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A model for multivariate longitudinal rank data with application to glioma patients

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Abstract: This paper proposes a model to analyze longitudinal rank responses using a Bayesian approach with a random effects framework. We consider rank responses that are implicitly determined by their latent variables. Further, the usual univariate model, as well as a multivariate model, is also considered for analyzing the multiple longitudinal rank responses. We use random effect vectors to evaluate the correlation between individual responses across time. Also, a Bayesian approach that is used to yield Bayesian estimates of the model's parameters. Some simulation studies are conducted to estimate the parameters of the considered models. The model is used for a neurocognitive data set of Glioma patients who underwent surgery. The results of the data analysis are presented to illustrate the method.

Keywords: Latent variable; Longitudinal rank data; Neurocognitive data set; Random effect.
Mathematics Subject Classification (2010): 62P10, 62H12, 62J05

1 Introduction

Ranking is an integral part of statistics, both in nonparametric analysis and in the analysis of objects ranking by judges. Ranking data usually comes from cases where one wants to rank a set of individuals or objects based on some criteria. Such data may observe directly or may obtain from ranking a set of assigned scores. The first case happens when only one order (or partial order) is placed without any quantitative superiority criteria. The second happens when quantitative measurements are available, but for other considerations it is preferable to use the rank transformation. Data

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ranking techniques have attracted the attention of many researchers. Many efforts have been made to determine methods for analysing rank data. There are unique challenges in ranking data analytic, especially in multivariate models. Also, the presence of longitudinal rank data is one of the most important and common issues in statistical applications such as medicine, clinical studies and public health. As a motivational example, examining cognitive function using neurocognitive test data on Glioma patients (Gliomas are primary brain tumors) was considered to illustrate the issue and the importance of the univariate and multivariate models studied in this paper and to determine the effects of some explanations on considered longitudinal rank variables of patients' cognitive functions. The cognitive performance of these patients was evaluated at four different time points during the week before and after the operation (pre-operation and post-operation), 3 and 6 months later. Therefore, these data were treated as a longitudinal rank response study.

1.1 Related works

The papers and discussions address these basic questions: How can ranking be best done? What goes on in the mind of a ranking? How does a statistician model and analyse such highly structured data? Thurstone models (Thurstone, 1927, 1931) are the basic and major models for modeling ranking data. Much recent work has focused on the estimation and application of Thurstonian ranking models (Chan and Bentler, 1998; Yao and Böckenholt, 1999). A Thurstonian approach is attractive because it facilitates the simple formulation of both linear models for the means and structural equations for the covariance matrix of preference judgments. Consequently, Thurstonian models provide a simple and interpretable representation of individual preference differences. However, rankings or weak orders contain a lot of information about how individuals differ in their evaluations of selected items.

A Thurstonian approach is appropriate for analysing these flavor differences without making strong assumptions about their determinants. Among others, Daniels (1950); Stern (1990); Böckenholt (1992) and Böckenholt (1993) introduced models, along with other methods for analysing rank data, see Marden (1995). Marden (1995) reviewed some classical techniques for analysis of rank data and introduced summary of considerations for data with ties, partial orderings, and incomplete rankings. One way to deal with ranking data is to build a model that describes how the ranking data was generated and use that model to create a ranking list (McFadden, 1980; Diaconis, 1988; Critchlow et al., 1991; Alvo and Yu, 2014). Thurstone assumes that a ranking data set is the result of ranking of latent continuous variables associated with each object to rank (Johnson and Kuhn, 2013).

Kendall and Smith (1940) and Marden (1995) introduced paired comparison models like a rank to the result of a paired comparison process. One of the most popular models that covers the situation of paired comparison is derived by Bradley and Terry (1952). The basic BT (Bradley and Terry, 1952) model has been extensively discussed in the literature (David, 1988) and various extensions have been suggested. To name just a few of those: ties (Rao and Kupper, 1967; Davidson, 1970; Kousgaard, 1976), random effects (Davidson and Beaver, 1977; Fienberg, 1979), the covariate (Matthews and Morris, 1995; Dittrich et al., 1998; Francis et al., 2002) and ordinal paired comparison models (Agresti, 2002). The covariate information can not only help to generate a full

ranking list but also possibility can specify inhomogeneity among these experts in their ranking "qualities" as well as their way of using the covariates. These are applied by Liu et al. (2019). Dunson (2000) and Gruhl et al. (2013), among others, who have used latent variable models to model multivariate observations with mixed response types. Bayesian latent variable models have been widely used for Bernoulli responses, while less notice has been paid to rank responses.

Hoff (2007) used rankings to estimate copula, thus giving the marginal distributions in multivariate data to be unknown while still modeling dependency. Murray et al. (2013) made based on Hoff (2007) to form a Gaussian copula factor model to jointly respond to rankings and other types of responses. While these methods allow ties by due to the data to be only partially ordered, they do not present a model for the probability that two outcomes will be tied. (Johnson et al., 2002) introduced a model based only on rank data, and they modeled the probability that any two given observations were tied. This approach, using a Bayesian perspective, is now restated with changes to the notation in the simplified case where data are available from only one assessment. We will discuss these in the next section.

In this paper, we consider univariate and multivariate models with the random effects for analysing longitudinal rank responses using latent variable models and Bayesian approach. The structure of the paper is as follows: In Section 2, we present the models and likelihood of the univariate and multivariate rank responses using latent variables and the prior and posterior distributions for the model's framework. Some simulation studies are conducted in Section 3. In Section 4, the proposed models are fitted to data of the neurocognition test of Glioma patient who underwent surgery and results of univariate and multivariate models obtained from real data of neurocognition test are given. Conclusions are given in Section 5.

2 Model specification

In this section, the univariate and the multivariate models of longitudinal rank data and their likelihood functions are presented. Also, we consider rank responses which are implicitly determined by their latent variables. A random effect vector is used to consider the correlation between individual responses across time, which lead to the conditional independence of vectors of responses in different occasions given subject-level random effect.

2.1 Model and likelihood

We suppose that a higher value of the rank denotes better (rather than worse) performance. Assume that N subjects are ranked based on an assessment, with y_i denoting the rank of the ith subject, $i = 1, \ldots, N$ and each observed rank variable y_i has related latent variable z_i. If no ties are permitted in rankings, y and z are connected through the condition that $y_i < y_{i'}$ if only if $z_i < z_{i'}$ for $\forall i \neq i'$. Therefore, the order of the latent variables corresponds to the order of the observed ranks. When a tie is allowed for observed ranks, Johnson et al. (2002) explicitly modeled the probability that two observations are tied based on the scaled distance between the latent variables with

the scale determined by the parameter κ. They supposed that

$$\begin{aligned} p\left(y_i = y_{i'} \left| z_i, z_{i'}, \kappa \right.\right) &= \exp\left(\frac{-\left|z_i - z_{i'}\right|}{\kappa}\right), \\ p\left(y_i < y_{i'} \left| z_i, z_{i'}, \kappa \right.\right) &= 1 - p\left(y_i = y_{i'} \left| z_i, z_{i'}, \kappa \right.\right) 1\left(z_i < z_{i'}\right). \end{aligned}$$

These formulas are used to define the likelihood of a set of observed of rankings based on quantities of the form $p_{(i)}\left(\kappa\right), i = 1, \ldots, N-1$ defied as

$$p_{(i)}\left(\kappa\right) = \begin{cases} \exp\left(\frac{-\left(z_{(i+1)} - z_{(i)}\right)}{\kappa}\right) & \text{if } y_{(i+1)} = y_{(i)} \\ 1 - \exp\left(\frac{-\left(z_{(i+1)} - z_{(i)}\right)}{\kappa}\right) & \text{if } y_{(i+1)} > y_{(i)}, \end{cases}$$

where $y_{(i)}$ and $z_{(i)}$ are the ith smallest observed value and latent variable, respectively. Then the likelihood can be summarized as

$$f\left(y \left| z, \kappa \right.\right) = \prod_{i=1}^{N-1} \left[p_{(i)}\left(\kappa\right) \prod_{i'=i+1}^{N} I\left(\left\{z_{i'} \leq z_i \cap y_{i'} \leq y_i\right\} \cup \left\{z_{i'} \geq z_i \cap y_{i'} \geq y_i\right\}\right) \right].$$

The ordering of the observed variables should be consistent with the ordering of the latent variables.

Now, data collected over time are called longitudinal data. In the following sections, longitudinal rank data, models and likelihood function corresponding to them have been discussed.

2.2 Univariate longitudinal rank data

Suppose that $y_i(t)$ denotes an observed longitudinal rank response for $i = 1, \ldots, N$ at follow-up time $t, t = 1, \ldots, T$. Also, assume that data are available from one assessment. For each observed outcome $y_i\left(t\right)$, we associate a latent variable $z_i(t)$. To handle ties in longitudinal rank data, the probability of observing the tie status of the i and $(i+1)$th order statistics for longitudinal rank variable, given their latent procedure variable is written as

$$p_{(it)}\left(\kappa\right) = \begin{cases} \exp\left(\frac{-\left(z_{(i+1)}(t) - z_{(i)}(t)\right)}{\kappa}\right) & \text{if } y_{(i+1)}(t) = y_{(i)}(t) \\ 1 - \exp\left(\frac{-\left(z_{(i+1)}(t) - z_{(i)}(t)\right)}{\kappa}\right) & \text{if } y_{(i+1)}(t) > y_{(i)}(t), \end{cases}$$

where $y_{(i)}(t)$ and $z_{(i)}(t)$ are the ith smallest observed value and latent variable, respectively. Also, the value of κ has important concepts for modeling rank data when there are large differences in the number of items ranked under different assessments, and in specific when large numbers of items are ranked simultaneously. In such cases, small value of κ account for case in which few ties are recorded, while more balance value of κ reflect the case in which the central portion of the distribution of ranked items are difficult to distinguish, and many more mid-range items are devoted tied values than are the extreme items. This pattern of ties is consistent with a latent distribution of

ties that is unimodal, whereas a disproportionate number of ties in the extremes of the ranked values proposes a multimodal distribution of latent traits or some other model deficiency.

To model the latent variable, let $X_i'(t)$ be ith rows of design matrices $X_{n\times T}$, and $W_i'(t)$ is the sub-vectors of $X_i'(t)$. With this notation, we suppose that

$$Z_i(t) = X_i'(t)\beta + W_i'(t)b_i + \varepsilon_i(t). \tag{1}$$

The vector of parameters β is fixed effects parameters, when $W_i'(t)$ is the sub-vectors of $X_i'(t)$, the model allows the regression coefficients for the covariates contained in $W_i'(t)$ to vary among subjects, while assuming that the remaining coefficients are fixed for all subjects. By permitting a subset of the regression coefficients to vary randomly, a very flexible, and yet quite parsimonious, class of random effect covariance structures becomes available (Fitzmaurice et al., 1988). b_i is the vector of unobserved subject-specific random effects for subject i with $MVN\,(0, \Sigma_b)$. b_i is used to consider the correlation between individual responses across time, which lead to the conditional independence of vectors of responses in different occasions given subject-level effect b_i. Also, let $\varepsilon_i = (\varepsilon_i(1), \ldots, \varepsilon_i(T))'$ for $i = 1, \ldots, N$, be vectors of errors where ε_i are independent and identically distributed with $MVN\,(0, \Sigma_\varepsilon)$. Let $\varepsilon_i(t)$ and $b_i(i = 1, \ldots, N)$ be mutually independent. Johnson et al. (2002) established scale in their analysis of rank data by fixing the variance for a particular effect at 1. The proposed analysis establishes scale in such a manner that the marginal variability of z is 1, regardless of the number of random effects. So, for identifiability of Model, we assume variance-covariance matrix Σ_ε to be fixed. Therefore, we have

$$Z_i(t) \vee b_i, X_i(t), \beta \;\; N\left(X_i'\left(t\right)\beta + W_i'\left(t\right)b_i, \left(\Sigma_\varepsilon\right)_i\right),$$

where $(\Sigma_\varepsilon)_i$ is the (i,i)th element of Σ_ε.

Given subject-level effect b_i, likelihood function for the rank responses can be written as

$$\begin{aligned} f\left(y\,|z,\kappa,\beta\,\right) &= \int f\left(y\,|z,b,\kappa,\beta\,\right)\varphi\left(b\right)db \\ &= \prod_i \int \prod_t f(y_i\left(t\right) \vee z_i\left(t\right), b_i, \kappa, \beta)\varphi(b_i)db_i \\ &= \prod_i \int \prod_t p_{(it)}\left(\kappa\right) \prod_{i':z_{i'}(t)<z_i(t)} I\left(y_{i'}\left(t\right) \leq y_i\left(t\right)\right)\varphi\left(b_i\right)db_i, \end{aligned}$$

where $\varphi(.)$ denotes the corresponding density function of the distribution of the random effects.

2.3 Multivariate longitudinal rank data

Suppose that data come from N subjects with M distinct assessments. Let $y_{ij}(t)$ denotes an observed longitudinal rank response for $i = 1, \ldots, N \wedge j = 1, \ldots, M$ at follow-up time t, $t = 1, \ldots, T$. For each observed responses $y_{ij}(t)$, we associate a latent variable $z_{ij}(t)$. For longitudinal rank response, we assume that $y_{ij}(t) > y_{kj}(t)$ only if

$z_{ij}(t) > z_{kj}(t)$. Given their latent procedure variable, the probability of observing the tie status of the i and $(i+1)$th order statistics for longitudinal rank variable for assessment j at time t is written as

$$p_{(ijt)}(\kappa_j) = \begin{cases} \exp\left(\frac{-\left(z_{(i+1),j}(t) - z_{(i),j}(t)\right)}{\kappa_j}\right) & \text{if } y_{(i+1),j}(t) = y_{(i),j}(t) \\ 1 - \exp\left(\frac{-\left(z_{(i+1),j}(t) - z_{(i),j}(t)\right)}{\kappa_j}\right) & \text{if } y_{(i+1),j}(t) > y_{(i),j}(t), \end{cases}$$

where $y_{(i)}(t)$ and $z_{(i)}(t)$ denote the corresponding ordered values of the observed rank and latent variable and κ_j is the jth assessment-specific value. To model the latent variable, let $X'_{ij}(t)$ be known vector of fixed effect covariates for the ith subject and jth assessment at time t and $W'_{ij}(t)$ be the sub-vector of $X'_{ij}(t)$. So, we have

$$Z_{ij}(t) = X'_{ij}(t)\beta + W'_{ij}(t)b_i + \varepsilon_{ij}(t), \tag{2}$$

where β is fixed effects parameters and b_i is the vector of unobserved subject-specific random effects for subject i with $MVN(0, \Sigma_b)$. b_i is used to consider the correlation between individual responses across time, which lead to the conditional independence of vectors of responses in different occasions given subject-level effect b_i. Also, let $\varepsilon_{ij} = (\varepsilon_{ij}(1), \ldots, \varepsilon_{ij}(T))'$ for $i = 1, \ldots, N$ and $j = 1, \ldots, M$ be vectors of errors where ε_{ij} are independent and identically distributed with $MVN(0, \Sigma_\varepsilon)$. Let $\varepsilon_{ij}(t)$ and $b_i (i = 1, \ldots, N)$ be mutually independent. For identifiability of Model, we assume that marginal variability of z is 1 and variance-covariance matrix Σ_ε to be fixed. Therefore, we have

$$Z_{ij}(t) \vee b_i, X_{ij}(t), \beta \; N\left(X'_{ij}(t)\beta + W'_{ij}(t)b_i, (\Sigma_\varepsilon)_{ij\times ij}\right),$$

where $(\Sigma_\varepsilon)_{(i,j)}$ is the (i,j) element of Σ_ε . Given subject-level effect b_i , likelihood function for the rank responses can be written as

$$\begin{aligned} f(y|z,\kappa,\beta) &= \int f(y|z,b,\kappa,\beta)\,\varphi(b)\,db \\ &= \prod_j \prod_i \int \prod_t f(y_{ij}(t) \vee z_{ij}(t), b_i, \kappa, \beta)\varphi(b_i)db_i \\ &= \prod_j \prod_i \int \prod_t p_{(ijt)}(\kappa_j) \prod_{i': z_{i'j}(t) < z_{ij}(t)} I\left(y_{i'j}(t) \leq y_{ij}(t)\right) \varphi(b_i)\,db_i, \end{aligned}$$

and $\varphi(.)$ denotes the corresponding density function of the distribution of the random effects.

3 Hierarchical model priors and posterior inference

We use Bayesian approach for modeling of longitudinal rank response data in this paper. In Section 2.1, we explained the model and the likelihood distribution that depend on several parameters. To complete the model framework, we now introduce the prior distributions of the parameters. Also, the posterior distribution can be obtained to fit the models using the likelihood distribution and prior distributions.

3.1 Univariate longitudinal rank data

According to model (1), we suppose that $\varepsilon_i = (\varepsilon_i(1), \ldots, \varepsilon_i(T))'$ for $i = 1, \ldots, N$, be vector of errors where ε_i are independent and identically distributed with $MVN(0, \Sigma_\varepsilon)$. For identifiability of the model, we assume that Σ_ε is fix. We suppose that b_i, be independent and identically distributed normal with mean zero and variance σ_b^2 for all subjects $i = 1, \ldots, N$. Also, ε_i and b_i are independent of each other. We use exponential density for the prior distribution of σ_b of random effect. Also, we let the parameters of regression coefficients in model be standard normal. We choose a prior density for κ, a right choice for a prior on κ is gamma distribution with parameters a and b ($Gamma\,(\kappa; a, b)$). So, the prior density for model is written by

$$\pi\,(z, b, \kappa, \beta, \sigma_b) = \pi\,(z\,|b, \beta, \sigma_b\,)\,\pi\,(b\,|\sigma_b\,)\,\pi\,(\sigma_b)\,\pi\,(\kappa)\,.$$

The posterior distribution for the univariate longitudinal rank responses can be expressed as

$$f\,(y\,|z, x, b, \kappa, \beta, \sigma_b\,)\,\pi\,(z\,|b, \beta, \sigma_b\,)\,\pi\,(b\,|\sigma_b\,)\,\pi\,(\sigma_b)\,\pi\,(\kappa)\,\pi\,(\beta) = \prod_i \int \prod_t p_{(it)}\,(\kappa)$$
$$\times \prod_{i':z_{i'}(t)<z_i(t)} I\,(y_{i'}\,(t) \leq y_i\,(t))\,\varphi\,(b_i)\,db_i \prod_i \prod_t \left(z_i(t); x_i'(t)\beta_t + w_i'(t)b_i, (\Sigma_\varepsilon)_{ij\times ij}\right)$$
$$\times \prod_{i=1}^{N} N\left(b_i, ; 0, \sigma_b^2\right) \times N\,(\beta_0; 0, 1) \times \prod_{t=1}^{T} N\,(\beta_t; 0, 1) \exp\,(\sigma_b; \alpha)\,Gamma(\kappa; a, b),$$

where $Gamma\,(\kappa; a, b)$ is gamma distribution with parameters a and b.

3.2 Multivariate longitudinal rank data

According to model (2), we suppose that $\varepsilon_{ij} = (\varepsilon_{ij}(1), \ldots, \varepsilon_{ij}(T))'$ for $i = 1, \ldots, N$ and $j = 1, \ldots, M$ be vectors of errors where ε_{ij} are independent and identically distributed with $MVN\,(0, \Sigma_\varepsilon)$. For identifiability of the model, we assume that Σ_ε is fix. We suppose that b_i be independent and identically distributed normal with mean zero and variance σ_b^2 for all subjects $i = 1, \ldots, N$. Also, $\varepsilon_{ij}(t)$ and $b_i (i = 1, \ldots, N)$ be mutually independent. We use exponential density for the prior distribution of σ_b of random effect. Also, we let the parameters of regression coefficients in model be standard normal. We choose a prior density for each κ_j. Let κ be the set of κ_j's for all j with rank responses, a right choice for a prior on κ is to assume that the κ_j s are mutually independent with a $Gamma(a_j; b_j)$. So, the prior density for model is written by

$$\pi\,(z, b, \kappa, \beta, \sigma_b) = \pi\,(z\,|b, \beta, \sigma_b\,)\,\pi\,(b\,|\sigma_b\,)\,\pi\,(\sigma_b)\,\pi\,(\kappa)\,.$$

The posterior distribution for the multivariate longitudinal rank responses can be expressed as

$$f\,(y\,|z, x, b, \kappa, \beta, \sigma_b\,)\,\pi\,(z\,|b, \beta, \sigma_b\,)\,\pi\,(b\,|\sigma_b\,)\,\pi\,(\sigma_b)\,\pi\,(\kappa)\,\pi\,(\beta) = \prod_j \prod_i \int \prod_t p_{(ijt)}\,(\kappa_j)$$

$$\times \prod_{i':z_{i'j}(t)<z_{ij}(t)} I\left(y_{i'j}\left(t\right) \leq y_{ij}\left(t\right)\right) \varphi\left(b_i\right) db_i$$

$$\times \prod_j \prod_i \prod_t \left(z_{ij}(t); x'_{ij}(t)\beta_{jt} + w'_{ij}(t)b_i, \left(\Sigma_\varepsilon\right)_{ij\times ij}\right) \prod_{i=1}^{N} N\left(b_i, ; 0, \sigma_b\right) \times N\left(\beta_0; 0, 1\right)$$

$$\times \prod_{j=1}^{M} \prod_{t=1}^{T} N\left(\beta_{jt}; 0, 1\right) \exp\left(\sigma_b; \alpha\right) \times \prod_{j=1}^{M} Gamma\left(\kappa_j; a_j, b_j\right).$$

When using Bayesian inferences, our goal is to calculate and use the posterior distribution on a set of random variables. But this often requires the calculation of complex integrals. In such cases, we may search solving the analytical equations and perform sampling methods based on the Monte Carlo Markov Chain (MCMC) method. When using MCMC methods, we estimate the posterior distribution and insolvable integrals using simulated samples of the posterior distribution. The algorithm updates parameters by sampling from full conditional distributions. The selection of priors mentioned was made with an eye towards making updates simple.

3.3 Simulation Study

In this Section, we conduct simulation study to determine the performance of the univariate and multivariate modeling of longitudinal rank responses and the proposed MCMC algorithm. For this goal, we consider two simulation study. The first one focuses on the univariate model and the second is based on multivariate model. Results of simulation study are obtained for effective sample size $N = 100$, 300 and 600 and with 20 iterations. The results are based on 2000 Monte Carlo replications for each effective sample size. We compare the results of our proposed approximated Bayesian model. Via the simulation studies, we want to assign accuracy of models. If the posterior estimates given by Monte Carlo are the same as true values that we assigned in the model, we can sure that the model works well. These models were fitted in software R.

3.3.1 Simulation study 1: univariate longitudinal rank model

We consider the longitudinal rank responses $Y_i(t)$, $i = 1, \ldots, N \wedge t = 1, 2$, with latent variables $Z_i(t)$. We begin by simulating data from the proposed model. In model (1)

$$Z_i\left(t\right) = \beta_0 + \beta_t X_i(t) + \sigma_b b_i + \varepsilon_i(t).$$

For generating the data: we generate i.i.d. $\varepsilon_i = (\varepsilon_{1i}(1), \varepsilon_{1i}(2))'$ for $i = 1, \ldots, N$, from the bivariate normal distribution with mean 0 and identical covariance matrix. Covariate matrices $(X_i(1), X_i(2))$ with dimension N is simulated from bivariate normal with zero mean and variance matrices $\begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$. We set $\beta = (\beta_0, \beta_1, \beta_2)' = (2, 1, -1)'$ and the vector of random effects, b_i, independently is generated from normal distribution with mean zero and $\sigma_b = 0.5$ for all subjects. Given random effects the correlation between individual responses across time is conditionally independent. Also, we set $\kappa = 1$. With these assumptions, we can generate $z_i(t)$ s. Johnson et al. (2002) assumed

that $y_i > y_{i'}$ only if $z_i > z_{i'}$, where the latent procedure variable is z_i. So, to generate the full ranking lists $y(t)$, for each $t = 1, 2$, we rank the vector $z(t)$.

Results of the simulation study for model of rank responses is given in Table 1. Table 1 shows that posterior estimates of parameters of univariate model are close to their true values, and with increasing N, the estimates become nearer to the real values. Also, the standard errors (se) of estimates reduce and mean square errors (MSE) of estimates of the model tend to zero with increasing N.

Table 1: Simulation results for 100, 300 and 600 samples using the univariate model (Est. and S.E. are estimation and standard error for parameter.

		$N = 100$			$N = 300$			$N = 600$		
Parameter	True value	EST.	S.E.	MSE	EST.	S.E.	MSE	EST.	S.E.	MSE
β_0	2	2.198	0.160	0.065	2.010	0.095	0.009	2.045	0.072	0.007
β_1	1	1.154	0.210	0.068	1.134	0.127	0.034	1.002	0.082	0.007
β_2	-1	-0.733	0.236	0.127	-0.984	0.123	0.015	-0.988	0.082	0.006
κ	1	0.77	0.019	0.050	0.834	0.004	0.020	0.93	0.002	0.004
σ_b	0.5	0.520	0.213	0.046	0.635	0.138	0.037	0.649	0.095	0.031

3.3.2 Simulation study 2: multivariate longitudinal rank model

We consider the longitudinal rank responses $Y_{ij}(t)$, $i = 1, \ldots, N, j = 1, 2 \wedge t = 1, 2$, with latent variables $Z_{ij}(t)$. We begin by simulating data from the proposed model. In model (2)

$$Z_{ij}(t) = \beta_0 + \beta_{jt} X_{ij}(t) + \sigma_b b_i + \varepsilon_{ij}(t).$$

For generating the data: we generate *i.i.d.* $\varepsilon_{ij}(t) = (\varepsilon_{ij}(1), \varepsilon_{ij}(2))'$ for $i = 1, \ldots, N$ and $j = 1, 2$ from the multivariate normal distribution with mean 0 and identical covariance matrix. Covariate matrices $(X_{ij}(1), X_{ij}(2)) \, for i = 1, \ldots, N \wedge j = 1, 2$ is simulated from multivariate normal distribution with mean $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and variance matrices $\begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$. We set $\beta = (\beta_0, \beta_{11}, \beta_{12}, \beta_{21}, \beta_{22})' = (-2, 2, -1, 1, 1)'$ and the vector of random effects independently is generated from identically distributed normal with mean 0 and variance $\sigma_b = 2$ for all subjects. Also, we set $\kappa_1 = 0.5$ and $\kappa_2 = 0.3$. With these assumptions, we can generate $z_{ij}(t)$s. Johnson et al. (2002) assumed that $y_{ij} > y_{kj}$ only if $z_{ij} > z_{kj}$, where the latent procedure variable is z_{ij}. So, to generate the full ranking lists, for each $j = 1, 2 \wedge t = 1, 2$ we rank the vector $z_j(t)$.

The results are based on 2000 Monte Carlo replications for each effective sample size. If the posterior estimates given by Monte Carlo are the same as true values that we assigned in the model, we can sure that the model works well. Results of the simulation study for model of rank responses is given in Table 2. Table 2 shows that posterior estimates of parameters of the joint linear model are close to their true values, and with increasing N, the estimates become nearer to the real values. Also, MSEs of estimates of the model tend to zero with increasing N.

Table 2: Simulation results for 100, 300 and 600 samples using the multivariate model.

		$N = 100$			$N = 300$			$N = 600$		
Parameter	True value	EST.	S.E.	MSE	EST.	S.E.	MSE	EST.	S.E.	MSE
β_0	-2	-2.371	0.325	0.244	-1.940	0.104	0.014	-1.927	0.090	0.013
β_{11}	2	2.406	0.103	0.176	2.208	0.070	0.048	2.052	0.168	0.031
β_{12}	-1	-1.118	0.197	0.053	-1.027	0.109	0.013	-1.012	0.077	0.006
β_{21}	1	1.262	0.103	0.079	1.022	0.184	0.034	1.002	0.078	0.006
β_{22}	1	1.335	0.185	0.146	1.154	0.102	0.034	1.006	0.081	0.007
κ_1	0.5	0.420	0.050	0.008	0.421	0.005	0.006	0.492	0.002	6.8e-05
κ_2	0.3	0.211	0.021	0.008	0.241	0.004	0.003	0.252	0.002	0.002
σ_b	2	2.063	0.467	0.222	1.853	0.109	0.034	1.884	0.111	0.026

4 Application

Glioma is a type of primary brain tumor that arises from brain tissue. Glioma is the most frequent type of central nervous system (CNS) neoplasm that occurs when glial cells proliferate uncontrollably. These cells support nerves and help your central nervous system work, (Mesfin and Al-Dhahir, 2023). The most widely used WHO brain tumor classification relies on traditional methods using morphology to classify gliomas, while recent progresses have shown that molecular diagnostic techniques are an exact way to better classify tumors using molecular abnormalities and signaling pathways involved in glioma development. These molecular subtypes have distinct prognoses and therapeutic responses (Jiao et al., 2012). Among the molecular alterations, two are specifically noteworthy because they happen early in glioma formation and are common in glioma. The first is Isocitrate dehydrogenase (IDH)-1 and the second is the telomerase reverse transcriptase (TERT). These two molecular alterations aid in diagnosis and prognosis of diffuse glioma. IDH-1 and TERT are divided into two groups: wild type and mutant ones. Isocitrate dehydrogenase (IDH)-1 mutation is an early event in glioma development and occurs prominently in low grade tumors. Introduction of mutated IDH into normal cells causes increased proliferation, increased colony formation, and inability to differentiate (Cohen et al., 2013). Generally, mutations in IDH-1 gene are found in human glioma but it is also not clear why tumors with this mutation generally have a better prognosis than IDH wild type tumors (Karpel-Massler et al., 2019). Also, the recent glioblastoma publication from the cancer genome atlas showed that the only subgroup with improved survival was tumors with IDH1 mutations. Initial reports suggested that the mutant protein functions in a dominate-negative fashion by heterodimerizing to wildtype IDH1 and impairing its activity (Yang et al., 2010). The telomerase reverse transcriptase (TERT) has been the subject of numerous studies on the grading and prognosis of glioma. In glioblastoma (glioblastoma is a type of glioma), Killela et al. (2013) found a trend for increased telomerase expression in cases with TERT promoter mutations. Interestingly, unlike IDH1-mutations, mutations in the TERT promoter, which lead to increased telomerase activity and telomere elongation, are seen in the most aggressive human glioma.

Neurological examination and brain scans are used to assign the existence of a tumor, its location and characteristics. Brain tumor symptoms can vary according to its location and size. Brain tumor location is an important factor in determining functional

status after brain tumor surgery. The existence of a brain tumor can damage to healthy brain tissue and disrupt the normal functioning of that area. The brain can be divided into five lobes: frontal, temporal, parietal, occipital, and insula. Various regions of the brain are responsible for different functions, which means that the symptoms of a brain tumor are determined by its location (Krajewski et al., 2022). Figure 1 shows a midline brain tumor extending to both frontal lobes in preoperative MRI.

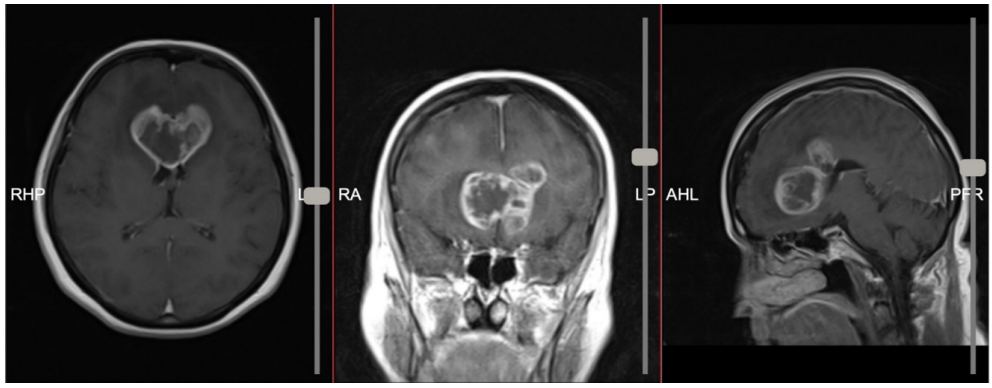

Figure 1: T1-weighted contrast in axial, coronal and sagittal planes (from left to right) MRI show a high-grade glioma in corpus callosum extending to both frontal lobes.

Furthermore, surgery is the first treatment for a glioma if its location is accessible via surgical routes. One of the most reliable prognostic factors for overall survival in brain tumor patients is the extent of resection (EOR). A surgeon may be able to remove all the tumor if it's easily reachable. The WHO classification of gliomas is used to guide glioma treatment. As indicated in the classification, most patients require surgical intervention for taking the sample via gross total resection or biopsy (Weller et al., 2017). Early surgery together with higher EOR can accompany by better overall survival than biopsy and watchful waiting policies. It has been explained in many studies that the removal rate is a positive prognostic factor (Sanai, 2012).

The Karnofsky Performance Scale (KPS) is a measure of the ability of cancer patients to perform normal tasks and evaluate the functional independence of patients, which ranges from 0 to 100. A higher score means that the patient is better able to perform normal activities.

This study was conducted on 13 glioma patients who underwent surgery, and data were collected from patients hospitalized in the neurosurgical department of Sina Hospital, Tehran since 2019 to 2021 by our team's neurosurgeon (A.P.). The cognitive function and KPS of these patients were examined at four different time points: during the week before and after the surgery (pre-operatively and post- operatively), 3 and 6 months later. Cognitive subdomains values are qualitative values, so they are collected as rank data. Neurocognition was evaluated using a standardized test battery. The Addenbrooke's Cognitive Examination (ACE) is a sensitive, accurate, yet brief screening test to assess cognitive functions, particularly language and speech, that has been widely used to assess cognition in health conditions such as dementia and glioma surgical settings. The data of this paper were used to evaluate the cognitive performance of the valid Persian version of Addenbrooke's Revised Cognitive Test (ACE-R).

ACE includes activities and tasks across via five cognitive subdomains, including attention (time and place orientation, 3-item repetition and word spelling), memory (3-item recall, progressive memory, retrograde memory, Recall and Recognition of Name and Address), verbal fluency (Letters fluency and categorization), language (comprehension, sentence writing, repeating single words, naming and reading objects), and visuospatial abilities (copying pentagons on top of each other, copying 3D wire cubes, drawing clocks, counting letters) (Tymowski et al., 2018). We chose the ACE (is obtained from the sum of 5 cognitive subdomains) variable over time as the longitudinal rank response to fit univariate rank model and chose attention, memory, verbal fluency, language and visuospatial abilities cognitive variables over time as the multiple longitudinal rank responses to fit multivariate rank model. In fact, at each of the four different times, patients were ranked based on their cognitive ability, separately. In the following, we described the statistical method used to analyse these data.

In this Section, we applied the methods and models introduced in the Section 2 to data on Neurocognition test of glioma patients. As introduced in Section 2, we started by fitting univariate and multivariate models with random effects. Some results of models and detailed results are presented in Section 4.2.

4.1 Data

As described, data from glioma patients are used as motivational applications for the development of the detailed model. We chose the ACE (is obtained from the sum of 5 cognitive subdomains) variable over time as the longitudinal rank response to fit univariate rank model and chose attention, memory, verbal fluency, language and visuospatial abilities cognitive variables over time as the multiple longitudinal rank responses to fit multivariate rank model. In fact, at each of the four different times points: during the week before and after the surgery (pre-operatively and post- operatively), 3 and 6 months later, patients were ranked based on their cognitive ability, separately. So, we have longitudinal rank responses. Table 3 contains descriptive statistics of these patients.

Figure 2 shows density estimates from the KPS data at four different time points: during the week before and after the surgery (pre-operatively and post- operatively), 3 and 6 months later. As it can be seen the amount of KPS in patients after surgery, 3 and also 6 months later has increased, which can indicate a significant effect of surgery and treatment on the ability of patients to perform ordinary tasks and normality activities.

Kendall and Smith introduced the Kendall tau rank distance, which is a metric that counts the number of pairwise disagreements between two ranking lists length. The larger the distance, the more different the two lists are. In this data, at each of the four different times, patients were ranked based on their cognitive ability in ACE, attention, memory, verbal fluency, language and visuospatial abilities cognitive, separately. The Kendall distances of patient ranking lists between 4 times in each cognitive subdomain were calculated in Table 4.

Based on ACE and attention of patients, the largest distance of the patient ranking list was between the week before and after the operation. Also, the largest distance for patients memory was between the week before and 3 months later operation and for verbal fluency, language and visuospatial abilities cognitive of patients was between the week before and 6 months later operation. The biggest distance in all cognitive

Table 3: The detailed information of descriptive statistics of glioma patients ($N = 13$).

	Mean (S.E.)
Age	43.615 (10.324)
KPS (Preoperative)	79.615 (8.530)
KPS (Preoperative)	82.692 (7.250)
KPS (3 month later)	87.692 (10.127)
KPS (6 month later)	92.692 (8.807)
Resection rate	%92.690 (8.808)
	Count (percent)
Sex	
Female	5 (%38.5)
Male	8 (%61.5)
IDH-1 mutation	
Mutant	11 (%84.6)
Wildtype	2 (%15.4)
TERT mutation	
Mutant	7 (%53.8)
Non-mutant	6 (%46.2)
Location	
Frontal	5 (%38.5)
Temporal	4 (%30.8))
Parietal	2 (%15.4)
Occipital	1 (%7.7)
Midline	1 (%7.7)

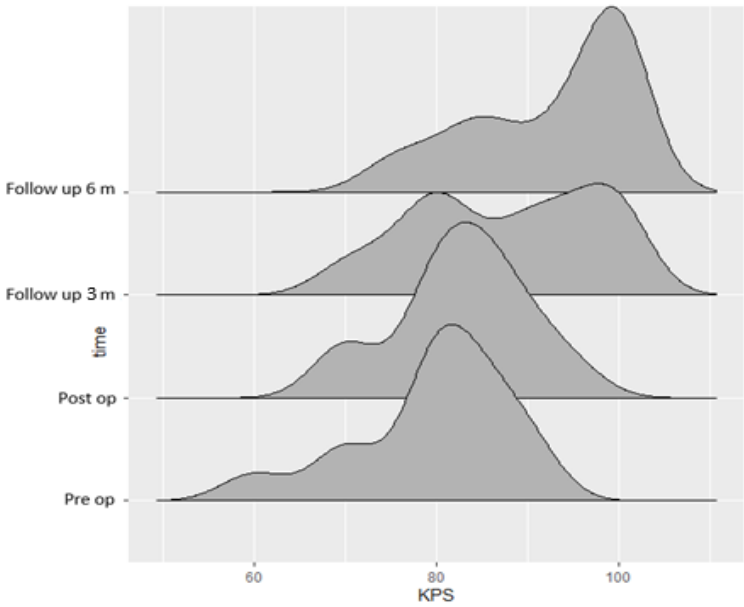

Figure 2: Density ridgeline plot of the Karnofsky Performance Scale (KPS) of patient at 4 times.

abilities is the distance between before surgery and other times after surgery, which

Table 4: The Kendall tau rank distance of patient ranking between different times in cognitive subdomains.

		Preoperative-				3 months -
	Postoperative	3 months	6 months	3 months	6 months	6 months
ACE	0.348	0.299	0.221	0.184	0.247	0.081
Attention	0.453	0.365	0.194	0.205	0.219	0.303
Memory	0.336	0.187	0.127	0.388	0.289	0.142
verbal fluency	0.281	0.320	0.357	0.261	0.227	0.089
Language	0.242	0.321	0.385	0.193	0.235	0.112
visuospatial	0.375	0.359	0.440	0.278	0.304	0.170

shows the effect of surgery and treatment. The most similar ranking lists in ACE, verbal fluency, language and visuospatial abilities cognitive are between months 3 and 6, which can be said that patients' conditions are almost stable and favorable after 3 months.

4.2 Model for data

We apply the univariate and multivariate modeling described in the last section to real data on neurocognitive test.

4.2.1 Univariate model for data

The univariate model is given by

$$\begin{aligned} Z_i(t) &= \beta_0 + \beta_{1t} KPS_i(t) + \beta_{21} Age_i + \beta_{22} IDH1_i + \beta_{23} TERT_i \\ &\quad + \beta_{24} Resection_i + \beta_{25} Location_i + \beta_{26} Gender_i + \sigma_b b_i + \varepsilon_i(t), \end{aligned}$$

That $Z_i(t)$ denotes the latent variable associated with rank response $ACE_i(t)$. The seven explanatory variables are $KPS_i(t\ )$, which is measured the week before and after surgery, 3 and 6 months later, $Age_i, Gender_i(1 = male, 2 = female)$ $IDH1_i$ (1=mutant, 2= wildtype), $TERT_i$ (1=mutant, 2=Non-mutant), $Location_i$ (1=Frontal, 2=Temporal, 3=Parietal, 4=Occipital and 5=Midline), and $Resection_i$, $i = 1, \ldots, 13$ and $t = 1, 2, 3, 4$. We assume the random effect is conditionally independent and has the normal distribution with mean 0 and variance σ_b^2, and $\varepsilon_i = (\varepsilon_i(1), \ldots, \varepsilon_i(4))'$ has the multivariate normal distribution with mean 0 and identical covariance matrix and $\varepsilon_i(t)$ and b_i be mutually independent. b_i is used to consider the correlation between individual responses across time, which lead to the conditional independence of vectors of responses in different occasions given subject-level effect b_i. Results of posterior estimates of parameters with standard deviations and mean square errors for analysing Neurocognition data using univariate modeling of longitudinal rank data are presented in Table 5.

Based on the results of Table 5, KPS, Age, IDH-1, TERTmutant, location and resection rate are significant at the level of $\alpha = 0.05$ on latent value of ACE. The ability of cancer patients to perform normal tasks and evaluate the functional independence of patients (KPS) is effective factor on Neurocognitive state (ACE) of patients at 4 different times. This result is consistent with the result obtained by Baba and

Table 5: Results of posterior estimates of parameters with standard deviations and mean square errors for analysing Neurocognition data using univariate modeling of longitudinal rank data. (Parameters significant at 5% level are highlighted in bold).

Parameter	EST.	S.E.	MSE	Test statistics
Intercept	0.887	1.070	1.157	0.828
Age	**-0.123**	**0.048**	**0.106**	**-2.562**
Location	**-0.141**	**0.255**	**0.181**	**-0.552**
Resection	**0.321**	**0.038**	**0.016**	**8.44**
IDH1	0.280	0.511	0.268	0.547
TERT	**-0.569**	**0.225**	**0.187**	**-2.528**
Gender	0.053	0.490	0.262	1.870
KPS (1)	0.091	0.048	0.014	1.895
KPS (2)	**0.201**	**0.046**	**0.002**	**4.369**
KPS (3)	**0.093**	**0.043**	**0.013**	**2.162**
KPS (4)	**0.198**	**0.041**	**0.002**	**4.829**
κ	**1.249**	**0.213**	**0.107**	**5.863**
σ_b	**2.274**	**0.188**	**0.110**	**12.095**

Adali (2021). The results obtained on the significance of IDH-1, TERTmutation on neurocognitive state of glioma patients in this paper are similar to the results obtained by Wefel et al. (2016). Also, the significance of age, resection and location tumor on neurocognitive state of glioma patients here are like the results obtained by Dallabona et al. (2017) and Hendriks et al. (2018).

The posterior means of latent variables of ACE based on the univariate model at the 4 different times are provided in columns of Figure 3. Each column of Figure 3 shows the ranking list of the patients according to the posterior means of latent values of ACE at each time.

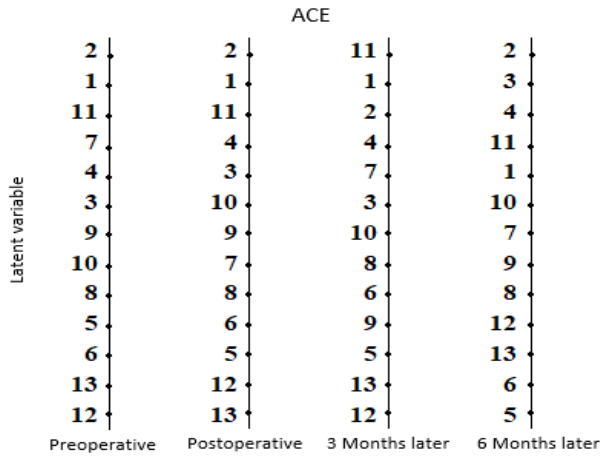

Figure 3: Posterior Means of Latent variables under univariate model.

Figure 4 shows a violin plot with boxplot to see both the distribution and its summary statistics of posterrior estimates of parameters in univariate model of data.

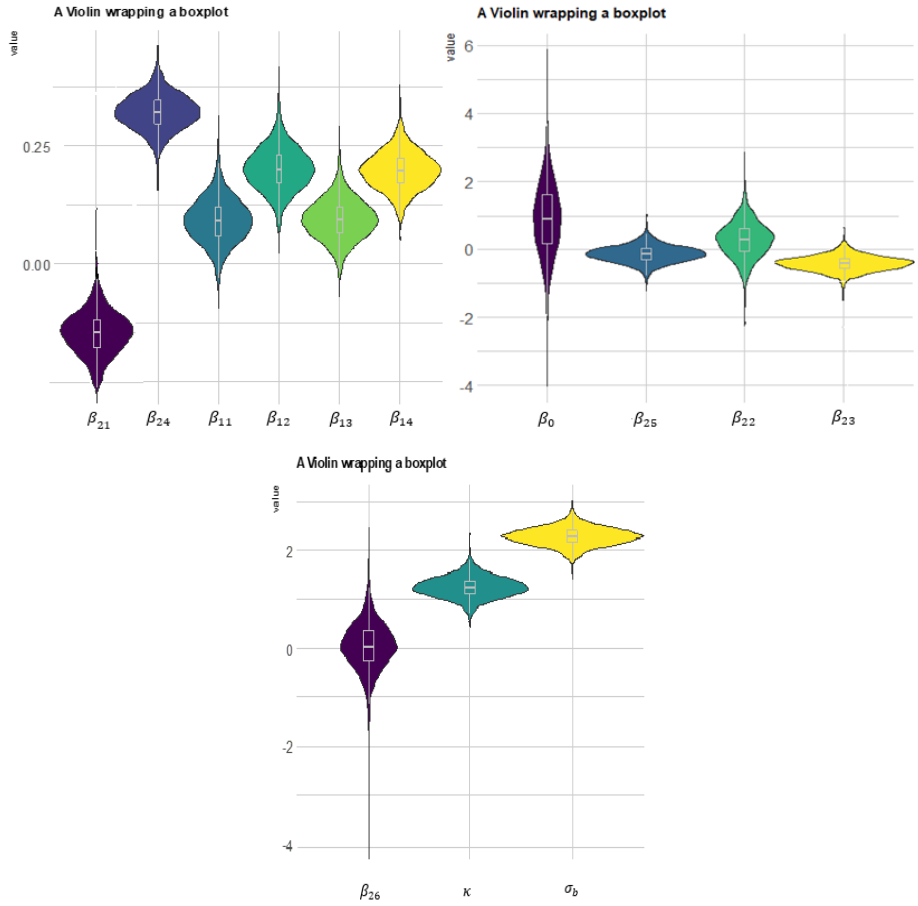

Figure 4: The violin boxplot of posterior estimates of parameters under univariate model ($n = 2000$).

4.3 Multivariate model for Data

The multivariate model is given by

$$\begin{aligned} Z_{ij}(t) &= \beta_0 + \beta_{1t} KPS_i(t) + \beta_{21} Age_i + \beta_{22} IDH1_i + \beta_{23} TERT_i \\ &\quad + \beta_{24} Resection_i + \beta_{25} Location_i + \beta_{26} Gender_i + \sigma_b b_i + \varepsilon_{ij}(t), \end{aligned}$$

That $Z_{ij}(t)$ denotes the latent variable associated with rank response $Y_{ij}(t)$ for $i = 1, \ldots, 13, j = 1, \ldots, 5$ and $t = 1, 2, 3, 4$, so that multiple responses include attention, memory, verbal fluency, language and visuospatial abilities cognitive. The seven explanatory variables and random effect are the same as that defined in univariate model. $\varepsilon_{ij} = (\varepsilon_{ij}(1), \ldots, \varepsilon_{ij}(4))'$ has the multivariate normal distribution with mean 0 and identical covariance matrix and $\varepsilon_{ij}(t)$ and b_i be mutually independent. Results of posterior estimates of parameters with standard deviations and mean square errors for analysing Neurocognition data using multivariate modeling of longitudinal rank data are presented in Table 6.

Table 6: Results of posterior estimates of parameters with standard deviations and mean square errors for analysing Neurocognition data using multivariate modeling of longitudinal rank data. (Parameters significant at 5% level are highlighted in bold)

Parameter	EST.	S.E.	MSE
Intercept	**1.088**	**0.479**	**0.237**
Age	**0.132**	**0.009**	**0.005**
Location	**0.800**	**0.070**	**0.364**
Resection	**0.158**	**0.008**	**0.002**
IDH1	**0.319**	**0.177**	**0.046**
TERT	**-0.253**	**0.341**	**0.119**
Gender	0.177	0.177	0.032
KPS (1)	**0.299**	**0.018**	**0.010**
KPS (2)	**0.292**	**0.018**	**0.009**
KPS (3)	**0.287**	**0.017**	**0.008**
KPS (4)	**0.280**	**0.016**	**0.007**
κ_1	**0.721**	**0.101**	**0.088**
κ_2	**1.169**	**0.184**	**0.063**
κ_3	**1.161**	**0.166**	**0.053**
κ_4	**0.754**	**0.087**	**0.068**
κ_5	**1.220**	**0.255**	**0.113**
σ_b	**2.379**	**0.056**	**0.147**

Based on the results of Table 6, KPS, Age, IDH-1, TERTmutant, location and resection rate are significant factors at the level of $\alpha = 0.05$ on all latent value of subdomain of ACE. Increasing KPS of patients will definitely improve their attention, memory, verbal fluency, language and visuospatial abilities cognitive. Dallabona et al. (2017) stated that longitudinal neuropsychological performance of patients with high-grade glioma depends on the complex interplay of tumor volume, surrounding edema volume, tumor localization, and patient age, among other factors. As, preoperative performances in verbal, language and memory tasks depended on the joint effect of tumor volume, surrounding edema volume, and tumor localization, with major deficits in patients with left-sided tumors, particularly insular and temporal. Preoperative performance in attention tasks and constructive abilities depends on the joint effect of tumor volume, surrounding edema volume, and patient age. Many studies have been conducted on the effect of surgery on cognitive subdomains. Satoer et al. (2016) expressed the significant effect of the operative, the immediate postoperative phase and follow-up on the domains of language, memory and attention and executive functioning under three studies. Also, Wefel et al. (2016) expressed the average performance of patients with IDH1-widetype was also significantly lower than that of patients with IDH1-mutant tumors in the measures of learning and memory, processing speed, language, executive function and skill. The posterior means of each latent variable based on the multivariate model for each cognitive subdomain at the 4 different times are provided in columns of Figure 5. As, the ranking list of patients according to the posterior means of each latent value of cognitive subdomains at different times is shown.

Figure 6 shows a violin plot with box plot to see both the distribution and its summary statistics of posterior estimates of parameters in multivariate model of data.

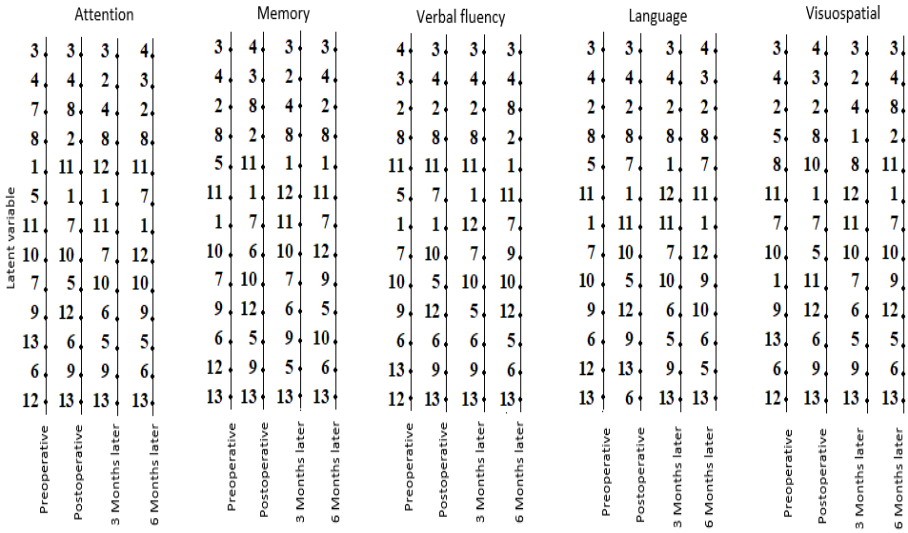

Figure 5: Posterior Means of Latent variables under multivariate model.

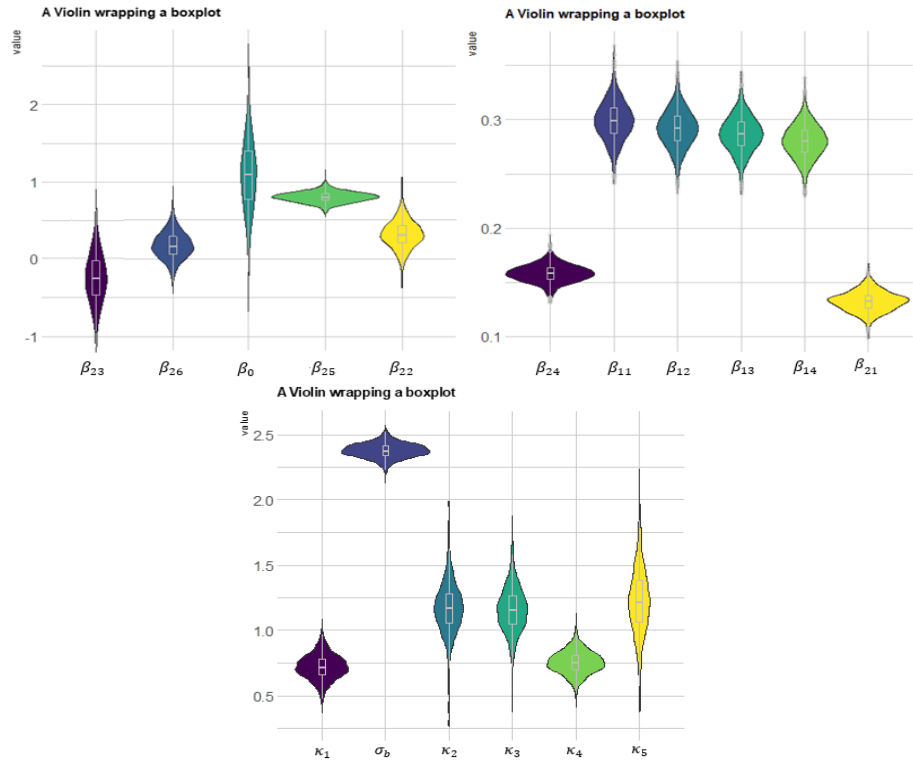

Figure 6: The violin box plot of posterior estimates of parameters under multivariate model ($n = 2000$).

5 Conclusion

We have provided a framework for analysing longitudinal rank data with univariate and multivariate model. We proposed a Bayesian latent variable model for analysing these data. Also, we applied Bayes approach via some simulation studies, and showed that its use significantly improved the efficiency, accuracy and required computational time. So, we have applied some simulation studies to study the performance of the proposed models. The cognitive function using neurocognitive data on glioma patients were analysed as an illustrative example. Also, we observed that the results obtained from the significant effect of the considered covariates on cognitive ability of glioma patients in our models are consistent with the result obtained in medical studies.

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References

Agresti, A. (2002). *Categorical Data Analysis*. New York: John Wiley.

Alvo, M. and Yu, P.L. (2014). *Statistical Methods for Ranking Data*. New York: Springer-Verlag.

Baba, M.A. and Adali, N. (2021). Neurocognitive state and quality of life of patients with glioblastoma in Mediterranean countries: a systematic review. *Annals of Palliative Medicine*, **10**(11):11980–11993.

Bradley, R.A. and Terry, M.E. (1952). The rank analysis of incomplete block designs: 1, the method of paired comparisons. *Biometrika*, **39**(3/4):324–345.

Böckenholt, U. (1992). Thurstonian Representation of Ranking Data. *British Journal of Mathematical and Statistical Psychology*, **45**(1):31–50.

Böckenholt, U. (1993). Applications of Thurstonian models to ranking data. In *Probability Models and Statistical Analyses for Ranking Data*. pp. 157–172, New York: Springer.

Chan, W. and Bentler, P.M. (1998). Covariance structure analysis of ordinal ipsative data. *Psychometrika*, **63**(4):369–399.

Critchlow, D.E., Fligner, M.A. and Verducci, J.S. (1991). Probability models on rankings. *Journal of Mathematical Psychology*, **35**(3):294–18.

Cohen, A.L., Holmen, S.L. and Colman, H. (2013). IDH1 and IDH2 mutations in gliomas. *Current Neurology and Neuroscience Reports*, **13**(5):345.

Dallabona, M., Sarubbo, S., Merler, S., Corsini, F., Pulcrano, G., Rozzanigo, U., Barbareschi, M. and Chioffi, F. (2017). Impact of mass effect, tumor location, age, and surgery on the cognitive outcome of patients with high-grade gliomas: a longitudinal study. *Neuro-Oncology Practice*, **4**(4):229–240.

Daniels, H.E. (1950). Rank correlation and population models. *Journal of the Royal Statistical Society*, **12**(2):171–181.

David, H.A. (1988). *The Method of Paired Comparisons*. London: Chapman and Hall.

Davidson, R.R. (1970). On extending the Bradley-Terry model to accommodate ties in paired comparison experiments. *Journal of the American Statistical Association*, **65**(329):317–328.

Davidson, R.R. and Beaver, R.J. (1977). On extending the Bradley-Terry model to incorporate within-pair order effects. *Biometrics*, **33**(4):693–702.

Diaconis, P. (1988). Group representations in probability and statistics. *Lecture Notes Monograph Series*, **11**:1–192.

Dittrich, R., Hatzinger, R., and Katzenbeisser, W. (1998). Modelling the effect of subject-specific covariates in paired comparison studies with an application to university rankings. *Journal of the Royal Statistical Society: Series C (Applied Statistics)*, **47**(4):511–525.

Dunson, D.B. (2000). Bayesian latent variable models for clustered mixed outcomes. *Journal of the Royal Statistical Society (Series B: Statistical Methodology)*, **62**(2):355–366.

Fienberg, S.E. (1979). Graphical Methods in Statistics. *The American Statistician*, **33**(4):165–178.

Fitzmaurice, G.M., Laird, N.M. and Ware, J.H. (2011). *Applied Longitudinal Analysis*. 2nd edition, New Jersey: Wiley & Sons.

Francis, B., Dittrich, R. and Hatzinger, R. (2002). Analysing partial ranks by using smoothed paired comparison methods: An investigation of value orientation in Europe. *Journal of the Royal Statistical Society Series C: Applied Statistics*, **51**(3):319–336.

Gruhl, J., Erosheva, E.A. and Crane, P.K. (2013). A semiparametric approach to mixed outcome latent variable models: Estimating the association between cognition and regional brain volumes. *The Annals of Applied Statistics*, **7**(4):2361–2383.

Hendriks, E.J., Habets, E.J., Taphoorn, M.J., Douw, L., Zwinderman, A.H., Vandertop, W.P., Barkhof, F., Klein, M. and De Witt Hamer, P.C. (2018). Linking late cognitive outcome with glioma surgery location using resection cavity maps. *Human Brain Mapping*, **39**(5):2064–2074.

Hoff, P.D. (2007). Extending the rank likelihood for semiparametric copula estimation. *The Annals of Applied Statistics*, **1**(1):265–283.

Jiao, Y., Killela, P.J. and Reitman, Z.J. (2012). Frequent ATRX, CIC, FUBP1 and IDH1 mutations refine the classification of malignant gliomas. *Oncotarget*, **3**(7):709–722.

Johnson, T.R., and Kuhn, K.M. (2013). Bayesian Thurstonian models for ranking data using JAGS. *Behavior Research Methods*, **45**(3):857–872.

Johnson, V.E., Deaner, R.O. and Van Schaik, C.P. (2002). Bayesian analysis of rank data with application to primate intelligence experiments. *Journal of the American Statistical Association*, **97**(457):8–17.

Karpel-Massler, G., Nguyen, T.T.T. and Shang, E. (2019). Novel IDH1-targeted glioma therapies. *CNS Drugs*, **33**(12):1155–1166.

Kendall, M. and Smith, B. (1940). On the method of paired comparisons. *Biometrika*, **31**(3/4):324–345.

Killela, P.J., Reitman, Z.J. and Jiao, Y. (2013). TERT promoter mutations occur frequently in gliomas and a subset of tumors derived from cells with low rates of self-renewal. *Proceedings of the National Academy of Sciences*, **110**(15):6021–6026.

Kousgaard, N. (1976). Models for paired comparisons with ties. *Scandinavian Journal of Statistics*, **3**(1):1–14.

Krajewski, S., Furtak, J., Zawadka-Kunikowska, M., Kachelski, M., Birski, M. and Harat, M. (2022). Comparison of the functional state and motor skills of patients after cerebral hemisphere, ventricular system, and cerebellopontine angle tumor surgery. *International Journal of Environmental Research and Public Health*, **19**(4):2308.

Liu, A., Zhao, Z., and Liao, C. (2019). Learning plackett-luce mixtures from partial preferences. *Proceedings of the Thirty-third AAAI Conference on Artificial Intelligence*, **33**(01):4328–4335.

Marden, J.I. (1995). *Analysing and Modeling Rank Data*. London: Chapman and Hall.

Matthews, J.N.S. and Morris, K.P. (1995). An application of Bradley-Terry-type models to the measurement of pain. *Journal of the Royal Statistical Society: Series C (Applied Statistics)*, **44**(2):243–255.

McFadden, D. (1980). Econometric models for probabilistic choice among products. *The Journal of Business*, **53**(3):S13–S29.

Mesfin, F.B. and Al-Dhahir, M.A. (2023). *Gliomas*. StatPearls Publishing, Treasure Island (FL).

Murray, J.S., Dunson, D.B. and Carin, L. (2013). Bayesian Gaussian copula factor models for mixed data. *Journal of the American Statistical Association*, **108**(502):656–665.

Rao, P.V. and Kupper, L.L. (1967). Ties in paired-comparison experiments: A generalization of the Bradley-Terry model. *Journal of the American Statistical Association*, **62**(317):194–204.

Sanai, N. (2012). Emerging operative strategies in neurosurgical oncology. *Current Opinion in Neurology*, **25**(6):756–766.

Satoer, D., Visch-Brink, E. and Dirven, C. (2016). Glioma surgery in eloquent areas: can we preserve cognition?. *Acta Neurochirurgica*, **158**(1):35–50.

Stern, H. (1990). Models for distributions on permutations. *Journal of the American Statistical Association*, **85**(410):558–564.

Thurstone, L.L. (1927). A law of comparative judgement. *Psychological Review*, **15**:273–286.

Thurstone, L.L. (1931). Rank order as a psycho-physical method. *Journal of Experimental Psychology*, **14**(3):187–201.

Tymowski, M., Kaspera, W. and Metta-Pieszka, J. (2018). Neuropsychological assessment of patients undergoing surgery due to low-grade glioma involving the supplementary motor area. *Clinical Neurology and Neurosurgery*, **175**:1–8.

Wefel, J.S., Noll, K.R., Rao, G. and Cahill, D.P. (2016). Neurocognitive function varies by IDH1 genetic mutation status in patients with malignant glioma prior to surgical resection. *Neuro-oncology*, **18**(12):1656–1663.

Weller, M., Van Den Bent, M., Tonn, J.C., Stupp, R., Preusser, M., Cohen-Jonathan-Moyal, E., Henriksson, R., Le Rhun, E., Balana, C., Chinot, O. and Bendszus, M. (2017). European Association for Neuro-Oncology (EANO) guideline on the diagnosis and treatment of adult astrocytic and oligodendroglial gliomas. *The lancet oncology*, **18**(6):e315–e329.

Yang, B., Zhong, C., Peng, Y., Lai, Z. and Ding, J. (2010). Molecular mechanisms of "off-on switch" of activities of human IDH1 by tumor-associated mutation R132H. *Cell research*, **20**(11):1188–200.

Yao, G. and Böckenholt, U. (1999). Bayesian estimation of Thurstonian ranking models based on the Gibbs sampler. *British Journal of Mathematical and Statistical Psychology*, **52**(1):79–92.