | -0.173 | -0.132 | -0.068 | -0.178 | -0.140 | -0.075 | -0.205 | -0.167 | -0.093 |
| | | | Prior 2 | 0.014 | 0.048 | 0.048 | 0.017 | 0.052 | 0.051 | 0.010 | 0.044 | 0.046 | 0.008 | 0.039 | 0.040 | -0.005 | 0.020 | 0.024 |
| | | MH | Prior 1 | -0.138 | -0.102 | -0.052 | -0.131 | -0.097 | -0.050 | -0.144 | -0.107 | -0.054 | -0.150 | -0.116 | -0.061 | -0.178 | -0.146 | -0.082 |
| | | | Prior 2 | 0.049 | 0.088 | 0.078 | 0.052 | 0.092 | 0.081 | 0.045 | 0.083 | 0.075 | 0.044 | 0.079 | 0.070 | 0.032 | 0.061 | 0.052 |

Table 4: Estimated risks of the Bayes estimators.

| | | | | BS | | | BLc1 | | | BLc2 | | | BGEp1 | | | BGEp2 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\theta$ | $m$ | | | $t_1$ | $t_2$ | $t_3$ | $t_1$ | $t_2$ | $t_3$ | $t_1$ | $t_2$ | $t_3$ | $t_1$ | $t_2$ | $t_3$ | $t_1$ | $t_2$ | $t_3$ |
| 0.5 | 3 | IS | Prior1 | 0.0314 | 0.0264 | 0.0148 | 0.0039 | 0.0033 | 0.0019 | 0.0040 | 0.0018 | 0.0033 | 0.0152 | 0.0264 | 0.0394 | 0.0192 | 0.0329 | 0.0477 |
| | | | Prior2 | 0.0227 | 0.0587 | 0.0631 | 0.0028 | 0.0072 | 0.0077 | 0.0029 | 0.0075 | 0.0081 | 0.0044 | 0.0168 | 0.0418 | 0.0048 | 0.0184 | 0.0476 |
| | | MH | Prior1 | 0.0195 | 0.0245 | 0.0192 | 0.0024 | 0.0031 | 0.0025 | 0.0025 | 0.0030 | 0.0023 | 0.0076 | 0.0151 | 0.0266 | 0.0103 | 0.0195 | 0.0313 |
| | | | Prior2 | 0.0282 | 0.0803 | 0.0918 | 0.0035 | 0.0097 | 0.0111 | 0.0036 | 0.0104 | 0.0119 | 0.0049 | 0.0210 | 0.0551 | 0.0052 | 0.0232 | 0.0655 |
| | 4 | IS | Prior1 | 0.0272 | 0.0238 | 0.0128 | 0.0033 | 0.0030 | 0.0016 | 0.0035 | 0.0030 | 0.0016 | 0.0119 | 0.0215 | 0.0327 | 0.0142 | 0.0254 | 0.0377 |
| | | | Prior2 | 0.0201 | 0.0473 | 0.0456 | 0.0025 | 0.0058 | 0.0056 | 0.0025 | 0.0060 | 0.0058 | 0.0041 | 0.0143 | 0.0338 | 0.0045 | 0.0157 | 0.0378 |
| | | MH | Prior1 | 0.0184 | 0.0235 | 0.0172 | 0.0023 | 0.0030 | 0.0022 | 0.0023 | 0.0029 | 0.0021 | 0.0065 | 0.0137 | 0.0241 | 0.0081 | 0.0162 | 0.0271 |
| | | | Prior2 | 0.0243 | 0.0642 | 0.0661 | 0.0030 | 0.0078 | 0.0080 | 0.0031 | 0.0083 | 0.0085 | 0.0043 | 0.0175 | 0.0438 | 0.0045 | 0.0192 | 0.0506 |
| | 5 | IS | Prior1 | 0.0217 | 0.0204 | 0.0112 | 0.0027 | 0.0026 | 0.0014 | 0.0028 | 0.0026 | 0.0014 | 0.0084 | 0.0159 | 0.0250 | 0.0100 | 0.0188 | 0.0289 |
| | | | Prior2 | 0.0176 | 0.0402 | 0.0363 | 0.0022 | 0.0049 | 0.0045 | 0.0022 | 0.0051 | 0.0046 | 0.0034 | 0.0122 | 0.0283 | 0.0037 | 0.0133 | 0.0314 |
| | | MH | Prior1 | 0.0153 | 0.0206 | 0.0147 | 0.0019 | 0.0026 | 0.0019 | 0.0019 | 0.0026 | 0.0018 | 0.0048 | 0.0107 | 0.0195 | 0.0058 | 0.0125 | 0.0217 |
| | | | Prior2 | 0.0214 | 0.0542 | 0.0518 | 0.0026 | 0.0066 | 0.0063 | 0.0027 | 0.0069 | 0.0066 | 0.0038 | 0.0151 | 0.0367 | 0.0040 | 0.0163 | 0.0415 |
| 1 | 3 | IS | Prior1 | 0.0369 | 0.0244 | 0.0092 | 0.0045 | 0.0030 | 0.0012 | 0.0047 | 0.0031 | 0.0011 | 0.0191 | 0.0310 | 0.0421 | 0.0235 | 0.0379 | 0.0511 |
| | | | Prior2 | 0.0204 | 0.0467 | 0.0432 | 0.0025 | 0.0057 | 0.0053 | 0.0026 | 0.0059 | 0.0055 | 0.0043 | 0.0146 | 0.0331 | 0.0047 | 0.0156 | 0.0360 |
| | | MH | Prior1 | 0.0222 | 0.0161 | 0.0073 | 0.0027 | 0.0020 | 0.0010 | 0.0029 | 0.0020 | 0.0009 | 0.0104 | 0.0170 | 0.0238 | 0.0142 | 0.0233 | 0.0319 |
| | | | Prior2 | 0.0246 | 0.0636 | 0.0618 | 0.0030 | 0.0078 | 0.0076 | 0.0031 | 0.0081 | 0.0079 | 0.0044 | 0.0177 | 0.0427 | 0.0047 | 0.0190 | 0.0477 |
| | 4 | IS | Prior1 | 0.0309 | 0.0222 | 0.0088 | 0.0038 | 0.0028 | 0.0011 | 0.0039 | 0.0028 | 0.0011 | 0.0139 | 0.0237 | 0.0332 | 0.0166 | 0.0281 | 0.0391 |
| | | | Prior2 | 0.0177 | 0.0382 | 0.0326 | 0.0022 | 0.0047 | 0.0040 | 0.0022 | 0.0048 | 0.0041 | 0.0037 | 0.0122 | 0.0267 | 0.0041 | 0.0132 | 0.0294 |
| | | MH | Prior1 | 0.0192 | 0.0164 | 0.0082 | 0.0023 | 0.0021 | 0.0011 | 0.0025 | 0.0020 | 0.0010 | 0.0077 | 0.0137 | 0.0204 | 0.0097 | 0.0172 | 0.0248 |
| | | | Prior2 | 0.0209 | 0.0512 | 0.0464 | 0.0026 | 0.0063 | 0.0057 | 0.0026 | 0.0065 | 0.0059 | 0.0038 | 0.0145 | 0.0341 | 0.0040 | 0.0156 | 0.0379 |
| | 5 | IS | Prior1 | 0.0262 | 0.0190 | 0.0074 | 0.0032 | 0.0024 | 0.0009 | 0.0033 | 0.0024 | 0.0009 | 0.0112 | 0.0194 | 0.0274 | 0.0129 | 0.0223 | 0.0313 |
| | | | Prior2 | 0.0149 | 0.0288 | 0.0222 | 0.0018 | 0.0036 | 0.0028 | 0.0019 | 0.0036 | 0.0028 | 0.0033 | 0.0100 | 0.0209 | 0.0036 | 0.0107 | 0.0222 |
| | | MH | Prior1 | 0.0170 | 0.0147 | 0.0070 | 0.0021 | 0.0018 | 0.0009 | 0.0022 | 0.0018 | 0.0009 | 0.0066 | 0.0119 | 0.0178 | 0.0079 | 0.0143 | 0.0209 |
| | | | Prior2 | 0.0176 | 0.0394 | 0.0323 | 0.0022 | 0.0049 | 0.0040 | 0.0022 | 0.0050 | 0.0041 | 0.0033 | 0.0119 | 0.0266 | 0.0035 | 0.0125 | 0.0285 |
| 2 | 3 | IS | Prior1 | 0.0639 | 0.0375 | 0.0108 | 0.0079 | 0.0047 | 0.0013 | 0.0080 | 0.0047 | 0.0014 | 0.0332 | 0.0519 | 0.0672 | 0.0396 | 0.0613 | 0.0790 |
| | | | Prior2 | 0.0127 | 0.0211 | 0.0143 | 0.0016 | 0.0027 | 0.0018 | 0.0016 | 0.0026 | 0.0018 | 0.0032 | 0.0084 | 0.0160 | 0.0037 | 0.0091 | 0.0164 |
| | | MH | Prior1 | 0.0513 | 0.0298 | 0.0084 | 0.0062 | 0.0036 | 0.0010 | 0.0066 | 0.0038 | 0.0011 | 0.0243 | 0.0374 | 0.0477 | 0.0319 | 0.0504 | 0.0658 |
| | | | Prior2 | 0.0124 | 0.0258 | 0.0189 | 0.0016 | 0.0033 | 0.0024 | 0.0015 | 0.0032 | 0.0023 | 0.0025 | 0.0084 | 0.0179 | 0.0026 | 0.0083 | 0.0168 |
| | 4 | IS | Prior1 | 0.0469 | 0.0287 | 0.0085 | 0.0058 | 0.0036 | 0.0011 | 0.0059 | 0.0036 | 0.0011 | 0.0219 | 0.0352 | 0.0463 | 0.0252 | 0.0404 | 0.0531 |
| | | | Prior2 | 0.0118 | 0.0186 | 0.0120 | 0.0015 | 0.0023 | 0.0015 | 0.0015 | 0.0023 | 0.0015 | 0.0030 | 0.0076 | 0.0141 | 0.0034 | 0.0082 | 0.0146 |
| | | MH | Prior1 | 0.0347 | 0.0209 | 0.0061 | 0.0042 | 0.0025 | 0.0008 | 0.0045 | 0.0027 | 0.0008 | 0.0150 | 0.0238 | 0.0310 | 0.0186 | 0.0304 | 0.0404 |
| | | | Prior2 | 0.0119 | 0.0234 | 0.0164 | 0.0015 | 0.0030 | 0.0021 | 0.0015 | 0.0029 | 0.0020 | 0.0024 | 0.0078 | 0.0162 | 0.0025 | 0.0078 | 0.0156 |
| | 5 | IS | Prior1 | 0.0402 | 0.0254 | 0.0077 | 0.0050 | 0.0032 | 0.0010 | 0.0051 | 0.0032 | 0.0010 | 0.0178 | 0.0293 | 0.0391 | 0.0197 | 0.0322 | 0.0428 |
| | | | Prior2 | 0.0120 | 0.0175 | 0.0107 | 0.0015 | 0.0022 | 0.0014 | 0.0015 | 0.0022 | 0.0013 | 0.0031 | 0.0076 | 0.0136 | 0.0035 | 0.0083 | 0.0145 |
| | | MH | Prior1 | 0.0287 | 0.0181 | 0.0055 | 0.0035 | 0.0022 | 0.0007 | 0.0037 | 0.0023 | 0.0007 | 0.0119 | 0.0194 | 0.0258 | 0.0139 | 0.0232 | 0.0311 |
| | | | Prior2 | 0.0116 | 0.0214 | 0.0144 | 0.0014 | 0.0027 | 0.0018 | 0.0015 | 0.0027 | 0.0018 | 0.0024 | 0.0074 | 0.0149 | 0.0026 | 0.0077 | 0.0150 |

Table 5: ALs and CPs for the 95% CSSW CrIs.

| $\theta$ | $m$ | | IS Prior 1 | | | IS Prior 2 | | | MH Prior 1 | | | MH Prior 2 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $t_1$ | $t_2$ | $t_3$ | $t_1$ | $t_2$ | $t_3$ | $t_1$ | $t_2$ | $t_3$ | $t_1$ | $t_2$ | $t_3$ |
| 0.5 | 3 | AL | 0.6317 | 0.6402 | 0.4619 | 0.3711 | 0.5267 | 0.5226 | 0.5940 | 0.6636 | 0.5285 | 0.2967 | 0.4739 | 0.5298 |
| | | CP | 0.937 | 0.927 | 0.925 | 0.719 | 0.718 | 0.776 | 0.967 | 0.960 | 0.961 | 0.566 | 0.568 | 0.633 |
| | 4 | AL | 0.5811 | 0.5900 | 0.4175 | 0.3801 | 0.5152 | 0.4756 | 0.5401 | 0.6091 | 0.4759 | 0.3128 | 0.4780 | 0.4938 |
| | | CP | 0.916 | 0.906 | 0.905 | 0.747 | 0.750 | 0.805 | 0.958 | 0.952 | 0.953 | 0.633 | 0.636 | 0.681 |
| | 5 | AL | 0.5348 | 0.5460 | 0.3833 | 0.3710 | 0.4953 | 0.4368 | 0.5061 | 0.5770 | 0.4436 | 0.3170 | 0.4757 | 0.4676 |
| | | CP | 0.913 | 0.910 | 0.915 | 0.766 | 0.774 | 0.824 | 0.946 | 0.944 | 0.952 | 0.665 | 0.676 | 0.735 |
| 1 | 3 | AL | 0.6542 | 0.6156 | 0.4020 | 0.3944 | 0.5361 | 0.4897 | 0.6494 | 0.6635 | 0.4634 | 0.3269 | 0.4983 | 0.5103 |
| | | CP | 0.908 | 0.903 | 0.902 | 0.792 | 0.788 | 0.839 | 0.960 | 0.955 | 0.954 | 0.655 | 0.659 | 0.727 |
| | 4 | AL | 0.5946 | 0.5641 | 0.3673 | 0.3940 | 0.5117 | 0.4376 | 0.5834 | 0.6065 | 0.4256 | 0.3359 | 0.4885 | 0.4650 |
| | | CP | 0.898 | 0.889 | 0.892 | 0.810 | 0.813 | 0.860 | 0.973 | 0.963 | 0.961 | 0.707 | 0.710 | 0.762 |
| | 5 | AL | 0.5497 | 0.5173 | 0.3313 | 0.3946 | 0.4893 | 0.3919 | 0.5463 | 0.5668 | 0.3903 | 0.3452 | 0.4841 | 0.4303 |
| | | CP | 0.914 | 0.909 | 0.909 | 0.836 | 0.846 | 0.879 | 0.965 | 0.958 | 0.959 | 0.750 | 0.765 | 0.830 |
| 2 | 3 | AL | 0.6711 | 0.5347 | 0.2988 | 0.4635 | 0.5506 | 0.4187 | 0.7053 | 0.5847 | 0.3346 | 0.4214 | 0.5510 | 0.4559 |
| | | CP | 0.813 | 0.781 | 0.773 | 0.939 | 0.938 | 0.961 | 0.923 | 0.912 | 0.906 | 0.931 | 0.934 | 0.980 |
| | 4 | AL | 0.6092 | 0.5022 | 0.2865 | 0.4409 | 0.5090 | 0.3746 | 0.6381 | 0.5573 | 0.3295 | 0.4019 | 0.5184 | 0.4152 |
| | | CP | 0.840 | 0.820 | 0.811 | 0.931 | 0.938 | 0.962 | 0.941 | 0.929 | 0.924 | 0.902 | 0.907 | 0.954 |
| | 5 | AL | 0.5496 | 0.4592 | 0.2661 | 0.4184 | 0.4660 | 0.3332 | 0.5812 | 0.5230 | 0.3155 | 0.3907 | 0.4880 | 0.3790 |
| | | CP | 0.816 | 0.799 | 0.799 | 0.899 | 0.905 | 0.930 | 0.935 | 0.918 | 0.907 | 0.888 | 0.892 | 0.931 |

## 6 Application to real data

To illustrate the practical application of the estimation methods discussed in this paper, we consider the dataset used by Crowder (2000) that shows the lifetimes of steel specimens in units of 1000000 cycles of 24 steel specimens under stress. Specifically, we focus on the stress level "32". The dataset under this stress level is given as follows: 1.144, 0.231, 0.523, 0.474, 4.510, 3.107, 0.815, 6.297, 1.580, 0.605, 1.786, 0.206, 1.943, 0.935, 0.283, 1.336, 0.727, 0.370, 1.056, 0.413, 0.619, 2.214, 1.826, 0.597

To verify that the inverse Lindley distribution is suitable for these data, a Kolmogorov-Smirnov test was performed, yielding a statistic of 0.0917 with a p-value of 0.9765, confirming the suitability of the inverse Lindley distribution for these data. The estimate of the distribution parameter was $\hat{\theta} = 0.9892$ which will be used to compute $t_1$, $t_2$, and $t_3$, where $t_i$ denotes the $i$-th quartile of the distribution.

Since our study is based on lower record values, the lower records from the aforementioned data were extracted as: 1.144, 0.231, and 0.206. The proposed methods of the paper were applied to these values, yielding the following results. The ML estimate of $\theta$ was obtained as 0.8466. Under the same settings stated in the simulation study for the priors and the parameters of the loss functions, the point estimates of the reliability function under different estimation methods are presented in Table 6. Besides, the 95% interval estimates of the reliability were also derived and are shown in Table 7. From Table 6, we see that the approximate Bayes point estimates under Prior 2 based on the IS method and the bootstrap estimates are the closest values to the real values of reliability functions. Table 7 reveals that most of the CSSW CrIs possess smaller lengths than those based on classical ones.

Table 6: Point estimates of the reliability function.

| Method | Prior | Estimator | $t_1$ | $t_2$ | $t_3$ |
|---|---|---|---|---|---|
| ML | | | 0.6748 | 0.4268 | 0.2047 |
| bootstrap | | | 0.6948 | 0.4806 | 0.25877 |
| IS | Prior 1 | BS | 0.5472 | 0.3416 | 0.1648 |
| | | BLc1 | 0.5570 | 0.3483 | 0.1671 |
| | | BLc2 | 0.5374 | 0.3350 | 0.1625 |
| | | BGEp1 | 0.5269 | 0.3210 | 0.1512 |
| | | BGEp2 | 0.4791 | 0.2758 | 0.1233 |
| | Prior 2 | BS | 0.7897 | 0.5716 | 0.3130 |
| | | BLc1 | 0.7947 | 0.5782 | 0.3166 |
| | | BLc2 | 0.7845 | 0.5650 | 0.3093 |
| | | BGEp1 | 0.7825 | 0.5590 | 0.3008 |
| | | BGEp2 | 0.7662 | 0.5308 | 0.2746 |
| MH | Prior 1 | BS | 0.6148 | 0.3985 | 0.1981 |
| | | BLc1 | 0.6240 | 0.4057 | 0.2008 |
| | | BLc2 | 0.6055 | 0.3914 | 0.1954 |
| | | BGEp1 | 0.5975 | 0.3790 | 0.1842 |
| | | BGEp2 | 0.5555 | 0.3344 | 0.1545 |
| | Prior 2 | BS | 0.89296 | 0.6174 | 0.3462 |
| | | BLc1 | 0.8331 | 0.6229 | 0.3496 |
| | | BLc2 | 0.8258 | 0.6118 | 0.3428 |
| | | BGEp1 | 0.8247 | 0.6075 | 0.3357 |
| | | BGEp2 | 0.8135 | 0.5852 | 0.3131 |

Table 7: 95% Interval estimates of the reliability function.

| | | $t_1$ | | $t_2$ | | $t_3$ | |
|---|---|---|---|---|---|---|---|
| | | lower | upper | lower | upper | lower | upper |
| Approximate | | 0.2107 | 0.9416 | 0.1222 | 0.7993 | 0.0546 | 0.5341 |
| Bootstrap | | 0.2755 | 0.9981 | 0.1403 | 0.9635 | 0.0577 | 0.7648 |
| IS | Prior 1 | 0.1434 | 0.8784 | 0.0682 | 0.6577 | 0.0204 | 0.3533 |
| | Prior 2 | 0.4939 | 0.9895 | 0.2562 | 0.8585 | 0.0893 | 0.5182 |
| MH | Prior 1 | 0.2724 | 0.9429 | 0.1151 | 0.7156 | 0.0379 | 0.4042 |
| | Prior 2 | 0.5838 | 0.9847 | 0.3492 | 0.8869 | 0.1467 | 0.5806 |

# 7 Discussion and conclusion

Record data, representing extreme values in a sequence of random variables, were vital in various fields. The inverse Lindley distribution offered a flexible framework for survival analysis. The paper provided theoretical foundations and practical tools for survival function estimation. Actually, the paper focused on estimating the survival function for the inverse Lindley distribution using lower records, a critical aspect in understanding lifetime phenomena. It explored estimation techniques including maximum likelihood, bootstrap, and Bayesian approaches like Metropolis-Hastings and importance sampling methods. The approximate confidence intervals and credible intervals were derived, and a simulation study evaluated the proposed estimators. Through a simulation study, we assessed the performance of the proposed estimators to aid researchers in method selection for survival analysis. A real data set example is also given to check the applicability of the theoretical results of the paper. The simulation study and the real data example establish the effect of the priors and methods that are used to obtain the estimates.

All the computations of this paper were done using the statistical software *R* (R Core Team, 2022) and the packages coda (Plummer et al., 2006, 2018), LindleyR (Mazucheli et al., 2016) and lamW (Adler, 2017) therein.

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