Journal of Statistical Modelling: Theory and Applications
Vol. 5, No. 1, 2024, pp. 65-93
Yazd University Press 2024

JSMTA

Research Paper

Stress-strength reliability of a non-identical-component-strengths system under the progressive censoring sample from the two parameter Rayleigh distribution

Akram Kohansal*[1], Zeinab Davoodi[2]
[1]Department of Statistics, Imam Khomeini International University, Qazvin, Iran [2]Department of Geology, Faculty of Science, Imam Khomeini International University, Qazvin, Iran

Received: April 23, 2024/ Revised: August 18, 2024/ Accepted: August 31, 2024

Abstract: This paper deals with the statistical inference of the stress-strength reliability of a multi-component system with non-identical-component strengths based on the progressively Type-II censored sample from the two-parameter Rayleigh distribution. Both stress and strength are assumed to have a Rayleigh distribution with different scale parameters yet similar location parameters. Its maximum likelihood estimate, asymptotic confidence interval, Bayes estimates, and highest posterior density are derived. The uniformly minimum-variance unbiased estimator and Bayes estimations for the reliability are obtained when the common location parameter is known. Different methods are compared using Monte Carlo simulations. The results demonstrate that Bayes estimates outperform maximum likelihood estimates, highest posterior density intervals outperform asymptotic intervals, and in Bayes estimates, informative priors outperform non-informative priors. Finally, a dataset is analyzed for illustrative purposes.

Keywords: Lindley's approximation; Markov Chain Monte Carlo method; Multi-component stress-strength reliability; Progressively censored.
Mathematics Subject Classification (2010): 62F10, 62F15, 62N02.

*Corresponding author: `kohansal@sci.ikiu.ac.ir`

1 Introduction

Type-I and Type-II censoring schemes are the most prominent censoring schemes in the data censoring theory. In both cases, the experiment is terminated when a pre-chosen number out of items are failed, or when a pre-determined time on the experiment has been reached, respectively. A hybrid censoring scheme, denoted by Epstein (1954) is a mixture of the two above schemes. Here, the experiment is terminated at $T^* = \min\{X_{m:n}, T\}$, where $X_{m:n}$ is the m-th failure times from n items and $T > 0$. Since none of the above schemes can omit active units during the experiment, the progressive scheme is introduced. Type-II progressive scheme is a mixture of Type-II and progressive censoring schemes, which has become very popular over the last decade among various censoring schemes. It can be described as follows: Consider N units are on a life test, and the experimenter determines the quantity n beforehand; then, the number of failures can be observed. At the time of the first failure, R_1 units are randomly removed from the experiment. At the time of the second failure, R_2 of the remaining $N - R_1 - 2$ units are randomly removed from the experiment, and so on. At the time of the n-th failure, all the remaining surviving units $R_n = N - n - R_1 - \cdots - R_{n-1}$ are removed from the experiment. Therefore, a progressive Type-II censoring scheme consists of n and $\{R_1, \ldots, R_n\}$, such that $R_1 + \cdots + R_n = N - n$. Clearly, this scheme includes the conventional Type-II right censoring scheme (when $R_n = N - n$ and $R_1 = \cdots = R_{n-1} = 0$) and complete sampling scheme (when $N = n$ and $R_1 = \cdots = R_n = 0$). Quite recently, Ahmed et al. (2024) considered the new and efficient estimators of reliability characteristics for a family of lifetime distributions under progressive censoring. Also, estimation for the reliability characteristics of a family of lifetime distributions under progressive censoring is considered by Safariyan and Arabi Belaghi (2021). For further details on progressively censoring and relevant references, the reader may refer to the book by Balakrishnan and Aggarwala (2000).

Herein, the two-parameter Rayleigh distribution (tRD) with one location parameter (μ) and one scale parameter (λ) has the probability density function (PDF) and cumulative distribution function (CDF) as follows

$$f(x) = 2\lambda(x-\mu)e^{-\lambda(x-\mu)^2}, \quad x > \mu, \tag{1}$$

$$F(x) = 1 - e^{-\lambda(x-\mu)^2}, \qquad x > \mu. \tag{2}$$

Henceforth, a tRD with the pdf (1) will be denoted by $tR(\mu, \lambda)$. tRD has an increasing failure rate function. Thus, if empirical research indicates an increasing failure rate function of the underlying distribution, the tRD may be used to analyze such datasets. Recently, Kohansal and Rezakhah (2019) have considered stress-strength parameter estimation in progressively censored data for tRD.

Inference on the stress-strength model is one of the most attractive topics in the reliability theory. The stress-strength parameter can be defined as follows

$$R = \int_{-\infty}^{\infty} \big(1 - F_X(y)\big) dF_Y(y), \tag{3}$$

where Y and X denote stress and strength, respectively. An active system is reliable when the applied stress is less than its strength. The idea of this model was introduced by Birnbaum (1956). Safariyan et al. (2019) studied improved point and interval estimation of the stress-strength reliability based on ranked set sampling.

Recently, multi-component stress-strength (MCSS) models have attracted great attention from researchers. A multi-component system is composed of k independent and identical strength components and a common stress component. The system fails when s $(1 \leq s \leq k)$ or more of k components simultaneously damage. This model was first developed by Bhattacharyya and Johnson (1974) and can be mathematically defined as follows

$$R_{s,k} = \sum_{p=s}^{k} \binom{k}{p} \int_{-\infty}^{\infty} \left(1 - F_X(y)\right)^p \left(F_X(y)\right)^{k-p} dF_Y(y), \tag{4}$$

where the independent and identically distributed random variables $(X_1, \ldots, X_k)$ are the strengths, and Y is the stress random variable. In what follows, several recent works on MCSS models are discussed. Nadar and Kizilaslan (2016) estimated reliability in an MCSS model based on the Marshall-Olkin bivariate Weibull Distribution. Moreover, Kizilaslan and Nadar (2018) estimated reliability in an MCSS model based on a bivariate Kumaraswamy distribution. Kohansal (2019) published the first paper on data censoring. She discussed reliability estimation in an MCSS model for the Kumaraswamy distribution based on a progressively censored sample. Also, Kohansal and Shoaee (2021) studied the Bayesian and classical estimation of reliability in an MCSS model under adaptive hybrid progressive censored data for the Weibull distribution. Very recently, Makhdoom et al. (2023) studied E-Bayesian and hierarchical Bayesian estimation of reliability in an MCSS model based on inverse Rayleigh distribution. Moreover, Singh et al. (2024) considered estimation in an MCSS model for progressive censored lognormal distribution. Kumari et al. (2024) studied the Bayesian and likelihood estimation of MCSS reliability from power Lindley distribution based on progressively censored samples.

$$R_{\boldsymbol{s},\boldsymbol{k}} = \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \left(\prod_{i=1}^{m} \binom{k_i}{p_i} \right) \int_{-\infty}^{\infty} \prod_{i=1}^{m} \left(\left(1 - F_i(y)\right)^{p_i} \left(F_i(y)\right)^{k_i - p_i} \right) dF_Y(y). \tag{5}$$

$$R_{\boldsymbol{s},\boldsymbol{k}} = \sum_{p_1=s_1}^{k_1} \sum_{p_2=s_2}^{k_2} \left(\prod_{i=1}^{2} \binom{k_i}{p_i} \right) \int_{-\infty}^{\infty} \prod_{i=1}^{2} \left(\left(1 - F_i(y)\right)^{p_i} \left(F_i(y)\right)^{k_i - p_i} \right) dF_Y(y). \tag{6}$$

This model can be studied without having to consider other cases, such as stress-strength and MCSS parameters. Indeed, it is a general model from which two essential stress-strength parameters can be derived:

- If $\boldsymbol{k} = (k, 0)$ then $R_{\boldsymbol{s},\boldsymbol{k}}$ is converted to $R_{s,k}$ in (4).
- If $\boldsymbol{k} = (1, 0)$ then $R_{\boldsymbol{s},\boldsymbol{k}}$ is converted to R in (3).

While the MCSS parameter with two non-identical-component strengths has been estimated by some researchers for uncensored samples, scant research has been done on its estimation for censored samples. Meanwhile, different censoring schemes should be dealt with in certain practical situations. Recently, Kohansal et al. (2021) has studied an MCSS model with two non-identical-component strengths in bathtub-shaped distribution under adaptive hybrid progressive censoring samples. This paper considers the MCSS parameter with two non-identical-component strengths in tRD under a progressive censoring scheme.

The remainder of this paper is structured as follows: Section 2 obtains the statistical inference of $R_{s,k}$ for common and unknown location parameters by computing its maximum likelihood estimate (MLE) and Bayes estimates. In Bayesian estimation, Lindley's approximation and Markov Chain Monte Carlo (MCMC) method are employed under the squared error loss function. Corresponding asymptotic confidence intervals (ACIs) and highest posterior density (HPD) credible intervals are also constructed. Section 3 obtains the statistical inference of $R_{s,k}$ for common and known location parameters by deriving its MLE, asymptotic confidence interval, exact Bayes estimate, HPD credible interval, and uniformly minimum-variance unbiased estimator (UMVUE). Section 4 presents simulation and data analysis results.

2 Inference on $R_{s,k}$ when location parameter is unknown

2.1 Maximum likelihood estimation of $R_{s,k}$

Let $X_1 \sim tR(\mu,\lambda)$, $X_2 \sim tR(\mu,\beta)$ and $Y \sim tR(\mu,\alpha)$ be independent random variables. Now, using (1) and (2), the MCSS reliability with two non-identical-component strengths can be obtained as follows

$$\begin{aligned}
R_{s,k} &= \sum_{p_1=s_1}^{k_1}\sum_{p_2=s_2}^{k_2}\binom{k_1}{p_1}\binom{k_2}{p_2}\int_0^\infty e^{-\lambda(y-\mu)^2p_1}(1-e^{-\lambda(y-\mu)^2})^{k_1-p_1} \\
&\quad\times e^{-\beta(y-\mu)^2p_2}(1-e^{-\beta(y-\mu)^2})^{k_2-p_2}2\alpha(y-\mu)e^{-\alpha(y-\mu)^2}dy \quad (\text{Put: } t=e^{-(y-\mu)^2}) \\
&= \sum_{p_1=s_1}^{k_1}\sum_{p_2=s_2}^{k_2}\binom{k_1}{p_1}\binom{k_2}{p_2}\alpha\int_0^1 t^{\lambda p_1+\beta p_2+\alpha-1}(1-t^\lambda)^{k_1-p_1}(1-t^\beta)^{k_2-p_2}dt \\
&= \sum_{p_1=s_1}^{k_1}\sum_{p_2=s_2}^{k_2}\sum_{q_1=0}^{k_1-p_1}\sum_{q_2=0}^{k_2-p_2}\binom{k_1}{p_1}\binom{k_2}{p_2}\binom{k_1-p_1}{q_1}\binom{k_2-p_2}{q_2} \\
&\quad\times(-1)^{q_1+q_2}\alpha\int_0^1 t^{\lambda(p_1+q_1)+\beta(p_2+q_2)+\alpha-1}dt \\
&= \sum_{p_1=s_1}^{k_1}\sum_{p_2=s_2}^{k_2}\sum_{q_1=0}^{k_1-p_1}\sum_{q_2=0}^{k_2-p_2}\binom{k_1}{p_1}\binom{k_2}{p_2}\binom{k_1-p_1}{q_1}\binom{k_2-p_2}{q_2} \\
&\quad\times\frac{(-1)^{q_1+q_2}\alpha}{\lambda(p_1+q_1)+\beta(p_2+q_2)+\alpha}. \qquad (7)
\end{aligned}$$

The MLE of $R_{s,k}$ can be obtained by initially measuring MLEs of λ, β, α and μ. Now, as we have n system on the life-testing experiment, the likelihood function can be constructed based on the following samples

$$\overset{\text{Observed stress variables}}{Y=\begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}} \quad \text{and} \quad \overset{\text{Observed strength variables}}{X_1=\begin{pmatrix} U_{11} & \dots & U_{1k_1} \\ \vdots & \ddots & \vdots \\ U_{n1} & \dots & U_{nk_1} \end{pmatrix}, X_2=\begin{pmatrix} V_{11} & \dots & V_{1k_2} \\ \vdots & \ddots & \vdots \\ V_{n1} & \dots & V_{nk_2} \end{pmatrix}}.$$

$$\begin{aligned} L(\lambda,\beta,\alpha,\mu|\text{data}) &= c_1\prod_{i=1}^{n}\Big(c_2\prod_{j_1=1}^{k_1} f_1(u_{ij_1})[1-F_1(u_{ij_1})]^{R_{ij_1}}\Big) \\ &\times\Big(c_3\prod_{j_2=1}^{k_2} f_2(v_{ij_2})[1-F_2(v_{ij_2})]^{Q_{ij_2}}\Big) f(y_i)[1-F(y_i)]^{S_i}. \end{aligned}$$

Generality is one of the most important advantages of this likelihood function. By making a few symbol modifications, the likelihood function can be obtained for $R_{\boldsymbol{s},\boldsymbol{k}}$ in Type-II censored and complete samples. It can also be derived for $R_{\boldsymbol{s},\boldsymbol{k}}$ and reliability parameter R in progressive censored, Type-II censored, and complete samples.

Based on the observed data, the likelihood function can be derived as follows

$$\begin{aligned} L(\lambda,\beta,\alpha,\mu|\text{data}) &\propto \lambda^{nk_1}\beta^{nk_2}\alpha^n\big(\prod_{i=1}^{n}\prod_{j_1=1}^{k_1}(u_{ij_1}-\mu)\big)\big(\prod_{i=1}^{n}\prod_{j_2=1}^{k_2}(v_{ij_2}-\mu)\big) \\ &\times\big(\prod_{i=1}^{n}(y_i-\mu)\big)e^{-\lambda A(\mu)-\beta B(\mu)-\alpha C(\mu)}, \end{aligned} \tag{8}$$

where

$$A(\mu) = \sum_{i=1}^{n}\sum_{j_1=1}^{J_2}(R_{ij_1}+1)(u_{ij_1}-\mu)^2, \tag{9}$$

$$B(\mu) = \sum_{i=1}^{n}\sum_{j_2=1}^{J_3}(Q_{ij_2}+1)(v_{ij_2}-\mu)^2, \tag{10}$$

$$C(\mu) = \sum_{i=1}^{n}(S_i+1)(y_i-\mu)^2. \tag{11}$$

Ignoring the constant value in (8), the log-likelihood function can be derived as

$$\begin{aligned} \ell(\lambda,\beta,\alpha,\mu|\text{data}) = {} & nk_1\log(\lambda)+nk_2\log(\beta)+n\log(\alpha)+\sum_{i=1}^{n}\sum_{j_1=1}^{k_1}\log(u_{ij_1}-\mu) \\ & +\sum_{i=1}^{n}\sum_{j_2=1}^{k_2}\log(v_{ij_2}-\mu)+\sum_{i=1}^{n}\log(y_i-\mu) \\ & -\lambda A(\mu)-\beta B(\mu)-\alpha C(\mu). \end{aligned}$$

Thus, $\hat{\lambda}$, $\hat{\beta}$ and $\hat{\alpha}$, MLEs of λ, β and α can be obtained by

$$\hat{\lambda}(\mu) = \frac{nk_1}{A(\mu)}, \quad \hat{\beta}(\mu) = \frac{nk_2}{B(\mu)}, \quad \hat{\alpha}(\mu) = \frac{n}{C(\mu)}.$$

Moreover, $\hat{\mu}$, MLE of μ is derived using the Newton-Raphson numerical on the following equation:

$$\begin{aligned}\frac{\partial \ell}{\partial \mu} =& 2\lambda \sum_{i=1}^{n}\sum_{j_1=1}^{k_1}(R_{ij_1}+1)(u_{ij_1}-\mu) + 2\beta \sum_{i=1}^{n}\sum_{j_2=1}^{k_2}(Q_{ij_2}+1)(v_{ij_2}-\mu)\\ &+2\alpha\sum_{i=1}^{n}(S_i+1)(y_i-\mu) - \sum_{i=1}^{n}\sum_{j_1=1}^{k_1}\frac{1}{u_{ij_1}-\mu} - \sum_{i=1}^{n}\sum_{j_2=1}^{k_2}\frac{1}{v_{ij_2}-\mu} - \sum_{i=1}^{n}\frac{1}{y_i-\mu} = 0.\end{aligned}$$

After obtaining $\hat{\lambda}$, $\hat{\beta}$, $\hat{\alpha}$ and $\hat{\mu}$, the MLE of $R_{\boldsymbol{s},\boldsymbol{k}}$, say $\hat{R}^{MLE}_{\boldsymbol{s},\boldsymbol{k}}$, is derived using the invariance property as follows

$$\begin{aligned}\hat{R}^{MLE}_{\boldsymbol{s},\boldsymbol{k}} =& \sum_{p_1=s_1}^{k_1}\sum_{p_2=s_2}^{k_2}\sum_{q_1=0}^{k_1-p_1}\sum_{q_2=0}^{k_2-p_2}\binom{k_1}{p_1}\binom{k_2}{p_2}\binom{k_1-p_1}{q_1}\binom{k_2-p_2}{q_2}\\ &\times\frac{(-1)^{q_1+q_2}\hat{\alpha}}{\hat{\lambda}(p_1+q_1)+\hat{\beta}(p_2+q_2)+\hat{\alpha}}. \qquad (12)\end{aligned}$$

2.2 Asymptotic confidence interval

This section derives ACIs for $R_{\boldsymbol{s},\boldsymbol{k}}$ using the asymptotic distribution of $\hat{R}_{\boldsymbol{s},\boldsymbol{k}}$. To this end, the asymptotic distribution of $\hat{\lambda}$, $\hat{\beta}$, $\hat{\alpha}$ and $\hat{\mu}$ should be derived using the inverse of the expected Fisher information matrix $J = [J_{ij}] = -[E(\frac{\partial^2 \ell}{\partial\theta_i\partial\theta_j})]$, where $i, j = 1, 2, 3, 4$ and $\Theta = (\theta_1, \theta_2, \theta_3, \theta_4) = (\lambda, \beta, \alpha, \mu)$. Dey et al. (2014) show that if $X \sim tR(\mu, \lambda)$, not all elements of matrix J are finite. Hence, the observed Fisher information matrix is used by dropping the expectation operator E, represented by $I = [I_{ij}] = -[\frac{\partial^2 \ell}{\partial\theta_i\partial\theta_j}]$, instead of J. The elements of symmetric matrix I can be obtained as follows

$$I_{11} = \frac{nk_1}{\lambda^2}, \quad I_{12} = 0, \quad I_{13} = 0, \quad I_{22} = \frac{nk_2}{\beta^2}, \quad I_{23} = 0, \quad I_{33} = \frac{n}{\alpha^2},$$

$$I_{14} = -2\sum_{i=1}^{n}\sum_{j_1=1}^{k_1}(R_{ij_1}+1)(u_{ij_1}-\mu),$$

$$I_{24} = -2\sum_{i=1}^{n}\sum_{j_2=1}^{k_2}(Q_{ij_2}+1)(v_{ij_2}-\mu)$$

$$I_{34} = -2\sum_{i=1}^{n}(S_i+1)(y_i-\mu),$$

$$I_{44} = 2\lambda\sum_{i=1}^{n}\sum_{j_1=1}^{k_1}(R_{ij_1}+1) + 2\beta\sum_{i=1}^{n}\sum_{j_2=1}^{k_2}(Q_{ij_2}+1) + 2\alpha\sum_{i=1}^{n}(S_i+1)$$

$$+\sum_{i=1}^{n}\sum_{j_1=1}^{k_1}\frac{1}{(u_{ij_1}-\mu)^2}+\sum_{i=1}^{n}\sum_{j_2=1}^{k_2}\frac{1}{(v_{ij_2}-\mu)^2}+\sum_{i=1}^{n}\frac{1}{(y_i-\mu)^2}.$$

Using the multivariate central limit theorem (CLT), the asymptotic distribution of $(\hat{\lambda},\hat{\beta},\hat{\alpha},\hat{\mu})$ can be obtained as follows

$$[(\hat{\lambda}-\lambda)\ \ (\hat{\beta}-\beta)\ \ (\hat{\alpha}-\alpha)\ \ (\hat{\mu}-\mu)]^T \overset{AppD}{\sim} N_4(0,\boldsymbol{I^{-1}}(\lambda,\beta,\alpha,\mu)),$$

where $\overset{AppD}{\sim}$ denotes "approximately distributed as" and $\boldsymbol{I}(\lambda,\beta,\alpha,\mu)$ and $\boldsymbol{I^{-1}}(\lambda,\beta,\alpha,\mu)$ are two symmetric matrices that can be derived by

$$\boldsymbol{I}(\lambda,\beta,\alpha,\mu) = \begin{pmatrix} I_{11} & 0 & 0 & I_{14}\\ & I_{22} & 0 & I_{24}\\ & & I_{33} & I_{34}\\ & & & I_{44}\end{pmatrix},$$

$$\boldsymbol{I^{-1}}(\lambda,\beta,\alpha,\mu) = \frac{1}{|\boldsymbol{I}(\lambda,\beta,\alpha,\mu)|}\begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14}\\ & b_{22} & b_{23} & b_{24}\\ & & b_{33} & b_{34}\\ & & & b_{44}\end{pmatrix},$$

where

$$b_{11}=I_{22}I_{33}I_{44}-I_{24}^2I_{33}-I_{22}I_{34}^2,\quad b_{12}=I_{14}I_{24}I_{33},\quad b_{13}=I_{14}I_{22}I_{34},\quad b_{14}=-I_{14}I_{22}I_{33},$$
$$b_{22}=I_{11}I_{33}I_{44}-I_{14}^2I_{33}-I_{11}I_{34}^2,\quad b_{23}=I_{11}I_{24}I_{34},\quad b_{24}=-I_{11}I_{24}I_{33},$$
$$b_{33}=I_{11}I_{22}I_{44}-I_{14}^2I_{22}-I_{11}I_{24}^2,\quad b_{34}=-I_{11}I_{22}I_{34},\ b_{44}=I_{11}I_{22}I_{33},$$
$$|\boldsymbol{I}(\lambda,\beta,\alpha,\mu)|=I_{11}I_{22}I_{33}I_{44}-I_{11}I_{24}^2I_{33}-I_{11}I_{22}I_{34}^2-I_{14}^2I_{22}I_{33}.$$

Theorem 2.1. *If $R_{\boldsymbol{s},\boldsymbol{k}}^{MLE}$ is the MLE of $R_{\boldsymbol{s},\boldsymbol{k}}$, then*

$$\hat{R}_{\boldsymbol{s},\boldsymbol{k}}^{MLE}-R_{\boldsymbol{s},\boldsymbol{k}} \overset{AppD}{\sim} N(0,V),$$

where

$$V = \frac{1}{|\boldsymbol{I}(\lambda,\beta,\alpha,\mu)|}\Big[(\frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial\lambda})^2 b_{11}+(\frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial\beta})^2 b_{22}+(\frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial\alpha})^2 b_{33}+2(\frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial\lambda})$$
$$(\frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial\beta})b_{12}+2(\frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial\lambda})(\frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial\alpha})b_{13}+2(\frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial\beta})(\frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial\alpha})b_{23}\Big].$$

Proof. Using the delta method, the asymptotic distribution of $\hat{R}_{\boldsymbol{s},\boldsymbol{k}}^{MLE}$ can be derived as:

$$\hat{R}_{\boldsymbol{s},\boldsymbol{k}}^{MLE}-R_{\boldsymbol{s},\boldsymbol{k}} \overset{AppD}{\sim} N(0,V),$$

where $V=\boldsymbol{b}^T\boldsymbol{I}^{-1}(\lambda,\beta,\alpha,\mu)\boldsymbol{b}$, where

$$\boldsymbol{b}=\left[\frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial\lambda}\ \ \frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial\beta}\ \ \frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial\alpha}\ \ \frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial\mu}\right]^T=\left[\frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial\lambda}\ \ \frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial\beta}\ \ \frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial\alpha}\ \ 0\right]^T,$$

with

$$\frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial\lambda} = \sum_{p_1=s_1}^{k_1}\sum_{p_2=s_2}^{k_2}\sum_{q_1=0}^{k_1-p_1}\sum_{q_2=0}^{k_2-p_2}\binom{k_1}{p_1}\binom{k_2}{p_2}\binom{k_1-p_1}{q_1}\binom{k_2-p_2}{q_2}$$

$$\times \frac{(-1)^{q_1+q_2+1}\alpha(p_1+q_1)}{\big(\lambda(p_1+q_1)+\beta(p_2+q_2)+\alpha\big)^2}, \tag{13}$$

$$\frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial \beta} = \sum_{p_1=s_1}^{k_1}\sum_{p_2=s_2}^{k_2}\sum_{q_1=0}^{k_1-p_1}\sum_{q_2=0}^{k_2-p_2}\binom{k_1}{p_1}\binom{k_2}{p_2}\binom{k_1-p_1}{q_1}\binom{k_2-p_2}{q_2}$$

$$\times \frac{(-1)^{q_1+q_2+1}\alpha(p_2+q_2)}{(\lambda(p_1+q_1)+\beta(p_2+q_2)+\alpha)^2}, \tag{14}$$

$$\frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial \alpha} = \sum_{p_1=s_1}^{k_1}\sum_{p_2=s_2}^{k_2}\sum_{q_1=0}^{k_1-p_1}\sum_{q_2=0}^{k_2-p_2}\binom{k_1}{p_1}\binom{k_2}{p_2}\binom{k_1-p_1}{q_1}\binom{k_2-p_2}{q_2}$$

$$\times \frac{(-1)^{q_1+q_2}\big(\lambda(p_1+q_1)+\beta(p_2+q_2)\big)}{(\lambda(p_1+q_1)+\beta(p_2+q_2)+\alpha)^2}. \tag{15}$$

Therefore,

$$V = \boldsymbol{b}^T \boldsymbol{I}^{-1}(\lambda,\beta,\alpha,\mu)\boldsymbol{b} = \frac{1}{|\boldsymbol{I}(\lambda,\beta,\alpha,\mu)|}\Big[(\frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial \lambda})^2 b_{11} + (\frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial \beta})^2 b_{22} + (\frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial \alpha})^2 b_{33}$$
$$+2(\frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial \lambda})(\frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial \beta})b_{12} + 2(\frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial \lambda})(\frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial \alpha})b_{13} + 2(\frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial \beta})(\frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial \alpha})b_{23}\Big].$$

Thus, the theorem is proved. □

By Theorem 2.1, the $100(1-\eta)\%$ ACI of $R_{\boldsymbol{s},\boldsymbol{k}}$ is constructed as:

$$(\hat{R}^{MLE}_{\boldsymbol{s},\boldsymbol{k}} - z_{1-\frac{\eta}{2}}\sqrt{\hat{V}}, \hat{R}^{MLE}_{\boldsymbol{s},\boldsymbol{k}} + z_{1-\frac{\eta}{2}}\sqrt{\hat{V}}),$$

where z_η is 100η-th percentile of $N(0,1)$.

2.3 Bayes estimation of $R_{\boldsymbol{s},\boldsymbol{k}}$

This section discusses the Bayesian inference of $R_{\boldsymbol{s},\boldsymbol{k}}$, when unknown parameters λ, β, α and μ are random variables. If the location parameter μ is known, the other parameters have gamma conjugate priors. However, joint conjugate priors are non-existent if all parameters are unknown. Thus, not all elements of the expected Fisher information matrix are finite, even for the complete sample data. Hence, Jeffreys prior does not exist in this case. So, the following priors are considered for λ, β, α and μ,

$$\begin{aligned}
\pi_1(\lambda) &\propto \lambda^{a_1-1}e^{-b_1\lambda}, \quad a_1,b_1,\lambda>0,\\
\pi_2(\beta) &\propto \beta^{a_2-1}e^{-b_2\beta}, \quad a_2,b_2,\beta>0,\\
\pi_3(\alpha) &\propto \alpha^{a_3-1}e^{-b_3\alpha}, \quad a_3,b_3,\alpha>0,\\
\pi_3(\mu) &\propto 1, \qquad\qquad\quad 0<\mu<t_1.
\end{aligned}$$

Besides, these random variables are assumed independent. Therefore, based on the observed sample, the joint posterior density function can be obtained as:

$$\pi(\lambda,\beta,\alpha,\mu|\text{data}) \propto L(\lambda,\beta,\alpha,\mu|\text{data})\pi_1(\lambda)\pi_2(\beta)\pi_3(\alpha)\pi_4(\mu). \tag{16}$$

From (16), the Bayes estimates can not be shown in closed form. Thus, they are estimated using two approximation methods:
• Lindley's approximation,
• MCMC method.

2.3.1 Lindley's approximation

Lindley's approximation, which proposed by Lindley (1980), is among the most widely used numerical methods to obtain Bayes estimates that can be explained as follows. If $U(\Theta)$ is a function of Θ, its Bayes estimate under the squared error loss function is as follows

$$\mathbb{E}(u(\Theta)|\text{data}) = \frac{\int u(\Theta)e^{Q(\Theta)}d\Theta}{\int e^{Q(\Theta)}d\Theta},$$

where $Q(\Theta) = \ell(\Theta)+\rho(\Theta)$, $\ell(\Theta)$ and $\rho(\Theta)$ are logarithms of the likelihood function and prior density Θ, respectively. Using Lindley's method, $\mathbb{E}(u(\Theta)|\text{data})$ is approximated by

$$\mathbb{E}(u(\Theta)|\text{data}) = u + \frac{1}{2}\sum_i\sum_j(u_{ij} + 2u_i\rho_j)\sigma_{ij} + \frac{1}{2}\sum_i\sum_j\sum_k\sum_p \ell_{ijk}\sigma_{ij}\sigma_{kp}u_p\Big|_{\Theta=\hat{\Theta}}, \tag{17}$$

where $\Theta = (\theta_1, \dots, \theta_m)$, $i, j, k, p = 1, \dots, m$, $\hat{\Theta}$ is the MLE of Θ, $u = u(\Theta)$, $u_i = \partial u/\partial\theta_i$, $u_{ij} = \partial^2 u/(\partial\theta_i\partial\theta_j)$, $\ell_{ijk} = \partial^3\ell/(\partial\theta_i\partial\theta_j\partial\theta_k)$, $\rho_j = \partial\rho/\partial\theta_j$, and $\sigma_{ij} = (i, j)$-th element in the inverse of the matrix $[-\ell_{ij}]$, all evaluated based on the MLE of the parameters.

For the four parameters $\Theta = (\theta_1, \theta_2, \theta_3, \theta_4)$, (17) can be simplified as follows:

$$\begin{aligned}\mathbb{E}(u(\Theta)|\text{data}) \quad = \quad & u + (u_1d_1 + u_2d_2 + u_3d_3 + u_4d_4 + d_5 + d_6)\\ &+\frac{1}{2}[A(u_1\sigma_{11} + u_2\sigma_{12} + u_3\sigma_{13} + u_4\sigma_{14})\\ &+B(u_1\sigma_{21} + u_2\sigma_{22} + u_3\sigma_{23} + u_4\sigma_{24})\\ &+C(u_1\sigma_{31} + u_2\sigma_{32} + u_3\sigma_{33} + u_4\sigma_{34})\\ &+D(u_1\sigma_{41} + u_2\sigma_{42} + u_3\sigma_{43} + u_4\sigma_{44})],\end{aligned}$$

where

$$\begin{aligned}d_i &= \rho_1\sigma_{i1} + \rho_2\sigma_{i2} + \rho_3\sigma_{i3} + \rho_4\sigma_{i4}, \quad i = 1, 2, 3, 4,\\ d_5 &= u_{12}\sigma_{12} + u_{13}\sigma_{13} + u_{14}\sigma_{14} + u_{23}\sigma_{23} + u_{24}\sigma_{24} + u_{34}\sigma_{34},\\ d_6 &= \frac{1}{2}(u_{11}\sigma_{11} + u_{22}\sigma_{22} + u_{33}\sigma_{33} + u_{44}\sigma_{44}),\\ A &= \ell_{111}\sigma_{11} + 2\ell_{121}\sigma_{12} + 2\ell_{131}\sigma_{13} + 2\ell_{141}\sigma_{14} + 2\ell_{231}\sigma_{23}\\ &\quad +2\ell_{241}\sigma_{24} + 2\ell_{341}\sigma_{34} + \ell_{221}\sigma_{22} + \ell_{331}\sigma_{33} + \ell_{441}\sigma_{44},\\ B &= \ell_{112}\sigma_{11} + 2\ell_{122}\sigma_{12} + 2\ell_{132}\sigma_{13} + 2\ell_{142}\sigma_{14} + 2\ell_{232}\sigma_{23}\\ &\quad +2\ell_{242}\sigma_{24} + 2\ell_{342}\sigma_{34} + \ell_{222}\sigma_{22} + \ell_{332}\sigma_{33} + \ell_{442}\sigma_{44},\\ C &= \ell_{113}\sigma_{11} + 2\ell_{123}\sigma_{12} + 2\ell_{133}\sigma_{13} + 2\ell_{143}\sigma_{14} + 2\ell_{233}\sigma_{23}\\ &\quad +2\ell_{243}\sigma_{24} + 2\ell_{343}\sigma_{34} + \ell_{223}\sigma_{22} + \ell_{333}\sigma_{33} + \ell_{443}\sigma_{44},\end{aligned}$$

$$\begin{aligned} D &= \ell_{114}\sigma_{11} + 2\ell_{124}\sigma_{12} + 2\ell_{134}\sigma_{13} + 2\ell_{144}\sigma_{14} + 2\ell_{234}\sigma_{23} \\ &\quad + 2\ell_{244}\sigma_{24} + 2\ell_{344}\sigma_{34} + \ell_{224}\sigma_{22} + \ell_{334}\sigma_{33} + \ell_{444}\sigma_{44}. \end{aligned}$$

$$\begin{aligned}
\rho_1 &= \frac{a_1-1}{\alpha} - b_1, \quad \rho_2 = \frac{a_2-1}{\beta} - b_2, \quad \rho_3 = \frac{a_3-1}{\lambda} - b_3, \quad \rho_4 = 0, \\
\ell_{11} &= -\frac{nk_1}{\lambda^2} \qquad \ell_{12} = 0 = \ell_{21}, \qquad \ell_{13} = 0 = \ell_{31}, \\
\ell_{22} &= -\frac{nk_2}{\beta^2}, \qquad \ell_{23} = 0 = \ell_{32}, \qquad \ell_{33} = -\frac{n}{\alpha^2}, \\
\ell_{14} &= 2\sum_{i=1}^{n}\sum_{j_1=1}^{k_1}(R_{ij_1}+1)(u_{ij_1}-\mu) = \ell_{41}, \\
\ell_{24} &= 2\sum_{i=1}^{n}\sum_{j_2=1}^{k_2}(Q_{ij_2}+1)(v_{ij_2}-\mu) = \ell_{42}, \\
\ell_{34} &= 2\sum_{i=1}^{n}(S_i+1)(y_i-\mu) = \ell_{43}, \\
\ell_{44} &= -2\lambda\sum_{i=1}^{n}\sum_{j_1=1}^{k_1}(R_{ij_1}+1) - 2\beta\sum_{i=1}^{n}\sum_{j_2=1}^{k_2}(Q_{ij_2}+1) - 2\alpha\sum_{i=1}^{n}(S_i+1) \\
&\quad - \sum_{i=1}^{n}\sum_{j_1=1}^{k_1}\frac{1}{(u_{ij_1}-\mu)^2} - \sum_{i=1}^{n}\sum_{j_2=1}^{k_2}\frac{1}{(v_{ij_2}-\mu)^2} - \sum_{i=1}^{n}\frac{1}{(y_i-\mu)^2}.
\end{aligned}$$

Moreover, $\sigma_{ij}, i,j = 1,2,3,4$ should be derived from $\ell_{ij}, i,j = 1,2,3,4$. We also have,

$$\ell_{111} = \frac{2nk_1}{\lambda^3}, \qquad \ell_{222} = \frac{2nk_2}{\beta^3}, \qquad \ell_{333} = \frac{2n}{\alpha^3},$$

$$\ell_{144} = -2\sum_{i=1}^{n}\sum_{j_1=1}^{k_1}(R_{ij_1}+1), \quad \ell_{244} = -2\sum_{i=1}^{n}\sum_{j_2=1}^{k_2}(Q_{ij_2}+1), \quad \ell_{344} = -2\sum_{i=1}^{n}(S_i+1),$$

$$\ell_{444} = -\sum_{i=1}^{n}\sum_{j_1=1}^{k_1}\frac{2}{(u_{ij_1}-\mu)^2} - \sum_{i=1}^{n}\sum_{j_2=1}^{k_2}\frac{2}{(v_{ij_2}-\mu)^2} - \sum_{i=1}^{n}\frac{2}{(y_i-\mu)^2}.$$

and other $\ell_{ijk} = 0$. Furthermore, u_1, u_2 and u_3 are represented by (13), (14) and (15), respectively, and $u_4 = u_{i4} = 0$, $i = 1,2,3,4$. Moreover,

$$\begin{aligned}
u_{11} &= \sum_{p_1=s_1}^{k_1}\sum_{p_2=s_2}^{k_2}\sum_{q_1=0}^{k_1-p_1}\sum_{q_2=0}^{k_2-p_2}\binom{k_1}{p_1}\binom{k_2}{p_2}\binom{k_1-p_1}{q_1}\binom{k_2-p_2}{q_2} \\
&\quad \times \frac{2(-1)^{q_1+q_2}\alpha(p_1+q_1)^2}{\big(\lambda(p_1+q_1)+\beta(p_2+q_2)+\alpha\big)^3}, \\
u_{22} &= \sum_{p_1=s_1}^{k_1}\sum_{p_2=s_2}^{k_2}\sum_{q_1=0}^{k_1-p_1}\sum_{q_2=0}^{k_2-p_2}\binom{k_1}{p_1}\binom{k_2}{p_2}\binom{k_1-p_1}{q_1}\binom{k_2-p_2}{q_2}
\end{aligned}$$

$$
\begin{aligned}
& \times \frac{2(-1)^{q_1+q_2}\alpha(p_2+q_2)^2}{(\lambda(p_1+q_1)+\beta(p_2+q_2)+\alpha)^3}, \\
u_{33} &= \sum_{p_1=s_1}^{k_1}\sum_{p_2=s_2}^{k_2}\sum_{q_1=0}^{k_1-p_1}\sum_{q_2=0}^{k_2-p_2}\binom{k_1}{p_1}\binom{k_2}{p_2}\binom{k_1-p_1}{q_1}\binom{k_2-p_2}{q_2} \\
& \times \frac{2(-1)^{q_1+q_2+1}\big(\lambda(p_1+q_1)+\beta(p_2+q_2)\big)}{(\lambda(p_1+q_1)+\beta(p_2+q_2)+\alpha)^3}, \\
u_{12} &= \sum_{p_1=s_1}^{k_1}\sum_{p_2=s_2}^{k_2}\sum_{q_1=0}^{k_1-p_1}\sum_{q_2=0}^{k_2-p_2}\binom{k_1}{p_1}\binom{k_2}{p_2}\binom{k_1-p_1}{q_1}\binom{k_2-p_2}{q_2} \\
& \times \frac{2(-1)^{q_1+q_2}\alpha(p_1+q_1)(p_2+q_2)}{\big(\lambda(p_1+q_1)+\beta(p_2+q_2)+\alpha\big)^3}, \\
u_{13} &= \sum_{p_1=s_1}^{k_1}\sum_{p_2=s_2}^{k_2}\sum_{q_1=0}^{k_1-p_1}\sum_{q_2=0}^{k_2-p_2}\binom{k_1}{p_1}\binom{k_2}{p_2}\binom{k_1-p_1}{q_1}\binom{k_2-p_2}{q_2} \\
& \times \frac{(-1)^{q_1+q_2+1}(p_1+q_1)\big(\lambda(p_1+q_1)+\beta(p_2+q_2)-\alpha\big)}{(\lambda(p_1+q_1)+\beta(p_2+q_2)+\alpha)^3}, \\
u_{23} &= \sum_{p_1=s_1}^{k_1}\sum_{p_2=s_2}^{k_2}\sum_{q_1=0}^{k_1-p_1}\sum_{q_2=0}^{k_2-p_2}\binom{k_1}{p_1}\binom{k_2}{p_2}\binom{k_1-p_1}{q_1}\binom{k_2-p_2}{q_2} \\
& \times \frac{(-1)^{q_1+q_2+1}(p_2+q_2)\big(\lambda(p_1+q_1)+\beta(p_2+q_2)-\alpha\big)}{(\lambda(p_1+q_1)+\beta(p_2+q_2)+\alpha)^3}.
\end{aligned}
$$

Hence, the Bayes estimate of $R_{\boldsymbol{s},\boldsymbol{k}}$ based on Lindley's approximation, denoted by $R^{Lin}_{\boldsymbol{s},\boldsymbol{k}}$, becomes

$$
\begin{aligned}
\hat{R}^{Lin}_{\boldsymbol{s},\boldsymbol{k}} &= R_{\boldsymbol{s},\boldsymbol{k}} + [u_1d_1+u_2d_2+u_3d_3+d_5+d_6] + \frac{1}{2}[A(u_1\sigma_{11}+u_2\sigma_{12}+u_3\sigma_{13}) \\
& +B(u_1\sigma_{21}+u_2\sigma_{22}+u_3\sigma_{23}) + C(u_1\sigma_{31}+u_2\sigma_{32}+u_3\sigma_{33}) \\
& +D(u_1\sigma_{41}+u_2\sigma_{42}+u_3\sigma_{43})]. \qquad (18)
\end{aligned}
$$

Remember that all parameters should be computed at $(\hat{\lambda},\hat{\beta},\hat{\alpha},\hat{\mu})$.

As shown, the credible interval cannot be constructed using Lindely's approximation. Therefore, another approximation method (i.e., MCMC) is applied to obtain the Bayes estimate and corresponding HPD credible interval.

2.3.2 MCMC method

The posterior PDFs of λ, β, α, and μ from (16) can be obtained as

$$
\begin{aligned}
\lambda|\mu,\text{data} &\sim \Gamma(nk_1+a_1, b_1+A(\mu)), \\
\beta|\mu,\text{data} &\sim \Gamma(nk_2+a_2, b_2+B(\mu)), \\
\alpha|\mu,\text{data} &\sim \Gamma(n+a_3, b_3+C(\mu)),
\end{aligned}
$$

$$
\pi(\mu|\lambda,\beta,\alpha,\text{data}) \propto \Big(\prod_{i=1}^{n}\prod_{j_1=1}^{k_1}(u_{ij_1}-\mu)\Big)\Big(\prod_{i=1}^{n}\prod_{j_2=1}^{k_2}(v_{ij_2}-\mu)\Big)\Big(\prod_{i=1}^{n}(y_i-\mu)\Big)
$$

$$\times e^{-\lambda A(\mu)-\beta B(\mu)-\alpha C(\mu)},$$

1. Start with an initial values $(\lambda_{(0)},\ \beta_{(0)},\ \alpha_{(0)},\ \mu_{(0)})$.
2. Set $t = 1$.
3. Generate $\mu_{(t)}$ from $\pi(\mu|\lambda_{(t-1)}, \beta_{(t-1)}, \alpha_{(t-1)}, \text{data})$ using the Metropolis-Hastings method, with $N(\mu_{(t-1)}, 1)$ as the proposal distribution.
4. Generate $\lambda_{(t)}$ from $\Gamma(nk_1 + a_1, b_1 + A(\mu_{(t-1)}))$.
5. Generate $\beta_{(t)}$ from $\Gamma(nk_2 + a_2, b_2 + B(\mu_{(t-1)}))$.
6. Generate $\alpha_{(t)}$ from $\Gamma(n + a_3, b_3 + C(\mu_{(t-1)}))$.
7. Evaluate the value:

$$\begin{aligned} R_{(t)\boldsymbol{s},\boldsymbol{k}} &= \sum_{p_1=s_1}^{k_1}\sum_{p_2=s_2}^{k_2}\sum_{q_1=0}^{k_1-p_1}\sum_{q_2=0}^{k_2-p_2}\binom{k_1}{p_1}\binom{k_2}{p_2}\binom{k_1-p_1}{q_1}\binom{k_2-p_2}{q_2} \\ &\times\frac{(-1)^{q_1+q_2}\alpha_{(t)}}{\lambda_{(t)}(p_1+q_1)+\beta_{(t)}(p_2+q_2)+\alpha_{(t)}}. \end{aligned}$$

8. Set $t = t + 1$.
9. Repeat steps 3-8, T_{bayes} times.

Thus, the Bayes estimate of $R_{\boldsymbol{s},\boldsymbol{k}}$ based on MCMC method, which presented by $\hat{R}^{MC}_{\boldsymbol{s},\boldsymbol{k}}$ becomes

$$\hat{R}^{MC}_{\boldsymbol{s},\boldsymbol{k}} = \frac{1}{T_{bayes}}\sum_{t=1}^{T_{bayes}} R_{(t)\boldsymbol{s},\boldsymbol{k}}. \tag{19}$$

Also, using the Chen and Shao (1999) method, the HPD credible interval of $R_{\boldsymbol{s},\boldsymbol{k}}$ is constructed.

3 Inference on $R_{\boldsymbol{s},\boldsymbol{k}}$ when location parameter is known

3.1 Maximum likelihood estimation of $R_{\boldsymbol{s},\boldsymbol{k}}$

Suppose $\{Y_1, \ldots, Y_n\}$ is a progressively censored sample from the $tR(\mu, \alpha)$ with the scheme$\{S_1, \ldots, S_n\}$, $\{U_{i1}, \ldots, U_{ik_1}\}$ and $\{V_{i1}, \ldots, V_{ik_2}\}$, $i = 1, \ldots, n$, are progressively censored samples from $tR(\mu, \lambda)$ and $tR(\mu, \beta)$ with the schemes $\{R_{i1}, \ldots, R_{ik_1}\}$ and $\{Q_{i1}, \ldots, Q_{ik_2}\}$, respectively. Now, $R_{\boldsymbol{s},\boldsymbol{k}}$ is considered when the location parameter μ is known. Like Subsection 2.1, the MLE of $R_{\boldsymbol{s},\boldsymbol{k}}$ can be obtained as follows:

$$\begin{aligned} \hat{R}^{MLE}_{\boldsymbol{s},\boldsymbol{k}} &= \sum_{p_1=s_1}^{k_1}\sum_{p_2=s_2}^{k_2}\sum_{q_1=0}^{k_1-p_1}\sum_{q_2=0}^{k_2-p_2}\binom{k_1}{p_1}\binom{k_2}{p_2}\binom{k_1-p_1}{q_1}\binom{k_2-p_2}{q_2}(-1)^{q_1+q_2} \\ &\times\left(\frac{k_1C(\mu)}{A(\mu)}(p_1+q_1)+\frac{k_2C(\mu)}{B(\mu)}(p_2+q_2)+1\right)^{-1}. \end{aligned} \tag{20}$$

From Theorem 2.1, the asymptotic distribution of $R^{MLE}_{\boldsymbol{s},\boldsymbol{k}}$, can be derived as

$$\hat{R}^{MLE}_{\boldsymbol{s},\boldsymbol{k}} - R_{\boldsymbol{s},\boldsymbol{k}} \overset{AppD}{\sim} N(0, V),$$

where $V = (\frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial \lambda})^2 \frac{1}{I_{11}} + (\frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial \beta})^2 \frac{1}{I_{22}} + (\frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial \alpha})^2 \frac{1}{I_{33}}$, and $\frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial \lambda}$, $\frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial \beta}$ and $\frac{\partial R_{\boldsymbol{s},\boldsymbol{k}}}{\partial \alpha}$ are presented in (13), (14) and (15), respectively. So, the $100(1-\eta)\%$ asymptotic confidence interval for $R_{\boldsymbol{s},\boldsymbol{k}}$ can be constructed as

$$(\hat{R}^{MLE}_{\boldsymbol{s},\boldsymbol{k}} - z_{1-\frac{\eta}{2}}\sqrt{\hat{V}}, \hat{R}^{MLE}_{\boldsymbol{s},\boldsymbol{k}} + z_{1-\frac{\eta}{2}}\sqrt{\hat{V}}),$$

where z_η is the 100η-th percentile of $N(0,1)$.

3.2 Bayes estimation of $R_{\boldsymbol{s},\boldsymbol{k}}$

This section discusses the Bayesian inference of $R_{\boldsymbol{s},\boldsymbol{k}}$ when unknown parameters λ, β and α are random variables. The following priors are considered for λ, β and α

$$\begin{aligned} \pi_1(\lambda) &\propto \lambda^{a_1-1}e^{-b_1\lambda}, \quad a_1, b_1, \lambda > 0, \\ \pi_2(\beta) &\propto \beta^{a_2-1}e^{-b_2\beta}, \quad a_2, b_2, \beta > 0, \\ \pi_3(\alpha) &\propto \alpha^{a_3-1}e^{-b_3\alpha}, \quad a_3, b_3, \alpha > 0. \end{aligned}$$

By the above selection, the joint posterior density function of λ, β and α can be derived as follows

$$\begin{aligned} \pi(\lambda, \beta, \alpha|\theta, \text{data}) &= \frac{(b_1 + A(\mu))^{nk_1+a_1}(b_2 + B(\mu))^{nk_2+a_2}(b_3 + C(\mu))^{n+a_3}}{\Gamma(nk_1+a_1)\Gamma(nk_2+a_2)\Gamma(n+a_3)}\lambda^{nk_1+a_1-1} \\ &\times \beta^{nk_2+a_2-1}\alpha^{n+a_3-1}e^{-\lambda(b_1+A(\mu))-\beta(b_2+B(\mu))-\alpha(b_3+C(\mu))}, \quad (21) \end{aligned}$$

where $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ are given by (9), (10) and (11), respectively. Under the squared error loss function, the Bayes estimate of $R_{\boldsymbol{s},\boldsymbol{k}}$ can be obtained by solving the following triple integral

$$\begin{aligned} \hat{R}^B_{\boldsymbol{s},\boldsymbol{k}} &= \int_0^\infty \int_0^\infty \int_0^\infty R_{\boldsymbol{s},\boldsymbol{k}}\pi(\lambda, \beta, \alpha|\mu, \text{data})d\lambda d\beta d\alpha \\ &= \sum_{p_1=s_1}^{k_1}\sum_{p_2=s_2}^{k_2}\sum_{q_1=0}^{k_1-p_1}\sum_{q_2=0}^{k_2-p_2}\binom{k_1}{p_1}\binom{k_2}{p_2}\binom{k_1-p_1}{q_1}\binom{k_2-p_2}{q_2}(-1)^{q_1+q_2} \\ &\times \int_0^\infty \int_0^\infty \int_0^\infty \frac{\alpha}{\lambda_(p_1+q_1)+\beta(p_2+q_2)+\alpha}\pi(\lambda, \beta, \alpha|\mu, \text{data})d\lambda d\beta d\alpha. \quad (22) \end{aligned}$$

Now, by applying $\pi(\lambda, \beta, \alpha|\mu, \text{data})$ from (21) to (22) and using the idea of Rasethuntsa and Nadar (2018), the part of triple integral in (22) can be solved as

$$M = \begin{cases} A_1 & |w_1| < 1, |w_2| < 1, \\ A_2 & w_1 < -1, w_2 < -1, \\ A_3 & |w_1| < 1, w_2 < -1, \\ A_4 & w_1 < -1, |w_2| < 1, \end{cases}$$

where

$$A_1 = \frac{\nu_3(1-w_1)^{\nu_1}(1-w_2)^{\nu_2}}{\nu_1+\nu_2+\nu_3} F_1(\nu_1+\nu_2+\nu_3, \nu_1, \nu_2, 1+\nu_1+\nu_2+\nu_3; w_1, w_2),$$

$$\begin{aligned}
A_2 &= \frac{\nu_3}{\nu_1+\nu_2+\nu_3}\ F_1(1,\nu_1,\nu_2,1+\nu_1+\nu_2+\nu_3;\frac{w_1}{w_1-1},\frac{w_2}{w_2-1}),\\
A_3 &= \frac{\nu_3(1-w_1)}{\nu_1+\nu_2+\nu_3}\ F_1(1,\nu_3+1,\nu_2,1+\nu_1+\nu_2+\nu_3;w_1,\frac{w_1-w_2}{1-w_2}),\\
A_4 &= \frac{\nu_3(1-w_2)}{\nu_1+\nu_2+\nu_3}\ F_1(1,\nu_1,\nu_3+1,1+\nu_1+\nu_2+\nu_3;\frac{w_2-w_1}{1-w_1},w_2),
\end{aligned}$$

and $\nu_1 = nk_1 + a_1$, $\nu_2 = nk_2 + a_2$, $\nu_3 = n + a_3$, $w_1 = 1 - \frac{b_1+A(\mu)}{(p_1+q_1)(b_3+C(\mu))}$ and $w_2 = 1 - \frac{b_2+B(\mu)}{(p_2+q_2)(b_3+C(\mu))}$. Also,

$$F_1(\alpha,\beta,\beta',\gamma;x,y) = \frac{1}{B(\alpha,\gamma-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-tx)^{-\beta}(1-ty)^{-\beta'}dt.$$

In this presentation, the function $F_1(\alpha,\beta,\beta',\gamma;x,y)$ is known as the hypergeometric series. It can be quickly and readily evaluated using standard software like MATLAB. Hence, the Bayes estimate of $R_{\boldsymbol{s},\boldsymbol{k}}$, represented by $\hat{R}^B_{\boldsymbol{s},\boldsymbol{k}}$, is obtained as

$$\hat{R}^B_{\boldsymbol{s},\boldsymbol{k}} = \sum_{p_1=s_1}^{k_1}\sum_{p_2=s_2}^{k_2}\sum_{q_1=0}^{k_1-p_1}\sum_{q_2=0}^{k_2-p_2}\binom{k_1}{p_1}\binom{k_2}{p_2}\binom{k_1-p_1}{q_1}\binom{k_2-p_2}{q_2}(-1)^{q_1+q_2}\times M. \quad (23)$$

Also, the $100(1-\eta)\%$ HPD credible interval of $R_{\boldsymbol{s},\boldsymbol{k}}$ can be constructed using the Chen and Shao (1999) method.

3.3 UMVUE of $R_{\boldsymbol{s},\boldsymbol{k}}$

Suppose $\{Y_1,\dots,Y_n\}$ is a progressively censored sample from $tR(\mu,\alpha)$ with the scheme $\{S_1,\dots,S_n\}$, $\{U_{i1},\dots,U_{ik_1}\}$ and $\{V_{i1},\dots,V_{ik_2}\}$, $i=1,\dots,n$, are progressively censored samples from the $tR(\mu,\lambda)$ and the $tR(\mu,\beta)$ with the schemes $\{R_{i1},\dots,R_{ik_1}\}$ and $\{Q_{i1},\dots,Q_{ik_2}\}$, respectively. When the location parameter μ is known, the likelihood function can be expressed as

$$\begin{aligned}
L(\lambda,\beta,\alpha|\theta,\text{data}) &\propto \lambda^{nk_1}\beta^{nk_2}\alpha^n\big(\prod_{i=1}^{n}\prod_{j_1=1}^{k_1}(u_{ij_1}-\mu)\big)\big(\prod_{i=1}^{n}\prod_{j_2=1}^{k_2}(v_{ij_2}-\mu)\big)\\
&\quad\times\big(\prod_{i=1}^{n}(y_i-\mu)\big)e^{-\alpha A(\mu)-\beta B(\mu)-\lambda C(\mu)}, \qquad (24)
\end{aligned}$$

where $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ are presented in (9), (10) and (11), respectively. When μ is known, from (24), it can be easily concluded that $A(\mu)$, $B(\mu)$ and $C(\mu)$ are complete sufficient statistics for λ, β and α, respectively. By transforming $Y_i^* = (Y-i-\mu)^2$, $i = 1,\dots,n$, a progressively censored sample is derived from the exponential distribution with mean $\frac{1}{\alpha}$. Now, put

$$\begin{aligned}
Z_1 &= NY_1^*,\\
Z_2 &= (N-S_1-1)(Y_2^*-Y_1^*),\\
&\ \vdots
\end{aligned}$$

$$Z_n = (N - \sum_{i=1}^{n} S_i - n + 1)(Y_n^* - Y_{n-1}^*).$$

Using the idea of Balakrishnan and Aggarwala (2000), it can be concluded that $Z_1, \cdots, Z_n$ are iid random variables from an exponential distribution with mean $\frac{1}{\alpha}$. So, $C(\mu) = \sum_{i=1}^{n} Z_i$ has a gamma distribution with the following PDF

$$f_{C(\mu)}(c) = \frac{\alpha^n}{\Gamma(n)} c^{n-1} e^{-\alpha c}, \quad c > 0.$$

Lemma 3.1. *Suppose $U_{ij_1}^* = (U_{ij_1} - \mu)^2$, $V_{ij_2}^* = (V_{ij_2} - \mu)^2$, $j_1 = 1, \ldots, k_1$, $j_2 = 1, \ldots, k_2$, $i = 1, \ldots, n$. By these transformations, the conditional pdfs of Y_1^* given $C(\mu) = c$, U_{11}^* given $A(\mu) = a$ and V_{11}^* given $B(\mu) = b$ are respectively as follows*

$$\begin{aligned} f_{Y_1^*|C(\mu)=c}(y) &= N(n-1)\frac{(c-Ny)^{n-2}}{c^{n-1}}, \qquad 0 < y < c/N, \\ f_{U_{11}^*|A(\mu)=a}(u) &= K_1(nk_1-1)\frac{(a-K_1u)^{nk_1-2}}{a^{nk_1-1}}, \quad 0 < u < a/K_1, \\ f_{V_{11}^*|B(\mu)=b}(v) &= K_2(nk_2-1)\frac{(b-K_2v)^{nk_2-2}}{b^{nk_2-1}}, \quad 0 < v < b/K_2. \end{aligned}$$

Proof. The lemma can be proved using a similar method as in Kohansal (2019). □

Theorem 3.2. *Based on the complete sufficient statistics $A(\mu)$, $B(\mu)$ and $C(\mu)$, for λ, β and α, respectively, the UMVUE of $\psi(\lambda, \beta, \alpha) = \dfrac{\alpha}{\lambda(p_1+q_1) + \beta(p_2+q_2) + \alpha}$, represented by $\hat{\psi}_U(\lambda, \beta, \alpha)$, is as follows*

$$\hat{\psi}_U(\lambda, \beta, \alpha) = \begin{cases} B_1 & \textit{Case I}, \\ B_2 & \textit{Case II}, \\ B_3 & \textit{Case III}, \end{cases}$$

where

$$\begin{aligned} B_1 &= \sum_{l_1=0}^{nk_1-1}\sum_{l_2=0}^{nk_2-1}(-1)^{l_1+l_2}\left(\frac{c(p_1+q_1)}{a}\right)^{l_1}\left(\frac{c(p_2+q_2)}{b}\right)^{l_2}\frac{\binom{nk_1-1}{l_1}\binom{nk_2-1}{l_2}}{\binom{l_1+l_2+n-1}{l_1+l_2}}, \\ B_2 &= \frac{n-1}{nk_1}\sum_{l_1=0}^{nk_2-1}\sum_{l_2=0}^{n-2}(-1)^{l_1+l_2}\left(\frac{a(p_2+q_2)}{b(p_1+q_1)}\right)^{l_1}\left(\frac{a}{c(p_1+q_1)}\right)^{l_2+1}\frac{\binom{nk_2-1}{l_1}\binom{n-2}{l_2}}{\binom{l_1+l_2+nk_1}{l_1+l_2}}, \\ B_3 &= \frac{n-1}{nk_2}\sum_{l_1=0}^{nk_1-1}\sum_{l_2=0}^{n-2}(-1)^{l_1+l_2}\left(\frac{b(p_1+q_1)}{a(p_2+q_2)}\right)^{l_1}\left(\frac{b}{c(p_2+q_2)}\right)^{l_2+1}\frac{\binom{nk_1-1}{l_1}\binom{n-2}{l_2}}{\binom{l_1+l_2+nk_2}{l_1+l_2}}, \end{aligned}$$

and

$$\textit{Case I:} \qquad \frac{c}{N} < \min\left\{\frac{a}{(p_1+q_1)N}, \frac{b}{(p_2+q_2)N}\right\},$$

$$\textit{Case II:}\qquad \frac{a}{(p_1+q_1)N} < \min\left\{\frac{c}{N}, \frac{b}{(p_2+q_2)N}\right\},$$
$$\textit{Case III:}\qquad \frac{b}{(p_2+q_2)N} < \min\left\{\frac{c}{N}, \frac{a}{(p_1+q_1)N}\right\}.$$

Proof. It can be easily proved that Y_1^*, U_{11}^* and V_{11}^* are random variables coming from the exponential distributions with means $\frac{1}{\alpha N}$, $\frac{1}{\lambda K_1}$ and $\frac{1}{\beta K_2}$, respectively. Therefore,

$$\phi(U_{11}^*, V_{11}^*, Y_1^*) = \begin{cases} 1 & K_1U_{11}^* > N(p_1+q_1)Y_1^*,\ K_2V_{11}^* > N(p_2+q_2)Y_1^* \\ 0 & \text{Otherwise,} \end{cases}$$

is an unbiased estimate of $\psi(\lambda, \beta, \alpha)$. So,

$$\begin{aligned}\hat{\psi}_U(\lambda,\beta,\alpha) &= E[\phi(U_{11}^*, V_{11}^*, Y_1^*)|A(\mu)=a, B(\mu)=b, C(\mu)=c] \\ &= \iiint_{\mathcal{A}} f_{U_{11}^*|A(\mu)=a}(u) f_{V_{11}^*|B(\mu)=b}(v) f_{Y_1^*|C(\mu)=c}(y)\,dudvdy,\end{aligned}$$

where

$$\begin{aligned}\mathcal{A} &= \Big\{(u,v,y): 0<y<\frac{c}{N}, 0<u<\frac{a}{K_1}, 0<v<\frac{b}{K_2}, K_1u > N(p_1+q_1)y, \\ &\quad K_2v > N(p_2+q_2)y\Big\}.\end{aligned}$$

Also, $f_{U_{11}^*|A(\mu)=a}(u)$, $f_{V_{11}^*|B(\mu)=b}(v)$ and $f_{Y_1^*|C(\mu)=c}(y)$ are defined in Lemma 3.1. For Case I, we have

$$\begin{aligned}\hat{\psi}_U(\alpha,\beta,\lambda) &= \int_0^{\frac{c}{N}} \int_{(p_2+q_2)\frac{N}{K_2}y}^{\frac{b}{K_2}} \int_{(p_1+q_1)\frac{N}{K_1}y}^{\frac{a}{K_1}} \frac{K_1(nk_1-1)(a-K_1u)^{nk_1-2}}{a^{nk_1-2}} \\ &\quad \times \frac{K_2(nk_2-1)(b-K_2v)^{nk_2-2}}{b^{nk_2-1}} \times \frac{N(n-1)(c-Ny)^{n-2}}{c^{n-1}} dudvdy \\ &= \int_0^{\frac{c}{N}} \left(\int_{(p_1+q_1)\frac{N}{K_1}y}^{\frac{b}{K_1}} \frac{K_1(nk_1-1)(a-K_1u)^{nk_1-2}}{a^{nk_1-1}} du\right) \\ &\quad \times \left(\int_{(p_2+q_2)\frac{N}{K_2}y}^{\frac{b}{K_2}} \frac{K_2(nk_2-1)(b-K_2v)^{nk_2-2}}{b^{nk_2-1}} dv\right) \frac{N(n-1)(c-Ny)^{n-2}}{c^{n-1}} dy \\ &= \frac{N(n-1)}{c} \int_0^{\frac{c}{N}} \left(1-(p_1+q_1)\frac{N}{a}y\right)^{nk_1-1} \left(1-(p_2+q_2)\frac{N}{b}y\right)^{nk_2-1} \\ &\quad \times \left(1-\frac{N}{c}y\right)^{n-2} dy \quad \left\{\text{Put: } t=\frac{Ny}{c}\right\} \\ &= (n-1)\int_0^1 (1-t)^{n-2} \left(1-(p_1+q_1)\frac{c}{a}t\right)^{nk_1-1} \left(1-(p_2+q_2)\frac{c}{b}t\right)^{nk_2-1} dt \\ &= (n-1)\int_0^1 (1-t)^{n-2} \left(\sum_{l_1=0}^{nk_1-1} (-1)^{l_1} \binom{nk_1-1}{l_1} ((p_1+q_1)\frac{c}{a}t)^{l_1}\right)\end{aligned}$$

$$\times\left(\sum_{l_2=0}^{nk_2-1}(-1)^{l_2}\binom{nk_2-1}{l_2}((p_2+q_2)\frac{c}{b}t)^{l_2}\right)dt$$
$$=\sum_{l_1=0}^{nk_1-1}\sum_{l_2=0}^{nk_2-1}(-1)^{l_1+l_2}\left(\frac{c(p_1+q_1)}{a}\right)^{l_1}\left(\frac{c(p_2+q_2)}{b}\right)^{l_2}\frac{\binom{nk_1-1}{l_1}\binom{nk_2-1}{l_2}}{\binom{l_1+l_2+n-1}{l_1+l_2}}.$$

For the other cases, the results can be obtained similarly. □

Hence, $\hat{R}^U_{\boldsymbol{s},\boldsymbol{k}}$, UMVUE of $R_{\boldsymbol{s},\boldsymbol{k}}$ is derived by

$$\hat{R}^U_{\boldsymbol{s},\boldsymbol{k}}=\sum_{p_1=s_1}^{k_1}\sum_{p_2=s_2}^{k_2}\sum_{q_1=0}^{k_1-p_1}\sum_{q_2=0}^{k_2-p_2}\binom{k_1}{p_1}\binom{k_2}{p_2}\binom{k_1-p_1}{q_1}\binom{k_2-p_2}{q_2}(-1)^{q_1+q_2}\hat{\psi}_U(\lambda,\beta,\alpha). \tag{25}$$

4 Data analysis and comparison study

This section compares different methods described in previous sections in terms of performance using Monte Carlo simulations. A real-world dataset is also analyzed and discussed.

4.1 Numerical experiments and simulations

This section evaluates the performance of different estimates obtained by Monte Carlo simulation by comparing point estimations based on MSEs. The performance of different interval estimations is also compared based on average confidence lengths and coverage percentages. A simulation study is implemented based on different censoring schemes, parameter values, and hyperparameters. The results are presented for 2000 repetitions. The censoring schemes employed in the simulation study are given in Table 1. In this Table, the notation a^{*b} shows that a is repeated b times.Also, K_i is the complete sample size for the strength i, and k_i is the observed sample size. For example, when $(k_1,K_1)=(5,10)$ and $R_1=(0^{*4},5)$, the complete sample is $(K_1=)10$ for X_1. Besides, using the censoring scheme $(0^{*4},5)$, 5 data points are omitted; therefore, the observed sample size becomes $(k_1=)5$. In the simulation study, when the common location parameter μ is unknown, the results are obtained based on the parameter values as $(\lambda,\beta,\alpha,\mu)=(2.5,2,1,3)$. In addition, three priors are assumed to consider the Bayesian inference as follows

$$\begin{aligned}&\text{Prior 1}: && a_i=0, && b_i=0, && i=1,2,3.\\&\text{Prior 2}: && a_i=1, && b_i=1, && i=1,2,3.\\&\text{Prior 3}: && a_i=2, && b_i=2, && i=1,2,3.\end{aligned}$$

In this case, the MLEs and Bayes estimates of $R_{\boldsymbol{s},\boldsymbol{k}}$ are calculated via Lindely's approximation and MCMC using (12), (18) and (19), respectively. The results are listed in Table 2. From Table 2, Bayes estimates outperform classical estimates, informative priors outperform non-informative priors in Bayes estimates, and MCMC outperforms

Table 1: Censoring schemes.

$(k_i, K_i),\ i = 1,2$		C.S.	(n, N)		C.S.
	R_1	$(0^{*4}$,5)		S_1	$(0^{*4}$,5)
(5,10)	R_2	(5,0^{*4})	(5,10)	S_2	(5,0^{*4})
	R_3	(1^{*5})		S_3	(1^{*5})
	R_4	$(0^{*9}$,10)		S_4	$(0^{*9}$,10)
(10,20)	R_5	(10,0^{*9})	(10,20)	S_5	(10,0^{*9})
	R_6	(1^{*10})		S_6	(1^{*10})

Lindley's approximation based on MSEs. Also, the 95% ACI and HPD credible intervals ($T_{bayes} = 3000$) for $R_{\boldsymbol{s},\boldsymbol{k}}$ are derived. Average lengths and corresponding coverage percentages are computed and given in Table 3. From Table 3, HPD credible intervals outperform ACIs in all cases. Also, in Bayes estimates, informative priors outperform non-informative priors based on the length of credible intervals and coverage percentages. Furthermore, from Tables 2 and 3, with increasing n for fixed $\boldsymbol{s}$ and $\boldsymbol{k}$ or $\boldsymbol{k}$ for fixed $\boldsymbol{s}$ and n, the performance of point and interval estimates has improved in all cases.

Now, when the common location parameter μ is known, the results are obtained based on parameter values as$(\lambda, \beta, \alpha, \mu) = (2, 3, 1, 2.5)$. Also, three priors are assumed to consider the Bayesian inference as follows

$$\begin{aligned} &\text{Prior 4}: \quad a_i = 0, \quad b_i = 0, \qquad i = 1,2,3. \\ &\text{Prior 5}: \quad a_i = 1, \quad b_i = 1, \qquad i = 1,2,3. \\ &\text{Prior 6}: \quad a_i = 2, \quad b_i = 2, \qquad i = 1,2,3. \end{aligned}$$

In this case, the MLEs, Bayes estimates, and UMVUE of $R_{\boldsymbol{s},\boldsymbol{k}}$ are estimated using (20), (23) and (25), respectively. The results are listed in Table 4. From Table 4, Bayes estimates outperform classical estimates, and informative priors outperform non-informative priors based on MSEs. Also, the 95% ACIs and HPD credible intervals are derived for $R_{\boldsymbol{s},\boldsymbol{k}}$. Besides, average lengths and corresponding coverage percentages are computed and given in Table 5. From Table 5, HPD credible intervals outperform ACIs in all cases. Also, in Bayes estimates, informative priors outperform non-informative priors based on the length of credible intervals and coverage percentages. Likewise, from Tables 4 and 5, with increasing n for fixed $\boldsymbol{s}$ and $\boldsymbol{k}$ or $\boldsymbol{k}$ for fixed $\boldsymbol{s}$ and n, the performance of point and interval estimates has improved in all cases.

Table 2: |Biases| and MSE of the MLE and Bayes estimates of $R_{\boldsymbol{s},\boldsymbol{k}}$ under various censoring schemes when μ is unknown.

(k_1,k_2,n,s_1,s_2)	C.S.	MLE		Lindley						Bayes					
				Prior 1		Prior 2		Prior 3		Prior 1		Prior 2		Prior 3	
		|Bias|	MSE	|Bias|	MSE	|Bias|	MSE	|Bias|	MSE	|Bias|	MSE	|Bias|	MSE	|Bias|	MSE
(5,5,5,2,2)	(R_1,R_1,S_1)	0.0134	0.0250	0.0143	0.0229	0.0151	0.0215	0.0100	0.0202	0.0118	0.0204	0.0149	0.0188	0.0114	0.0154
	(R_1,R_1,S_2)	0.0102	0.0248	0.0141	0.0220	0.0130	0.0215	0.0121	0.0206	0.0141	0.0201	0.0120	0.0198	0.0099	0.0149
	(R_1,R_1,S_3)	0.0090	0.0256	0.0151	0.0226	0.0114	0.0219	0.0143	0.0201	0.0154	0.0207	0.0125	0.0195	0.0151	0.0162
	(R_2,R_2,S_2)	0.009	0.0236	0.0111	0.0225	0.0135	0.0210	0.0125	0.0207	0.0144	0.0208	0.0144	0.0189	0.0124	0.0147
	(R_2,R_2,S_3)	0.0107	0.0257	0.0157	0.0224	0.0105	0.0212	0.0159	0.0204	0.0157	0.0213	0.0153	0.0195	0.0132	0.0159
	(R_3,R_3,S_3)	0.0140	0.0256	0.0110	0.0226	0.0103	0.0219	0.0144	0.0202	0.0136	0.0207	0.0105	0.0182	0.0144	0.0154
(5,5,10,2,2)	(R_1,R_1,S_4)	0.0101	0.0189	0.0104	0.0163	0.0124	0.0125	0.0124	0.0091	0.0152	0.0137	0.0153	0.0082	0.0144	0.0066
	(R_1,R_1,S_5)	0.0152	0.0185	0.0106	0.0144	0.0117	0.0136	0.0102	0.0098	0.0094	0.0127	0.0135	0.0102	0.0132	0.0052
	(R_1,R_1,S_6)	0.0127	0.0174	0.0113	0.0146	0.0134	0.0138	0.0098	0.0094	0.0155	0.0128	0.0121	0.0085	0.0159	0.0060
	(R_2,R_2,S_5)	0.0157	0.0177	0.0102	0.0156	0.0090	0.0134	0.0113	0.0108	0.0149	0.0133	0.0097	0.0089	0.0151	0.0078
	(R_2,R_2,S_6)	0.0115	0.0171	0.0140	0.0150	0.0133	0.0128	0.0093	0.0091	0.0121	0.0135	0.0122	0.0100	0.0127	0.0069
	(R_3,R_3,S_6)	0.0143	0.0176	0.0157	0.0164	0.0155	0.0131	0.0128	0.0109	0.0140	0.0126	0.0127	0.0097	0.0159	0.0057
(10,10,5,2,2)	(R_4,R_4,S_1)	0.0114	0.0202	0.0154	0.0173	0.0120	0.0130	0.0113	0.0106	0.0126	0.0125	0.0101	0.0094	0.0147	0.0070
	(R_4,R_4,S_2)	0.0136	0.0196	0.0096	0.0165	0.0138	0.0135	0.0104	0.0112	0.0143	0.0122	0.0141	0.0090	0.0092	0.0079
	(R_4,R_4,S_3)	0.0149	0.0193	0.0139	0.0159	0.0136	0.0140	0.0096	0.0116	0.0132	0.0133	0.0117	0.0103	0.0107	0.0072
	(R_5,R_5,S_2)	0.0090	0.0206	0.0156	0.0177	0.0143	0.0139	0.0140	0.0102	0.0116	0.0124	0.0151	0.0093	0.0148	0.0075
	(R_5,R_5,S_3)	0.0144	0.0198	0.0140	0.0172	0.0151	0.0149	0.0121	0.0109	0.0103	0.0130	0.0130	0.0095	0.0143	0.0088
	(R_6,R_6,S_3)	0.0122	0.0207	0.0138	0.0181	0.0136	0.0136	0.0151	0.0107	0.0105	0.0121	0.0155	0.0101	0.0106	0.0083
(10,10,10,2,2)	(R_4,R_4,S_4)	0.0144	0.0138	0.0117	0.0115	0.0103	0.0080	0.0146	0.0073	0.0092	0.0093	0.0145	0.0067	0.0112	0.0046
	(R_4,R_4,S_5)	0.0141	0.0131	0.0151	0.0112	0.0153	0.0092	0.0152	0.0079	0.0119	0.0091	0.0118	0.0065	0.0132	0.0031
	(R_4,R_4,S_6)	0.0145	0.0135	0.0135	0.0127	0.0146	0.0082	0.0150	0.0071	0.0093	0.0087	0.0148	0.0060	0.0115	0.0037
	(R_5,R_5,S_5)	0.0110	0.0132	0.0136	0.0100	0.0102	0.0089	0.0102	0.0075	0.0134	0.0099	0.0127	0.0067	0.0156	0.0049
	(R_5,R_5,S_6)	0.0095	0.0130	0.0123	0.0127	0.0140	0.0098	0.0125	0.0074	0.0098	0.0096	0.0113	0.0063	0.0094	0.0032
	(R_6,R_6,S_6)	0.0132	0.0135	0.0131	0.0114	0.0145	0.0080	0.0137	0.0072	0.0113	0.0089	0.0114	0.0059	0.0148	0.0046
(5,5,5,4,4)	(R_1,R_1,S_1)	0.0120	0.0366	0.0141	0.0243	0.0158	0.0227	0.0154	0.0181	0.0154	0.0236	0.0156	0.0200	0.0120	0.0170
	(R_1,R_1,S_2)	0.0124	0.0371	0.0128	0.0252	0.0136	0.0215	0.0121	0.0188	0.0106	0.0230	0.0155	0.0195	0.0123	0.0156
	(R_1,R_1,S_3)	0.0145	0.0282	0.0119	0.0240	0.0157	0.0222	0.0101	0.0179	0.0098	0.0227	0.0108	0.0185	0.0130	0.0159
	(R_2,R_2,S_2)	0.0102	0.0371	0.0131	0.0244	0.0127	0.0218	0.0122	0.0152	0.0111	0.0238	0.0136	0.0215	0.0128	0.0159
	(R_2,R_2,S_3)	0.0134	0.0278	0.0119	0.0255	0.0158	0.0220	0.0150	0.0178	0.0137	0.0224	0.0120	0.0207	0.0123	0.0168
	(R_3,R_3,S_3)	0.0143	0.0302	0.0107	0.0257	0.0102	0.0225	0.0138	0.0191	0.0138	0.0234	0.0147	0.0194	0.0149	0.0163

Continuation of Table 2.

(k_1, k_2, n, s_1, s_2)	C.S.	MLE		Lindley						Bayes					
				Prior 1		Prior 2		Prior 3		Prior 1		Prior 2		Prior 3	
		|Bias|	MSE	|Bias|	MSE	|Bias|	MSE	|Bias|	MSE	|Bias|	MSE	|Bias|	MSE	|Bias|	MSE
(5,5,10,4,4)	(R_1,R_1,S_4)	0.0120	0.0144	0.0104	0.0107	0.0118	0.0093	0.0157	0.0057	0.0104	0.0073	0.0126	0.0060	0.0129	0.0034
	(R_1,R_1,S_5)	0.0101	0.0150	0.0103	0.0116	0.0091	0.0091	0.0113	0.0056	0.0108	0.0079	0.0110	0.0045	0.0140	0.0036
	(R_1,R_1,S_6)	0.0128	0.0158	0.0148	0.0101	0.0109	0.0085	0.0109	0.0059	0.0153	0.0094	0.0137	0.0046	0.0100	0.0038
	(R_2,R_2,S_5)	0.0131	0.0142	0.0139	0.0113	0.0123	0.0087	0.0118	0.0062	0.0150	0.0088	0.0111	0.0051	0.0100	0.0039
	(R_2,R_2,S_6)	0.0111	0.0158	0.0119	0.0119	0.0155	0.0094	0.0123	0.0065	0.0141	0.0080	0.0144	0.0064	0.0091	0.0030
	(R_3,R_3,S_6)	0.0122	0.0148	0.0139	0.0121	0.0111	0.0084	0.0156	0.0058	0.0147	0.0093	0.0135	0.0046	0.0111	0.0035
(10,10,5,4,4)	(R_4,R_4,S_1)	0.0137	0.0171	0.0091	0.0143	0.0109	0.0110	0.0133	0.0089	0.0100	0.0100	0.0090	0.0096	0.0110	0.0066
	(R_4,R_4,S_2)	0.0099	0.0167	0.0106	0.0136	0.0155	0.0115	0.0151	0.0087	0.0105	0.0103	0.0150	0.0081	0.0155	0.0052
	(R_4,R_4,S_3)	0.0112	0.0173	0.0128	0.0148	0.0128	0.0120	0.0102	0.0093	0.0129	0.0107	0.0147	0.0098	0.0152	0.0062
	(R_5,R_5,S_2)	0.0105	0.0170	0.0118	0.0147	0.0133	0.0114	0.0144	0.0097	0.0108	0.0114	0.0130	0.0086	0.0098	0.0061
	(R_5,R_5,S_3)	0.0109	0.0163	0.0099	0.0150	0.0103	0.0117	0.0108	0.0096	0.0133	0.0100	0.0100	0.0089	0.0130	0.0052
	(R_6,R_6,S_3)	0.0150	0.0180	0.0090	0.0157	0.0098	0.0116	0.0159	0.0094	0.0149	0.0112	0.0125	0.0087	0.0147	0.0072
(10,10,10,4,4)	(R_4,R_4,S_4)	0.0127	0.0038	0.0127	0.0032	0.0144	0.0029	0.0142	0.0022	0.0159	0.0029	0.0127	0.0020	0.0128	0.0017
	(R_4,R_4,S_5)	0.0126	0.0035	0.0098	0.0031	0.0093	0.0027	0.0127	0.0021	0.0118	0.0028	0.0143	0.0024	0.0143	0.0011
	(R_4,R_4,S_6)	0.0156	0.0037	0.0108	0.0031	0.0122	0.0028	0.0146	0.0025	0.0112	0.0027	0.0135	0.0021	0.0108	0.0015
	(R_5,R_5,S_5)	0.0112	0.0037	0.0128	0.0032	0.0090	0.0027	0.0111	0.0023	0.0096	0.0026	0.0140	0.0023	0.0142	0.0012
	(R_5,R_5,S_6)	0.0104	0.0034	0.0137	0.0032	0.0091	0.0028	0.0140	0.0026	0.0139	0.0026	0.0091	0.0022	0.0140	0.0017
	(R_6,R_6,S_6)	0.0097	0.0039	0.0144	0.0033	0.0094	0.0029	0.0118	0.0024	0.0105	0.0027	0.0090	0.0020	0.0129	0.0015

Table 3: Average confidence/credible length and coverage percentage for estimators of $R_{\boldsymbol{s},\boldsymbol{k}}$ under various censoring schemes when μ is unknown.

(k_1, k_2, n, s_1, s_2)	C.S.	MLE		Bayes					
				Prior 1		Prior 2		Prior 3	
		Length	CP	Length	CP	Length	CP	Length	CP
(5,5,5,2,2)	(R_1,R_1,S_1)	0.5639	0.895	0.5376	0.943	0.5038	0.948	0.4785	0.952
	(R_1,R_1,S_2)	0.5600	0.897	0.5266	0.945	0.5043	0.949	0.4769	0.952
	(R_1,R_1,S_3)	0.5541	0.890	0.5388	0.942	0.5032	0.948	0.4852	0.950
	(R_2,R_2,S_2)	0.5904	0.893	0.5246	0.943	0.5100	0.949	0.4832	0.951
	(R_2,R_2,S_3)	0.5550	0.894	0.5276	0.945	0.5025	0.948	0.4889	0.950
	(R_3,R_3,S_3)	0.5852	0.895	0.5290	0.946	0.5004	0.949	0.4887	0.952
(5,5,10,2,2)	(R_1,R_1,S_4)	0.4156	0.902	0.3998	0.951	0.3753	0.954	0.3454	0.957
	(R_1,R_1,S_5)	0.4002	0.907	0.3892	0.950	0.3719	0.954	0.3444	0.958
	(R_1,R_1,S_6)	0.4135	0.907	0.3817	0.951	0.3734	0.954	0.3553	0.958
	(R_2,R_2,S_5)	0.4126	0.905	0.3826	0.950	0.3609	0.955	0.3438	0.959
	(R_2,R_2,S_6)	0.4037	0.906	0.3861	0.950	0.3680	0.954	0.3431	0.958
	(R_3,R_3,S_6)	0.4006	0.905	0.3859	0.951	0.3708	0.955	0.3414	0.958
(10,10,5,2,2)	(R_4,R_4,S_1)	0.4465	0.912	0.4286	0.946	0.3994	0.951	0.3644	0.961
	(R_4,R_4,S_2)	0.4411	0.918	0.4227	0.947	0.3917	0.950	0.3791	0.960
	(R_4,R_4,S_3)	0.4497	0.916	0.4378	0.947	0.4001	0.952	0.3612	0.962
	(R_5,R_5,S_2)	0.4465	0.917	0.4334	0.945	0.3962	0.951	0.3628	0.962
	(R_5,R_5,S_3)	0.4442	0.918	0.4269	0.945	0.3914	0.952	0.3696	0.960
	(R_6,R_6,S_3)	0.4473	0.916	0.4250	0.946	0.4063	0.952	0.3600	0.962
(10,10,10,2,2)	(R_4,R_4,S_4)	0.4083	0.925	0.3716	0.951	0.3363	0.958	0.3068	0.968
	(R_4,R_4,S_5)	0.3938	0.924	0.3772	0.952	0.3454	0.959	0.3052	0.967
	(R_4,R_4,S_6)	0.3965	0.928	0.3754	0.950	0.3400	0.959	0.3087	0.968
	(R_5,R_5,S_5)	0.3979	0.927	0.3675	0.951	0.3355	0.958	0.3133	0.967
	(R_5,R_5,S_6)	0.4086	0.927	0.3770	0.950	0.3405	0.959	0.3181	0.968
	(R_6,R_6,S_6)	0.3831	0.928	0.3795	0.951	0.3395	0.958	0.3069	0.967
(5,5,5,4,4)	(R_1,R_1,S_1)	0.5456	0.895	0.5204	0.942	0.4899	0.948	0.4699	0.953
	(R_1,R_1,S_2)	0.5454	0.896	0.5326	0.940	0.4816	0.949	0.4619	0.952
	(R_1,R_1,S_3)	0.5582	0.896	0.5224	0.942	0.4995	0.948	0.4504	0.953
	(R_2,R_2,S_2)	0.5599	0.891	0.5280	0.941	0.4913	0.949	0.4685	0.953
	(R_2,R_2,S_3)	0.5438	0.891	0.5249	0.943	0.4994	0.948	0.4531	0.953
	(R_3,R_3,S_3)	0.5508	0.890	0.5374	0.944	0.4873	0.949	0.4584	0.954
(5,5,10,4,4)	(R_1,R_1,S_4)	0.4073	0.905	0.3409	0.948	0.3171	0.952	0.2856	0.958
	(R_1,R_1,S_5)	0.4158	0.906	0.3387	0.949	0.3116	0.953	0.2869	0.958
	(R_1,R_1,S_6)	0.4190	0.906	0.3407	0.948	0.3116	0.950	0.2866	0.958
	(R_2,R_2,S_5)	0.4132	0.907	0.3418	0.946	0.3138	0.952	0.2902	0.959
	(R_2,R_2,S_6)	0.4118	0.907	0.3320	0.948	0.3147	0.953	0.2822	0.959
	(R_3,R_3,S_6)	0.4180	0.906	0.3401	0.947	0.3206	0.953	0.2852	0.957
(10,10,5,4,4)	(R_4,R_4,S_1)	0.4215	0.918	0.3769	0.950	0.3435	0.955	0.3033	0.961
	(R_4,R_4,S_2)	0.4260	0.915	0.3775	0.951	0.3366	0.954	0.3020	0.960
	(R_4,R_4,S_3)	0.4238	0.918	0.3753	0.952	0.3488	0.955	0.3019	0.961
	(R_5,R_5,S_2)	0.4220	0.916	0.3767	0.950	0.3371	0.954	0.3179	0.964
	(R_5,R_5,S_3)	0.4249	0.913	0.3611	0.951	0.3412	0.955	0.3170	0.962
	(R_6,R_6,S_3)	0.4195	0.913	0.3660	0.952	0.3414	0.955	0.3090	0.962
(10,10,10,4,4)	(R_4,R_4,S_4)	0.3336	0.926	0.3079	0.956	0.2795	0.958	0.2487	0.968
	(R_4,R_4,S_5)	0.3312	0.927	0.3045	0.956	0.2860	0.959	0.2480	0.967
	(R_4,R_4,S_6)	0.3384	0.927	0.2971	0.957	0.2775	0.958	0.2547	0.968
	(R_5,R_5,S_5)	0.3484	0.927	0.3091	0.955	0.2759	0.959	0.2478	0.967
	(R_5,R_5,S_6)	0.3458	0.928	0.3088	0.957	0.2825	0.959	0.2535	0.968
	(R_6,R_6,S_6)	0.3392	0.926	0.2933	0.956	0.2866	0.959	0.2463	0.967

Table 4: |Biases| and MSE of the MLE and Bayes estimates of $R_{\boldsymbol{s},\boldsymbol{k}}$ under various censoring schemes when μ is known.

(k_1, k_2, n, s_1, s_2)	C.S.	MLE		Bayes							
				Prior 4		Prior 5		Prior 6		UMVUE	
		|Bias|	MSE	|Bias|	MSE	|Bias|	MSE	|Bias|	MSE	|Bias|	MSE
(5,5,5,2,2)	(R_1,R_1,S_1)	0.0132	0.0248	0.0097	0.0203	0.0123	0.0196	0.0123	0.0159	0.0112	0.0250
	(R_1,R_1,S_2)	0.0100	0.0252	0.0144	0.0213	0.0103	0.0189	0.0149	0.0152	0.0104	0.0258
	(R_1,R_1,S_3)	0.0104	0.0244	0.0091	0.0204	0.0132	0.0186	0.0147	0.0165	0.0094	0.0249
	(R_2,R_2,S_2)	0.0152	0.0255	0.0152	0.0206	0.0122	0.0189	0.0156	0.0148	0.0130	0.0260
	(R_2,R_2,S_3)	0.0148	0.0243	0.0154	0.0215	0.0140	0.0199	0.0150	0.0157	0.090	0.0249
	(R_3,R_3,S_3)	0.0098	0.0246	0.0150	0.0216	0.0135	0.0180	0.0154	0.0165	0.0142	0.0252
(5,5,10,2,2)	(R_1,R_1,S_4)	0.0139	0.0173	0.0147	0.0119	0.0153	0.0088	0.0090	0.0067	0.0120	0.0182
	(R_1,R_1,S_5)	0.0109	0.0159	0.0149	0.0117	0.0147	0.0100	0.0095	0.0075	0.0098	0.0180
	(R_1,R_1,S_6)	0.0156	00163	0.0135	0.0123	0.0095	0.0105	0.0107	0.0061	0.0106	0.0190
	(R_2,R_2,S_5)	0.0141	0.0167	0.0134	0.0116	0.0090	0.0106	0.0127	0.0063	0.0134	0.0175
	(R_2,R_2,S_6)	0.0134	0.0165	0.0102	0.0127	0.0136	0.0107	0.0095	0.0069	0.0102	0.0182
	(R_3,R_3,S_6)	0.0109	0.0170	0.0096	0.0119	0.0121	0.0101	0.0146	0.0075	0.0120	0.0183
(10,10,5,2,2)	(R_4,R_4,S_1)	0.0145	0.0222	0.0098	0.0148	0.0141	0.0101	0.0125	0.0081	0.0108	0.0241
	(R_4,R_4,S_2)	0.0131	0.0227	0.0093	0.0132	0.0120	0.0107	0.0106	0.0077	0.0135	0.0239
	(R_4,R_4,S_3)	0.0128	0.0220	0.0158	0.0130	0.0094	0.0106	0.0127	0.0088	0.0143	0.0230
	(R_5,R_5,S_2)	0.0135	0.0217	0.0096	0.0122	0.0119	0.0102	0.0115	0.0072	0.0153	0.0228
	(R_5,R_5,S_3)	0.0137	0.0218	0.0142	0.0144	0.0112	0.0100	0.0128	0.0076	0.0159	0.0228
	(R_6,R_6,S_3)	0.0120	0.0216	0.0095	0.0135	0.0128	0.0116	0.0116	0.0083	0.0108	0.0230
(10,10,10,2,2)	(R_4,R_4,S_4)	0.0112	0.0131	0.0104	0.0082	0.0112	0.0074	0.0109	0.0048	0.0127	0.0140
	(R_4,R_4,S_5)	0.0094	0.0136	0.0153	0.0094	0.0107	0.0058	0.0141	0.0040	0.0128	0.0142
	(R_4,R_4,S_6)	0.0155	0.0134	0.0102	0.0090	0.0136	0.0052	0.0091	0.0046	0.0134	0.0145
	(R_5,R_5,S_5)	0.0090	0.0130	0.0136	0.0092	0.0137	0.0060	0.0119	0.0031	0.0147	0.0140
	(R_5,R_5,S_6)	0.0116	0.0129	0.0151	0.0095	0.0143	0.0074	0.0155	0.0048	0.0128	0.0144
	(R_6,R_6,S_6)	0.0129	0.0133	0.0093	0.0099	0.0146	0.0055	0.0099	0.0046	0.0133	0.0143
(5,5,5,4,4)	(R_1,R_1,S_1)	0.0137	0.0314	0.0108	0.0234	0.0130	0.0192	0.0128	0.0176	0.0123	0.0331
	(R_1,R_1,S_2)	0.0151	0.0295	0.0150	0.0220	0.0115	0.0187	0.0157	0.0152	0.0106	0.0340
	(R_1,R_1,S_3)	0.0114	0.0323	0.0097	0.0222	0.0148	0.0217	0.0092	0.0157	0.0094	0.0330
	(R_2,R_2,S_2)	0.0158	0.0284	0.0135	0.0234	0.0134	0.0187	0.0109	0.0160	0.0147	0.0322
	(R_2,R_2,S_3)	0.0136	0.0317	0.0115	0.0224	0.0113	0.0210	0.0132	0.0150	0.0150	0.0325
	(R_3,R_3,S_3)	0.0098	0.0325	0.0151	0.0222	0.0149	0.0190	0.0114	0.0176	0.0114	0.0345

Continuation of Table 4.

(k_1, k_2, n, s_1, s_2)	C.S.	MLE		Bayes						UMVUE	
				Prior 4		Prior 5		Prior 6			
		|Bias|	MSE	|Bias|	MSE	|Bias|	MSE	|Bias|	MSE	|Bias|	MSE
(5,5,10,4,4)	(R_1,R_1,S_4)	0.0113	0.0157	0.0117	0.0083	0.0108	0.0062	0.0103	0.0034	0.0133	0.0165
	(R_1,R_1,S_5)	0.0110	0.0159	0.0107	0.0085	0.0146	0.0047	0.0116	0.0032	0.0154	0.0168
	(R_1,R_1,S_6)	0.0109	0.0154	0.0159	0.0089	0.0127	0.0040	0.0143	0.0034	0.0096	0.0163
	(R_2,R_2,S_5)	0.0111	0.0148	0.0117	0.0070	0.0091	0.0052	0.0103	0.0038	0.0092	0.0162
	(R_2,R_2,S_6)	0.0141	0.0154	0.0102	0.0072	0.0152	0.0067	0.0108	0.0031	0.0144	0.0167
	(R_3,R_3,S_6)	0.0103	0.0151	0.0136	0.0077	0.0122	0.0055	0.0116	0.0036	0.0102	0.0160
(10,10,5,4,4)	(R_4,R_4,S_1)	0.0129	0.0165	0.0120	0.0106	0.0128	0.0083	0.0142	0.0077	0.0135	0.0174
	(R_4,R_4,S_2)	0.0105	0.0169	0.0152	0.0105	0.0129	0.0080	0.0098	0.0078	0.0143	0.0179
	(R_4,R_4,S_3)	0.0148	0.0154	0.0136	0.0121	0.0107	0.0081	0.0158	0.0077	0.0114	0.0170
	(R_5,R_5,S_2)	0.0145	0.0163	0.0146	0.0109	0.0152	0.0083	0.0110	0.0059	0.0111	0.0176
	(R_5,R_5,S_3)	0.0091	0.0175	0.0093	0.0112	0.0108	0.0087	0.0143	0.0076	0.0095	0.0182
	(R_6,R_6,S_3)	0.0099	0.0173	0.0123	0.0118	0.0143	0.0080	0.0149	0.0066	0.0132	0.0189
(10,10,10,4,4)	(R_4,R_4,S_4)	0.0143	0.0030	0.0101	0.0028	0.0138	0.0021	0.0107	0.0011	0.0090	0.0050
	(R_4,R_4,S_5)	0.0115	0.0031	0.0125	0.0027	0.0147	0.0020	0.0151	0.0012	0.0101	0.0053
	(R_4,R_4,S_6)	0.0100	0.0034	0.0093	0.0028	0.0136	0.0023	0.0144	0.0014	0.0094	0.0049
	(R_5,R_5,S_5)	0.0133	0.0037	0.0105	0.0030	0.0132	0.0027	0.0152	0.0016	0.0117	0.0057
	(R_5,R_5,S_6)	0.0124	0.0035	0.0142	0.0029	0.0140	0.0024	0.0104	0.0011	0.0143	0.0045
	(R_6,R_6,S_6)	0.0096	0.0034	0.0091	0.0031	0.0153	0.0020	0.0121	0.0010	0.0147	0.0049

Table 5: Average confidence/credible length and coverage percentage for estimators of $R_{\boldsymbol{s},\boldsymbol{k}}$ under various censoring schemes when μ is known.

(k_1, k_2, n, s_1, s_2)	C.S.	MLE		Bayes					
				Prior 4		Prior 5		Prior 6	
		Length	CP	Length	CP	Length	CP	Length	CP
	(R_1,R_1,S_1)	0.5842	0.896	0.5399	0.941	0.5019	0.948	0.4767	0.950
	(R_1,R_1,S_2)	0.5820	0.894	0.5323	0.940	0.5046	0.947	0.4724	0.952
(5,5,5,2,2)	(R_1,R_1,S_3)	0.5744	0.890	0.5334	0.940	0.5092	0.948	0.4613	0.952
	(R_2,R_2,S_2)	0.5754	0.895	0.5250	0.942	0.5059	0.947	0.4735	0.951
	(R_2,R_2,S_3)	0.5779	0.896	0.5356	0.941	0.5004	0.948	0.4747	0.950
	(R_3,R_3,S_3)	0.5867	0.894	0.5354	0.940	0.5080	0.948	0.4607	0.952
	(R_1,R_1,S_4)	0.4028	0.902	0.3845	0.950	0.3655	0.955	0.3454	0.957
	(R_1,R_1,S_5)	0.4081	0.903	0.3970	0.950	0.3628	0.954	0.3469	0.958
(5,5,10,2,2)	(R_1,R_1,S_6)	0.4040	0.902	0.3802	0.951	0.3668	0.955	0.3468	0.957
	(R_2,R_2,S_5)	0.4122	0.905	0.3915	0.951	0.3669	0.954	0.3400	0.957
	(R_2,R_2,S_6)	0.4002	0.905	0.3819	0.951	0.3708	0.954	0.3434	0.958
	(R_3,R_3,S_6)	0.4192	0.902	0.3828	0.951	0.3685	0.955	0.3498	0.958
	(R_4,R_4,S_1)	0.4427	0.915	0.4236	0.945	0.3975	0.950	0.3692	0.962
	(R_4,R_4,S_2)	0.4487	0.960	0.4247	0.946	0.4056	0.951	0.3630	0.960
(10,10,5,2,2)	(R_4,R_4,S_3)	0.4489	0.917	0.4275	0.945	0.3989	0.951	0.3657	0.962
	(R_5,R_5,S_2)	0.4483	0.915	0.4222	0.942	0.3967	0.950	0.3667	0.961
	(R_5,R_5,S_3)	0.4477	0.915	0.4269	0.944	0.4020	0.952	0.3701	0.961
	(R_6,R_6,S_3)	0.4412	0.915	0.4294	0.945	0.3981	0.952	0.3687	0.962
	(R_4,R_4,S_4)	0.4014	0.924	0.3635	0.950	0.3305	0.957	0.3073	0.967
	(R_4,R_4,S_5)	0.4001	0.926	0.3622	0.952	0.3335	0.958	0.3062	0.968
(10,10,10,2,2)	(R_4,R_4,S_6)	0.3962	0.926	0.3747	0.952	0.3329	0.957	0.3077	0.967
	(R_5,R_5,S_5)	0.4070	0.924	0.3685	0.950	0.3315	0.957	0.3151	0.968
	(R_5,R_5,S_6)	0.4068	0.926	0.3784	0.951	0.3373	0.957	0.3010	0.967
	(R_6,R_6,S_6)	0.3941	0.926	0.3622	0.952	0.3351	0.958	0.3032	0.968
	(R_1,R_1,S_1)	0.5498	0.895	0.5335	0.942	0.4890	0.947	0.4598	0.951
	(R_1,R_1,S_2)	0.5420	0.895	0.5301	0.940	0.4845	0.948	0.4609	0.952
(5,5,5,4,4)	(R_1,R_1,S_3)	0.5586	0.896	0.5351	0.941	0.4944	0.947	0.4530	0.952
	(R_2,R_2,S_2)	0.5565	0.892	0.5252	0.940	0.4872	0.948	0.4590	0.950
	(R_2,R_2,S_3)	0.5504	0.892	0.5314	0.941	0.4855	0.949	0.4635	0.951
	(R_3,R_3,S_3)	0.5592	0.895	0.5310	0.942	0.4929	0.948	0.4575	0.952
	(R_1,R_1,S_4)	0.4039	0.901	0.3368	0.947	0.3250	0.950	0.2820	0.957
	(R_1,R_1,S_5)	0.4034	0.905	0.3325	0.948	0.3190	0.953	0.2936	0.958
(5,5,10,4,4)	(R_1,R_1,S_6)	0.4105	0.903	0.3472	0.948	0.3157	0.953	0.2900	0.957
	(R_2,R_2,S_5)	0.4084	0.905	0.3380	0.949	0.3277	0.952	0.2817	0.958
	(R_2,R_2,S_6)	0.4075	0.905	0.3337	0.948	0.3004	0.953	0.2969	0.959
	(R_3,R_3,S_6)	0.4035	0.903	0.3330	0.949	0.3156	0.952	0.2864	0.958
	(R_4,R_4,S_1)	0.4163	0.914	0.3838	0.950	0.3304	0.955	0.3116	0.962
	(R_4,R_4,S_2)	0.4151	0.915	0.3868	0.951	0.3351	0.954	0.3017	0.962
(10,10,5,4,4)	(R_4,R_4,S_3)	0.4170	0.918	0.3845	0.950	0.3323	0.955	0.3104	0.960
	(R_5,R_5,S_2)	0.4201	0.917	0.3715	0.951	0.3377	0.954	0.3045	0.961
	(R_5,R_5,S_3)	0.4188	0.917	0.3828	0.950	0.3382	0.955	0.3109	0.962
	(R_6,R_6,S_3)	0.4213	0.914	0.3767	0.951	0.3325	0.956	0.3192	0.962
	(R_4,R_4,S_4)	0.3449	0.923	0.3180	0.956	0.2772	0.959	0.2494	0.967
	(R_4,R_4,S_5)	0.3381	0.924	0.3009	0.955	0.2732	0.958	0.2410	0.968
(10,10,10,4,4)	(R_4,R_4,S_6)	0.3398	0.922	0.3140	0.955	0.2767	0.959	0.2406	0.968
	(R_5,R_5,S_5)	0.3394	0.922	0.2987	0.956	0.2812	0.958	0.2334	0.967
	(R_5,R_5,S_6)	0.3478	0.923	0.3045	0.955	0.2783	0.959	0.2420	0.968
	(R_6,R_6,S_6)	0.3458	0.923	0.3095	0.956	0.2766	0.958	0.2338	0.967

4.2 Data analysis

This section analyzes monthly water capacity of the Shasta reservoir in California, USA, for illustrative purposes. Required data can be found at http://cdec.water.ca.gov/cgi-progs/queryMonthly?SHA, which have been used by some authors like Kizilaslan and Nadar (2016, 2018), Kohansal (2019) and Kohansal and Shoaee (2021). It is crucial to determine the probability of drought occurrences as they cause significant harm to agricultural productivity. In what follows, we conclude that there is no drought when a region's reservoir capacity in August and July for at least two of the following five years exceeds the total amount of water reached in December of the preceding year. Hence, the MCSS reliability with two non-identical-component strengths can be considered the probability of drought non-occurrence.

In this scenario, we consider $U_{11}, \ldots, U_{15}$ and $V_{11}, \ldots, V_{15}$ as the capacities of July and August from 1976 to 1980, $U_{21}, \ldots, U_{25}$ and $V_{21}, \ldots, V_{25}$ as the capacities of July and August from 1982 to 1986, and so on $U_{71}, \ldots, U_{75}$ and $V_{71}, \ldots, V_{75}$ as the capacities of July and August from 2012 to 2016. Also, $Y_1, Y_2, \ldots, Y_7$ are the capacity of December 1975, 1981 up to 2011. For simplicity the calculations, all data points are divided by the total capacity of Shasta reservoir, 4552000 acre-foot. We noted that this work will not have any effect on statistical inference. It should be initially checked whether tRD can be fitted to the data. This point can be further enhanced by including a detailed goodness of fit analysis. For this purpose, some statistical goodness of fit tests (e.g., Kolmogorov-Smirnov test, Cramer-von Mises test, and Anderson-Darling test) are first provided, whose results are listed in Table 6. The p-values suggest that tRD provides an adequate fit for the data in three cases. Second, charts and graphs are utilized to clearly illustrate how well the model fits the data. Accordingly, Q-Q and P-P plots were provided to visually assess the fit of tRD to the data. Histograms, density plots overlaid, and empirical CDF plots with fitted distribution curves were also given. The results are presented in Figs. 1-3.

Table 6: Statistical test for goodness of fit test.

	KS		CvM		AD	
	Statistics	p-value	Statistics	p-value	Statistics	p-value
X_1	0.1200	0.4987	0.0564	0.5454	0.4649	0.4836
X_2	0.1142	0.5163	0.0562	0.5472	0.4614	0.4880
Y	0.1857	0.8307	0.0360	0.7686	0.2372	0.7704

For complete data sets, we set $\boldsymbol{s} = (2, 2)$ and $\boldsymbol{k} = (5, 5)$. All results are provided in Table 7. Moreover, we consider two progressive censoring schemes as follows:

$$\text{Scheme 1:} \quad R = Q = [0, 1, 0, 0], \; S = [0, 0, 0, 1, 0, 0], \; (\boldsymbol{k} = (4, 4), \boldsymbol{s} = (2, 2)).$$
$$\text{Scheme 2:} \quad R = Q = [1, 1, 0], \quad S = [1, 0, 0, 0, 1], \quad (\boldsymbol{k} = (3, 3), \boldsymbol{s} = (1, 1)).$$

For these censoring schemes, we provide the results in Table 7. To observe the effect of hyper-parameters, Bayes estimates are obtained with informative priors. The hyper-parameters are obtained as $a_1 = 7.8, \; b_1 = 2, \; a_2 = 8, \; b_2 = 1.5, \; a_3 = 7.7, \; b_3 = 2.3$ using the resampling method. Accordingly, Bayesian results are derived again and listed in Table 7.

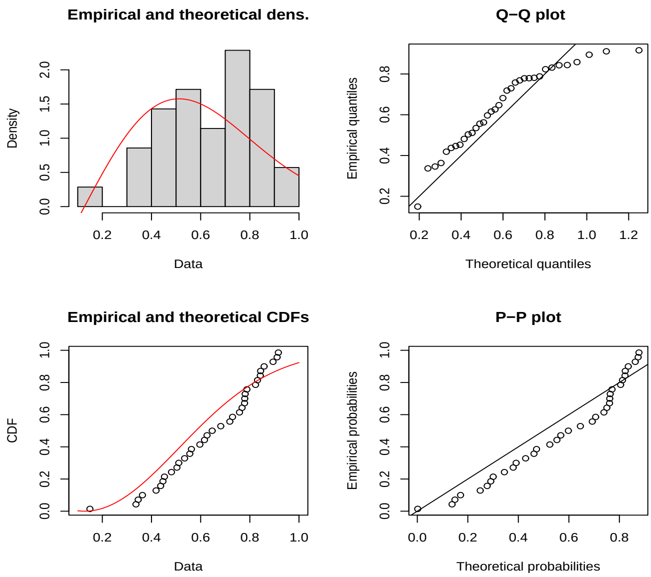

Figure 1: Charts and graphs for goodness of fit, for X_1.

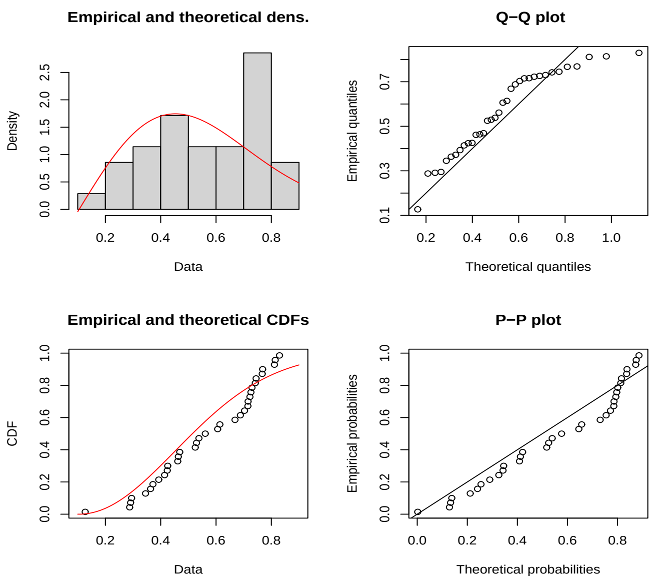

Figure 2: Charts and graphs for goodness of fit, for X_2.

A comparison of the results shows that some useful information will be missed if data is censored. It can also be observed that the Bayesian inference depends on hyper-parameters. Hence, informative priors should be used if they are available.

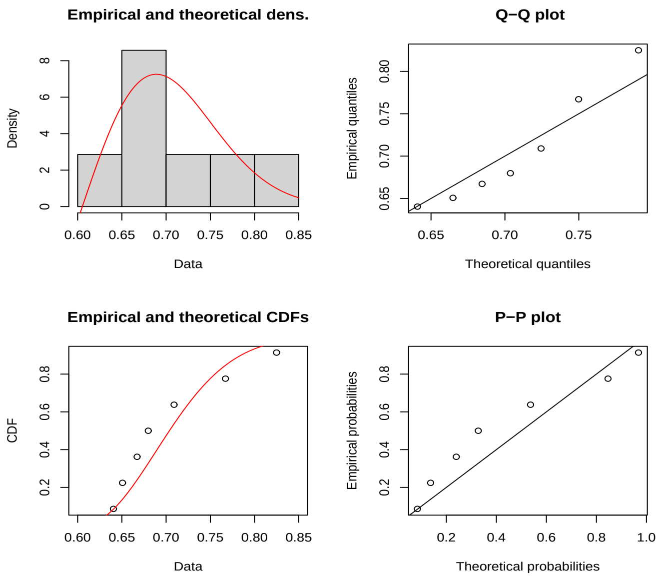

Figure 3: Charts and graphs for goodness of fit, for Y.

Table 7: Results of Section 4.2.

	Complete	Scheme 1	Scheme 2
MLE	0.4708	0.4130	0.6093
Asy. CI	(0.2261,0.7155)	(0.1668,0.6593)	(0.3130,0.9057)
Bayes (infor.)			
Lindley	0.4610	0.3980	0.6171
MCMC	0.4591	0.3996	0.5748
HPD	(0.2418,0.7009)	(0.1690,0.6367)	(0.2721,0.8312)
Bayes(non-infor.)			
Lindley	0.4957	0.4499	0.6479
MCMC	0.4862	0.3914	0.6120
HPD	(0.3211,0.6573)	(0.2312,0.5725)	(0.3902,0.7986)

5 Discussion and conclusions

This paper obtained different estimates of $R_{\boldsymbol{s},\boldsymbol{k}}$ for tRD under the progressive censoring scheme. When the common location parameter μ was unknown, the MLEs of $R_{\boldsymbol{s},\boldsymbol{k}}$ were obtained using the numerical method. Also, the ACI of this parameter was derived using its asymptotic distribution. Moreover, the Bayes estimate of $R_{\boldsymbol{s},\boldsymbol{k}}$ was approximated using Lindley's approximation and MCMC due to the lack of explicit forms. In addition to, when the common location parameter μ was known, MLEs, exact Bayes estimates, UMVUE, ACI, and HPD credible intervals were derived for $R_{\boldsymbol{s},\boldsymbol{k}}$. According to the simulation results, Bayes estimates outperformed classical estimates, informative priors outperformed non-informative priors, and MCMC outperformed Lindley's approximation. This work could be applied in the context of reliability theory and censored data analysis. Further studies can be conducted in this regard by extending the progressive censoring schemes to the progressive hybrid and adaptive progressive hybrid censoring schemes.

Acknowledgement

The authors want to say thank you to the editor and anonymous reviewers for their helpful comments that improved the work.

References

Ahmed, S.E., Belaghi, R.A., Hussein, A. and Safariyan, A. (2024). New and efficient estimators of reliability characteristics for a family of lifetime distributions under progressive censoring. *Mathematics*, **12**(10):1599.

Balakrishnan, N. and Aggarwala, R. (2000). *Progressive Censoring: Theory, Methods and Applications*, Boston: Birkhäuser.

Bhattacharyya, G.K. and Johnson, R.A. (1974). Estimation of reliability in multicomponent stress-strength model. *Journal of the American Statistical Association*, **69**(348):966–970.

Birnbaum, Z.W. (1956). On a use of the Mann-Whitney statistic. *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*, **1**:13–17.

Chen, M.H. and Shao, Q.M. (1999). Monte Carlo estimation of Bayesian Credible and HPD intervals. *Journal of Computational and Graphical Statistics*, **8**(1):69–92.

Dey, S., Dey, T. and Kundu, D. (2014). Two-parameter Rayleigh distribution: different methods of estimation. *American Journal of Mathematical and Management Sciences*, **33**(1):55–74.

Epstein, B. (1954). Truncated life tests in the exponential case. *The Annals of Mathematical Statistics*, **25**:555–564.

Kizilaslan, F. and Nadar, M. (2016). Estimation and prediction of the Kumaraswamy distribution based on record values and inter-record times. *Journal of Statistical Computation and Simulation*, **86**(12):2471–2493.

Kizilaslan, F. and Nadar, M. (2018). Estimation of reliability in a multicomponent stress-strength model based on a bivariate Kumaraswamy distribution. *Statistical Papers*, **59**:307–340.

Kohansal, A. and Rezakhah, S. (2019). Inference of $R = P(Y < X)$ for two-parameter Rayleigh distribution based on progressively censored samples. *Statistics*, **53**(1):81–100.

Kohansal, A. (2019). On estimation of reliability in a multicomponent stress-strength model for a Kumaraswamy distribution based on progressively censored sample. *Statistical Papers*, **60**:2185–2224.

Kohansal, A. and Shoaee, S. (2021). Bayesian and classical estimation of reliability in a multicomponent stress-strength model under adaptive hybrid progressive censored data. *Statistical Papers*, **62**(1):309–359.

Kohansal, A., Fernández, A.J. and Pérez-González, C.J. (2021). Multi-component stress–strength parameter estimation of a non-identical-component strengths system under the adaptive hybrid progressive censoring samples. *Statistics*, **55**(4):925–962.

Kotz, S., Lumelskii, I. and Pensky, M. (2003). *The Stress-Strength Model and Its Generalizations: Theory and Applications.* Singapore: World Scientific Publishing Company.

Kumari, A., Ghosh, I. and Kumar, K. (2024). Bayesian and likelihood estimation of multicomponent stress–strength reliability from power Lindley distribution based on progressively censored samples. *Journal of Statistical Computation and Simulation*, **94**(5):923–964.

Lindley, D.V. (1980). Approximate Bayesian methods. *Trabajos de Estadistica Y de Investigacion Operativa*, **3**(4):281–288.

Makhdoom, I., Yaghoobzadeh Shahrastani, S. and Pak, A. (2023). E-Bayesian and hierarchical Bayesian estimation of reliability in multicomponent stress-strength model based on inverse Rayleigh distribution. *Journal of Statistical Modelling: Theory and Applications*, **4**(1):95–106.

Nadar, M. and Kizilaslan, F. (2016). Estimation of reliability in a multicomponent stress-strength model based on a Marshall-Olkin bivariate weibull distribution. *IEEE Transactions on Reliability*, **65**(1):370–380.

Rasethuntsa, T.R. and Nadar, M. (2018). Stress-strength reliability of a non-identical-component strengths system based on upper record values from the family of Kumaraswamy generalized distributions. *Statistics*, **52**(3):684–716.

Safariyan, A., Arashi, M. and Arabi Belaghi, R. (2019). Improved point and interval estimation of the stress-strength reliability based on ranked set sampling. *Statistics*, **53**(1):101–125.

Safariyan, A. and Arabi Belaghi, R. (2021). Estimation for the reliability characteristics of a family of lifetime distributions under progressive censoring. *Journal of Data Science and Modeling*, **1**(2):71–86.

Singh, K., Mahto, A.K., Tripathi, Y.M. and Wang, L. (2024). Estimation in a multicomponent stress-strength model for progressive censored lognormal distribution. *Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability*, **238**(3):622–642.