Journal of Statistical Modelling: Theory and Applications
Vol. 5, No. 1, 2024, pp. 165-180
Yazd University Press 2024

JSMTA

Research Paper

Exploring the Burr type III distribution: Properties, stochastic orders, and information measures

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Received: May 27, 2024/ Revised: December 18, 2024/ Accepted: January 04, 2025

Abstract: The Burr Type III distribution, also known as the Singh-Maddala distribution, is a versatile probability distribution widely used in various fields such as hydrology, finance, reliability engineering, and actuarial science. This paper explores the Burr Type III distribution, discussing its properties, moment calculations, stochastic orders, and entropy measures tailored for this distribution.

Keywords: Beta Distribution, Divergence, Entropy, Extropy, Stochastic orders.
Mathematics Subject Classification (2010): 60E05, 60E15, 94A17.

1 Introduction

Burr (1942) introduced twelve families of cumulative distributions for modeling real-world data, with the Burr Type III distribution being one of the most versatile Yakubu and Doguwa (2017). Known for its heavy skewness and flexibility in shape, the Burr Type III distribution can model a wide variety of data characteristics. It also incorporates elements of non-normal distributions like the gamma, logistic, and exponential distributions, enhancing its applicability. For example, the Kumaraswamy-Burr Type III distribution combines the Kumaraswamy and Burr Type III distributions Behairy et al. (2017). The Burr Type III distribution reduces to the Lomax distribution when $\alpha = 1$, and transforms into an exponentiated exponential distribution when $\dfrac{x}{s} = e^{cy}$ Nasir et al. (2017).

Domma (2010) expanded on certain findings regarding the correlation patterns of the bivariate Burr Type III distribution. Azizi and Sayyareh (2020) Studied the characteristics of the Marshal-Olkin bivariate model using the Burr Type III distribution

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in the presence of random left censoring. The estimation and prediction issues were examined by Altinda et al. (2017) and Hassan et al. (2023) for the Burr Type III distribution when dealing with Type II censored data. Jamal et al. (2017) suggest a novel group of probability distributions known as the odd Burr Type III distribution family, which is derived from the logit transformation of the Burr Type III random variable. Asadi et al. (2023) conducted a study on estimating the SSRe model using the Burr Type III distribution in conjunction with an enhanced adaptive progressive Type II censoring scheme. Hassan et al. (2021) create a maximum likelihood estimator for confidence level within a Burr type III distribution using a progressive Type II censored sample.

The Burr Type III distribution, also known as the inverse Burr distribution or Dagum Type distribution, is a statistical distribution commonly used for modeling in statistics. According to Johnson et al. (1995) it is considered a simple distribution and can be obtained from the probability density function (PDF) of the Burr Type II distribution by replacing X with $ln(x)$. A random variable, X, follows the scaled Burr III (BIII) distribution if it's obtained by subtracting one from a random variable following the Burr XII (BXII) distribution. This relationship holds true for positive values ($x > 0$) of X Gomes et al. (2013) and Olobatuyi (2017) described the mathematical formula for the BIII distribution's cumulative distribution function (CDF) in this context Cordeiro et al. (2017)

$$G_{\alpha,\beta,s}(x) = \left[1 + \left(\frac{x}{s}\right)^{-\alpha}\right]^{-\beta} = \left[\frac{(x/s)^{\alpha}}{1 + (x/s)^{\alpha}}\right]^{\beta}.$$

The Burr Type III distribution is characterized by closed-form expressions for both its Probability Density Function (PDF) and CDF, that is a significant advantage of the Burr Type III distribution, as it simplifies both theoretical analysis and practical computation. The PDF can be obtained by differentiating the CDF with respect to X, as shown below:

$$g_{\alpha,\beta,s}(x) = \frac{\alpha\beta}{s}\left(\frac{x}{s}\right)^{-\alpha-1}\left[1 + \left(\frac{x}{s}\right)^{-\alpha}\right]^{-\beta-1} = \frac{\alpha\beta}{s\left(\frac{x}{s}\right)^{\alpha+1}}\left[\frac{(x/s)^{\alpha}}{1 + (x/s)^{\alpha}}\right]^{\beta+1}.$$

In a recent paper we studied the characterization and properties of Burr Type III distribution. Section 2 introduces the general properties of the Burr Type III distribution. In this section, we present and prove various characterization results of the Cumulative Distribution Function as well as transformed distributions. Furthermore, we present calculation methods for determining $E(X^k)$ and $E(\frac{1}{X}^k)$ for the Burr Type III distribution, where k is a positive arbitrary number. In Section 3, we establish various properties and boundaries for the Burr Type III distribution in relation to stochastic orders, including the usual stochastic order, likelihood ratio order, hazard rate order, and reversed hazard rate order. Finally, Section 4 delves into the properties and boundaries of entropies and extropies. In this section, we present solutions for various entropies and extropies, including Shannon entropy, extropy, Cumulative and Cumulative residual entropies and extropies, as well as Renyi, Tsallis, and Sharma-Mital entropies specifically for Burr Type III distribution.

2 General properties

In the context of probability, the CDF over a specific interval provides information about the probability of the random variable falling within that interval. Furthermore, for the lifetime variables, there is a relationship between the CDF and the expectation of the random variable X. Specifically, for a non-negative random variable X representing lifetime, the relationship is given by

$$E(X) = \int_0^\infty (1 - G(x))dx,$$

where $\bar{G}(X) = 1 - G(x)$ is the survival function (complementary cumulative distribution function).

However, the integral of the cumulative distribution function of the random variable is not provided.

Propositions 2.1 and 2.2 outline the key properties of the cumulative distribution function (CDF) for X following a Burr Type III distribution.

Proposition 2.1. *The integral of the CDF, $(G(x))$ of the random variable X, which follows a Burr Type III distribution, is denoted as*

$$\int_0^\infty G(x)dx = \frac{E(X^{\alpha+1})}{\alpha(\beta-1)s^\alpha}. \tag{1}$$

Proof. It is derived from

$$\int_0^\infty \left[\frac{(x/s)^\alpha}{1+(x/s)^\alpha}\right]^\beta dx = \frac{1}{\alpha(\beta-1)s^\alpha}\int_0^\infty \frac{\alpha(\beta-1)}{s(\frac{x}{s})^{\alpha+1}} x^{\alpha+1}\left[\frac{(x/s)^\alpha}{1+(x/s)^\alpha}\right]^\beta dx.$$

□

Proposition 2.2. *The integral of the cumulative distribution function raised to the power of k $(CDI^{(k)})$ of the random variable X following a Burr Type III distribution is calculated as*

$$CDI^{(k)} = \int_0^\infty G^k(x)dx = \frac{E(X^{\alpha+1})}{\alpha(k\beta-1)s^\alpha}.$$

Proof. Building upon the proof of Proposition 2.1, we can demonstrate

$$\int_0^\infty G^k(x)dx = \int_0^\infty \left[\frac{(x/s)^\alpha}{1+(x/s)^\alpha}\right]^{k\beta} dx = \frac{E(X^{\alpha+1})}{\alpha(k\beta-1)s^\alpha}.$$

□

Corollary 2.3. *The cumulative extropy, denoted by $CJ(X)$ defined by Nair and Sathar (2020), is proportional to the second-order Cumulative Distribution Index $(CDI^{(2)})$ as $-2CJ(X) = CDI^{(2)}$. This relationship was proven in Proposition 4.5.*

Transforming a distribution to a Beta distribution enhances the simplification process. Essentially, converting the Burr Type III distribution to the Beta distribution serves as a bridge that enables us to leverage the advantages of parameter estimation in the Beta distribution. This transformation provides a more practical and efficient method for parameter estimation when working with the Burr Type III distribution.

Proposition 2.4. *The Burr Type III distribution with the parameters α, β and s can be transformed into a Beta distribution with parameters $(1, \beta)$.*

Proof. Let us substitute the term $(\frac{x}{s})^\alpha$ into $\frac{u}{1-u}$. Then we would have

$$\begin{aligned}\int_0^\infty \frac{\alpha\beta}{s\left(\frac{x}{s}\right)^{\alpha+1}}\left[\frac{(x/s)^\alpha}{1+(x/s)^\alpha}\right]^{\beta+1} dx &= \int_1^0 \frac{\alpha\beta}{s}(\frac{u}{1-u})^{-\frac{\alpha+1}{\alpha}}(\frac{u}{1-u})^{\beta+1}u^{\beta+1}\\ &\times\frac{s}{\alpha}(\frac{u}{1-u})^{\frac{1-\alpha}{\alpha}}(1-u)^{-2}du\\ &=\int_0^1 \beta u^{\beta-1}du = \int_0^1 \frac{\Gamma(\beta+1)}{\Gamma(1)\Gamma(\beta)}u^{\beta-1}du = 1.\end{aligned}$$

□

Propositions 2.5 and 2.6 have demonstrated a clear connection between two distinct distributions. In Proposition 2.5, we present a linear transformation applied to X, involving scaling by a factor of a and adding a constant offset of b. In Proposition 2.6, we introduce the Lehmann model that incorporates the power of θ for the Burr Type III distribution.

Proposition 2.5. *consider a random variable X that follows this distribution with parameters α, β and s. Then the CDF of $Y = aX + b$, where a and b are constants, is given by*

$$G_{\alpha,\beta,as}(x) = \left[1+\left(\frac{x}{a.s}\right)^{-\alpha}\right]^{-\beta}, \quad x > b.$$

Proof. The CDF of Y is given by

$$F_Y(y) = P(Y \le y) = P\left(X \le \frac{y-b}{a}\right) = \left[\frac{(\frac{(y-b)}{as})^\alpha}{1+(\frac{(y-b)}{a.s})^\alpha}\right]^\beta, \quad y > b.$$

This expression should match the form of a Burr Type III CDF, but with potentially different parameter c into $a.c$. In other hand, the scale parameter s would be scaled by a, and the location would shift by b. However, the shape parameters α and β would remain unchanged because they are scale-invariant. □

Proposition 2.6. *Suppose $G(x)$ is a CDF of Burr type III distribution, with the parameters α, β and s. Consider the function $F(x) = (G(x))^\theta$, where θ is a positive real number. Then $F(x)$is distributed by Burr Type III with the parameters α, $\theta\beta$ and s.*

Proof. The proof is clear and the details are readily apparent. □

Remark 2.7. *Let X be a random variable distributed by Burr Type III. Then the density for the ith order statistic is given by*

$$g_{(i)}(x) = \binom{n}{i}\frac{\alpha\beta}{s\left(\frac{x}{s}\right)^{\alpha+1}}\left[\frac{(x/s)^\alpha}{1+(x/s)^\alpha}\right]^{i\beta+1}\left(1-\left[\frac{(x/s)^\alpha}{1+(x/s)^\alpha}\right]^\beta\right)^{n-i}. \quad (2)$$

Proof. The density for the ith order statistic formula is given by

$$g_{(i)}(x) = \frac{n!}{(i-1)!(n-i)!}[G(x)]^{i-1}[\bar{G}(x)]^{n-i}g(x),$$

that is estated as (2).

It is obvious that substituting the context $G(x)$ into the term u results the binomial distribution. □

Proposition 2.8 by utilizing a transformation from Burr Type III to Beta distribution, provides a streamlined approach to more effectively estimate the parameters of the original Burr Type III distribution.

Proposition 2.8. *The $k\alpha^{th}$ moment of the inverse Burr Type III lifetime distribution with the parameters α, β nd s is given by*

$$E(\frac{1}{x})^{k\alpha} = s^{-k\alpha}\beta\binom{\beta}{k}, \qquad 0 \leq k\alpha \leq \beta - 1. \tag{3}$$

Proof. Letting Proposition 2.4 the expectation of $E(X^{-k\alpha})$ is equal to $s^{-k\alpha}E(\frac{u}{1-u})^{-k}$ in which is given by

$$\begin{aligned} E(\frac{u}{1-u})^{-k} &= \int_0^1 \beta(\frac{u}{1-u})^{-k}u^{\beta-1}du = \int_0^1 \beta(1-u)^k u^{\beta-k-1}du \\ &= \beta\frac{\Gamma(\beta-k)\Gamma(k+1)}{\Gamma(\beta+1)} = \frac{\beta}{(\beta-k)\binom{\beta}{k}}. \end{aligned}$$

□

While Proposition 2.8 provides a formula for some central moments about zero, it has limitations. It only works when $0 \leq k\alpha \leq \beta - 1$. (where k is the moment order). There's a more powerful approach, though. By leveraging the properties of the integral of the CDF raised to the power of r, we can potentially derive more general expressions for central moments.

Corollary 2.9. *Assume that the random variable X following a Burr Type III distribution with the parameters $\alpha = 1$, β and s. Then*

- $E(X^2) = s\mu$.
- $\sigma^2(X) = \mu(s-\mu)$.
- $s = \mu(CV(X)+1) = \frac{E(X^2)}{\mu} = \frac{\sigma^2(X)}{\mu} + \mu$.
- $\beta = \frac{2CJ(X)-\mu}{4CJ(X)}$.

Proof. Letting $\alpha = 1$ we have

$$E(X) = \int_0^\infty \frac{\beta x}{s(\frac{x}{s})}\left[\frac{(x/s)}{1+(x/s)}\right]^{\beta+1}dx = \beta\int_0^\infty \left(\frac{(x/s)}{1+(x/s)}\right)^{\beta+1}dx = \frac{E(X^2)}{s}.$$

Therefore,

$$Var(X) = E(X^2) - E^2(X) = E(X)(s - E(X)).$$

Then

$$CV(X) = \frac{Var(X)}{E^2(X)} = \frac{(s - E(X))}{E(X)}.$$

Furthermore, based on the characteristics $CDI^{(1)}$ and $CDI^{(2)}$ we can conclude that

$$E(X^2) = s\mu = (2\beta - 1)sCDI^{(2)} = -2(2\beta - 1)sCJ(X),$$

that obtained that $\beta = \frac{2CJ(X)-\mu}{4CJ(X)}$. □

Remark 2.10. *The integral of the cumulative distribution function raised to the power of k ($CDI^{(k)}$) of the random variable X following a Burr Type III distribution can be transformed into a Beta distribution with the parameters $(\beta, 1)$.*

Proof. Similar to the Proposition 2.4 by substituting $(\frac{x}{s})^\alpha$ into $\dfrac{u}{1-u}$ we would have

$$\begin{aligned} CDI^{(k)} &= \int_0^\infty \left[\frac{(x/s)^\alpha}{1+(x/s)^\alpha}\right]^{k\beta} dx = \int_0^1 \beta(1-u)^{3-\frac{1}{\alpha}} u^{k\beta+\frac{1}{\alpha}-1} du \\ &= \beta \frac{\Gamma(k\beta+4)}{\Gamma(4-\frac{1}{\alpha})\Gamma(k\beta+\frac{1}{\alpha})} = \beta \binom{k\beta+3}{3-\frac{1}{\alpha}}. \end{aligned}$$

□

Corollary 2.11. *Solving* (2) *and* (1) *yields the following cors*

$$E(X^{\alpha+1}) = \alpha s^\alpha \beta(\beta-1)\binom{\beta+3}{3-\frac{1}{\alpha}}. \tag{4}$$

For example,

$$E(X^2) = \frac{s(\beta+3)(\beta+2)\beta(\beta-1)}{2}.$$

Remark 2.12. *Proposition 2.8 represents the generalized negative power mean, while Corollary 2.11 represents the k^{th} moment of the Burr Type III Lifetime distribution.*

Corollary 2.13. *Let $X_{(i)}$ is the ith order statistic in Burr Type III distribution data. Then, we have*

$$E(X_{(i)}) = nG(x) = n\left[\frac{(x/s)^\alpha}{1+(x/s)^\alpha}\right]^\beta dx = \frac{nE(X^{\alpha+1})}{\alpha(\beta-1)s^\alpha} = n\beta\binom{\beta+3}{3-\alpha^{-1}}.$$

Corollary 2.14. *Let $X_{(i)}$ is the ith order statistic in Burr Type III distribution data. Then the $g_i(x)$ is symmetric if $G(x) = \frac{1}{2}$ that yields*

$$\beta\binom{\beta+3}{3-\alpha^{-1}} = \frac{1}{2}.$$

3 Stochastic orders

Stochastic orders provide a framework for comparing random variables based on their likelihood of being larger or smaller. These orders include the Usual Stochastic Order (st), which compares variables using cumulative distribution functions; the Likelihood Ratio Order (lr), which compares based on probability density functions; the Hazard Rate Order (hr), which focuses on failure rates; and the Reversed Hazard Rate Order (rhr), which considers cumulative probabilities. These orders are commonly used in reliability analysis, statistical inference, actuarial science, and decision-making under uncertainty.
In our analysis, Proposition 3.1 explores the st order, while Propositions 3.2 and 3.3 delve into the lr order. Additionally, Propositions 3.5 and 3.6 examine the characteristics of the hr order. Furthermore, Propositions 3.8 and 3.9 shed light on the impact of random orderings on the rhr order.

Proposition 3.1. *Let X and Y be two continuous lifetime random variables with β_1 and β_2 respectively and $\beta_1 \leq (\geq)\beta_2$. Then $X \overset{st}{\leq (\geq)} Y$.*

Proof. It is obvious that when $\beta_1 \leq (\geq)\beta_2$ we would have $G_{\alpha,\beta_1,s}(x) \geq (\leq)G_{\alpha,\beta_2,s}(y)$ because of

$$\left[1+\left(\frac{x}{s}\right)^{-\alpha}\right]^{-\beta_1} \geq (\leq) \left[1+\left(\frac{x}{s}\right)^{-\alpha}\right]^{-\beta_2}.$$

Hence from the definition of Stochastic order (st) Pakgohar et al. (2019) the proof is completed. □

Proposition 3.2. *Let X and Y are two random variables with the Burr Type III distribution with the parameters β_1 and β_2 respectively. Then X is less (greater) than Y in likelihood ratio order $X \overset{lr}{\leq} (\overset{lr}{\geq})Y$ if $\beta_1 \leq (\geq)\beta_2$.*

Proof. The likelihood ratio order (denoted by $X \overset{lr}{\leq} Y$) holds if the ratio $\frac{g_X(t)}{g_Y(t)}$ is decreasing in t Di Crescenzo and Longobardi (2001). Therefore, we have

$$f(t) = \frac{g_X(t)}{g_Y(t)} = \frac{\beta_1}{\beta_2}\left[\frac{(t/s)^\alpha}{1+(t/s)^\alpha}\right]^{\beta_1-\beta_2}.$$

Taking the derivative of $f(t)$ with respect to t, we get

$$f(t) = \frac{d}{dt}\left[\frac{\beta_1}{\beta_2}\left[\frac{(t/s)\alpha}{1+(t/s)\alpha}\right]^{\beta_1-\beta_2}\right].$$

To simplify the differentiation, let's denote the inner function as $(t/s)^\alpha = u$ also $k(u) = \left[\frac{u}{1+u}\right]$, and $k(u) = \frac{1}{(1+u)^2}$. Then

$$f(u) = \frac{\beta_1}{\beta_2}(\beta_1-\beta_2)k(u)^{\beta_1-\beta_2-1}k(u) = \frac{\beta_1}{\beta_2}(\beta_1-\beta_2)\left[\frac{u}{1+u}\right]^{\beta_1-\beta_2-1}\frac{1}{(1+u)^2}.$$

Hence, $f(u) \leq (\geq)0$ if $\beta_1 \leq (\geq)\beta_2$. Therefore $\frac{g_X(t)}{g_Y(t)}$ is decreasing (increasing) with the condition $\beta_1 \leq (\geq)\beta_2$ and the proof is completed. □

Proposition 3.3. *Let X and Y are two random variables with the Burr Type III distribution with the parameters s_1 and s_2 respectively. Then X is greater (less) than Y in likelihood ratio order $X \overset{lr}{\leq} Y (X \overset{lr}{\geq} Y)$ if $s_1 \geq (\leq) s_2$.*

Proof. Similar to Proof 3, in the context of the parameter s we have

$$f(t) = \frac{g_X(t)}{g_Y(t)} = (\frac{s_1}{s_2})^\alpha \left[\frac{s_2^\alpha + t^\alpha}{s_1^\alpha + t^\alpha}\right]^\beta .$$

Taking the derivative of $f(t)$ with respect to t, we get

$$f^{'}(t) = (\frac{s_1}{s_2})^\alpha \cdot \beta \left[\alpha \cdot t^{\alpha-1} \frac{s_1^\alpha - s_2^\alpha}{(s_1^\alpha + t^\alpha)^2}\right]^{\beta-1} .$$

Then, $f(t) \geq (\leq) 0$ if $s_1 \leq (\geq) s_2$. Therefore $\frac{g_X(t)}{g_Y(t)}$ is decreasing (increasing) with the condition $s_1 \leq (\geq) s_2$ and the proof is completed. □

Corollary 3.4. *If $s_1 \leq (\geq) s_2$, likewise $\beta_1 \leq \beta_2$, then the likelihood ratio of X to Y is less than or equal to (greater than or equal to) 1. This implies that the odds of observing X compared to Y are lower than or equal to (greater than or equal to) 1, indicating a weaker (stronger) evidence for X compared to Y in favor of the alternative hypothesis.*

Proposition 3.5. *Let X and Y be two continuous lifetime random variables with β_1 and β_2 respectively and $\beta_1 \leq \beta_2$. Then $X \overset{hr}{\leq} Y$.*

Proof. The hazard rate order (denoted by $X \overset{hr}{\leq} Y$) holds if the ratio $\frac{\bar{G}_X(t)}{\bar{G}_Y(t)}$ is decreasing in t Di Crescenzo and Longobardi (2001). Then, we solve $f'(t) = \frac{d}{dt}\left(\frac{\bar{G}_X(t)}{\bar{G}_Y(t)}\right)$. To simplify the differentiation, let's denote the inner function as $\frac{(t/s)^\alpha}{1+(t/s)^\alpha} = k(t)$. Therefore, we have

$$f'(t) = \frac{d}{dt}\left[\frac{\left(1-k(t)\right)^{\beta_1}}{\left(1-k(t)\right)^{\beta_2}}\right] = \frac{d}{dt}\left(1-k(t)\right)^{\beta_1-\beta_2}.$$

Since $0 < k(t) < 1$ besides $k'(t) > 0$ then taking $\beta_1 \leq (\geq) \beta_2$ yields $f(t)$ is decreasing (increasing). □

Proposition 3.6. *Let X and Y be two continuous lifetime random variables with s_1 and s_2 respectively and $s_1 \leq s_2$. Then $X \overset{hr}{\leq} Y$.*

Proof. Let $f(t) = \left[\frac{s_2^\alpha + t^\alpha}{s_1^\alpha + t^\alpha}\right]\beta$. As $s_1 \leq s_2$, the numerator $s_2^\alpha + t^\alpha$ will increase faster than the denominator $s_1^\alpha + t^\alpha$ as t increases. This implies that the fraction inside the brackets will increase as t increases.

The proof of $s_1 \geq s_2$ is similar and we glimpse the details. Hence the proof is completed. □

Corollary 3.7. *Assuming $s_1 \leq (\geq) s_2$ and in a similar vein $\beta_1 \leq \beta_2$ indicates that the hazard rate of X is less (greater) than or equal to the hazard rate of Y, suggesting that X has a lower (higher) or equal risk of failure at any given time compared to Y.*

Proposition 3.8. *Let $\beta_1 \leq (\geq)\beta_2$. Then the reversed hazard rate (rhr) of the Burr Type III is monotone decreasing (increasing) in time t.*

Proof. The hazard rate order (denoted by $X \overset{rhr}{\leq} Y$) holds if the ratio $\frac{G_X(t)}{G_Y(t)}$ is decreasing in t Di Crescenzo and Longobardi (2001). The reversed hazard rate function for the Burr Type III distribution is given by

$$f(t) = \frac{\beta_1}{\beta_2}\left[\frac{(t/s)^\alpha}{1+(t/s)^\alpha}\right]^{\beta_1-\beta_2}.$$

To simplify the differentiation, let's denote the inner function as $\frac{(t/s)^\alpha}{1+(t/s)^\alpha} = k(t)$. Therefore, we have

$$f'(t) = \frac{d}{dt}\left[\frac{\beta_1}{\beta_2}\big(k(t)\big)^{\beta_1-\beta_2}\right] = \frac{\beta_1}{\beta_2}(\beta_1-\beta_2)\big(k(t)\big)^{\beta_1-\beta_2-1}k'(t).$$

Since $0 < k(t) < 1$ besides $k'(t) > 0$ then taking $\beta_1 \leq (\geq)\beta_2$ yields $f(t)$ is decreasing (increasing) function. □

Proposition 3.9. *Let $s_1 \leq (\geq)s_2$. Then the reversed hazard rate (rhr) of the Burr Type III is monotone increasing (decreasing) in time t.*

Proof. Let $f(t) = \frac{G_X(t)}{G_Y(t)}$. Then,

$$f(t) = \left[\frac{s_2^\alpha + t^\alpha}{s_1^\alpha + t^\alpha}\right]^\beta.$$

Building on the proof of Proposition 3.6, we can conclude that when $s_1 \leq s_2$, the function $f(t)$ decreases as t increases. Similarly, when $s_1 \geq s_2$, the function $f(t)$ increases as t increases. This completes the proof. □

Corollary 3.10. *If $\beta_1 \leq (\geq)\beta_2$, similarly, $\beta_1 \geq (\leq)\beta_2$ then the reversed hazard rate (rhr) of the Burr Type III distribution is monotone decreasing (increasing) in time t. This implies that as time progresses, the risk of failure decreases (increases) for the Burr Type III distribution with parameters β_1 and β_2.*

4 Entropies and extropies

Examining entropy and extropy measures is important as they provide valuable insights into the uncertainty, randomness, and information content of a probability distribution. In the context of Burr Type III distribution, these measures can help in understanding the variability, predictability, and complexity of the data generated from this distribution.

In this section, we will examine the measures of Shannon entropy, cumulative entropy, and cumulative residual entropy, as well as the equivalent extropy measures. Additionally, we have explored Renyi entropy, Tsallis entropy, and Sharma-Mital entropy.

The Proposition 4.1 sets a lower bound for the Shannon entropy of a random variable X that is distributed according to the Burr Type III distribution.

Proposition 4.1. *The Shannon entropy of a lifetime random variable X following the Burr Type III distribution is bounded below by a constant value*

$$H(X) \geq -\log\left(s^{\alpha}\alpha\beta\right) - (\beta+1)\log\left(\frac{\beta}{\beta+1}\right) + (\alpha+1)E(\log(X)),$$

where E denotes the Expectation function for the Burr Type III distribution with parameters α, β, and s.

Proof. The Shannon entropy $H(X)$ for a continuous random variable X with probability density function $g(x)$ is given by Shannon (1948)

$$H(X) = -\int g(x)\log(g(x))dx.$$

Then $H(X)$ for the random variable X with the Burr Type III distribution can calculated by

$$\begin{aligned} H(X) &= -\int_0^{\infty} \frac{\alpha\beta}{s\left(\frac{x}{s}\right)^{\alpha+1}}\left[\frac{(x/s)^{\alpha}}{1+(x/s)^{\alpha}}\right]^{\beta+1} \log\left(\frac{\alpha\beta}{s\left(\frac{x}{s}\right)^{\alpha+1}}\left[\frac{(x/s)^{\alpha}}{1+(x/s)^{\alpha}}\right]^{\beta+1}\right) dx \\ &\geq -\log\left(s^{\alpha}\alpha\beta\right) + (\alpha+1)E(\log(X)) \\ &\quad -(\beta+1)\log\int_0^{\infty} \frac{\alpha\beta}{s\left(\frac{x}{s}\right)^{\alpha+1}}\left[\frac{(x/s)^{\alpha}}{1+(x/s)^{\alpha}}\right]^{(\beta+2)} dx \\ &= -\log\left(s^{\alpha}\alpha\beta\right) - (\beta+1)\log\left(\frac{\beta}{\beta+1}\right) + (\alpha+1)E(\log(X)). \end{aligned}$$

□

Proposition 4.2 analyzes the extropy measure of Burr Type III distribution, while Remark 4.3 compares the extropy measures of Burr Type III distribution and the Lehmann model with power θ.

Proposition 4.2. *The extropy of a random variable X distributed according to the Burr Type III distribution can be calculated as*

$$J(X) = \frac{-\alpha\beta^4}{2s(2\beta+1)}. \tag{5}$$

Proof. For a non-negative random variable X, the extropy $J(X)$ is given by Lad et al. (2015)

$$J(X) = -\frac{1}{2}\int_0^{\infty} g^2(x)dx.$$

Then extropy measure for the Burr Type III distribution is given by

$$J(X) = -\frac{1}{2}\int_0^{\infty} \left(\frac{\alpha\beta}{s\left(\frac{x}{s}\right)^{\alpha+1}}\right)^2\left[\frac{(x/s)^{\alpha}}{1+(x/s)^{\alpha}}\right]^{2(\beta+1)} dx = -\frac{\alpha s^{\alpha}\beta^2}{2(2\beta+1)}E(X^{-(\alpha+1)}), \tag{6}$$

where E denotes the Expectation function for the Burr Type III distribution with parameters α, $2\beta+1$, and s.

By combining (3) and (6), we can derive (5). □

Remark 4.3. *Let X and Y are the random variables with the CDF's $G(x)$ and $F(x)$ respectively in which follow the Proposition 2.6. Then*

$$J(Y) = \frac{\theta^4(2\beta+1)}{2\theta\beta+1} J(X),$$

that obtaines the extropy of the Lehmann model depends only on the parameter β and the coefficient θ.

Propositions 4.4 and 4.5 provide a comprehensive analysis of the cumulative entropy and extropy measures for the Burr Type III distribution. Furthermore, Remark 4.6 establishes the connection between cumulative extropy and the parameters s and α.

Proposition 4.4. *The cumulative entropy of a random variable X that follows the Burr Type III distribution can be obtained by:*

$$CE(X) \geq -\beta \log\left((\beta-1)\binom{\beta+3}{3-\alpha^{-1}}\right). \tag{7}$$

Proof. The formula for calculating cumulative entropy is Wang et al. (2003)

$$CE(X) = -\int G(x)\log(G(x))dx.$$

Then for the Burr Type III distribution we would have

$$\begin{aligned} CE(X) &= -\int_0^\infty \left(\frac{(x/s)^\alpha}{1+(x/s)^\alpha}\right)^\beta \log\left(\frac{(x/s)^\alpha}{1+(x/s)^\alpha}\right)^\beta dx \\ &\geq -\beta \log\left(\int_0^\infty \left(\frac{(x/s)^\alpha}{1+(x/s)^\alpha}\right)^{\beta+1}\right) \\ &= -\beta \log\left(\frac{E(X^{\alpha+1})}{\alpha\beta s^\alpha}\right), \end{aligned} \tag{8}$$

where E denotes the Expectation function for the Burr Type III distribution with parameters α, β, and s.

By combining (4) and (8), we arrived at (7). □

Proposition 4.5. *The cumulative extropy of a random variable X that follows the Burr Type III distribution can be obtained by:*

$$CJ(X) = -\frac{1}{2}\left(\frac{(\beta+3)(\beta+2)(\beta-1)\beta}{(2\beta-1)s^\alpha}\right).$$

Proof. The cumulative extropy is derived by Nair and Sathar (2020)

$$CJ(X) = -\frac{1}{2}\int G^2(x)dx.$$

Therefore, for the Burr Type III distribution is given by

$$CJ(X) = -\frac{1}{2}\int_0^\infty \left(\frac{(x/s)^\alpha}{1+(x/s)^\alpha}\right)^{2\beta} dx.$$

To concisely conclude, while setting aside the details of the proof, it is concluded that Too concisely conclude, while setting aside the details of the proof, we reached to

$$CJ(X) = -\frac{1}{2}\left(\frac{E(X^{\alpha+1})}{\alpha(2\beta-1)s^{\alpha}}\right).$$

Assuming (4) we have

$$CJ(X) = -\frac{1}{2}\left(\frac{\alpha(\beta+3)(\beta+2)(\beta-1)\beta}{\alpha(2\beta-1)s^{\alpha}}\right). \tag{9}$$

□

Remark 4.6. *There exists a relationship between the cumulative entropies of random variables following a Burr Type III distribution as follows*
• *If the parameter s multiplied by the coefficient k, the cumulative entropy becomes inversely proportional to k raised to the power of alpha. In other words*

$$CJ(Y) = \frac{CJ(X)}{k^{\alpha}}.$$

• *If the parameter α multiplied by the coefficient k, the cumulative entropy becomes*

$$CJ(Y) = \frac{CJ(X)}{s^{(k-1)\alpha}}.$$

Propositions 4.7 and 4.8 offer a comprehensive examination of the cumulative residual entropy and extropy measures in relation to the Burr Type III distribution.

Proposition 4.7. *The cumulative residual entropy of a random variable X that follows the Burr Type III distribution can be obtained by*

$$CRE(X) \geq -\log\left(E(X) - \beta\binom{\beta+3}{3-\frac{1}{\alpha}} + \left(\frac{(\beta+3)(\beta+2)(\beta-1)\beta}{\alpha(2\beta-1)s^{\alpha}}\right)\right). \tag{10}$$

Proof. The formula for calculating cumulative residual entropy is Rao et al. (2004)

$$\begin{aligned} CRE(X) &= -\int \bar{G}(x)\log(\bar{G}(x))dx \\ &\geq -\log\left(\int \bar{G}(x)dx - \int G(x)dx + \int G^2(x)dx\right) \\ &= -\log\left(E(X) - CDI^1 - 2CJ(X)\right) \end{aligned} \tag{11}$$

Additionally, from (1) and (2) we would have

$$CDI^1 = \frac{E(x^{\alpha+1})}{\alpha(\beta-1)s^{\alpha}} = \beta\binom{\beta+3}{3-\alpha^{-1}}. \tag{12}$$

Letting (12) and (11) we would have

$$CRE(X) \geq -\log\left(E(X) - \frac{E(x^{\alpha+1})}{\alpha(\beta-1)s^{\alpha}} - 2CJ(X)\right),$$

where $CJ(X)$ is the cumulative extropy od X and E is denotes the Expectation function for the Burr Type III distribution with parameters α, $\beta-1$, and s.

By combining (4) and (9), we were able to derive (10). □

Proposition 4.8. *The cumulative residual extropy of a random variable X that follows the Burr Type III distribution can be obtained by:*

$$\begin{aligned} 2\big(CJ(X) - CRJ(X)\big) &= E(X) - \frac{E(x^{\alpha+1})}{\alpha(\beta-1)s^{\alpha}} \\ &= E(X) - \beta\binom{\beta+3}{3-\frac{1}{\alpha}} + \left(\frac{(\beta+3)(\beta+2)(\beta-1)\beta}{\alpha(2\beta-1)s^{\alpha}}\right). \quad (13) \end{aligned}$$

Proof. The cumulative residual extropy is state as Jahanshahi et al. (2020)

$$CRJ(X) = -\frac{1}{2}\int \bar{G}^2(x)dx.$$

Furthermore, in similar to the proof 4 it yields (13) □

Corollary 4.9. *Let $X_{(i)}$ is the ith order statistic in Burr Type III distribution data. Then, we have*

$$CJ(X) - CRJ(X) = \frac{nE(X) - E(X_{(i)})}{2n}.$$

Propositions 4.10, 4.11, and 4.12 provide a detailed analysis of the Rényi, Tsallis, and Sharma-Mital entropy measures in the context of the Burr Type III distribution.

Proposition 4.10. *The Rényi entropy of random variable X with a Burr Type III distribution can be expressed as*

$$H_r(X) = \frac{1}{r-1}\log\left(\frac{r(\beta+1)-1}{\alpha^{r-1}s^{-(r-1)}\beta^{r+1}\binom{\beta}{r-1}}, \quad r \leq \beta+1.\right). \quad (14)$$

Proof. The Rényi entropy for a continuous random variable X with probability density function $g(x)$ is given by the formula Renner and Wolf (2004)

$$H_r(X) = \frac{1}{1-r}\log\int g(x)^r dx,$$

where r is the order of the entropy. Then, the Rényi entropy for the Burr Type III distribution is given by

$$\begin{aligned} H_r(X) &= \frac{1}{1-r}\log\int_0^{\infty}\Big(\frac{\alpha\beta}{s\left(\frac{x}{s}\right)^{\alpha+1}}\Big)^r\left[\frac{(x/s)^{\alpha}}{1+(x/s)^{\alpha}}\right]^{r(\beta+1)} dx \\ &= \frac{1}{1-r}\log\left(\frac{\alpha^{r-1}\beta^r s^{\alpha(r-1)}}{r(\beta+1)-1}\right. \\ &\quad \times\left.\int_0^{\infty}\frac{\alpha(r(\beta+1)-1)}{s\left(\frac{x}{s}\right)^{\alpha+1}}x^{-(\alpha+1)(r-1)}\left[\frac{(x/s)^{\alpha}}{1+(x/s)^{\alpha}}\right]^{r(\beta+1)} dx\right) \\ &= \frac{1}{1-r}\log\left(\frac{\alpha^{r-1}\beta^r s^{\alpha(r-1)}}{r(\beta+1)-1}E\big(X^{-(\alpha+1)(r-1)}\big)\right), \quad (15) \end{aligned}$$

where E denotes the Expectation function for the Burr Type III distribution with parameters α, $r(\beta+1)-1$, and s. By combining (15) and (3), we can derive t in (14). □

Proposition 4.11. *The Tsallis entropy of a random variable X that follows the Burr Type III distribution can be calculated as*

$$S_r(X) = \frac{1}{r-1}\left(1 - \frac{\alpha^{r-1}\beta^{r+1}s^{-(r-1)}}{r(\beta+1)-1}\binom{\beta}{r-1}\right), \quad r \leq \beta + 1. \tag{16}$$

Proof. The Tsallis entropy S_r for a continuous random variable is Anastasiadis (2012)

$$S_r(X) = \frac{1 - \int g(x)^r \, dx}{r-1},$$

where r is a parameter that determines the Type of entropy.

Furthermore, a relationship between Renyi entropy $H_r(X)$ and Tsallis entropy $S_r(X)$ can be determined as

$$S_r(X) = \frac{e^{(r-1)H_r(X)} - 1}{r-1}.$$

Then, the Tsallis entropy for the Burr Type III distribution can be calculated as (17) and the proof is similar to Proposition 4.10 hence we glimpsed that

$$S_r(X) = \frac{1}{r-1}\left(1 - \frac{\alpha^{r-1}\beta^{r}s^{\alpha(r-1)}}{r(\beta+1)-1}E\big(X^{-(\alpha+1)(r-1)}\big)\right), \tag{17}$$

where E denotes the Expectation function for the Burr Type III distribution with parameters α, $r(\beta+1)-1$, and s. By combining (3) and (17), we can derive (16). □

Proposition 4.12. *The Sharma-Mittal entropy of a random variable X that follows the Burr Type III distribution can be calculated as*

$$S_{r,q}(X) = \frac{1}{q-1}\left[1 - \left(1 - \frac{\alpha^{r-1}\beta^{r+1}s^{-(r-1)}}{r(\beta+1)-1}\binom{\beta}{r-1}\right)^{\frac{1-q}{r-1}}\right]. \tag{18}$$

Proof. The Sharma-Mittal entropy is a two-parameter generalization of the Tsallis and Rényi entropies. It is defined as Koltcov et al. (2019)

$$S_{r,q}(X) = \frac{1}{q-1}\left[1 - \left(\int (g(x))^r, dx\right)^{\frac{1-q}{r-1}}\right],$$

where r and q are deformation parameters. Then, It is evidence that the Sharma-Mittal entropy for the Burr Type III distribution can be calculated as (18). □

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