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Characteristics of elasticity functions in economics and reliability

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Abstract: In recent decades, there has been a significant increase in research focused on dependability across various fields, including survival analysis, failure rates, generalized measures of failure, and related characteristics. A strong connection exists between the generalized failure rate and the economic concept of elasticity. Sensitivity analysis employs this ratio to assess how changes in an input variable affect an output variable. We apply this ratio to evaluate the sensitivity of an output variable in response to changes in an input variable. In this article, we explain the elasticity function of an economy and its implications for reliability. We will also explore how this function relates to dependable and unequal economic indicators. Furthermore, we will discuss various extensions of Lorenz curves, which are instrumental in understanding measures of inequality. Our focus has been on Lorenz curve elasticity and its connections to distorted and extended forms of warped curves.

Keywords: Income distribution, Inequality measurement, Truncated variables.
Mathematics Subject Classification (2010): 91B15, 91B80, 62N05.

1 Introduction

Functions such as the characteristic function and the cumulative distribution function can assist in identifying the distribution of a random variable. Additionally, this includes the survival function, several types of failure rates (including reversed), mean

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residual life, generalized failure rates, and elasticity functions, all of which are significant in the fields of dependability and economics.

In recent years, the statistical community has embraced and redefined the well-known idea of elasticity from economics and physics, resulting in the creation of the elasticity function. This function has shown its use in estimating and assessing dangers and is applicable across several domains, like financial risk management and public health. Metrics of aging, such as the failure rate and reversed failure rate, are crucial in scientific inquiry. These principles are thoroughly examined in several sources, including Block et al. (1998) and Barlow and Proschan (1996) along with several reliability textbooks. Lariviere and Porteus proposed the generalized failure rate (IGFR) in 2001 (Lariviere and Porteus, 2001), and Lariviere recognized the increasing generalized failure rate (IGFR) in 2006 (Lariviere, 2006). These failure rates can help solve problems with pricing and supply chain contracts.

New studies, like those by Chechile (2011), Banciu and Mirchandai (2013), and Torrado and Oliviera (2013), have looked into similar extensions of the reversed hazard rate. Furthermore, the generalized failure rate (or reversed failure rate) can be defined as an elasticity function, which has significant economic implications. Oliveira and Torrado (2015), Pavia and Veres-Ferrer (2016), and Nanda et al. (2003) have looked at the reliability aspects of these ideas, mainly looking at the reversed hazard rate and its features. Elasticity is a crucial instrument in economics, measuring the responsiveness of an output variable to variations in an input variable. As the values of a random variable fluctuate within its support, the cumulative distribution function correspondingly changes. This link arises from the variable's elasticity, which indicates how the relative accumulation of probability functions throughout the variable's support. It evaluates the probability of advancing to greater values proportionally. This comprehension facilitates the analysis of fundamental processes, systems, or action protocols, yielding significant insights for efficient risk management in domains such as economics and health.

Two authors, Veres-Ferrer and Pavia, looked at the elasticity of continuous distributions for random variables that are not negative in papers released in 2012 (Veres-Ferrer et al., 2012) and 2014 (Veres-Ferrer et al., 2014) and 2017 (Veres-Ferrer and Pavia, 2017). Given the concerns regarding dependability and economic applications, they proposed various interpretations of elasticity and its characteristics.The Lorenz curve (Lorenz, 1905) and the Gini index (Gini, 1912) are effective instruments for examining inequality and income distributions. A lot of research has looked into the types, levels, and uses of income inequality, as well as how they relate to factors of reliability (Klefsjo, 1984; Chandra and Singpurwalla, 1981; Mohtashami Borzadaran, 2018; Pham and Turkkan, 1994; Behdani et al., 2020, 2018, 2020). Various dimensions of distorting inequality metrics have been presented by Sordo et al. (2014), Sordo et al. (2017), and Wang et al. (2011). This research looks at the generalized failure rate (reversed failure) and elasticity functions. It ends with a more detailed version of the distorted Lorenz curve and the elasticity criteria that go with it.

This paper presents several key innovations that contribute to the understanding of dependability and economic dynamics. This study delineates the major innovations: This research demonstrates a substantial correlation between generalized failure rates and the economic principle of elasticity. This study elucidates the relationship between fluctuations in input factors and their impact on output variables, offering a fresh viewpoint on failure analysis in economic situations. The article presents a novel methodology using elasticity for sensitivity analysis. This approach evaluates the sensitivity of output variables to variations in input factors, providing critical insights for decision-making in diverse economic and management contexts.

This work's essential novelty is the analysis of the elasticity of the Lorenz curve and its association with distorted and extended curve shapes. This approach enhances the comprehension of inequality metrics, providing researchers and policymakers with improved instruments to assess and evaluate economic disparities. The study also includes the computation of the elasticity function itself. This novel methodology offers insights into the effects of elasticity variations on economic variables; hence, it enhances the investigation of reliability in economic situations.The paper clarifies the connection between economic elasticity and dependable but disparate economic indicators. This approach enhances the comprehension of economic processes, emphasizing the use of elasticity functions in analyzing economic inequalities.

We outline the paper's structure as follows: Section 2 delineates essential terminology and ideas, including elasticity, Lorenz curves, and convexity. In Section 3, we look at the role of elasticity in reliability, including when the elasticity function stays the same and how changes in this function affect reliability metrics. Ultimately, we examine the correlation between economic inequality indicators and the elasticity function, computing the elasticity functions of inequality curves and analyzing transformations of Lorenz curves. Finally, we devote a small part of the article to summarizing and concluding the paper.

2 Preliminaries

This section delineates essential terms and terminology used in the article. Initially, we will delineate the notions of convex and log-convex distributions. Subsequently, we shall provide a comprehensive definition of the utility function.

Definition 2.1. *A random variable Y is said to have a concave distribution if, for every y_1, y_2 and $\theta \in [0,1]$, the density g of Y satisfies the following relation*

$$g(\theta y_1 + (1-\theta)y_2) \geq \theta g(y_1) + (1-\theta)g(y_2).$$

The function g is said to be convex if $-g$ is concave.

Definition 2.2. *A random variable Y is said to have a log-concave distribution if, for any y_1, y_2 and any $\theta \in [0,1]$, the following holds*

$$g(\theta y_1 + (1-\theta)y_2) \geq g^{\theta}(y_1)g^{(1-\theta)}(y_2). \tag{1}$$

Assuming that g is positive, taking the logarithm of both sides of (1) yields

$$\log(g(\theta y_1 + (1-\theta)y_2)) \geq \theta \log(g(y_1)) + (1-\theta)\log(g(y_2)).$$

The function g is considered to be log-convex if the inequality in (1) is reversed. Please note that g is continuous on its domain if it is log-concave, and continuously differentiable if it is differentiable. We should consider both of these facts. Moreover, if $h(y)$

is differentiable on $D^* \subset (0, \infty)$, then $h(y)$ is log-concave (or log-convex) on D^*. The monotonicity of $\frac{h'(y)}{h(y)}$ on D^* indicates that it is either decreasing (for log-concave) or increasing (for log-convex).

In economics, the concept of utility represents worth or value, and its application has evolved significantly over time. Neoclassical economics, which dominates contemporary economic theory, has redefined the term to refer to a utility function that expresses a single consumer's preference ordering over a set of options, though it is not comparable across individuals. This refined notion of utility is more strictly defined than its original concept, but it is less useful (and more contentious) for ethical judgments, as it is subjective and dependent on choice rather than on experienced pleasure. The utility function $v(\cdot)$ measures a decision-maker's risk preference. A utility function assigns a numerical value to the various outcomes that may result from different investment decisions, typically quantifying the outcome in terms of resulting wealth. The outcome is usually quantified in wealth, and the utility function can be expressed as $v(w)$, where w denotes wealth. Absolute risk aversion (ARA) and relative risk aversion (RRA) measures are defined by Arrow (1965) as follows

Let $A(w) = -\frac{v''(w)}{v'(w)}$ and $R(w) = -w\frac{v''(w)}{v'(w)} = wA(w)$.

In these expressions, $v'(w)$ and $v''(w)$ represent the first and second derivatives of $v(\cdot)$ with respect to w. Increasing absolute risk aversion is indicated by $A'(w) > 0$, while decreasing absolute risk aversion is indicated by $A'(w) < 0$. The degree of risk aversion correlates with the curvature of $v(c)$. However, since anticipated utility functions are not uniquely defined (they are specified only up to affine transformations), a measure that remains constant under these transformations is necessary, rather than relying solely on the second derivative of $v(c)$.

For example, if $v(w) = a + b\log(w)$, then $v'(w) = \frac{b}{w}$ and $v''(w) = -\frac{b}{w^2}$, leading to $A(w) = \frac{1}{w}$. Since $A(w)$ is independent of a and b, the utility function $v(w)$ remains unaffected by affine transformations. Relative risk aversion (RRA) is a dimensionless quantity, unlike ARA, which has units of dollars^{-1}, making RRA applicable universally. The equivalent terminologies for relative risk aversion include constant relative risk aversion (CRRA) and decreasing/increasing relative risk aversion (DRRA/IRRA), similar to absolute risk aversion. Because utility is not absolutely convex or concave across all w, this measure continues to reliably indicate risk aversion even when the utility function shifts from risk-averse to risk-loving as w fluctuates. While the opposite is not always true, a constant RRA suggests a decreasing ARA. The utility function $v(w)$ provides a concrete illustration of continuous relative risk aversion, with $R(w) = 1$ implied by $v(w) = \log(w)$. Risk measures are related to the failure rate, inverse failure rate, odds, and Glaser's function, expressed as follows:

- $r(w) = \frac{g(w)}{\overline{G}(w)}$,
- $\widetilde{r}(w) = \frac{g(w)}{G(w)}$,
- $O(w) = \frac{G(w)}{\overline{G}(w)}$,
- $\eta(w) = -\frac{g'(w)}{g(w)}$.

In this context, $g(w)$ represents the probability density function (pdf), while $G(w)$ denotes the cumulative distribution function (cdf) of the random variable Y. The generalized failure rate (or reversed failure rate) is defined as $h(w) = wr(w)$ and $\widetilde{h}(w) =$

$w\widetilde{r}(w)$.

When $v(w) = kG(w)$ with $k > 0$, indicating a utility function associated with G, it follows that $A(w) = \eta(w)$ and $R(w) = w\eta(w)$.

Remark 2.3. *The property of increasing generalized failure rate (IGFR) holds significant importance in pricing and revenue management, as noted by (Paul, 2005; Lariviere, 2006). It remains invariant under both left and right truncations.*

Let Y be a random variable that is not negative and has a distribution function G and a positive finite mean μ. This is how you describe the Lorenz curve that goes with Y

$$L(p) = \frac{1}{\mu}\int_0^p G^{-1}(t)dt, \quad p \in [0,1],$$

where $G^{-1}(t) = \inf\{y : G(y) \geq t\}$ with $0 \leq t \leq 1$. The function $L(p)$ represents the cumulative percentage of total income held by individuals in the lowest $100p$ percent of incomes for each $p \in (0,1)$. It satisfies the conditions $L(0) = 0$ and $L(1) = 1$. The Lorenz curve is a continuous, increasing, convex, and differentiable function within its domain.

In contrast, the Leimkuhler curve is defined as the reverse-mirror reflection of the Lorenz curve across the diagonal 45-degree line. It is expressed as

$$K(p) = 1 - L(1-p) = \overline{L}(1-p),$$

where $K(p)$ indicates the percentage of total income received by individuals with the highest incomes. This curve provides insights into income distribution from the perspective of wealth concentration among the affluent. Another type of inequality is expressed by the functions $B(p) = \frac{L(p)}{p}$ for $0 < p < 1$ and $Z(p) = 1 - \frac{L(p)}{p}\frac{1-p}{1-L(p)}$. These expressions represent the Bonferroni curve and Zenga curve, which are significant in economics, particularly in the context of poverty and inequality. In addition to establishing a correlation between the Lorenz curve and reliability concepts, we can also compute these two curves. We define the mean residual life, also known as mean inactivity, as follows

$$\begin{aligned}\mu(t) &= E(Y - t|Y \geq t) = \frac{\int_t^\infty \overline{G}(y)dy}{\overline{G}(t)}, \\ \widetilde{\mu}(t) &= E(t - Y|Y \leq t) = \frac{\int_0^t G(y)dy}{G(t)}.\end{aligned}$$

It is evident that these two functions are special cases of the function $\frac{1}{\eta(y)}$. We can use a Lorenz (Bonferroni, Zenga) curve as an example

$$\widetilde{\mu}(t) = t - \frac{\mu L(G(t))}{G(t)} = t - \mu B(G(t)) = t - \frac{\mu[1 - Z(G(t))]}{1 - G(t)Z(G(t))}.$$

Let Y be $IFR(DFR)$, then $L(p) \geq (\leq) G^{-1}(p)(1-p) - \frac{pG^{-1}(p)}{\log(1-p)}$.

3 Elasticity function and reliability measures

Elasticity is a crucial concept in economics, measuring the sensitivity of one variable to changes in another. While elasticity can take negative values in many contexts within economics, the elasticity of probability itself is always non-negative. This is because probabilities are constrained between 0 and 1, meaning that any change in probability must reflect a non-negative relationship.

In this section, we present several properties of the elasticity function. To achieve this objective, it is essential to review a few definitions of elasticity. The elasticity is defined as $E = \frac{\Delta Q}{Q} / \frac{\Delta P}{P}$ for the demand function $Q = f(P)$. The expressions ΔP and ΔQ reduce to differentials dP and dQ when the change in P is infinitesimal, resulting in a point elasticity of demand. This concept is applicable to various functions beyond just the demand function.

In monopoly theory, a monopolist sells a product where the quantity demanded is a function $D(p)$ of the monopolist's price p. In this context, the concepts of elasticity of demand are frequently utilized to understand how changes in price affect the quantity demanded. Specifically, the price elasticity of demand measures the responsiveness of the quantity demanded to changes in price, which is crucial for the monopolist when setting prices to maximize profit.

The monopolist's revenue in dollars is given by $R(p) = pD(p)$. The relationship to price elasticity can be explored by examining how the revenue changes with price. Specifically, the revenue $R(p)$ increases (or decreases) if and only if the derivative $\frac{d \log R(p)}{d \log p}$ is positive (or negative). The revenue is directly proportional to p if the price elasticity of demand $\varepsilon_D(p)$ is greater than 1, indicating inelastic demand. Conversely, if $\varepsilon_D(p) < 1$, the revenue is inversely proportional to p, indicating elastic demand.

The lost-sales rate (LSR) can be calculated using the formula $q(p,t) = 1 - G(p,t)$, where $G(p,t) = P(D(p) \leq t)$. Here, $G(p,t)$ represents the distribution function of demand or the probability of not experiencing a lost sale. The measure of the rate at which sales are lost, denoted as q, is determined by the price increase p and the quantity level t. The relationship can be expressed as

$$\varepsilon_q(p) = -p \frac{\frac{\partial G(p,t)}{\partial p}}{G(p,t)},$$

which is referred to as LSR elasticity.

The elasticity of a function $g(t)$ at a certain value of t is given by

$$\varepsilon_g(t) = t \frac{g'(t)}{g(t)} = \frac{d \log g(t)}{d \log t},$$

where g is a differentiable function. Lariviere and Porteus (2001) defined the generalized failure rate (also known as the reversed failure rate) as

$$h(x) = xr(x) = -\varepsilon_{\overline{F}}(x) \quad \text{and} \quad \widetilde{h}(x) = x\widetilde{r}(x) = \varepsilon_F(x),$$

where F and $\overline{F}$ are the distribution function and survival function, respectively.

The elasticity of a random variable indicates how probability accumulates relative to the variable's support. It describes how the cumulative distribution function changes as its log-scale values change. In simpler terms, it assesses the likelihood of progressing to higher values proportionately. This allows for the analysis of the underlying process, system, or action protocol, providing vital information for effective risk management in areas such as economics or health.

Studies by Pavia and Veres-Ferrer (2016) and Veres-Ferrer and Pavia (2021) say that if the elasticity value is greater than 1 (or slightly lower), it's clear that the risk is going up in a time-dependent stochastic process that counts failures, injuries, or deaths. This observation emphasizes the decline in the variable's progression. From this perspective, the probability distribution can demonstrate the likelihood of a random process exhibiting elasticities greater than one, or falling within a certain range of interest. This knowledge would aid in proactive risk management. To address the aforementioned problem, this research examines the concept of elasticity in relation to the associated random variable. For each specific probability model, it is straightforward to calculate the likelihood of encountering elastic or inelastic circumstances (i.e., with elasticities greater than or less than 1). This has significant implications. On one hand, it enables accurate resource sizing a priori to address risk materialization. On the other hand, by altering or modifying the risk structure, we can intelligently intervene in the process and maintain risk control at appropriate levels, as determined by a cost-benefit analysis. We can achieve this by anticipatorily assessing the effects of various system or protocol designs or the ramifications of implementing a set of restrictive or preventive actions (in terms of the probability of risk materialization).

Not only does this study consider elasticity as a random variable, but it also makes other important contributions. As indicated by the previous discussion and demonstrated in other research projects (see, for example, Pavia and Veres-Ferrer (2016); Veres-Ferrer and Pavia (2021)), unit elasticities play a crucial role in risk management as they serve as change points. Therefore, we also dedicate a significant portion of our work to examining their relationship with other isolated points within the distribution. In addition to these two key contributions, this study also presents some additional, potentially less significant results.

The interpretation of $\varepsilon(x)$ is similar to the traditional economic concept of elasticity. The condition of perfect inelasticity is represented by a null elasticity value, $\varepsilon(x) = 0$. In some cases, minute adjustments do not affect how probability accumulates. Inelastic situations have values between 0 and 1, where slight increases in x lead to significantly smaller increases in the accumulation of probabilities. When changes in x that are infinitesimally small result in changes of the same quantity in the accumulation of probabilities, this is known as unit elasticity, or $\varepsilon(x) = 1$. Finally, elastic conditions result in elasticities greater than one, or $\varepsilon(x) > 1$. In these scenarios, even the smallest increments in x lead to larger increases in the probability accumulation. When $\varepsilon(x)$ approaches infinity in the limit, a perfect elasticity scenario is achieved. In such cases, an infinitesimal increase in x results in a theoretically infinite increase in the accumulation of probability. We can synthesize the elasticity's cumulative distribution function, $Fe(y)$, by considering it as a function of the random variable rather than its values. It is advantageous to examine the specific scenario in which the elasticity of $f(x)$ in relation to x remains constant. We have $\log f(x) = e \log x + a$ where e and a are constants. Therefore, it can be concluded that the function f demonstrates constant elasticity exclusively when it takes the form of a power function. Generally, the sensitivity of f to variations in x is contingent upon the particular value of x. For instance, if the function $f(x) = a - bx$, then the elasticity $\varepsilon_{a-bx}(x)$ can be expressed as $\frac{-bx}{a-bx}$. If the function f is either strictly decreasing or increasing, then there exists a unique inverse function f^{-1} such that $f^{-1}(t) = x$ if and only if $f(x) = t$. If $\varepsilon(x)$ represents the elasticity of $f(x)$ with respect to x, then $\frac{1}{\varepsilon(x)}$ represents the elasticity of $f^{-1}(t)$ with respect to t. Kocabiyikoglu and Popescu (2011) study on elasticity provides further details. This concept is comparable to a generalized rate of failure. Lariviere (2006) discovered that one can use the increasing probability of success, represented as $pr(p)$, to maximize the revenue function. The function $R(p)$ represents the product of the price p and the demand $D(p)$, where $D(p)$ is the complement of the function $F(p)$.

Definition 3.1. *A random variable Y has increasing elasticity (IE) if the elasticity function $E(x)$ is weakly increasing for all x. Decreasing elasticity (DE) distributions can be defined analogously.*

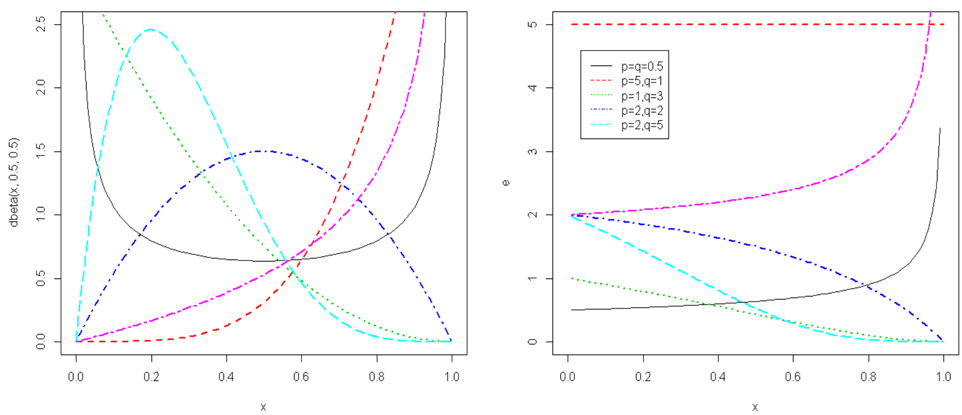


Figure 1: Density function and elasticity characteristics of the beta distribution

There is a connection between the parameters and the beta elasticity. For $0 \leq x \leq 1$ and parameters $p, q > 0$, the probability density function (PDF) of the beta distribution is given by

$$f(x; p, q) = \frac{x^{p-1}(1-x)^{q-1}}{\int_0^1 u^{p-1}(1-u)^{q-1} du}.$$

Figure 1 shows the graphs of the probability density function (left side) and the elasticity function graph (right side) of the beta distribution for different parameters. In this figure, the role and influence of the distribution parameters on the elasticity function and the density function are evident.

A random variable Y is referred to as an exponentiated random variable based on the base distribution H if its distribution function is given by $[H(t)]^{\mu}$ for some $\mu > 0$. Let Y and X denote random variables with distribution functions $H(t)$ and $K(t)$, respectively. If these random variables exhibit proportional hazards, it means that there exists a positive constant $\mu > 0$ such that

$$\overline{K}(t) = [\overline{H}(t)]^{\mu}, \quad Y > 0.$$

We can then write the reversed hazard rate function of K. Table 2 shows some continuous distributions with increasing survival function elasticity.

It is easy to prove the following relationships in special cases where the distribution function depends on other distribution functions:
Let $Z \sim F$ and $W \sim G$ where $G(x) = (F(x))^{\lambda}$ (proportional reversed hazard), then $\varepsilon_G(t) = \lambda \varepsilon_F(t)$.

Let $Z \sim F$ and $W \sim G$ where $\overline{G}(x) = (\overline{F}(x))^{\beta}$ (proportional hazard), then $\varepsilon_{\overline{G}}(t) = \beta \varepsilon_{\overline{F}}(t)$. Z is said to be an AL model ($Z \sim AL(\lambda)$) if its cumulative distribution function (CDF) can be expressed as $H(x) = F(\lambda t), \lambda > 0$, where F is the CDF of Z. Then, $\varepsilon_H(t) = \lambda \varepsilon_F(t)$.

Let $\widehat{Z}_w$ be a weighted random variable of Z with the weight function $w(x)$ where

$$\widehat{F}_w(x) = \frac{1}{E[w(Z)]} \int_{-\infty}^{x} w(t) dF(t),$$

(denoting $\widehat{F}_w(x)$ as the CDF of $\widehat{Z}$, called the weighted distribution function), then

$$\varepsilon_{\widehat{F}_w}(t) = \frac{w(x)F(x)}{\int_{-\infty}^{x} w(t)dF(t)} \varepsilon_{\widehat{F}}(t).$$

Some special cases of weighted families are equivalent to definitions in Table 1.

Table 1: Specific weight distributions and corresponding (Behdani et al., 2018).

$w(x)$	Name of $w(x)$	$\widehat{f}_w(x)$	$\varepsilon_{\widehat{F}_w}(t)$
x	Length (size)-biased	$\frac{xf(x)}{E(Z)}$	$\frac{x^2 f(x)}{L(F(x))} = \frac{xF(x)}{L(F(x))}\varepsilon_F(x)$
$[\overline{F}(x)]^{k-1}$	Proportional hazard rate	$kf(x)[\overline{F}(x)]^{k-1}$, $k > 0$	$-k\varepsilon_F(x)\frac{F(x)}{\overline{F}(x)}$
$[F(x)]^{k-1}$	Proportional reversed hazard rate	$kf(x)[F(x)]^{k-1}$, $k > 0$	$k\varepsilon_F(x)$
$\frac{1}{r_F(x)}$	Equilibrium distribution	$\frac{\overline{F}(x)}{E(Z)}$	$\frac{1}{1+\frac{L(F(x))}{x\frac{\overline{F}(x)}{E(Z)}}}$
$I_{(0,w)}$	Right truncated	$\frac{f(x)}{F(w)}$	$\varepsilon_F(x)$
$I_{(l,\infty)}$	Left truncated	$\frac{f(x)}{\overline{F}(l)}$	$\varepsilon_F(x)$

The first case can be interpreted as follows: if λ is an integer, then Z can be considered the lifetime of a parallel system consisting of a certain number of components, each having a lifetime distribution given by $F(x)$. On the other hand, if λ is not an integer, Z can be viewed as the lifetime of a parallel system with n components (where n is any positive integer), with each component having a lifetime distribution of $[F(x)]^{\lambda/n}$.

Suppose $Z_i \sim F_i(x)$ and $W_i \sim G_i(x) = (F_i(x))^{\theta_i}$, then $\varepsilon_{G_2}(t) \leq \frac{\theta_2}{\theta_1}\varepsilon_{G_1}(t)$, when $Z_1 \geq_{rh} Z_2$. Thus, $\theta_1 = \theta_2$ implies $\varepsilon_{G_2}(t) \leq \varepsilon_{G_1}(t), \forall t$.

The following note also specifies the relationship between the reverse hazard order and the elasticity function in certain situations.

Remark 3.2. *For example*
1. If $F_i(t) = (1 - e^{-t/\beta_i})^{\alpha_i}$, $i = 1, 2$, then $Z_1 \geq_{rh} Z_2$ if $\beta_1 \geq \beta_2$ and $\alpha_1 \geq \alpha_2$. If $\beta_1 = \beta_2 = \beta$, then $Z_1 \geq_{rh} Z_2$ iff $\alpha_1 \geq \alpha_2$ and for $\alpha_1 = \alpha_2 = \alpha$, $Z_1 \geq_{rh} Z_2$ iff $\beta_1 \geq \beta_2$, so $\varepsilon_{Z_1}(t) \geq \varepsilon_{Z_2}(t)$, $\forall t$.
2. If $F_i(x) = 1 - e^{-(t/\lambda_i)^{\alpha}}$ (Gupta and Nanda, 2001), then $Z_1 \geq_{rh} Z_2$ iff $\lambda_1 \geq \lambda_2$ and thus $\varepsilon_{Z_1}(t) \geq \varepsilon_{Z_2}(t)$, $\forall t$, which does not depend on α.

Remark 3.3. *Assume that $M(Z, p)$ is a random variable that represents a combination of objects, and that its CDF is $F_{M(Z,p)}(t) = \sum_{i=1}^{n} p_i F_{Z_i}(t)$ where $\sum p_i = 1$. Then*

$$\varepsilon_{F_{M(Z,p)}}(t) = \frac{1}{F_{M(Z,p)}(t)} \left[\sum_{i=1}^{n} p_i \varepsilon_{F_i}(t) F_{Z_i}(t) \right].$$

Table 2: Behavior of the elasticity function in increasing survival scenarios.

Name of distributions	Distribution function	$\varepsilon_{\overline{F}}$
Uniform (a, b)	$\frac{x-a}{b-a}$	$\frac{x}{b-x}$
Exponential $[0, \infty)$	$1 - e^{-\lambda x}$	λx
Normal $[0, \infty)$	$2\Phi(x) - 1$	$\frac{x}{\sqrt{2\pi}[1-\Phi(x)]} e^{-x^2/2}$
Logistic $[0, \infty)$	$\frac{2}{1+e^{-x}} - 1$	$\frac{x}{1+e^{-x}}$
Power function $[0, 1]$	x^k	$kx^{k-1}(1 - x^k)^{-1}$
Pareto $[1, \infty)$	$1 - x^{-k}$	k
Gumbel min $[0, \infty)$	$1 - e^{1-e^x}$	xe^x
Weibull $[0, \infty)$	$1 - e^{-x^k}$	kx^{k-1}

The claims listed below are apparent:
• Suppose that $\varepsilon_{\overline{F}_1}$ and $\varepsilon_{\overline{F}_2}$ are the elasticities of F_1 and F_2, respectively, which are increasing. Their mixture (M) does not necessarily imply that ε_M is increasing. For example, when $Z_1 \sim \text{Exp}(4)$ and $Z_2 \sim G(2, 1)$, $\varepsilon_M = \alpha Z_1 + (1 - \alpha) Z_2$ is not increasing.
• For price p and demand $D(p) = \overline{F}(p)$, the revenue $R = pD(p)$ is maximized at p^* which satisfies $\varepsilon_{\overline{F}}(p^*) = 1$.
• Let $Z \sim f$ and $W = h(Z)$ be a one-to-one transformation, then

$$\varepsilon_{F_W}(t) = \varepsilon_{h^{-1}}(t) \varepsilon_{F_{Z}(h^{-1})}(t).$$

- If $W = aZ + b$, then $\varepsilon_W(t) = \frac{t}{t-b} \varepsilon_Z\left(\frac{t-b}{a}\right)$.
- If $W = aZ$, then $\varepsilon_W(t) = \frac{1}{a} \varepsilon_Z\left(\frac{t}{a}\right)$.
- If $W = Z + b$, then $\varepsilon_W(t) = \frac{t}{t-b} \varepsilon_Z(t - b)$.
• The conditions $\varepsilon_F(x) = 0$, $0 < \varepsilon_F(x) < 1$, $\varepsilon_F(x) = 1$, and $\varepsilon_F(x) > 1$ correspond to perfect inelasticity, unit elasticity, and elastic behavior, respectively.
• Suppose that $\varepsilon_{\overline{F}_1}$ and $\varepsilon_{\overline{F}_2}$ have increasing elasticity for the survival function of random variables Z_1 and Z_2, respectively. Then the elasticity of the survival function of $T = Z_1 Z_2$ is also increasing.

Remark 3.4. *In this remark, we express the linking of the elasticity function from different functions for continuous variables.*
• $\varepsilon_F(x) = x\frac{f(x)}{F(x)} > \varepsilon_{\overline{F}}(x) = x\frac{-f(x)}{\overline{F}(x)}$ *for* $x > x_{median} > 0$ *and* $\varepsilon_F(x) < \varepsilon_{\overline{F}}(x)$ *for*

$0 < x < x_{median}$. *Additionally,* $r(x_{median}) > \lambda(x_{median})$ *and* $f(x_{median}) = \frac{\lambda(x_{median})}{2} = \frac{r(x_{median})}{2}$.

• *For all continuous symmetric distributions, for two points of symmetry about the median* x_a *and* x_b*, we have* $r(x_a) = \lambda(x_b)$ *and* $\lambda(x_a) = r(x_b)$.

• $F(x) = \frac{\varepsilon_{\overline{F}(x)}}{\varepsilon_{\overline{F}}(x)+\varepsilon_F(x)}$.

• $\overline{F}(x) = \frac{\varepsilon_F(x)}{\varepsilon_{\overline{F}}(x)+\varepsilon_F(x)}$.

• $f(x) = \frac{\varepsilon_{\overline{F}}(x)\varepsilon_F(x)}{x(\varepsilon_{\overline{F}}(x)+\varepsilon_F(x))}$.

• $O(x) = \frac{F(x)}{\overline{F}(x)} = \frac{\varepsilon_{\overline{F}(x)}}{\varepsilon_F(x)}$.

• $\frac{d\log\omega(x)}{dx} = r(x) + \lambda(x)$.

4 The relationship between elasticity function and inequality indices

We illustrate the responsiveness of the dual Lorenz curve $L(p)$ to variations in p, particularly emphasizing its elasticity, which is defined as

$$e_L(p) = \frac{d\log L(p)}{d\log p} = \frac{pL'(p)}{L(p)}, \quad 0 < p < 1.$$

This is equivalent to

$$e_L(p) = \frac{pF^{-1}(p)}{\int_0^p F^{-1}(t)dt}.$$

The elasticity of the Lorenz curve refers to changes in the income or wealth ratio against changes in the population ratio. In other words, elasticity shows how changes in the distribution of income or wealth can affect inequality. In general, one of the following scenarios may occur:

• **Elasticity greater than one**: If the elasticity is greater than 1, it indicates that income is disproportionately distributed among the population. In other words, small changes in the population can have a significant impact on cumulative income, indicating high inequality.

• **Elasticity less than 1**: If the elasticity is less than 1, it indicates that the income distribution is relatively more equal. Changes in the population have a lesser impact on cumulative income.

• **Elasticity equal to 1**: If the elasticity is equal to 1, it indicates a perfectly equal distribution of income. This means that each percentage of the population receives the same share of the income.

Lorenz curve elasticity is a powerful tool for understanding and analyzing income and wealth inequality. Considering the elasticity, it is possible to assess the inequality situation in a society and formulate effective policies to improve income distribution.

To better understand these concepts, we have provided the following example.

Example 4.1. *We simulated societies with 20 members as follows:*

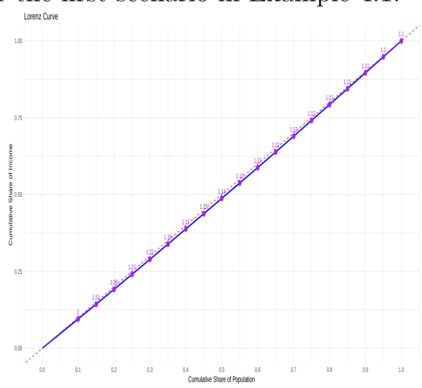
First scenario: In this case, all members of the community receive almost similar incomes. To this end, we simulated incomes from a uniform distribution in the range of (1000 and 1100). In Figure 2, we calculated the Lorenz curve of this data as well as the elasticity values at specific points on the curve, which are indicated on the figure. As you can see, the elasticity values are directly related to the level of inequality, such that in cases of lower inequality, the elasticity value is close to one.

Second scenario: In this case, 5 people are selected between 1000 and 1100, and 15 people are selected between 50,000 and 100,000. Figure 2 shows the Lorenz curve of this data as well as the elasticity values at specific points on the curve. As you can see, the elasticity values are directly related to the level of inequality, such that when inequality is lower, the elasticity value is close to one.

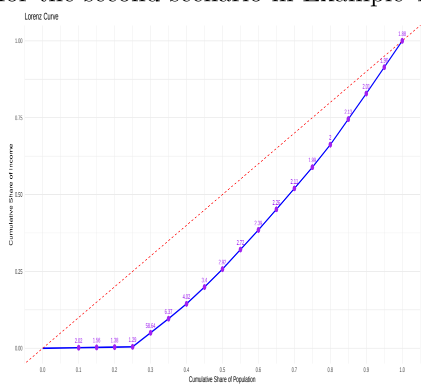
Third scenario: In this case, 15 people earned between 1000 and 1100, and the remaining 5 people earned between 50000 and 100000. Figure 2 shows the Lorenz curve of this data as well as the elasticity values at specific points on the curve.

Fourth scenario: In the end, in the fourth scenario, 20 people are selected between 1000 and 20000.

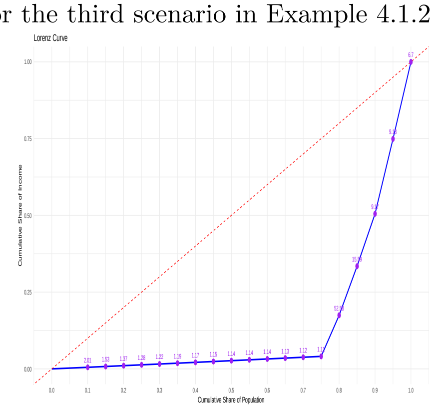
Lorenz curve and elasticity values for the first scenario in Example 4.1.



Lorenz curve and elasticity values for the second scenario in Example 4.1.



Lorenz curve and elasticity values for the third scenario in Example 4.1.2



Lorenz curve and elasticity values for the fourth scenario in Example 4.1.

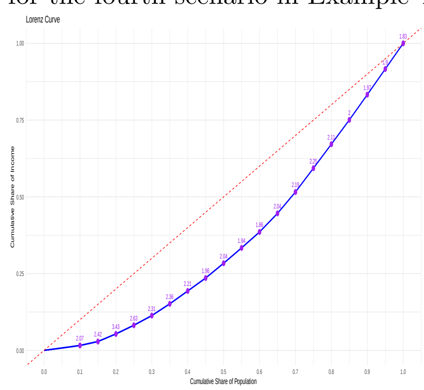


Figure 2: Lorenz curve and elasticity values for different scenarios (with data with different disparities) in Example 4.1.

As can be seen, in all these cases, the numerical elasticity is greater than one. In points where significant changes occur, the strain has a larger value, but in points where there are no significant changes, the strain will not be very large. Additionally, in the table, the values of strain and the corresponding hysteresis curve for each of the 4 described states have been calculated. Also, the calculated values for these scenarios are presented in Table 3. It is clear that some of the large stretch values in the table and results are due to the small sample size and the sudden change in the Lorenz curve. If the number of data is very large, very large values will not be observed in the stretching of the Lorenz curve.

Table 3: Comparison of Lorenz curve and elasticity for four simulated data sets with same percentiles in Example 4.1.

	First scenario		Second scenario		Third scenario		Fourth scenario	
p	$L(p)$	$\epsilon(p)$	$L(p)$	$\epsilon(p)$	$L(p)$	$\epsilon(p)$	$L(p)$	$\epsilon(p)$
0.1	0.095	2	0.002	2.023	0.005	2.009	0.016	2.074
0.2	0.192	1.36	0.004	1.382	0.01	1.368	0.029	2.420
0.3	0.289	1.22	0.050	58.64	0.016	1.223	0.113	2.312
0.4	0.388	1.19	0.144	6.37	0.021	1.167	0.193	2.212
0.5	0.488	1.15	0.257	2.923	0.027	1.143	0.284	2.041
0.6	0.588	1.13	0.385	2.386	0.032	1.136	0.386	1.862
0.7	0.689	1.120	0.520	2.107	0.038	1.121	0.518	2.186
0.8	0.793	1.110	0.662	1.998	0.174	52.58	0.672	2.111
0.9	0.896	1.10	0.828	2.011	0.505	9.175	0.833	1.973
1.0	1.0	1.09	1.0	1.876	1.0	6.701	1.0	1.825

The elasticity of the Lorenz curve indicates the effect of truncating the income distribution on inequality. In 2017, Sordo et al. (2017) showed that for left-truncated variables, the Lorenz order ($X_{(l,\infty)} \leq_L X_{(l',\infty)}$) only applies if the Leimkuhler elasticity e_K is rising in the range $(0, 1-F(l_0))$, $l_0 \in [0,1]$.For variables truncated from the right, the Lorenz order ($X_{(0,r)} \leq_L X_{(0,r')}$) is satisfied if and only if the Lorenz elasticity e_L is increasing in the interval $[F(r_0), 1]$, where $r_0 \in [0,1]$. It was also noted that an increase (or decrease) in the elasticity of $\overline{F}^{-1}$ implies $X_{(t,\infty)} \leq_* X_{(t',\infty)}$ (or $\geq_*$), and conversely. They also showed that truncations change the expected proportional shortfall order by changing the elasticities of its dual Leimkuhler curve, just like they did with the Lorenz order.

4.1 Distorted inequality curves and elasticity function

Ogwang and Rao (2000) defined the necessary conditions for

$$L_\alpha(p) = (L_1(p))^\alpha (L_2(p))^\beta,$$

to qualify as a Lorenz curve. This formulation represents a weighted product model in which $L_i(p)$, $\alpha \geq 1$, $\beta \geq 1$, $i = 1, 2$ are recognized as Lorenz curves. Wang et al. (2011) came up with a complete way to make Lorenz curves that can be used in a wider range of situations. They used the formula $L_1(p) = h(L(p))$, where h is a distortion function with $h > 0$ and L is a Lorenz curve. Sordo et al. (2017), presented a note that delineated a possible pathway for the enhancement of their concept.

Researchers Sordo et al. (2014, 2017) investigated the fundamental shapes of Lorenz curves and demonstrated how a distorted version of the Lorenz curve can be applied. A distortion function is defined as an increasing function represented by $h : [0,1] \to [0,1]$, which satisfies the properties $h(0) = 0$ and $h(1) = 1$. If we think of $L(p)$ as the Lorenz curve for the distribution F and h as a distortion function, then $L_*(p) = h(L(p))$ for $0 \le p \le 1$ is another Lorenz curve (see Sordo et al. (2017)). Assuming that $L(p)$ has a second derivative $L''(p)$ and is defined and continuous over the interval $[0,1]$, the following conditions must be satisfied for $L(p)$ to qualify as a Lorenz curve: $L(0) = 0$, $L(1) = 1$, $L'(0^+) > 0$, and $L''(p) \ge 0$ for $p \in [0,1]$. The following assertion is considered an extension of Theorem 3 presented in the research conducted by Sordo et al. (2017).

Theorem 4.2. *Let h_1 and h_2 be two distortion functions with $h_i'(t) \ge 0$, $t \in [0,1]$, $i = 1,2$ and twice differentiable. When both L_1 and L_2 are Lorenz curves, the function*

$$\widetilde{L}(p) = h_1(L_1(p))h_2(L_2(p)),$$

creates a Lorenz curve.

Proof. Assuming that L_i, $i = 1,2$ and h_i, $i = 1,2$ are respective Lorenz curves and distortion functions, it is straightforward to demonstrate that $\widetilde{L}(0) = 0$ and $\widetilde{L}(1) = 1$. The first derivative of $\widetilde{L}$ is represented by

$$\widetilde{L}'(p) = L_1'(p)h_1'(L_1(p))h_2(L_2(p)) + L_2'(p)h_2'(L_2(p))h_1(L_1(p)).$$

Since $L_i'(0^+) > 0$ and $h_i'(0) > 0$, it follows that $\widetilde{L}'(0^+) > 0$. Finding the second derivative yields

$$\begin{aligned}\widetilde{L}''(p) &= h_1''(p)h_1'(L_1(p))h_2(L_2(p)) + (L_1''(p))^2 h_1''(L_1(p))h_2(L_2(p)) \\ &\quad +2L_1'(p)L_2'(p)h_1'(L_1(p))h_2'(L_2(p)) + L_2''(p)h_2'(L_2(p))h_1(L_1(p)) \\ &\quad +(L_2''(p))^2 h_2''(L_2(p))h_1(L_1(p)).\end{aligned}$$

Given that $L_i''(p) > 0$ and h_i, $i = 1,2$ are increasing, it follows that $\widetilde{L}''(p) > 0$. As a result, $\widetilde{L}$ meets the requirements of a Lorenz curve. □

The following notes can be derived:

- The Bonferroni curve of $\widetilde{L}(p)$ can be expressed as

$$\widetilde{B}(p) = \frac{\widetilde{L}(p)}{p}, \quad p \in (0,1).$$

- By setting $h_2(x) = 1$ for all x, we obtain Theorem 4.2, as stated in Theorem 3 of Sordo et al. (2017).
- The expression

$$\widetilde{L}(p) = L_3(L_1(p))L_4(L_2(p)) \tag{2}$$

where L_i, $i = 1,2,3,4$ are Lorenz curves, is also a Lorenz curve; where $(L_i(p) = (L_j(p))^\gamma$, $\gamma \ge 1$, $L_i(p) = 1 - (1 - L_j(p))^\delta$, $\delta \in (0,1]$, $L_i(p) = \frac{\theta L_j(p)}{1-(1-\theta)L_j(p)}$, $0 \le p \le 1$ for $i = 3,4$ and $j = 1,2$ are special cases of (2). Also, Aggarwal (1984), Rohde (2009), and Sarabia et al. (2010) obtained special cases of $\widetilde{L}(p) = L_3(L_1(p))$.

• Let $G(t)$ denote the probability generating function of the random variable X. Then, the expression $\widetilde{L}(p) = (L(p))^\alpha G(L(p))$ also represents a Lorenz curve.
• Define $h_i(p) = (L_i^{-1}(p))^{\alpha_i} p^{\gamma_i}$, $i = 1, 2$, $\alpha_i > 0$, $\gamma_i \geq 1$. It follows that

$$\widetilde{L}(p) = h_1(L_1(p))h_2(L_2(p)) = p^{\alpha_1+\alpha_2}(L_1(p))^{\gamma_1}(L_2(p))^{\gamma_2},$$

which serves as an extension of the Lorenz curves characterized by the Sarabia-Castillo-Slottje class.
• Let $h_i(p) = 1 - (1-p)\exp\{-\gamma_i[1-(1-p)^{\lambda/\beta_i}]\}$, $\beta \in (0, 1]$; $\gamma_i > 0$. Then, we have

$$\widetilde{L}(p) = 1 - (1-p)^{\beta_1+\beta_2}\exp\{-(\gamma_1+\gamma_2)p\},$$

which acts as an extension of the Lorenz curve presented by Zuxiang et al. (2009).
• Define $h_i(p) = (L_1(p))^{\alpha_i}(L_2(p))^{\beta_i}$, $\alpha_i, \beta_i \geq 1$. This formulation represents an extension of the Lorenz curve introduced by Wang et al. (2011).
• The expression $\widetilde{L}(p) = p^\alpha[1 - L_\lambda(1-p)^\beta]^\eta$, where $L_\lambda(p) = \frac{e^{\lambda p}-1}{e^\lambda - 1}$, $\lambda > 0$, corresponds to the Lorenz curve proposed by Chotikapanich (1993). The function $\widetilde{L}(p)$ is closely related to $L^*(p) = p^\alpha[1-(1-p)^\beta]^\eta$, which can serve as a generator for efficient Lorenz curve models as discussed in Wang and Smyth (2007).

The elasticity of the Lorenz curve $L(p)$ can be defined in terms of elasticity as follows

$$\varepsilon_L(p) = p\frac{L'(p)}{L(p)}, \quad p \in (0, 1).$$

For the Lorenz curve $\widetilde{L}(p)$ introduced in equation (2), we can state the following theorem.

Theorem 4.3. *Let $L_i(p)$ for $i = 1, 2$ be Lorenz curves, and let h_i for $i = 1, 2$ be convex distortions. The distorted Lorenz curve is defined as follows*

$$\widetilde{L}(p) = \varepsilon_{L_1}(p)\varepsilon_{h_1}(L_1(p)) + \varepsilon_{L_2}(p)\varepsilon_{h_2}(L_2(p)). \tag{3}$$

Proof. By utilizing the definition of the elasticity function in conjunction with equation (2), we arrive at the following conclusions

$$\begin{aligned}
\varepsilon_{\widetilde{L}}(p) &= p\frac{\widetilde{L}'(p)}{\widetilde{L}(p)} \\
&= p\frac{L_1'(p)h_1'(L_1(p))h_2(L_2(p)) + L_2'(p)h_2'(L_2(p))h_1(L_1(p))}{h_1(L_1(p))h_2(L_2(p))} \\
&= p\frac{L_1'(p)}{L_1(p)}L_1(p)\frac{h_1'(L_1(p))}{h_1(L_1(p))} + p\frac{L_2'(p)}{L_2(p)}L_2(p)\frac{h_2'(L_2(p))}{h_2(L_2(p))} \\
&= \varepsilon_{L_1}(p)\varepsilon_{h_1}(L_1(p)) + \varepsilon_{L_2}(p)\varepsilon_{h_2}(L_2(p)),
\end{aligned}$$

which explicitly achieves (3). □

Remark 4.4. *For all the specific cases outlined in Sordo et al. (2017), we can determine the elasticity function for the distorted versions. For instance, if we have $\widetilde{L}(p) = (L_1(p))^\alpha(L_2(p))^\beta$, then the elasticity function can be expressed as*

$$\varepsilon_{\widetilde{L}}(p) = \alpha\varepsilon_{L_1}(p) + \beta\varepsilon_{L_2}(p).$$

Additionally, the Bonferroni curve can also be represented using the distorted elasticity function.

Theorem 4.5. *Let $B_i(p)$ for $i = 1, 2$ represent Bonferroni curves, and let h_i for $i = 1, 2$ denote convex distortions. Then,*

$$\varepsilon_{\widetilde{B}}(p) = \varepsilon_{\widetilde{L}}(p) - 1.$$

Proof. Noting that $\widetilde{B}(p) = \frac{\widetilde{L}(p)}{p}$ where $\widetilde{L}(p)$ is defined in (2), we have

$$\varepsilon_{\widetilde{B}}(p) = p\frac{\frac{d\widetilde{B}(p)}{dp}}{\widetilde{B}(p)} = p\frac{\left(\frac{\widetilde{L}(p)}{p}\right)'}{\frac{\widetilde{L}(p)}{p}} = p\frac{\widetilde{L}'(p)}{\widetilde{L}(p)} - 1 = \varepsilon_{\widetilde{L}}(p) - 1.$$

□

Remark 4.6. *A special case of Theorem 4.5 can be easily identified. For example, when $\widetilde{B}(p) = (B(p))^{\alpha}G(B(p))$, then*

$$\varepsilon_{\widetilde{B}}(p) = [\alpha + \varepsilon_G(B(p))]\varepsilon_B(p).$$

Remark 4.7. *Let the random variable Z have a cumulative distribution function (CDF) F and a probability density function (PDF) f, and let $w(x)$ be a weight depending on the observed value x such that the density of f_w is given by*

$$f_w(x) = \frac{w(x)f(x)}{E[w(X)]}.$$

• If $w(.)$ is increasing (decreasing) and $w(x)\tilde{h}(x)$ is decreasing (increasing), then F_w is DGFR (IGFR). If $w(x)\tilde{h}(x)$ is decreasing (increasing), then F_w is DGRFR (IGFR).
• Consider

$$\tilde{h}_w(x) = x\tilde{r}_w(x) = \frac{w(x)}{E[w(X)]}\tilde{h}(x)\frac{F(x)}{L_w(F(x))}.$$

We also calculated the elasticity of the elasticity function, which led us to the interesting result below.

$$\epsilon(\epsilon(f)) = 1 + \epsilon'_f(p) - \epsilon_f(p).$$

$\epsilon(\epsilon(f))$ represents the elasticity of the elasticity function. This indicates how the elasticity of a function responds to changes in its own elasticity. $\epsilon'_f(p)$ denotes the derivative of the elasticity function. In other words, this value shows how the elasticity of the main function responds to changes in price. $\epsilon_f(p)$ value represents the elasticity of the main function, indicating how sensitive demand is to changes in price.

Thus, the equation shows that the elasticity of the elasticity function ($\epsilon(\epsilon(f))$) is equal to one plus the derivative of the elasticity function minus the elasticity of the main function.

This result can be shows, the elasticity of the elasticity function depends on changes in the elasticity of the main function. In other words, if the elasticity of the main function changes, this change affects not only the value of the elasticity itself but also how the elasticity function responds to changes in price. Essentially, this equation tells us that the elasticity of the elasticity function is, in a way, a feedback from the elasticity of the main function and its variations.

5 Discussion and conclusions

In conclusion, this study highlights the intricate relationship between elasticity functions and their applications in both economic theory and reliability measures. By examining the characteristics of elasticity, we have demonstrated its significance in understanding how output variables respond to changes in input variables, particularly in the context of generalized failure rates and economic inequality. The insights gained from analyzing the elasticity of Lorenz curves and other related functions provide valuable tools for assessing risk management strategies across various fields, including economics and public health. As we move forward, further exploration of the interplay between elasticity and reliability measures will undoubtedly yield deeper insights, fostering a more robust understanding of these critical concepts in both theoretical and practical applications.

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