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Bayesian estimation in the $M/M/1$ queueing model under a Type II censoring scheme based on fuzzy data

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Abstract: In this article, the Bayesian estimation of the traffic intensity parameter for the $M/M/1$ queuing system is performed when it follows a beta prior distribution. The analysis is conducted under the general entropy and linear exponential loss functions, considering a Type II censoring scheme based on fuzzy data. Furthermore, the obtained estimators compared under these two loss functions using an extensive Monte Carlo simulation study. A real-world data set is also utilized to determine the more appropriate estimator.

Keywords: Entropy loss function; Fuzzy data; Linear exponential loss function; Traffic intensity parameter.
Mathematics Subject Classification (2010): 60K25, 62F15.

1 Introduction

The estimation using maximum likelihood (ML) and Bayesian methods has received considerable attention in queueing theory. The parameters of the $M/M/1$ queueing model under the steady-state condition were obtained using the ML method by Clarke (1957). Bayesian estimation of the arrival rate (λ) and service rate (μ) in the queueing models $M/M/1$ and $M/M/\infty$ was conducted by Muddapur (1972). Empirical Bayesian estimation of parameters in the queueing models $M/M/1$ and $M/M/\infty$ was carried out by Thiruvaiyaru and Basawa (1992), and their ancillary properties were investigated. Estimation of waiting time in the $M/M/1$ queueing model when its distribution is right-skewed was achieved by Chowdhury and Mukherjee (2011). Furthermore, the ML and Bayesian estimations of the traffic intensity parameter in the $M/M/1$ queueing model were obtained by Chowdhury and Mukherjee (2013). The equivalence of

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results in parameter estimation for the $M/M/1$ queueing model using Bayesian and ML methods was demonstrated by Singh and Acharya (2019). Nonparametric estimation of service time in queueing models with infinite servers and Poisson input was obtained by Goldenshluger and Koops (2019). Nonparametric estimation of the service time distribution in discrete-time queueing models was achieved by Schweer and Wichelhaus (2020). The ML and Bayesian estimations of the traffic intensity parameter in the $M/D/1$ queueing model were obtained by Chandrasekhar et al. (2021). Yaghoobzadeh (2023) implemented the estimation of traffic intensity parameters and the reliability probability of the $M/M/c$ queuing model under a system downtime. Recently, Makhdoom and Shahrastani (2023) improved the quality of $M/M/m/K$ queuing systems using the system cost function optimization.

Censoring is a common practice in lifetime experiments and reliability studies. If the experimenter decides that when r experimental units fail, the experiment terminates, then it is said that Type II censoring has occurred. For further study in this area, refer to Cohen (1991).

Moreover, fuzzy sets have been employed in the theory of estimation by several authors. Some applications of Bayesian methods in statistical analysis were elucidated by Coppi et al. (2006). A novel approach for determining the membership function for estimating parameters and the reliability function for multi-parameter lifetime distributions was proposed by Huang et al. (2006). A new method for fuzzy point estimation, minimizing variance uniformly, was presented by Akbari and Rezaei (2007). Extensive studies on statistical inference methods for fuzzy lifetime distributions based on fuzzy numbers were conducted by Pak et al. (2013, 2014b, 2020, 2014a). Expected Bayesian (E-Bayesian) and hierarchical Bayesian estimation of scalar parameter in Gompertz distribution based on fuzzy data were obtained by Yaghoobzadeh Shahrastani (2019). Now, based on Dubois and Prade (1978), several definitions are presented.

Definition 1.1. If X is a reference set, then for each x in X, the degree of membership of x in the fuzzy set $\tilde{A}$ with the membership function $\mu_{\tilde{A}}(x)$ is defined as $\mu_{\tilde{A}}(x) : R \to [0,1]$.

Definition 1.2. The fuzzy set $\tilde{A}$ in the reference set X is called normal if and only if $\sup_{x\in X} \mu_{\tilde{A}}(x) = 1$.

Definition 1.3. The fuzzy set $\tilde{A}$ in the reference set X is convex if and only if for every $x, y \in X$ and every $\lambda \in [0,1]$, the following condition holds

$$\mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq \min\{\mu_{\tilde{A}(x)}, \mu_{\tilde{A}}(y)\}.$$

Definition 1.4. If X is a reference set, then the fuzzy set $\tilde{A}$ is called a fuzzy number if $\tilde{A}$ is normal and convex in X.

Definition 1.5. Let $L : \mathbb{R}^+ \to [0,1]$ and $R : \mathbb{R}^+ \to [0,1]$ be two continuous functions with the properties

$$\begin{aligned} L(-x) &= L(x), \quad R(-x) = R(x), \\ L(0) &= R(0) = 1, \\ \lim_{x\to\infty} L(x) &= \lim_{x\to\infty} R(x) = 0. \end{aligned}$$

If L and R are decreasing in the interval $[0,\infty]$, then the fuzzy number $\tilde{M}$ is called an LR-type fuzzy number if and only if the relationship

$$\mu_{\tilde{M}}(x) = \begin{cases} L\left(\frac{m-x}{\alpha}\right), & x \leq m, \alpha > 0, \\ R\left(\frac{m-x}{\beta}\right), & x > m, \beta > 0, \end{cases}$$

holds, in which m is the mean of the fuzzy number $\tilde{M}$, and α and β are referred to as the left and right bounds of the fuzzy number, respectively. Additionally, an LR-type fuzzy number is denoted by $\tilde{M} = (m, \alpha, \beta)$.

Also, in this paper, the term "withdrawn applicant" is utilized, referring to an applicant who, for their own reasons, decides to withdraw from the queue without receiving the service offered by the system.

The traffic intensity parameter, also known as the utilization factor or system reliability factor, for the $M/M/1$ queuing system is measured by a researcher (or an experimenter) and concludes when the system experiences the departure of the r-th withdrawn applicant. Consequently, the time of the system's abandonment by the r-th withdrawn applicant is considered as the completion time of the experiment by the researcher. Furthermore, for $i = 1, 2, \ldots, r$, let X_i denote the count of applicants within the $M/M/1$ system, characterized by an average arrival rate of λ and an average service rate of $\frac{1}{\mu}$, when the i-th withdrawn applicant is identified. As outlined on pages 72 to 74 in Medhi (2002)'s work, the probability function for this is expressed as

$$f(x_i, \rho) = (1-\rho)\rho^{x_i}. \tag{1}$$

If $X_1, \ldots, X_r$ are independent random variables with the probability density function (1), and precise information about the times of departure for the r withdrawn applicants is not available, but rather information about them is presented in the form of fuzzy numbers $\tilde{x}i = (m_i, \alpha_i, \beta_i)$ with membership functions $\mu_{\tilde{x}_1}(x_1), \ldots, \mu_{\tilde{x}_r}(x_r)$, then assuming $m(r)$ as the maximum averages of these r fuzzy numbers, and letting $N = \sum_{i=1}^{n} X_i + r$, the times of the remaining $N-r$ applicants are represented as fuzzy numbers $\tilde{x}_{r+1}, \tilde{x}_{r+2}, \ldots, \tilde{x}_N$ with membership functions

$$\mu_{\tilde{x}_j}(x) = \begin{cases} 0 & x \leq m_{(r)}, \\ 1 & x > m_{(r)}, \end{cases} \quad j = r+1, r+2, \ldots, N. \tag{2}$$

Therefore, considering $\tilde{\boldsymbol{x}} = (\tilde{x}_1, \ldots, \tilde{x}_N)$ as the fuzzy data vector with the membership function $\boldsymbol{\mu}_{\tilde{\boldsymbol{x}}}(\boldsymbol{x}) = \mu_{\tilde{x}_1}(x_1) \times \ldots \times \mu_{\tilde{x}_n}(x_N)$, and based on the definition of fuzzy event probability Zadeh (1968), the likelihood function of fuzzy observations is given by

$$\begin{aligned} L(\rho, \tilde{\boldsymbol{x}}) &= \left(\int f(x,\rho)\mu_{\tilde{x}_1}(x)dx\right) \ldots \left(\int f(x,\rho)\mu_{\tilde{x}_N}(x)dx\right) = \prod_{j=1}^{N} \int_0^\infty (1-\rho)\rho^x \mu_{\tilde{x}_j}(x)dx \\ &= \left(\prod_{j=1}^{r} \int_0^\infty (1-\rho)\rho^x \mu_{\tilde{x}_j}(x)dx\right) \left(\prod_{j=r+1}^{N} \int_0^\infty (1-\rho)\rho^x \mu_{\tilde{x}_j}(x)dx\right) \\ &= \left(\prod_{j=1}^{r} \int_0^\infty (1-\rho)\rho^x \mu_{\tilde{x}_j}(x)dx\right) \left[\prod_{j=r+1}^{N} \left(\int_0^{m_{(r)}} (1-\rho)\rho^x \mu_{\tilde{x}_j}(x)dx\right.\right. \end{aligned}$$

$$+\int_{m_{(r)}}^{\infty}(1-\rho)\rho^{x}\mu_{\tilde{x}_j}(x)dx\Bigg)\Bigg].$$

Considering the relationship in (2), the likelihood function for fuzzy observations is obtained as follows:

$$L(\rho,\tilde{\boldsymbol{x}})=(1-\rho)^{N}\left(-\frac{1}{\log\rho}\right)^{N-r}\rho^{(N-r)m_{(r)}}\left(\prod_{j=1}^{r}\int_{0}^{\infty}\rho^{x}\mu_{\tilde{x}_j}(x)dx\right). \qquad (3)$$

The structure of the article is such that in the second section, the Bayesian estimation of the traffic intensity parameter in the $M/M/1$ queue model is obtained under the general entropy functions and linear exponential (LINEX) based on the second type censoring scheme and fuzzy data. The third section is dedicated to the application with assigned data. In this section, using Monte Carlo simulation and a set of real datasets, the estimates of the traffic intensity parameter under general entropy and LINEX functions are compared. The article concludes in the fourth section with a discussion and summary of the findings.

2 Bayesian estimation of ρ

In this section, the Bayesian estimation of the parameter ρ is obtained under the general entropy and LINEX loss functions. The general entropy loss function is defined as

$$L(\hat{\theta},\theta)\propto(\frac{\hat{\theta}}{\theta})^{p}-p\log(\frac{\hat{\theta}}{\theta})-1,\quad p\neq 0,$$

This generalized entropy loss function is an extension of the entropy loss function used by authors like Dey et al. (1986) and Dey (1992). The LINEX loss function, introduced by Varian (1975), is defined as

$$L(\hat{\theta},\theta)=be^{k\frac{\hat{\theta}}{\theta}}-k(\frac{\hat{\theta}}{\theta})-1,\quad k\neq 0,$$

where b and k are constants, and k determines the shape of the loss function, while b is used for scaling. According to Chang and Hung (2007), for $k=1$, the loss function is asymmetric around zero and tends to overestimate. For $k=-1$, the loss function is asymmetric and tends to underestimate. For small positive values of k, i.e., k^j for $j\geq 3$, the loss function is symmetric and similar to the quadratic loss function. For more information on the selection of loss functions, refer to Berger (2013) (Alizadeh Noughabi and Ahmadi, 2013). Also, in this section, the prior distribution for ρ is assumed to be

$$\pi(\rho)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\rho^{a-1}(1-\rho)^{b-1},\quad 0<\rho<1,\quad a>0,\quad b>0,$$

with a and b known.

2.1 Estimation under general entropy loss function

In general, Bayesian estimation of a parameter θ under the general entropy loss function, as suggested by Dey et al. (1986), is obtained as

$$\hat{\theta}_{Bay} = \left[E(\theta^{-p}|\boldsymbol{X})\right]^{-\frac{1}{p}}. \tag{4}$$

Therefore, based on Equation (3), the posterior distribution for ρ given fuzzy data is derived as

$$\pi(\rho|\tilde{\boldsymbol{x}}) = \frac{(1-\rho)^{N+b-r-1}\rho^{(N-r)m(r)+a-1}\left(-\frac{1}{\log\rho}\right)^{N-r}u(\rho)}{\int_0^1(1-\rho)^{N+b-r-1}\rho^{(N-r)m_{(r)}+a-1}\left(-\frac{1}{\log\rho}\right)^{N-r}u(\rho)d\rho},$$

where $u(\rho) = \prod_{j=1}^{r}\int_0^\infty \rho^x\mu_{\tilde{x}j}(x)dx$. The Bayesian estimate of ρ under the general entropy loss function, according to Equation (4), is given by

$$\begin{aligned}\hat{\rho}_{BS} &= \{E(\rho^{-p}|\tilde{\boldsymbol{X}})\}^{-\frac{1}{p}}\\ &= \left\{\frac{\int_0^1(1-\rho)^{N+b-r-1}\rho^{(N-r)m_{(r)}+a-p-1}\left(-\frac{1}{\log\rho}\right)^{N-r}u(\rho)d\rho}{\int_0^1(1-\rho)^{N+b-r-1}\rho^{(N-r)m_{(r)}+a-1}\left(-\frac{1}{\log\rho}\right)^{N-r}u(\rho)d\rho}\right\}^{-\frac{1}{p}}.\end{aligned} \tag{5}$$

It is found that its calculation is not feasible in general. For its calculation, the approximation method Lindley (1980) has been used. Generally, for arbitrary functions $h(\theta)$ and $w(\theta)$, according to the Lindley method, the following approximation is obtained

$$\frac{\int h(\theta)e^{w(\theta)}d\theta}{\int e^{w(\theta)}d\theta} \simeq (h(\theta) + \frac{1}{2}h_{11}\delta_{11} + v_1h_1\delta_{11} + \frac{1}{2}w_3\delta_{11}^2h_1)_{\theta=\hat{\theta}}. \tag{6}$$

The expression (6) is an approximation method, where

$$h_1 = \frac{dh(\theta)}{d\theta}, \quad h_{11} = \frac{d^2h(\theta)}{d\theta^2}, \quad v_1 = \frac{dv(\theta)}{d\theta}, \quad w_3 = \frac{d^3w(\theta)}{d\theta^3}, \quad \delta_{11} = (-\frac{d^2w(\theta)}{d\theta^2})^{-1}, \tag{7}$$

in which $\hat{\theta}$ is taken as the ML estimate of θ. Assuming

$$\begin{aligned} w(\rho) &= \log L(\rho,\tilde{\mathbf{x}}) + \log\pi(\rho) = L(\rho) + v(\rho),\\ L(\rho) &= N\log(1-\rho) + (N-r)m_{(r)}\log\rho - (N-r)\log(-\log\rho) + \log u(\rho),\\ v(\rho) &\propto (a-1)\log\rho + (b-1)\log(1-\rho).\end{aligned}$$

The relation (5) can be expressed as

$$\hat{\rho}_{BS} = \left\{\frac{\int_0^1\rho^{-p}e^{w(\rho)}d\rho}{\int_0^1 e^{w(\rho)}d\rho}\right\}^{-\frac{1}{p}}.$$

It is rewritten accordingly. Therefore, assuming $h(\rho) = \rho^{-p}$ and using the relation (7), the relationships can be expressed as

$$\hat{h}_1 = -p\hat{\rho}^{-(p+1)}, \quad \hat{h}_{11} = p(p+1)\hat{\rho}^{-(p+2)}, \quad \hat{v}_1 = \frac{a+(2-a-b)\hat{\rho}-1}{\hat{\rho}(1-\hat{\rho})},$$

$$
\begin{aligned}
\hat{\delta}_{11} &= \left(\frac{N}{(1-\hat{\rho})^2} - \frac{(N-r)m_{(r)}}{\hat{\rho}^2} - \frac{(N-1)(1+\log\hat{\rho})}{(\hat{\rho}\log\hat{\rho})^2} + \frac{(a-1)}{\hat{\rho}^2} + \frac{b-1}{(1-\hat{\rho})^2} \right. \\
&\quad \left. -\frac{1}{\hat{\rho}^2} U_1(\hat{\rho}) \right)^{-1}, \\
\hat{w}_3 &= \frac{N}{(1-\hat{\rho})^3} + \frac{(N-r)m_{(r)}}{\hat{\rho}^3} + \frac{(1+\log\hat{\rho})(2-\log\hat{\rho})}{(\hat{\rho}\log\hat{\rho})^3} - \frac{2(a-1)}{(\hat{\rho})^3} \\
&\quad + \frac{2(b-1)}{(1-\hat{\rho})^3} + \frac{1}{\hat{\rho}^3}(U_2(\hat{\rho}) + U_3(\hat{\rho})),
\end{aligned} \tag{8}
$$

in which

$$
\begin{aligned}
U_1(\rho) &= \sum_{j=1}^{r} \frac{I_j^0(\rho)I_j^1(\rho) - I_j^1(\rho)I_j^2(\rho) + (I_j^1(\rho))^2}{(I_j^0(\rho))^2}, \\
U_2(\rho) &= \sum_{j=1}^{r} \frac{I_j^0(\rho)I_j^1(\rho) - I_j^0(\rho)I_j^2(\rho) + (I_j^1(\rho))^2}{(I_j^0(\rho))^2}, \\
U_3(\rho) &= \sum_{j=1}^{r} \frac{I_j^3(\rho)(I_j^0(\rho))^2 - 3I_j^0(\rho)I_j^1(\rho)I_j^2(\rho) - 2I_j^2(\rho)(I_j^0(\rho))^2}{(I_j^0(\rho))^3}, \\
I_j^n(\rho) &= \int_0^\infty x^n \rho^x \mu_{\tilde{x}}(x)dx.
\end{aligned}
$$

$\hat{\rho}$ as the ML estimation of ρ is the solution to the equation $\rho = \frac{K(\rho)}{N+K(\rho)}$, where

$$
K(\rho) = \frac{N-r}{\log\rho} + (N-r)m_{(r)} + \sum_{j=1}^{r} \frac{I_j^1(\rho)}{I_j^0(\rho)}.
$$

and using the iterative method $\rho^{(j+1)} = K(\rho^{(j)})$, the solution to the equation $\rho = \frac{K(\rho)}{N+K(\rho)}$ is obtained. Therefore, based on the relations (5), (6), and (8), the Bayesian estimate ρ is given by

$$
\hat{\rho}_{BS} = (\hat{\rho}^{-p} + \frac{1}{2}\hat{h}_{11}\hat{\delta}_{11} + \hat{v}_1\hat{h}_1\hat{\delta}_{11} + \frac{1}{2}\hat{w}_3\hat{\delta}_{11}^2\hat{h}_1)^{-\frac{1}{p}}. \tag{9}
$$

2.2 Estimation under the LINEX loss function

The Bayesian estimation of the parameter θ, such as ρ, according to Varian (1975), is obtained as:

$$
\hat{\theta}_{BL} = -\frac{1}{k}\log(E(e^{-k\theta}|\boldsymbol{X}).
$$

Therefore, the Bayesian estimate ρ under the LINEX loss function is given by the equation:

$$
\hat{\rho}_{BL} = -\frac{1}{k}\log(E(e^{-k\rho}|\tilde{\boldsymbol{X}})
$$

$$= -\frac{1}{k}\log\left\{\frac{\int_0^1 e^{-k\rho}(1-\rho)^{N+b-r-1}\rho^{(N-r)m_{(r)}+a-1}\left(-\frac{1}{\log\rho}\right)^{N-r}u(\rho)d\rho}{\int_0^1(1-\rho)^{N+b-r-1}\rho^{(N-r)m_{(r)}+a-1}\left(-\frac{1}{\log\rho}\right)^{N-r}u(\rho)d\rho}\right\}.$$

The obtained relations, assuming $h(\rho) = e^{-k\rho}$ and consequently $h_1 = -ke^{-k\rho}$ and $h_{11} = k^2e^{-k\rho}$, result in the Bayesian estimation of ρ under the LINEX loss function using the Lindley's method as follows:

$$\hat{\rho}_{BL} = -\frac{1}{k}\log(e^{-k\hat{\rho}} + \frac{1}{2}\hat{h}_{11}\hat{\delta}_{11} + \hat{v}_1\hat{h}_1\hat{\delta}_{11} + \frac{1}{2}\hat{w}_3\hat{\delta}_{11}^2\hat{h}_1). \quad (10)$$

Here, δ_{11}, v_1, and w_3 have been obtained in Section 2.1.

3 Numerical study

In this section, Bayesian estimation of ρ is compared under general entropy and LINEX loss functions using the Monte Carlo simulation method and a real dataset.

3.1 Simulation study

In this section, Bayesian estimation of the traffic intensity parameter in the $M/M/1$ queuing model under the general entropy and LINEX loss functions is compared using the Monte Carlo simulation method. The simulation pattern is presented as follows.
(1) The random samples are generated from the geometric distribution (1) for $\rho = 0.3$, $\rho = 0.4$, $\rho = 0.5$, $\rho = 0.6$, and $\rho = 0.7$. Using the R software, 100 random samples are generated and considered as $\boldsymbol{X} = (X_1, \ldots, X_n)$.
(2) Assuming $r = 8$, a Type-II censored sample is generated from the samples obtained in the first step. This censored sample is considered as $\boldsymbol{X}_r = (X_1, \ldots, X_r)$.
(3) From the uniform distribution $U(0.03, 0.05)$, two random numbers u_1 and u_2 are generated. Then, the generated X_i values from the second step are transformed into fuzzy numbers $\tilde{X}_i = (X_i, X_i \times u_1, X_i \times u_2)$. These fuzzy numbers form a fuzzy sample $\tilde{\boldsymbol{X}} = (\tilde{X}_1, \ldots, \tilde{X}_r)$ based on a fuzzy system with membership functions

$$\mu_{\tilde{x}_1}(x) = \begin{cases} 1, & x \le 0.25, \\ \frac{0.5-x}{0.25}, & 0.25 \le x \le 0.5, \\ 0, & otherwise, \end{cases}, \qquad \mu_{\tilde{x}_2}(x) = \begin{cases} \frac{x-0.25}{0.25}, & 0.25 \le x \le 0.5, \\ \frac{0.75-x}{0.25}, & 0.5 \le x \le 0.75, \\ 0, & otherwise, \end{cases}$$

$$\mu_{\tilde{x}_3}(x) = \begin{cases} \frac{x-0.5}{0.25}, & 0.5 \le x \le 0.75, \\ \frac{1-x}{0.25}, & 0.75 \le x \le 1, \\ 0, & otherwise, \end{cases}, \qquad \mu_{\tilde{x}_4}(x) = \begin{cases} \frac{x-0.75}{0.25}, & 0.75 \le x \le 1, \\ \frac{1.25-x}{0.25}, & 1 \le x \le 1.25, \\ 0, & otherwise, \end{cases}$$

$$\mu_{\tilde{x}_5}(x) = \begin{cases} \frac{x-1}{0.25}, & 1 \le x \le 1.25, \\ \frac{1.5-x}{0.25}, & 1.25 \le x \le 1.5, \\ 0, & otherwise, \end{cases}, \qquad \mu_{\tilde{x}_6}(x) = \begin{cases} \frac{x-1.25}{0.25}, & 1.25 \le x \le 1.5, \\ \frac{1.75-x}{0.25}, & 1.5 \le x \le 1.75, \\ 0, & otherwise, \end{cases}$$

$$\mu_{\tilde{x}_7}(x) = \begin{cases} \frac{x-1.5}{0.25}, & 1.5 \le x \le 1.75, \\ \frac{2-x}{0.25}, & 1.75 \le x \le 2, \\ 0, & otherwise, \end{cases}, \qquad \mu_{\tilde{x}_8}(x) = \begin{cases} \frac{x-1.75}{0.25}, & 1.75 \le x \le 2, \\ 1, & x \ge 2, \\ 0, & otherwise, \end{cases}$$

are considered.
(4) In this stage, first, the Newton-Raphson iterative method is used to obtain $\hat{\rho} = MLE(\rho)$. Then, for (a, b) values of $(2, 3)$, $(3, 2)$, and $(1.5, 1.5)$, and for $p = k = -300(-100) - 800$, Bayesian estimates of ρ under general entropy and LINEX functions are obtained using equations (9) and (10), respectively. It is worth mentioning that, using the simulation program for calculating Bayesian estimators of ρ, it was determined that the values of k and p should be between -300 and -800, and the best estimation case is when $k = p$.

The first to third stages are repeated 5000 times, and the average of the estimators and the sum of their squared errors are obtained. The simulation results are presented in Tables 1 and 2. The results in these tables indicate that the Bayesian estimator of ρ under the general entropy function is always better than the estimator under the LINEX function for different values of $k = p$ and (a, b). As $k = p$ decrease, the estimators $\hat{\rho}_{BS}$ and $\hat{\rho}_{BL}$, particularly $\hat{\rho}_{BS}$, become closer to the true value of ρ. However, for $\rho < 0.5$, as ρ decreases, the distance of the estimator $\hat{\rho}_{BL}$ from the true value of ρ decreases. But for $\rho \geq 0.5$, as ρ increases, the distance of the estimator $\hat{\rho}_{BL}$ from ρ becomes very large (underestimation), and it does not seem to be a suitable estimator. Also, in all cases, when $k = p$ decreases, the mean squared errors of the estimators are smaller.

In Tables 1 and 2, in addition to the estimation of the traffic intensity parameter, which is one of the key parameters in any queueing model, the estimation was performed under the Generalized Entropy and LINEX loss functions. Furthermore, the mean squared error (MSE) of the estimators under the mentioned loss functions was also calculated. The MSE is a criterion used to determine the optimal estimate of the traffic intensity parameter. An estimator is considered optimal if its MSE is smaller. The results presented in Tables 1 and 2 are based on identifying the optimal estimate according to this criterion. Additionally, all calculations included in Tables 1 and 2 were performed using R software. The computational steps are also described in the simulation section of the article. The results in these Tables, also show that the estimators $\hat{\rho}_{BS}$ and $\hat{\rho}_{BL}$ are not affected by the values of a and b, but they are influenced by the values of k and p.

3.2 Real dataset

In this section, a dataset is used by Abiar and Badamchizadeh (2017) is employed. They investigated the behavior of the self-service machine of Bank Melli located at Mehrabad Airport during office hours, considering it as an $M/M/1$ model. For this purpose, the specified machine was monitored for ten consecutive days from 8 AM to 4 PM, and the number of visitors to the machine, both unsuccessful and successful in receiving services, was recorded for each hour. In this model, the time from the start of the visitor's work until the end of their work with the self-service machine has been taken into account. Now, the data is transformed into triangular fuzzy data. For this purpose, two random numbers, u_1 and u_2, are selected from the uniform distribution $U(0.03, 0.05)$. Then, each number of the dataset, denoted by a, is transformed into a triangular fuzzy number using the pattern $(a, a \times u_1, a \times u_2)$. These fuzzy data are presented in Table 3. Assuming $r = 8$, eight numbers, i.e., 12, 19, 20, 25, 30, 44, 48, and 70, are randomly selected from the numbers 1 to 80. These numbers represent the visitor IDs who were unsuccessful in obtaining a service from the self-service machine

Table 1: Average estimators of ρ with their mean squared errors for simulated data.

ρ	(a,b)	$k=p$	$\hat{\rho}_{BS}$	$\hat{\rho}_{BL}$	$MSE(\hat{\rho}_{BS})$	$MSE(\hat{\rho}_{BL})$
0.3	(2,3)	−300	0.3387	0.2438	0.0089	0.0153
		−400	0.3288	0.2725	0.0075	0.0130
		−500	0.3230	0.2898	0.0065	0.0115
		−600	0.3191	0.3014	0.0059	0.0104
		−700	0.3121	0.3097	0.0048	0.0096
		−800	0.3046	0.3160	0.0033	0.0088
	(3,2)	−300	0.3387	0.2439	0.0084	0.0145
		−400	0.3288	0.2725	0.0075	0.0131
		−500	0.3230	0.2898	0.0065	0.0115
		−600	0.3191	0.3014	0.0058	0.0102
		−700	0.3121	0.3097	0.0048	0.0099
		−800	0.3046	0.3160	0.0033	0.0092
	(1.5,1.5)	−300	0.3387	0.2438	0.0087	0.0149
		−400	0.3288	0.2725	0.0074	0.0129
		−500	0.3230	0.2898	0.0067	0.0118
		−600	0.3191	0.3014	0.0058	0.0103
		−700	0.3121	0.3097	0.0048	0.0096
		−800	0.3046	0.3160	0.0033	0.0087
0.4	(2,3)	−300	0.4513	0.2490	0.0072	0.0107
		−400	0.4382	0.2775	0.0059	0.0089
		−500	0.4305	0.2949	0.0054	0.0082
		−600	0.4254	0.3065	0.0047	0.0072
		−700	0.4218	0.3148	0.0045	0.0069
		−800	0.4190	0.3211	0.0041	0.0064
	(3,2)	−300	0.4514	0.2488	0.0073	0.0107
		−400	0.4382	0.2775	0.0061	0.0091
		−500	0.4305	0.2949	0.0053	0.0081
		−600	0.4254	0.3065	0.0048	0.0073
		−700	0.4218	0.3148	0.0044	0.0069
		−800	0.4191	0.3211	0.0041	0.0063
	(1.5,1.5)	−300	0.4514	0.2488	0.0070	0.0104
		−400	0.4382	0.2775	0.0060	0.0091
		−500	0.4305	0.2949	0.0054	0.0082
		−600	0.4254	0.3065	0.0050	0.0077
		−700	0.4218	0.3148	0.0044	0.0068
		−800	0.4190	0.3211	0.0042	0.0066
0.5	(2,3)	−300	0.5651	0.2266	0.0056	0.0074
		−400	0.5484	0.2558	0.0047	0.0063
		−500	0.5386	0.2735	0.0042	0.0057
		−600	0.53222	0.2854	0.0039	0.0054
		−700	0.5276	0.2939	0.0036	0.0049
		−800	0.5241	0.3003	0.0033	0.0046
	(3,2)	−300	0.5651	0.2266	0.0055	0.0074
		−400	0.5484	0.2558	0.0047	0.0064
		−500	0.5386	0.2735	0.0042	0.0057
		−600	0.5322	0.2854	0.0038	0.0052
		−700	0.5276	0.2939	0.0034	0.0047
		−800	0.5241	0.3003	0.0033	0.0046
	(1.5,1.5)	−300	0.5651	0.2265	0.0056	0.0074
		−400	0.5484	0.2558	0.0047	0.0063
		−500	0.5386	0.2735	0.0041	0.0055
		−600	0.5322	0.2854	0.0039	0.0053
		−700	0.5276	0.2939	0.0036	0.0049
		−800	0.5241	0.3003	0.0033	0.0046

Table 2: Average estimators of ρ with their mean squared errors for simulated data.

ρ	(a,b)	$k=p$	$\hat{\rho}_{BS}$	$\hat{\rho}_{BL}$	$MSE(\hat{\rho}_{BS})$	$MSE(\hat{\rho}_{BL})$
0.6	$(2,3)$	−300	0.6806	0.1821	0.0043	0.0052
		−400	0.6599	0.2125	0.0037	0.0045
		−500	0.6478	0.2309	0.0032	0.0039
		−600	0.6398	0.2432	0.0030	0.0038
		−700	0.6341	0.2520	0.0027	0.0034
		−800	0.6298	0.2586	0.0025	0.0032
	$(3,2)$	−300	0.6806	0.1821	0.0041	0.0050
		−400	0.6599	0.2125	0.0036	0.0045
		−500	0.6478	0.2309	0.0033	0.0041
		−600	0.6398	0.2432	0.0029	0.0036
		−700	0.6341	0.2520	0.0028	0.0035
		−800	0.6298	0.2586	0.0026	0.0033
	$(1.5,1.5)$	−300	0.6806	0.1821	0.0042	0.0051
		−400	0.6599	0.2125	0.0036	0.0044
		−500	0.6478	0.2309	0.0033	0.0041
		−600	0.6398	0.2432	0.0029	0.0037
		−700	0.6341	0.2520	0.0027	0.0033
		−800	0.6298	0.2586	0.0025	0.0032
0.7	$(2,3)$	−300	0.7993	0.1182	0.0035	0.0039
		−400	0.7738	0.1503	0.0030	0.0034
		−500	0.7489	0.1828	0.0024	0.0028
		−700	0.7419	0.1921	0.0022	0.0025
		−800	0.7366	0.1991	0.0021	0.0024
	$(3,2)$	−300	0.7993	0.1182	0.0035	0.0039
		−400	0.7738	0.1503	0.0030	0.0034
		−500	0.7588	0.1698	0.0026	0.0030
		−600	0.7489	0.1828	0.0024	0.0027
		−700	0.7419	0.1921	0.0022	0.0025
		−800	0.7366	0.1991	0.0021	0.0024
	$(1.5,1.5)$	−300	0.7993	0.1182	0.0034	0.0038
		−400	0.7738	0.1503	0.0030	0.0034
		−500	0.7589	0.1698	0.0026	0.0030
		−600	0.7489	0.1828	0.0025	0.0028
		−700	0.7419	0.1921	0.0022	0.0025
		−800	0.7366	0.1991	0.0021	0.0024

and abandoned it. Therefore, considering $(\tilde{X}_1,\ldots,\tilde{X}_r)=(\tilde{X}_1,\ldots,\tilde{X}_8)$, where

$$\tilde{X}_1=(10,0.4517,0.3705),\quad \tilde{X}_2=(12,0.5420,0.4446),\quad \tilde{X}_3=(9,0.4065,0.3346)$$
$$\tilde{X}_4=(12,0.5420,0.4446),\quad \tilde{X}_5=(10,0.4517,0.3705),\quad \tilde{X}_6=(5,0.2258,0.1852)$$
$$\tilde{X}_7=(17,0.7678,0.6298),\quad \tilde{X}_8=(14,0.6323,0.5187)$$

Using the data $(\tilde{X}_1,\ldots,\tilde{X}_8)$, $m_{(r)}=m_{(8)}=6.132633$ is obtained. According to $MLE(\rho)=0.9119912$, Bayesian estimation of ρ was performed with helping equations (9) and (10) with the assumption of $(a,b)=(2,3)$ and various values of $k=p$ mentioned in Section 3.1. The results are presented in Table 4. These results indicate that the numerical value of $\hat{\rho}_{BS}$ is very close to $MLE(\rho)$ for different values of $k=p$. However, the numerical value of $\hat{\rho}_{BL}$ for the same values of $k=p$ does not approach $MLE(\rho)$ closely, and its distance is significantly large. Therefore, Bayesian estimation of the traffic intensity parameter under the general entropy loss function appears to be more suitable.

Table 3: Triangular fuzzy data generated from the dataset.

$(16, 0.7227, 0.5928)$	$(11, 0.4968, 0.4076)$	$(6, 0.2710, 0.2223)$	$(13, 0.5872, 0.4817)$
$(6, 0.2710, 0.2223)$	$(11, 0.4968, 0.4076)$	$(7, 0.3162, 0.2594)$	$(13, 0.5872, 0.4817)$
$(7, 0.3162, 0.2594)$	$(4, 0.1801, 0.1482)$	$(7, 0.3162, 0.2594)$	$(10, 0.4517, 0.3705)$
$(10, 0.4517, 0.3705)$	$(9, 0.4065, 0.3346)$	$(7, 0.3162, 0.2594)$	$(12, 0.5420, 0.4446)$
$(10, 0.4517, 0.3705)$	$(14, 0.6323, 0.5187)$	$(12, 0.5420, 0.4446)$	$(9, 0.4065, 0.3346)$
$(9, 0.4065, 0.3346)$	$(8, 0.3613, 0.2964)$	$(8, 0.3613, 0.2964)$	$(9, 0.4065, 0.3346)$
$(12, 0.5420, 0.4446)$	$(14, 0.6323, 0.5187)$	$(6, 0.2710, 0.2223)$	$(8, 0.3613, 0.2964)$
$(10, 0.4517, 0.3705)$	$(10, 0.4517, 0.3705)$	$(17, 0.7678, 0.6298)$	$(9, 0.4065, 0.3346)$
$(19, 0.8582, 0.7039)$	$(10, 0.4517, 0.3705)$	$(12, 0.5420, 0.4446)$	$(13, 0.5872, 0.4817)$
$(7, 0.3162, 0.2594)$	$(9, 0.4065, 0.3346)$	$(12, 0.5420, 0.4446)$	$(6, 0.2710, 0.2223)$
$(8, 0.3613, 0.2964)$	$(12, 0.5420, 0.4446)$	$(10, 0.4517, 0.3705)$	$(5, 0.2258, 0.1852)$
$(11, 0.4968, 0.4076)$	$(23, 1.039, 0.8522)$	$(7, 0.3162, 0.2594)$	$(8, 0.3613, 0.2964)$
$(11, 0.4968, 0.4076)$	$(13, 0.5872, 0.4817)$	$(18, 0.8130, 0.6669)$	$(8, 0.3613, 0.2964)$
$(17, 0.7678, 0.6298)$	$(12, 0.5420, 0.4446)$	$(10, 0.4517, 0.3705)$	$(6, 0.2710, 0.2223)$
$(16, 0.7227, 0.5928)$	$(5, 0.2258, 0.1852)$	$(15, 0.6775, 0.5558)$	$(9, 0.4065, 0.3346)$
$(5, 0.2258, 0.1852)$	$(15, 0.6775, 0.5558)$	$(11, 0.4968, 0.4076)$	$(6, 0.2710, 0.2223)$
$(9, 0.4065, 0.3346)$	$(19, 0.8582, 0.7039)$	$(6, 0.2710, 0.2223)$	$(11, 0.4968, 0.4076)$
$(12, 0.5420, 0.4446)$	$(14, 0.6323, 0.5187)$	$(6, 0.2710, 0.2223)$	$(6, 0.2710, 0.2223)$
$(8, 0.3613, 0.2964)$	$(6, 0.2710, 0.2223)$	$(19, 0.8582, 0.7039)$	$(12, 0.5420, 0.4446)$
$(6, 0.2710, 0.2223)$	$(11, 0.4968, 0.4076)$	$(12, 0.5420, 0.4446)$	$(9, 0.4065, 0.3346)$

Table 4: Bayesian estimations of ρ under generalized entropy and LINEX loss functions for the data of bank Melli ATM.

$\hat{\rho}/\ k = p$	-300	-400	-500	-600	-700	-800
$\hat{\rho}_{BS}$	0.9993	0.9738	0.9588	0.9489	0.9319	0.9266
$\hat{\rho}_{BL}$	0.1315	0.1598	0.1799	0.1969	0.2576	0.2905

4 Discussion and conclusion

In this paper, the Bayesian estimation of the traffic intensity parameter (ρ) in the M/M/1 queuing model in the steady state, based on Type II censoring and using fuzzy data with a beta prior distribution, was obtained. The estimation was performed under general entropy and LINEX loss functions. The results, obtained through Monte Carlo simulation and a set of real-world data, demonstrated that Bayesian estimation under the general entropy loss function is better than to the LINEX loss function. It was also observed that when $\rho < 0.5$, the estimation distance of ρ under the LINEX loss function is closer to the true value of ρ, but in the case of $\rho \geq 0.5$, this distance becomes large. In all cases, the estimator under the general entropy loss function performed better. Furthermore, the estimators were not sensitive to the parameters of the beta prior distribution, but were influenced by the parameters of the general entropy loss function, denoted by p, and the LINEX loss function parameter k. It was determined that $-300 \leq k = p \leq -800$ for suitable estimators of ρ to be obtained.

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