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# Stress-strength reliability of the proportional hazard rate models under Type-I progressively hybrid censored samples

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**Abstract:** This study focuses on estimating the stress-strength parameter $R$, utilizing two independent Type-I progressively hybrid censored samples derived from populations governed by the proportional hazard rate model. The maximum likelihood and Bayes estimators are obtained under some well-known loss functions and the assumption that the priors are independently gamma-distributed. The asymptotic confidence interval and Bayesian and highest posterior density credible intervals are also presented. A Monte Carlo simulation study is used to evaluate the performances of the obtained point estimators and confidence and credible intervals. Finally, a pair of real data sets is analyzed for illustrative purposes.

**Keywords:** Bayes estimator; Maximum likelihood estimator; Proportional hazard rate model; Stress-strength parameter; Type-I progressive hybrid censoring.
**Mathematics Subject Classification (2010):** 62NO1, 62NO2.

## 1 Introduction

Stress-strength modelling is a critical aspect of reliability analysis, providing a measure of a component's reliability when exposed to random stress $X$ and possessing strength $Y$. A component fails if its strength is insufficient to withstand the stress. Thus, $R = P(X < Y)$ represents the component's reliability. This concept finds extensive applications in fields such as engineering and medical sciences. For more details and applications, refer to Kotz et al. (2003). Many researchers considered the problem of estimating $R$ in some distributions; such as Govidarajulu (1967), Enis and Geisser (1971), Downtown (1973), Awad et al. (1981), Sathe and Shah (1981), Awad and

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Gharraf (1986), Constantine et al. (1986), Gupta and Gupta (1990), McCool (1991), Nandi and Aich (1996), Surles and Padgett (1998), Gupta et al. (1999), Gupta and Brown (2001), Kundu and Gupta (2005), Raqab and Kundu (2005), Mokhlis (2005), Kundu and Gupta (2006), Raqab et al. (2008), Kundu and Raqab (2009), Gupta et al. (2010), Asgharzadeh et al. (2013, 2011), Saraçoğlu et al. (2012), Rao et al. (2013), Asgharzadeh and Kazemi (2014), Rao et al. (2016), Nadeb et al. (2019) and Khalifeh et al. (2020).

In recent years, numerous studies have focused on progressive Type-II censored samples. For an in-depth exploration of this topic, readers may refer to Balakrishnan and Aggarwala (2000). In progressive Type-II censoring, it is assumed that the removal of still-operating units occurs at observed failure times, and the censoring scheme $\boldsymbol{r} = (r_1, ..., r_m)$ is predetermined. Additionally, both the total number of units ($n$) and the number of observed failure times ($m$) are fixed in advance. Starting all $n$ units at the same time, the first progressive censoring step takes place at the observation of the first failure time $X_{1:m:n}$, at this time, $r_1$ units are randomly chosen from the still operating units and withdrawn from the experiment. Then, the experiment continues with the reduced sample size $n-r_1-1$. After observing the next failure at time $X_{2:m:n}$, $r_2$ units are randomly removed from $n-r_1-2$ active units. This process continued until the $m$th failure was observed. Then, the experiment ends and $X_{1:m:n} \leq \ldots \leq X_{m:m:n}$ are said to be progressive Type-II censored order statistics.

Type-I progressive hybrid censoring scheme that is a combination of the usual Type-I and the progressive Type-II censoring was proposed by Kundu and Joarder (2006) by introducing a stopping time $\min\{X_{m:m:n}, t_0\}$. This approach is based on progressively Type-II censored order statistics $X_{1:m:n} \leq \ldots \leq X_{m:m:n}$, and it guarantees that the life test would not last beyond time $t_0$. $t_0$ is pre-fixed time and is named threshold time. Indeed, the Type-I progressively hybrid censoring arises if the termination time of the life-test is chosen to be $\min\{X_{m:m:n}, t_0\}$. We denote this censoring scheme by $(\boldsymbol{r}, t_0)$. Under this censoring scheme, we have one of the two following types of order statistics:

$$\begin{aligned} \text{Case I}: &\quad \{X_{1:m:n} \leq \ldots \leq X_{m:m:n}\} \quad \text{if} \quad X_{m:m:n} \leq t_0, \\ \text{Case II}: &\quad \{X_{1:m:n} \leq \ldots \leq X_{D:m:n}\} \quad \text{if} \quad D < m, \quad X_{D:m:n} < t_0 < X_{D+1:m:n}, \end{aligned}$$

where $D$ is the number of failures before the time $t_0$ and it is a random variable with support $\{0, 1, \ldots, m\}$.

Many researchers made inferences about some distributions based on Type-I progressively hybrid censored data. For instance, Nadeb and Torabi (2016) presented a method for exact hypothesis testing and obtained confidence interval for mean of the exponential distribution. Shi and Wu (2016) studiesd the dependent competing risks model from Gompertz distribution. Wang and Liu (2017) established the estimation for the unknown scale parameter of the half-logistic distribution. Noori Asl et al. (2018) considered the problem of estimating and predicting the unknown parameters of the Lomax distribution. Arabi Belaghi and Noori Asl (2019) estimated the unknown parameters of the Burr XII distribution under classical and Bayesian frameworks. Singh et al. (2019) addressed the problems of estimating and predicting in the Type III Burr distribution. Sen et al. (2019) investigated the problems of estimating and predicting by classical and Bayesian approaches when lifetime data following a lognormal distribution. Yadav and Panwar (2024) developed estimation procedures for the inverse Maxwell distribution.

The proportional hazard rate (PHR) model is an important model in reliability theory and some other fields; for instance see Cox (1992), Kumar and Klefsjö (1994) and Finkelstein (2008). $X$ is said to follow the PHR model, denoted by $X \sim \text{PHR}(\bar{F}, \theta)$, if its survival function can be expressed as $\bar{F}^{\theta}(x)$, where $\bar{F}(x)$ is the baseline survival function and $\theta$ is a positive parameter. It is clear that, if the lifetimes of all components in a series system are independently, identically distributed and belong to the PHR model, the lifetime of the system also belongs to this model.

Making inference on the stress-strength parameter has been considered when the populations follow the PHR models. For instance Basirat et al. (2015) considered the statistical inference for stress-strength in the PHR models under progressive Type-II censoring; Basirat et al. (2016) studied this parameter using record values from the PHR models; and Bai et al. (2019) discussed on inference on the stress-strength parameter for the truncated PHR models under progressively Type-II censored samples.

Golparvar and Parsian (2016) made inference about the unknown parameter in the proportional hazard rate model under Type-I progressive hybrid censoring. Therefore, we consider the estimation of $R = P(X < Y)$ when $X$ and $Y$ follow the PHR model under Type-I progressive hybrid censoring. Let $X \sim \text{PHR}(\bar{F}, \theta_1)$ and $Y \sim \text{PHR}(\bar{F}, \theta_2)$ be independent random variables. Then it can be easily seen that

$$R = P(X < Y) = \frac{\theta_1}{\theta_1 + \theta_2}. \tag{1}$$

# 2 Point estimation

In this section, the maximum likelihood and Bayes estimators of $R$ are obtained. In the Subsection 2.1, we consider the maximum likelihood estimation (MLE) and then the Subsection 2.2 considers the Bayes estimation under some loss functions.

## 2.1 MLE of $R$

Our interest is estimating $R$ based on Type-I progressively hybrid censored data on both variables. To derive the MLE of $R$, first we get the MLEs of $\theta_1$ and $\theta_2$. Suppose $\boldsymbol{X} = (X_{1:m_1:n_1}, \ldots, X_{D_1:m_1:n_1})$ is a Type-I progressively hybrid censored sample from $\text{PHR}(\bar{F}, \theta_1)$ with censoring scheme $(\boldsymbol{r}_1, t_0)$ and $\boldsymbol{Y} = (Y_{1:m_2:n_2}, \ldots, Y_{D_2:m_2:n_2})$ is a Type-I progressively hybrid censored sample from $\text{PHR}(\bar{F}, \theta_2)$ with censoring scheme $(\boldsymbol{r}_2, t'_0)$, where $\boldsymbol{r}_i = (r_{i1}, \ldots, r_{im_i})$ and $\sum_{j=1}^{m_i} r_{ij} = n_i$, for $i = 1, 2$. For convenience, we will write $(X_1, \ldots, X_{D_1})$ instead of $(X_{1:m_1:n_1}, \ldots, X_{D_1:m_1:n_1})$, and $(Y_1, \ldots, Y_{D_2})$ instead of $(Y_{1:m_1:n_1}, \ldots, Y_{D_2:m_2:n_2})$. According to Cramer and Balakrishnan (2013), the likelihood function corresponding to $(\theta_1, \theta_2)$ can be written as

$$\begin{aligned} L(\theta_1, \theta_2) &= \left( \prod_{j=1}^{D_1} \gamma_j f(X_j) \right) \theta_1^{D_1} \bar{F}^{\theta_1 \gamma_{D_1+1}}(t_0) \prod_{j=1}^{D_1} \bar{F}^{\theta_1(1+r_{1j})-1}(X_j) \\ &\quad \times \left( \prod_{j=1}^{D_2} \gamma'_j f(X_j) \right) \theta_2^{D_2} \bar{F}^{\theta_2 \gamma'_{D_2+1}}(t'_0) \prod_{j=1}^{D_2} \bar{F}^{\theta_2(1+r_{2j})-1}(Y_j), \end{aligned} \tag{2}$$

where, $\gamma_j = \sum_{k=j}^{m_1}(1 + r_{1j})$ and $\gamma'_j = \sum_{k=j}^{m_2}(1 + r_{2j})$.

In view of Equation (2) and by omitting the normalizing constant, the log-likelihood function for the Type-I progressively hybrid censored samples is given by

$$\begin{aligned} l(\theta_1, \theta_2) &= D_1 \log(\theta_1) + \theta_1 \gamma_{D_1+1} \log \bar{F}(t_0) + \sum_{j=1}^{D_1} (\theta_1(1 + r_{1j}) - 1) \log \bar{F}(X_j) \\ &\quad + D_2 \log(\theta_2) + \theta_2 \gamma'_{D_2+1} \log \bar{F}(t'_0) + \sum_{j=1}^{D_2} (\theta_2(1 + r_2 j) - 1) \log \bar{F}(Y_j). \quad (3) \end{aligned}$$

The MLEs of $\theta_1$ and $\theta_2$ are denoted by $\hat{\theta}_1$ and $\hat{\theta}_2$, respectively; where they are obtained as the solution to the following equations

$$\frac{\partial l(\theta_1, \theta_2)}{\partial \theta_1} = \frac{D_1}{\theta_1} + \gamma_{D_1+1} \log \bar{F}(t_0) + \sum_{j=1}^{D_1}(1 + r_{1j}) \log \bar{F}(X_j) = 0, \quad (4)$$

$$\frac{\partial l(\theta_1, \theta_2)}{\partial \theta_2} = \frac{D_2}{\theta_2} + \gamma'_{D_2+1} \log \bar{F}(t'_0) + \sum_{j=1}^{D_2}(1 + r_{2j}) \log \bar{F}(Y_j) = 0. \quad (5)$$

If $D_1 > 0$ and $D_2 > 0$, by the solution of Equations (4) and (5), we have $\hat{\theta}_i = \frac{D_i}{B_i}$ where $B_1 = -\sum_{j=1}^{D_1}(1 + r_{1j}) \log \bar{F}(X_j) - \gamma_{D_1+1} \log \bar{F}(t_0)$ and $B_2 = -\sum_{j=1}^{D_2}(1 + r_{2j}) \log \bar{F}(Y_j) - \gamma'_{D_2+1} \log \bar{F}(t'_0)$. Therefore, we compute the MLE of $R$ as

$$\hat{R} = \frac{\hat{\theta}_1}{\hat{\theta}_1 + \hat{\theta}_2}. \quad (6)$$

But using the likelihood function (2), if $D_1 = 0$ or $D_2 = 0$, then $\hat{\theta}_1$ or $\hat{\theta}_2$ does not exist and therefore $\hat{R}$ does not exist.

## 2.2 Bayes estimation of $R$

In this section, we get the Bayes estimator of $R$ under Type-I progressive hybrid censoring. An important function which is needed for this purpose, is hypergeometric function, where is given by

$$F_{2,1}(a, b, c, z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 \frac{t^{b-1}(1 - t)^{c-b-1}}{(1 - zt)^a} dt,$$

such that $c > b > 0$. For a comprehensive discussion on this topic, one may refer to Baily (1935).

The following lemma presents a useful relation for $\mathrm{E}[R^k(1 - R)^l | \boldsymbol{X}, \boldsymbol{Y}]$, where E[.] denotes the expectation.

**Lemma 2.1.** *Let* $\theta_1 \sim \text{gamma}(\alpha_1, \beta_1)$ *and* $\theta_2 \sim \text{gamma}(\alpha_2, \beta_2)$ *be independent, where* $\alpha_1$ *and* $\alpha_2$ *are the shape parameters and* $\beta_1$ *and* $\beta_2$ *are the rate parameters. Then, for some constants* $k$ *and* $l$*, we have*

$$\mathrm{E}[R^k(1-R)^l|\boldsymbol{X},\boldsymbol{Y}] = \frac{\Gamma(\alpha_1+D_1+k)}{\Gamma(\alpha_1+D_1)}\frac{\Gamma(\alpha_2+D_2+l)}{\Gamma(\alpha_2+D_2)}\frac{\Gamma(S)}{\Gamma(S+k+l)}\left(\frac{\beta_1+W_1}{\beta_2+W_2}\right)^{\alpha_1+D_1} \times F_{2,1}\left(S, \alpha_1+D_1+k, S+k+l, 1-\frac{\beta_1+W_1}{\beta_2+W_2}\right), \quad (7)$$

*where* $S = \alpha_1 + D_1 + \alpha_2 + D_2$ *and*

$$W_1 = -\sum_{j=1}^{D_1}(1+r_{1j})\log \bar{F}(X_j) - \gamma_{D_1+1}\log \bar{F}(t_0),$$

$$W_2 = -\sum_{j=1}^{D_2}(1+r_{2j})\log \bar{F}(Y_j) - \gamma'_{D_2+1}\log \bar{F}(t'_0).$$

*Proof.* To obtain the posterior distributions of $\theta_1$ and $\theta_2$, we have

$$\pi(\theta_1|\boldsymbol{X}) \propto \theta_1^{\alpha_1+D_1-1}\, e^{-\left(\beta_1-\gamma_{D_1+1}\log\bar{F}(t_0)-\sum_{j=1}^{D_1}(1+r_{1j})\log\bar{F}(X_j)\right)\theta_1},$$

$$\pi(\theta_2|\boldsymbol{Y}) \propto \theta_2^{\alpha_2+D_2-1}\, e^{-\left(\beta_2-\gamma'_{D_1+1}\log\bar{F}(t'_0)-\sum_{j=1}^{D_2}(1+r_{2j})\log\bar{F}(Y_j)\right)\theta_2}.$$

Thus,

$$\theta_1|\boldsymbol{X} \sim \text{gamma}(\alpha_1+D_1, \beta_1+W_1), \quad (8)$$

$$\theta_2|\boldsymbol{Y} \sim \text{gamma}(\alpha_2+D_2, \beta_2+W_2). \quad (9)$$

For $\theta_2 > 0$, $0 < r < 1$ and using the independence of $\theta_1$ and $\theta_2$, the joint density function of $R = \frac{\theta_1}{\theta_1+\theta_2}$ and $\theta_2$ given the data can be obtained:

$$\begin{aligned} f_{(R,\theta_2)|\boldsymbol{X},\boldsymbol{Y}}(r,\theta_2) &= \frac{\theta_2}{(1-r)^2}\, f_{(\theta_1,\theta_2)|\boldsymbol{X},\boldsymbol{Y}}\left(\frac{r\theta_2}{1-r}, \theta_2\right) \\ &= \frac{\theta_2}{(1-r)^2}\frac{(\beta_1+W_1)^{\alpha_1+D_1}}{\Gamma(\alpha_1+D_1)}\left(\frac{r\theta_2}{1-r}\right)^{\alpha_1+D_1-1} e^{-(\beta_1+W_1)\frac{r\theta_2}{1-r}} \\ &\quad \times\frac{(\beta_2+W_2)^{\alpha_2+D_2}}{\Gamma(\alpha_2+D_2)}\theta_2^{\alpha_2+D_2-1}e^{-(\beta_2+W_2)\theta_2} \\ &= \frac{(\beta_1+W_1)^{\alpha_1+D_1}}{\Gamma(\alpha_1+D_1)}\frac{(\beta_2+W_2)^{\alpha_2+D_2}}{\Gamma(\alpha_2+D_2)}\frac{r^{\alpha_1+D_1-1}}{(1-r)^{\alpha_1+D_1+1}} \\ &\quad \times\theta_2^{S-1}e^{-\left(\beta_2+W_2+\frac{r}{1-r}(\beta_1+W_1)\right)\theta_2}. \end{aligned}$$

Thus, the density function of $R$ given the data can be obtained by integrating regarding $\theta_2$. Therefore, for $0 < r < 1$, we have

$$f_{R|\boldsymbol{X},\boldsymbol{Y}}(r) = \frac{(\beta_1+W_1)^{\alpha_1+D_1}}{\Gamma(\alpha_1+D_1)}\frac{(\beta_2+W_2)^{\alpha_2+D_2}}{\Gamma(\alpha_2+D_2)}\frac{r^{\alpha_1+D_1-1}}{(1-r)^{\alpha_1+D_1+1}}$$

$$
\begin{aligned}
&\times \int_0^\infty \theta_2^{S-1} e^{-\left[\beta_2+W_2+\frac{r}{1-r}(\beta_1+W_1)\right]\theta_2} \mathrm{d}\theta_2 \\
= &\frac{(\beta_1+W_1)^{\alpha_1+D_1}}{\Gamma(\alpha_1+D_1)} \frac{(\beta_2+W_2)^{\alpha_2+D_2}}{\Gamma(\alpha_2+D_2)} \frac{r^{\alpha_1+D_1-1}}{(1-r)^{\alpha_1+D_1+1}} \\
&\times \frac{\Gamma(S)}{\left(\beta_2+W_2+\frac{r}{1-r}(\beta_1+W_1)\right)^S} \\
= &\frac{(\beta_1+W_1)^{\alpha_1+D_1}}{\Gamma(\alpha_1+D_1)} \frac{(\beta_2+W_2)^{\alpha_2+D_2}}{\Gamma(\alpha_2+D_2)} \Gamma(S) \\
&\times \frac{r^{\alpha_1+D_1-1}(1-r)^{\alpha_2+D_2-1}}{\Big((1-r)(\beta_2+W_2)+r(\beta_1+W_1)\Big)^S}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
E[R^k(1-R)^l|\boldsymbol{X},\boldsymbol{Y}] &= \int_0^1 r^k(1-r)^l f_{R|(\boldsymbol{X},\boldsymbol{Y})}(r)\mathrm{d}r \\
&= \frac{(\beta_1+W_1)^{\alpha_1+D_1}}{\Gamma(\alpha_1+D_1)} \frac{(\beta_2+W_2)^{\alpha_2+D_2}}{\Gamma(\alpha_2+D_2)} \Gamma(S) \\
&\times \int_0^1 \frac{r^{\alpha_1+D_1+k-1}(1-r)^{\alpha_2+D_2+l-1}}{\Big((1-r)(\beta_2+W_2)+r(\beta_1+W_1)\Big)^S} \mathrm{d}r \\
&= \frac{(\beta_1+W_1)^{\alpha_1+D_1}}{\Gamma(\alpha_1+D_1)} \frac{(\beta_2+W_2)^{\alpha_2+D_2}}{\Gamma(\alpha_2+D_2)} \frac{\Gamma(S)}{(\beta_2+W_2)^S} \\
&\times \int_0^1 \frac{r^{\alpha_1+D_1+k-1}(1-r)^{\alpha_2+D_2+l-1}}{\left(1-\frac{\beta_2+W_2-(\beta_1+W_1)}{\beta_2+W_2} r\right)^S} \mathrm{d}r \\
&= \frac{(\beta_1+W_1)^{\alpha_1+D_1}}{\Gamma(\alpha_1+D_1)} \frac{(\beta_2+W_2)^{\alpha_2+D_2}}{\Gamma(\alpha_2+D_2)} \frac{\Gamma(S)}{(\beta_2+W_2)^S} \\
&\times \frac{\Gamma(\alpha_1+D_1+k)\Gamma(\alpha_2+D_2+l)}{\Gamma(S+k+l)} \\
&\times F_{2,1}\left(S, \alpha_1+D_1+k, S+k+k, 1-\frac{\beta_1+W_1}{\beta_2+W_2}\right) \\
&= \left(\frac{\beta_1+W_1}{\beta_2+W_2}\right)^{\alpha_1+D_1} \frac{\Gamma(\alpha_1+D_1+k)}{\Gamma(\alpha_1+D_1)} \frac{\Gamma(\alpha_2+D_2+l)}{\Gamma(\alpha_2+D_2)} \frac{\Gamma(S)}{\Gamma(S+k+l)} \\
&\times F_{2,1}\left(S, \alpha_1+D_1+k, S+k+l, 1-\frac{\beta_1+W_1}{\beta_2+W_2}\right).
\end{aligned}
$$

Thus, the proof is completed. □

We know that a Bayes estimator strongly depends on the loss function. The following theorem considers the Bayes estimators of $R$ under some well known loss functions.

**Theorem 2.2.** *Let* $\theta_1 \sim \text{gamma}(\alpha_1, \beta_1)$ *and* $\theta_2 \sim \text{gamma}(\alpha_2, \beta_2)$ *be independent, where* $\alpha_1$ *and* $\alpha_2$ *are the shape parameters and* $\beta_1$ *and* $\beta_2$ *are the rate parameters.*

*(i) Under the squared error loss function, $L(r,\delta)=(r-\delta)^2$, the Bayes estimator of $R$, denoted by $\hat{R}_{\rm sq}$, is given by*

$$\hat{R}_{\rm sq}=\frac{\alpha_1+D_1}{S}\left(\frac{\beta_1+W_1}{\beta_2+W_2}\right)^{\alpha_1+D_1}F_{2,1}\left(S,\alpha_1+D_1+1,S+1,1-\frac{\beta_1+W_1}{\beta_2+W_2}\right),$$

*(ii) Under the weighted squared error loss function, $L(r,\delta)=r^k(1-r)^l(r-\delta)^2$, the Bayes estimator of $R$, denoted by $\hat{R}_{\rm wsq}$, is given by*

$$\hat{R}_{\rm wsq}=\frac{\alpha_1+D_1+k}{S+k+l}\;\frac{F_{2,1}\left(S,\alpha_1+D_1+k+1,S+k+l+1,1-\frac{\beta_1+W_1}{\beta_2+W_2}\right)}{F_{2,1}\left(S,\alpha_1+D_1+k,S+k+l,1-\frac{\beta_1+W_1}{\beta_2+W_2}\right)},$$

*(iii) Under the Stein's loss function, $L(r,\delta)=-\log\frac{\delta}{r}+\frac{\delta}{r}-1$, the Bayes estimator of $R$, denoted by $\hat{R}_{\rm St}$, is given by*

$$\hat{R}_{\rm St}=\frac{\alpha_1+D_1-1}{S-1}\left(\frac{\beta_2+W_2}{\beta_1+W_1}\right)^{\alpha_1+D_1}\frac{1}{F_{2,1}\left(S,\alpha_1+D_1-1,S-1,1-\frac{\beta_1+W_1}{\beta_2+W_2}\right)},$$

*(iv) Under the 0-1 loss function,*

$$L(r,\delta)=\begin{cases}0, & \text{if } |r-\delta|\le c,\\ 1, & \text{if } |r-\delta|>c,\end{cases}$$

*when $c$ is a small constant, the approximate Bayes estimator of $R$, denoted by $\hat{R}_{0-1}$, is given by*

$$\hat{R}_{0-1}=\begin{cases}\frac{A_1B_2+A_2B_1+2(B_1-B_2)-\sqrt{\Delta}}{4(B_1-B_2)}, & \text{if } B_1>B_2,\\ \frac{A_1B_2+A_2B_1+2(B_1-B_2)+\sqrt{\Delta}}{4(B_1-B_2)}, & \text{if } B_1<B_2,\end{cases}$$

*where, $A_1=\alpha_1+D_1-1$, $A_2=\alpha_2+D_2-1$, $B_1=\beta_1+W_1$, $B_2=\beta_2+W_2$, and*

$$\Delta=\Big(A_1B_2+A_2B_1+2(B_1-B_2)\Big)^2-8A_1B_2(B_1-B_2).$$

*Proof.* (i) We know that $\hat{R}_{\rm sq}=E[R|\boldsymbol{X},\boldsymbol{Y}]$. Thus, substituting $k=1$ and $l=0$ in Equation (7), implies the required result.
(ii) We know that $\hat{R}_{\rm wsq}=\frac{E[R^{k+1}(1-R)^l|\boldsymbol{X},\boldsymbol{Y}]}{E[R^k(1-R)^l|\boldsymbol{X},\boldsymbol{Y}]}$. Hence, the Equation (7) immediately completes the proof.
(iii) We know that $\hat{R}_{\rm St}=\frac{1}{E[R^{-1}|\boldsymbol{X},\boldsymbol{Y}]}$. Hence, by substituting the appropriate constants $k$ and $l$ in Equation (7), and some simple computations, we have the desired result.
(iv) According to Lehmann and Casella (1998), Page 228, $\hat{R}_{0-1}$ is the midpoint of the interval $I$ of length $2c$ which maximizes $P(R\in I|\boldsymbol{X},\boldsymbol{Y})$. Thus, the posterior mode is an approximate Bayes estimator of $R$. It can be easily verified that

$$\frac{\mathrm{d}}{\mathrm{d}r}f_{R|\boldsymbol{X},\boldsymbol{Y}}(r)=\frac{B_1^{A_1+1}}{\Gamma(A_1+1)}\frac{B_2^{A_2+1}}{\Gamma(A_2+1)}\Gamma(A_1+A_2+2)\frac{r^{A_1-1}(1-r)^{A_2-1}}{\Big((1-r)B_2+rB_1\Big)^{A_1+A_2+3}}g(r),$$

where $g(r) = 2(B_1 - B_2)r^2 - \Big(A_1B_2 + A_2B_1 + 2(B_1 - B_2)\Big)r + A_1B_2$. Since $g(0) = A_1B_2 > 0$ and $g(1) = -A_2B_1 > 0$, it implies that $f_{R|\boldsymbol{X},\boldsymbol{Y}}(r)$ has a unique mode in $(0,\ 1)$ and the posterior mode can be obtained as the unique root of the quadratic equation $g(r) = 0$ over $0 < r < 1$. Clearly, $g(r)$ is a parabola, and it is monotonically decreasing on $(0,\ 1)$, which changes sign from positive to negative on this interval and it has two real roots on $(-\infty,\ \infty)$. Thus, if the coefficient of $r^2$ be positive, then the smaller root is desired and else, the larger root is desired. This fact completes the proof. □

# 3 Confidence interval

This section provides some confidence intervals for the parameter $R$. Subsection 3.1 presents an asymptotic confidence interval for $R$. Also, the Bayesian credible intervals are presented in Subsection 3.2.

## 3.1 Asymptotic confidence interval of $R$

In this subsection, we propose the asymptotic confidence interval for $R$, through computing the inverse of the observed Fisher information matrix of $(\theta_1, \theta_2)$ using the Cramér's theorem.

Using the log-likelihood function (3), the Fisher information matrix of $(\theta_1, \theta_2)$ conditioned on $D_1 \geq 1$ and $D_2 \geq 1$ can be obtained as follows

$$I(\theta_1, \theta_2) = -\mathrm{E}\begin{pmatrix} \frac{\partial^2 l(\theta_1,\theta_2)}{\partial\theta_1^2} & \frac{\partial^2 l(\theta_1,\theta_2)}{\partial\theta_1\partial\theta_2} \\ \frac{\partial^2 l(\theta_1,\theta_2)}{\partial\theta_1\partial\theta_2} & \frac{\partial^2 l(\theta_1,\theta_2)}{\partial\theta_2^2} \end{pmatrix} = \begin{pmatrix} \frac{\mathrm{E}[D_1|D_1>0]}{\theta_1^2} & 0 \\ 0 & \frac{\mathrm{E}[D_2|D_2>0]}{\theta_2^2} \end{pmatrix} = \begin{pmatrix} I_{11} & 0 \\ 0 & I_{22} \end{pmatrix}.$$

According to Kamps and Cramer (2001) and Cramer and Balakrishnan (2013), the density function of $D_1$ can be written as

$$f_{D_1}(d) = \begin{cases} \bar{F}^{n_1\theta_1}(t_0), & d = 0, \\ \prod_{j=1}^{d} \gamma_j \sum_{i=1}^{d+1} a_{i,d+1}\bar{F}^{\gamma_i\theta_1}(t_0), & 1 \leq d \leq m_1 - 1, \\ 1 - \prod_{j=1}^{m_1} \gamma_j \sum_{i=1}^{m_1} \frac{a_{i,m_1}}{\gamma_i}\bar{F}^{\gamma_i\theta_1}(t_0), & d = m_1, \end{cases}$$

where $a_{i,j} = \prod_{\ell=1,\ell\neq i}^{j} (\gamma_\ell - \gamma_i)^{-1}$. Consequently, we have

$$f_{D_1|D_1>0}(d) = \begin{cases} \frac{1}{\bar{F}^{n_1\theta_1}(t_0)} \prod_{j=1}^{d} \gamma_j \sum_{i=1}^{d+1} a_{i,d+1}\bar{F}^{\gamma_i\theta_1}(t_0), & 1 \leq d \leq m_1 - 1, \\ \frac{1}{\bar{F}^{n_1\theta_1}(t_0)} \left(1 - \prod_{j=1}^{m_1} \gamma_j \sum_{i=1}^{m_1} \frac{a_{i,m_1}}{\gamma_i}\bar{F}^{\gamma_i\theta_1}(t_0)\right), & d = m_1. \end{cases}$$

Thus, $I_{11} = \frac{1}{\theta_1^2} \sum_{d=1}^{m_1} d f_{D_1|D_1>0}(d)$. Similarly, $I_{22} = \frac{1}{\theta_2^2} \sum_{d=1}^{m_2} d f_{D_2|D_2>0}(d)$, where

$$f_{D_2|D_2>0}(d) = \begin{cases} \frac{1}{\bar{F}^{n_2\theta_2}(t_0')} \prod_{j=1}^{d} \gamma_j' \sum_{i=1}^{d+1} a'_{i,d+1} \bar{F}^{\gamma_i'\theta_2}(t_0'), & 1 \le d \le m_2 - 1, \\ \frac{1}{\bar{F}^{n_2\theta_2}(t_0')} \left(1 - \prod_{j=1}^{m_2} \gamma_j' \sum_{i=1}^{m_2} \frac{a'_{i,m_2}}{\gamma_i'} \bar{F}^{\gamma_i'\theta_2}(t_0)\right), & d = m_2, \end{cases}$$

and $a'_{i,j} = \prod_{\ell=1,\ell\neq i}^{j} (\gamma_\ell' - \gamma_i')^{-1}$. Therefore, the asymptotic variance-covariance matrix, $A = [a_{ij}]$, is obtained by inverting the Fisher information matrix as the following:

$$A = I^{-1}(\theta_1, \theta_2) = \begin{pmatrix} I_{11}^{-1} & 0 \\ 0 & I_{22}^{-1} \end{pmatrix}.$$

Now, the variance of $\hat{R}$, denoted by $B$, can be obtained using the Cramér's theorem; see Ferguson (1996) or Shao (2003). We have $\hat{R} = g(\hat{\theta}_1, \hat{\theta}_2)$, where $g(\theta_1, \theta_2) = \frac{\theta_1}{\theta_1+\theta_2}$. Therefore, $B = \boldsymbol{b}^t A \boldsymbol{b}$, where

$$\boldsymbol{b} = \begin{pmatrix} \frac{\partial g}{\partial \theta_1} \\ \frac{\partial g}{\partial \theta_2} \end{pmatrix} = \frac{1}{(\theta_1 + \theta_2)^2} \begin{pmatrix} \theta_2 \\ -\theta_1 \end{pmatrix}.$$

Thus, it can be easily verified that

$$B = \boldsymbol{b}^t A \boldsymbol{b} = R^2(1-R)^2 \left( \frac{1}{\mathrm{E}[D_1|D_1 > 0]} + \frac{1}{\mathrm{E}[D_2|D_2 > 0]} \right).$$

To compute the confidence interval of $R$, it is enough to estimate $B$. Therefore, we have immediately that

$$\hat{B} = \hat{R}^2(1-\hat{R})^2 \left( \frac{1}{\widehat{\mathrm{E}}[D_1|D_1 > 0]} + \frac{1}{\widehat{\mathrm{E}}[D_2|D_2 > 0]} \right),$$

which $\widehat{\mathrm{E}}[D_1|D_1 > 0]$ and $\widehat{\mathrm{E}}[D_2|D_2 > 0]$ are obtained by replacing $\hat{\theta}_1$ and $\hat{\theta}_2$ instead of $\theta_1$ and $\theta_2$ in $\mathrm{E}[D_1|D_1 > 0]$ and $\mathrm{E}[D_2|D_2 > 0]$, respectively. Hence, a $100(1-\alpha)\%$ asymptotic confidence interval of $R$ is given by

$$\left(\max(0, \hat{R} - Z_{1-\alpha/2}\sqrt{\hat{B}}), \min(1, \hat{R} + Z_{1-\alpha/2}\sqrt{\hat{B}})\right), \quad (10)$$

where $Z_\alpha$ is $100\alpha-$th percentile of standard normal distribution.

## 3.2 Bayesian credible interval of $R$

In this subsection, we obtain the Bayesian and the highest posterior density (HPD) intervals for $R$. We consider $\theta_1 \sim \text{gamma}(\alpha_1, \beta_1)$ and $\theta_2 \sim \text{gamma}(\alpha_2, \beta_2)$ as the prior distributions.

According to Shao (2003), for any $\alpha \in (0,1)$ a level credible set for $R$ is any $C(\boldsymbol{X},\boldsymbol{Y})$ with

$$P(R \in C(\boldsymbol{X},\boldsymbol{Y})) = \int_{C(\boldsymbol{X},\boldsymbol{Y})} f_{R|\boldsymbol{X},\boldsymbol{Y}}(r)\mathrm{d}r \geq 1-\alpha.$$

A level $1-\alpha$ HPD credible set for $R$ is defined to be the event

$$C(\boldsymbol{X},\boldsymbol{Y}) = \{r : f_{R|\boldsymbol{X},\boldsymbol{Y}}(r) \geq c_\alpha\},$$

where $c_\alpha$ is choosen so that $\int_{C(\boldsymbol{X},\boldsymbol{Y})} f_{R|\boldsymbol{X},\boldsymbol{Y}}(r)\mathrm{d}r \geq 1-\alpha$. According to Shao (2003), if $f_{R|\boldsymbol{X},\boldsymbol{Y}}(r)$ be continuous and unimodal function, then the HPD credible set is an interval having the shortest length within the class of intervals $[a,b]$ satisfying $\int_a^b f_{R|\boldsymbol{X},\boldsymbol{Y}}(r)\mathrm{d}r = 1-\alpha$. On the other hand, in the proof of Theorem 2.2 (iv), we showed that $f_{R|\boldsymbol{X},\boldsymbol{Y}}(r)$ has a unique mode in (0,1). Thus, there exists a HPD credible interval for $R$. The following algorithm can be used for these purposes.

**Algorithm 3.1.** *(i) Given $\boldsymbol{X}$ and $\boldsymbol{Y}$, generate $\theta_1$ and $\theta_2$ using the posterior distributions* (8) *and* (9)*, respectively.*
*(ii) Compute $R_1$ by substituting the generated $\theta_1$ and $\theta_2$ in Equation* (1)*.*
*(iii) Repeat the steps (i) and (ii) for $B$ times to get $R_1,\ldots,R_B$.*
*(iv) Arrange $R_1,\ldots,R_B$ increasingly, such that $R_{(1)} < \ldots < R_{(B)}$.*
*(v) A $100(1-\alpha)\%$ Bayesian credible interval is $\left(R_{([B\frac{\alpha}{2}])}, R_{([B(1-\frac{\alpha}{2})])}\right)$; where $[k]$ denotes the floor of $k$.*
*(vi) The HPD for $R$ is the shortest interval of the form $\left(R_{(j)}, R_{(j+[B(1-\alpha)])}\right)$.*

# 4 Simulation study

In this section, Monte Carlo simulations are carried out to evaluate the performances of the MLEs, Bayes estimators, asymptotic confidence interval, Bayesian and HPD credible intervals for different censoring schemes. We mainly evaluate the performances of the MLEs and Bayes estimators in terms of bias and mean of squared errors (MSE). Also, we evaluate the performances all of the mentioned intervals in terms of average lengths (AL) and coverage probabilities (CP).

For this purpose, we consider different Type-I progressive hybrid censoring schemes $(\boldsymbol{r}_1, t_0)$ and $(\boldsymbol{r}_2, t_0')$. The Type-II progressive censoring schemes that are employed in computations, have been represented in Table 1.

From the sample, we compute the MLE and Bayes estimators $\hat{R}_{\mathrm{sq}}$, $\hat{R}_{\mathrm{wsq}}$ (for $k = l = 1$), $\hat{R}_{\mathrm{ST}}$ and $\hat{R}_{0-1}$ using (6) and existing equations in Theorem 2.2. We compute the MLEs and Bayes estimators of $R$ for different Type-I progressive hybrid censoring schemes $(\boldsymbol{r}_1, t_0)$ and $(\boldsymbol{r}_2, t_0')$, and we report the biases and MSEs of the MLEs and Bayes estimators of $R$ by 10,000 replications. The results are represented in the Table 2 and Table 3. The computations corresponding to point estimators are performed using `Mathematica` software. Table 2 considers the case that $\theta_1 = 1$, $\theta_2 = 2$, $\alpha_1 = 0.9$, $\beta_1 = 1.1$, $\alpha_2 = 2.1$, $\beta_2 = 1.2$, and Table 3 considers the case $\alpha_1 = 0.5$, $\beta_1 = 1.5$, $\alpha_2 = 2$, $\beta_2 = 2$ with unchanged $\theta_1$ and $\theta_2$. Note that, in the tables, $F^{-1}$ denotes the inverse

function of the baseline distribution function $F$. From Table 2 and Table 3, we observe that
i. for fixed $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, when $t_0$ or $t_0'$ increases ($D_1$ or $D_2$ stochastically increases), the MSE decreases;
ii. in all considered cases, the Bayes estimators under different losses are better than the MLE in terms of the MSE criterion;
iii. among the Bayes estimators, $\hat{R}_{\text{wsq}}$ has the least MSE.
iv. in the most considered cases, for the fixed values of $m_1$, $m_2$, $t_0$ and $t_0'$, the censoring schemes with the late censoring perform well.

Overall, we suggest applying a Type-II censoring scheme and using $\hat{R}_{\text{wsq}}$ to estimate $R$.

We also got 95% asymptotic confidence interval (AC) of $R$ by simulating 10,000 samples under different Type-I progressive hybrid censoring schemes, and computed their ALs and CPs. The ALs and CPs of the Bayesian and HPD credible intervals of $R$ are also obtained by simulating 10,000 samples and computing the Bayesian and HPD confidence intervals with $B = 1,000$ using Algorithm 3.1. These results are reported in the Table 4 and Table 5. The computations corresponding to confidence and credible intervals are performed using `R` software.

In view of Table 4 and Table 5, we see that
i. in the all considered cases, we observe that CPs of ACs are less than 0.95;
ii. in the all situations, for fixed $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, as $t_0$ or $t_0'$ increases, the AL decreases;
iii. the ALs of the Bayesian and HPD intervals are shorter than the ALs of the ACs.
iv. the HPD intervals perform well in terms of CP criterion.
v. in the most considered cases, for the fixed values of $m_1$, $m_2$, $t_0$ and $t_0'$, the censoring schemes with the late censoring are better than the other schemes in terms of AL and AC criteria.

Overall, we suggest applying a Type-II censoring scheme and HPD approach to construct a confidence interval for $R$.

Table 1: Type-II progressive censoring schemes.

| $n$ | $m$ | $\boldsymbol{r}$ | scheme number |
|---|---|---|---|
| 30 | 5 | (5,5,5,5,5) | {1} |
| | 5 | (25,0,0,0,0) | {2} |
| | 10 | (0,0,0,0,0,0,0,0,0,20) | {3} |
| | 10 | (0,0,0,0,20,0,0,0,0,0) | {4} |
| 40 | 5 | (7,7,7,7,7) | {5} |
| | 5 | (35,0,0,0,0) | {6} |
| | 10 | (0,0,0,0,0,0,0,0,0,30) | {7} |
| | 10 | (6,0,6,0,6,0,6,0,6,0) | {8} |

Table 2: The bias and MSE of the MLEs and Bayes estimators of $R$ when $\theta_1 = 1$, $\theta_2 = 2$, $\alpha_1 = 0.9$, $\beta_1 = 1.1$, $\alpha_2 = 2.1$, $\beta_2 = 1.2$.

| $(\boldsymbol{r}_1, t_0)$ | $(\boldsymbol{r}_2, t_0')$ | $\hat{R}$ | | $\hat{R}_{\text{sq}}$ | | $\hat{R}_{\text{wsq}}$ | | $\hat{R}_{\text{St}}$ | | $\hat{R}_{0-1}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | bias | MSE | bias | MSE | bias | MSE | bias | MSE | bias | MSE |
| $(\{1\}, F^{-1}(0.4))$ | $(\{5\}, F^{-1}(0.6))$ | 0.0097 | 0.0199 | 0.0163 | 0.0102 | 0.0332 | 0.0091 | -0.0298 | 0.0110 | -0.0114 | 0.0139 |
| | $(\{6\}, F^{-1}(0.6))$ | 0.0209 | 0.0230 | 0.0246 | 0.0115 | 0.0408 | 0.0104 | -0.02348 | 0.0115 | -0.0044 | 0.0151 |
| | $(\{7\}, F^{-1}(0.6))$ | 0.0170 | 0.0166 | 0.0068 | 0.0092 | 0.0222 | 0.0082 | -0.0318 | 0.0103 | -0.0097 | 0.0123 |
| | $(\{8\}, F^{-1}(0.6))$ | 0.0200 | 0.0168 | 0.0095 | 0.0093 | 0.0247 | 0.0084 | -0.0294 | 0.0102 | -0.0071 | 0.0123 |
| $(\{2\}, F^{-1}(0.4))$ | $(\{5\}, F^{-1}(0.6))$ | -0.0051 | 0.0250 | -0.0066 | 0.0115 | 0.0206 | 0.0089 | -0.0763 | 0.0192 | -0.0439 | 0.0201 |
| | $(\{6\}, F^{-1}(0.6))$ | 0.0048 | 0.0267 | 0.0003 | 0.0120 | 0.0270 | 0.0095 | -0.0717 | 0.0187 | -0.0383 | 0.0203 |
| | $(\{7\}, F^{-1}(0.6))$ | 0.0025 | 0.0215 | -0.0151 | 0.0107 | 0.0107 | 0.0081 | -0.0771 | 0.0184 | -0.0415 | 0.0179 |
| | $(\{8\}, F^{-1}(0.6))$ | 0.0023 | 0.0210 | -0.0149 | 0.0105 | 0.0111 | 0.0079 | -0.0781 | 0.0181 | -0.0416 | 0.0176 |
| $(\{3\}, F^{-1}(0.4))$ | $(\{5\}, F^{-1}(0.6))$ | -0.0019 | 0.0430 | 0.0207 | 0.0085 | 0.0332 | 0.0079 | -0.0125 | 0.0084 | -0.0075 | 0.0101 |
| | $(\{6\}, F^{-1}(0.6))$ | 0.0072 | 0.0171 | 0.0270 | 0.0099 | 0.0391 | 0.0091 | -0.0083 | 0.0092 | -0.0033 | 0.0111 |
| | $(\{7\}, F^{-1}(0.6))$ | 0.0042 | 0.0102 | 0.0101 | 0.0071 | 0.0207 | 0.0067 | -0.0154 | 0.0074 | -0.0067 | 0.0085 |
| | $(\{8\}, F^{-1}(0.6))$ | 0.0040 | 0.0106 | 0.0101 | 0.0074 | 0.0208 | 0.0069 | -0.0157 | 0.0076 | -0.0077 | 0.0088 |
| $(\{4\}, F^{-1}(0.4))$ | $(\{5\}, F^{-1}(0.6))$ | -0.0015 | 0.0157 | 0.0149 | 0.0088 | 0.0302 | 0.0080 | -0.0252 | 0.0095 | -0.0140 | 0.0115 |
| | $(\{6\}, F^{-1}(0.6))$ | 0.0093 | 0.0186 | 0.0227 | 0.0102 | 0.0374 | 0.0093 | -0.0195 | 0.0102 | -0.0080 | 0.0126 |
| | $(\{7\}, F^{-1}(0.6))$ | 0.0048 | 0.0120 | 0.0050 | 0.0079 | 0.0185 | 0.0071 | -0.0276 | 0.0088 | -0.0125 | 0.0100 |
| | $(\{8\}, F^{-1}(0.6))$ | 0.0049 | 0.0117 | 0.0052 | 0.0077 | 0.0188 | 0.0069 | -0.0276 | 0.0086 | -0.0127 | 0.0097 |
| $(\{1\}, F^{-1}(0.5))$ | $(\{5\}, F^{-1}(0.7))$ | 0.0116 | 0.0192 | 0.0183 | 0.0098 | 0.0348 | 0.0089 | -0.0270 | 0.0102 | -0.0088 | 0.0131 |
| | $(\{6\}, F^{-1}(0.7))$ | 0.0207 | 0.0219 | 0.0243 | 0.0111 | 0.0401 | 0.0101 | -0.0220 | 0.0110 | -0.0032 | 0.0145 |
| | $(\{7\}, F^{-1}(0.7))$ | 0.0215 | 0.0160 | 0.0107 | 0.0088 | 0.0255 | 0.0080 | -0.0272 | 0.0094 | -0.0051 | 0.0115 |
| | $(\{8\}, F^{-1}(0.7))$ | 0.0209 | 0.0161 | 0.0103 | 0.0088 | 0.0251 | 0.0081 | -0.0277 | 0.0095 | -0.0058 | 0.0116 |
| $(\{2\}, F^{-1}(0.5))$ | $(\{5\}, F^{-1}(0.7))$ | -0.0040 | 0.0031 | -0.0027 | 0.0111 | 0.0220 | 0.0089 | -0.0658 | 0.0169 | -0.0374 | 0.0183 |
| | $(\{6\}, F^{-1}(0.7))$ | 0.0027 | 0.0246 | 0.0023 | 0.0117 | 0.0267 | 0.0095 | -0.0618 | 0.0170 | -0.0329 | 0.0189 |
| | $(\{7\}, F^{-1}(0.7))$ | 0.0020 | 0.0198 | -0.0122 | 0.0105 | 0.0111 | 0.0082 | -0.0679 | 0.0164 | -0.0360 | 0.0165 |
| | $(\{8\}, F^{-1}(0.7))$ | 0.0047 | 0.0203 | -0.0100 | 0.0106 | 0.0131 | 0.0084 | -0.0658 | 0.0164 | -0.0337 | 0.0167 |

Continuation of Table 2.

| $(\boldsymbol{r}_1, t_0)$ | $(\boldsymbol{r}_2, t'_0)$ | $\hat{R}$ | | $\hat{R}_{\text{sq}}$ | | $\hat{R}_{\text{wsq}}$ | | $\hat{R}_{\text{St}}$ | | $\hat{R}_{0-1}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | bias | MSE | bias | MSE | bias | MSE | bias | MSE | bias | MSE |
| $(\{3\}, F^{-1}(0.5))$ | $(\{5\}, F^{-1}(0.7))$ | -0.0010 | 0.0138 | 0.0216 | 0.0081 | 0.0338 | 0.0076 | -0.0114 | 0.0078 | -0.0066 | 0.0096 |
| | $(\{6\}, F^{-1}(0.7))$ | 0.0067 | 0.0158 | 0.0271 | 0.0092 | 0.0389 | 0.0086 | -0.0068 | 0.0084 | -0.0018 | 0.0104 |
| | $(\{7\}, F^{-1}(0.7))$ | 0.0082 | 0.0101 | 0.0107 | 0.0070 | 0.0238 | 0.0067 | -0.0115 | 0.0070 | -0.0027 | 0.0082 |
| | $(\{8\}, F^{-1}(0.7))$ | 0.0062 | 0.0102 | 0.0122 | 0.0071 | 0.0225 | 0.0067 | -0.0130 | 0.0071 | -0.0045 | 0.0083 |
| $(\{4\}, F^{-1}(0.5))$ | $(\{5\}, F^{-1}(0.7))$ | -0.0043 | 0.0149 | 0.0142 | 0.0084 | 0.0291 | 0.0077 | -0.0243 | 0.0091 | -0.0149 | 0.0108 |
| | $(\{6\}, F^{-1}(0.7))$ | 0.0023 | 0.0166 | 0.0189 | 0.0023 | 0.0334 | 0.0085 | -0.0207 | 0.0095 | -0.0110 | 0.0116 |
| | $(\{7\}, F^{-1}(0.7))$ | 0.0008 | 0.0110 | 0.0033 | 0.0074 | 0.064 | 0.0066 | -0.0276 | 0.0083 | 0.0144 | 0.0093 |
| | $(\{8\}, F^{-1}(0.7))$ | 0.0056 | 0.0113 | 0.0071 | 0.0076 | 0.0200 | 0.0069 | -0.0239 | 0.0083 | -0.0104 | 0.0094 |

Table 3: The bias and MSE of the MLEs and Bayes estimators of $R$ when $\theta_1 = 1$, $\theta_2 = 2$, $\alpha_1 = 0.5$, $\beta_1 = 1.5$, $\alpha_2 = 2$, $\beta_2 = 2$.

| $(\boldsymbol{r}_1, t_0)$ | $(\boldsymbol{r}_2, t'_0)$ | $\hat{R}$ | | $\hat{R}_{\text{sq}}$ | | $\hat{R}_{\text{wsq}}$ | | $\hat{R}_{\text{St}}$ | | $\hat{R}_{0-1}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | bias | MSE | bias | MSE | bias | MSE | bias | MSE | bias | MSE |
| $(\{1\}, F^{-1}(0.4))$ | $(\{5\}, F^{-1}(0.6))$ | 0.0108 | 0.0200 | 0.0314 | 0.0092 | 0.0479 | 0.0089 | -0.0179 | 0.0090 | 0.0052 | 0.0122 |
| | $(\{6\}, F^{-1}(0.6))$ | 0.0191 | 0.0227 | 0.0404 | 0.0112 | 0.0560 | 0.0106 | -0.0108 | 0.01008 | 0.0132 | 0.0140 |
| | $(\{7\}, F^{-1}(0.6))$ | 0.0190 | 0.0169 | 0.0044 | 0.0079 | 0.0210 | 0.0071 | -0.0369 | 0.0094 | -0.0131 | 0.0109 |
| | $(\{8\}, F^{-1}(0.6))$ | 0.0182 | 0.0164 | 0.00508 | 0.0078 | 0.0217 | 0.0070 | -0.0366 | 0.0092 | -0.0129 | 0.0107 |
| $(\{2\}, F^{-1}(0.4))$ | $(\{5\}, F^{-1}(0.6))$ | -0.0039 | 0.0242 | -0.0129 | 0.0108 | 0.0182 | 0.0078 | -0.0921 | 0.0220 | -0.0563 | 0.0214 |
| | $(\{6\}, F^{-1}(0.6))$ | 0.0057 | 0.0274 | -0.0029 | 0.0120 | 0.0272 | 0.0091 | -0.0847 | 0.0221 | -0.0470 | 0.0228 |
| | $(\{7\}, F^{-1}(0.6))$ | 0.0032 | 0.0217 | -0.0374 | 0.0114 | -0.0065 | 0.0074 | -0.1073 | 0.0238 | -0.0722 | 0.0215 |
| | $(\{8\}, F^{-1}(0.6))$ | 0.0027 | 0.0212 | -0.0369 | 0.0111 | -0.0059 | 0.0072 | -0.1076 | 0.0237 | -0.0723 | 0.0212 |
| $(\{3\}, F^{-1}(0.4))$ | $(\{5\}, F^{-1}(0.6))$ | -0.0029 | 0.0147 | 0.0503 | 0.0097 | 0.0608 | 0.0096 | -0.0160 | 0.0078 | 0.0248 | 0.0100 |
| | $(\{6\}, F^{-1}(0.6))$ | 0.0090 | 0.0172 | 0.0622 | 0.0122 | 0.0716 | 0.0119 | 0.0260 | 0.0093 | 0.0357 | 0.0122 |
| | $(\{7\}, F^{-1}(0.6))$ | 0.0029 | 0.0102 | 0.0206 | 0.0069 | 0.0310 | 0.0066 | -0.0059 | 0.0067 | 0.0043 | 0.0079 |
| | $(\{8\}, F^{-1}(0.6))$ | 0.0039 | 0.0104 | 0.0225 | 0.0071 | 0.0328 | 0.0068 | -0.0041 | 0.0067 | 0.0060 | 0.0080 |

Continuation of Table 3.

| $(\boldsymbol{r}_1, t_0)$ | $(\boldsymbol{r}_2, t_0')$ | $\hat{R}$ | | $\hat{R}_{\text{sq}}$ | | $\hat{R}_{\text{wsq}}$ | | $\hat{R}_{\text{St}}$ | | $\hat{R}_{0-1}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | bias | MSE | bias | MSE | bias | MSE | bias | MSE | bias | MSE |
| $(\{4\}, F^{-1}(0.4))$ | $(\{5\}, F^{-1}(0.6))$ | -0.0001 | 0.0157 | 0.0381 | 0.0091 | 0.0520 | 0.0088 | -0.0041 | 0.0081 | 0.0115 | 0.0105 |
| | $(\{6\}, F^{-1}(0.6))$ | 0.0075 | 0.0183 | 0.0467 | 0.0110 | 0.0598 | 0.0066 | 0.0025 | 0.0092 | 0.0188 | 0.0123 |
| | $(\{7\}, F^{-1}(0.6))$ | 0.0059 | 0.0119 | 0.0097 | 0.0072 | 0.0237 | 0.0066 | -0.0245 | 0.0080 | -0.0077 | 0.0091 |
| | $(\{8\}, F^{-1}(0.6))$ | 0.0064 | 0.0121 | 0.0109 | 0.0073 | 0.0248 | 0.0067 | -0.0236 | 0.0081 | -0.0068 | 0.0092 |
| $(\{1\}, F^{-1}(0.5))$ | $(\{5\}, F^{-1}(0.7))$ | 0.0104 | 0.0194 | 0.0323 | 0.0089 | 0.0484 | 0.0086 | -0.0161 | 0.0083 | 0.0063 | 0.0115 |
| | $(\{6\}, F^{-1}(0.7))$ | 0.0215 | 0.0215 | 0.0413 | 0.0104 | 0.0564 | 0.0100 | -0.0081 | 0.0089 | 0.0156 | 0.0128 |
| | $(\{7\}, F^{-1}(0.7))$ | 0.0203 | 0.0165 | 0.0065 | 0.0075 | 0.0226 | 0.0068 | -0.0339 | 0.0086 | -0.0104 | 0.0102 |
| | $(\{8\}, F^{-1}(0.7))$ | 0.0214 | 0.0166 | 0.0078 | 0.0077 | 0.0237 | 0.0070 | -0.0328 | 0.0087 | -0.0092 | 0.0104 |
| $(\{2\}, F^{-1}(0.5))$ | $(\{5\}, F^{-1}(0.7))$ | -0.0021 | 0.0237 | -0.0022 | 0.0108 | 0.0249 | 0.0084 | -0.0724 | 0.0183 | -0.0400 | 0.0190 |
| | $(\{6\}, F^{-1}(0.7))$ | 0.0067 | 0.0255 | 0.0051 | 0.0116 | 0.0314 | 0.0093 | -0.0663 | 0.0182 | -0.0324 | 0.0197 |
| | $(\{7\}, F^{-1}(0.7))$ | 0.0045 | 0.0200 | -0.0277 | 0.0105 | -0.0007 | 0.0074 | -0.0893 | 0.0195 | -0.0571 | 0.0182 |
| | $(\{8\}, F^{-1}(0.7))$ | 0.0027 | 0.0203 | -0.0288 | 0.0107 | -0.0014 | 0.0075 | -0.0909 | 0.0201 | -0.0589 | 0.0188 |
| $(\{3\}, F^{-1}(0.5))$ | $(\{5\}, F^{-1}(0.7))$ | 0.0005 | 0.0139 | 0.0532 | 0.0094 | 0.0632 | 0.0094 | 0.0193 | 0.0072 | 0.0280 | 0.0093 |
| | $(\{6\}, F^{-1}(0.7))$ | 0.0068 | 0.0158 | 0.0603 | 0.0111 | 0.0696 | 0.0110 | 0.0255 | 0.0083 | 0.0348 | 0.0109 |
| | $(\{7\}, F^{-1}(0.7))$ | 0.0073 | 0.0098 | 0.0251 | 0.0065 | 0.0350 | 0.0064 | -0.0008 | 0.0060 | 0.0094 | 0.0073 |
| | $(\{8\}, F^{-1}(0.7))$ | 0.0074 | 0.0101 | 0.0256 | 0.0068 | 0.0354 | 0.0067 | -0.0005 | 0.0062 | 0.0097 | 0.0076 |
| $(\{4\}, F^{-1}(0.5))$ | $(\{5\}, F^{-1}(0.7))$ | -0.0034 | 0.0150 | 0.0389 | 0.0089 | 0.0523 | 0.0087 | -0.0014 | 0.0079 | 0.0123 | 0.0101 |
| | $(\{6\}, F^{-1}(0.7))$ | 0.0039 | 0.0170 | 0.0458 | 0.0104 | 0.0584 | 0.0101 | 0.0045 | 0.0088 | 0.0188 | 0.0115 |
| | $(\{7\}, F^{-1}(0.7))$ | 0.0045 | 0.0113 | 0.0120 | 0.0070 | 0.0252 | 0.0065 | -0.0203 | 0.0076 | -0.0051 | 0.0087 |
| | $(\{8\}, F^{-1}(0.7))$ | 0.0045 | 0.0114 | 0.0127 | 0.0071 | 0.0258 | 0.0066 | -0.0198 | 0.0076 | -0.0047 | 0.0088 |

Table 4: The average lengths and coverage probabilities of the asymptotic confidence interval and the Bayesian and HPD credible intervals of $R$ when $\theta_1 = 1$, $\theta_2 = 2$, $\alpha_1 = 0.9$, $\beta_1 = 1.1$, $\alpha_2 = 2.1$, $\beta_2 = 1.2$.

| $(\boldsymbol{r}_1, t_0)$ | $(\boldsymbol{r}_2, t'_0)$ | AC | | Bayesian | | HPD | |
|---|---|---|---|---|---|---|---|
| | | AL | CP | AL | CP | AL | CP |
| $(\{1\}, F^{-1}(0.4))$ | $(\{5\}, F^{-1}(0.6))$ | 0.5138 | 0.8807 | 0.4562 | 0.9835 | 0.4460 | 0.9659 |
| | $(\{6\}, F^{-1}(0.6))$ | 0.5422 | 0.8753 | 0.4719 | 0.9787 | 0.4613 | 0.9608 |
| | $(\{7\}, F^{-1}(0.6))$ | 0.4638 | 0.8955 | 0.4107 | 0.9725 | 0.4038 | 0.9523 |
| | $(\{8\}, F^{-1}(0.6))$ | 0.4660 | 0.9002 | 0.4125 | 0.9737 | 0.4056 | 0.9549 |
| $(\{2\}, F^{-1}(0.4))$ | $(\{5\}, F^{-1}(0.6))$ | 0.5616 | 0.8586 | 0.5035 | 0.9851 | 0.4887 | 0.9642 |
| | $(\{6\}, F^{-1}(0.6))$ | 0.5822 | 0.8622 | 0.5170 | 0.9822 | 0.5017 | 0.9613 |
| | $(\{7\}, F^{-1}(0.6))$ | 0.5346 | 0.8811 | 0.4652 | 0.9755 | 0.4540 | 0.9514 |
| | $(\{8\}, F^{-1}(0.6))$ | 0.5365 | 0.8819 | 0.4670 | 0.9766 | 0.4556 | 0.9550 |
| $(\{3\}, F^{-1}(0.4))$ | $(\{5\}, F^{-1}(0.6))$ | 0.4480 | 0.8848 | 0.4067 | 0.9775 | 0.3972 | 0.9656 |
| | $(\{6\}, F^{-1}(0.6))$ | 0.4819 | 0.8866 | 0.4242 | 0.9720 | 0.4143 | 0.9598 |
| | $(\{7\}, F^{-1}(0.6))$ | 0.3799 | 0.9099 | 0.3503 | 0.9646 | 0.3445 | 0.9469 |
| | $(\{8\}, F^{-1}(0.6))$ | 0.3833 | 0.9122 | 0.3529 | 0.9631 | 0.3469 | 0.9473 |
| $(\{4\}, F^{-1}(0.4))$ | $(\{5\}, F^{-1}(0.6))$ | 0.4802 | 0.8959 | 0.4343 | 0.9813 | 0.4245 | 0.9688 |
| | $(\{6\}, F^{-1}(0.6))$ | 0.5089 | 0.8894 | 0.4493 | 0.9776 | 0.4390 | 0.9632 |
| | $(\{7\}, F^{-1}(0.6))$ | 0.4185 | 0.9164 | 0.3825 | 0.9729 | 0.3762 | 0.9568 |
| | $(\{8\}, F^{-1}(0.6))$ | 0.4217 | 0.9143 | 0.3851 | 0.9691 | 0.3786 | 0.9544 |
| $(\{1\}, F^{-1}(0.5))$ | $(\{5\}, F^{-1}(0.7))$ | 0.5139 | 0.8866 | 0.4561 | 0.9837 | 0.4461 | 0.9699 |
| | $(\{6\}, F^{-1}(0.7))$ | 0.5284 | 0.8754 | 0.4633 | 0.9823 | 0.4531 | 0.9654 |
| | $(\{7\}, F^{-1}(0.7))$ | 0.4583 | 0.8969 | 0.4074 | 0.9721 | 0.4007 | 0.9530 |
| | $(\{8\}, F^{-1}(0.7))$ | 0.4624 | 0.9060 | 0.4103 | 0.9764 | 0.4036 | 0.9599 |
| $(\{2\}, F^{-1}(0.5))$ | $(\{5\}, F^{-1}(0.7))$ | 0.5501 | 0.8653 | 0.4919 | 0.9830 | 0.4783 | 0.9606 |
| | $(\{6\}, F^{-1}(0.7))$ | 0.5627 | 0.8588 | 0.4987 | 0.9806 | 0.4849 | 0.9589 |
| | $(\{7\}, F^{-1}(0.7))$ | 0.5166 | 0.8815 | 0.4514 | 0.9745 | 0.4414 | 0.9476 |
| | $(\{8\}, F^{-1}(0.7))$ | 0.5175 | 0.8833 | 0.4521 | 0.9768 | 0.4420 | 0.9424 |
| $(\{3\}, F^{-1}(0.5))$ | $(\{5\}, F^{-1}(0.7))$ | 0.4488 | 0.8946 | 0.4074 | 0.9797 | 0.3982 | 0.9708 |
| | $(\{6\}, F^{-1}(0.7))$ | 0.4671 | 0.8893 | 0.4161 | 0.9764 | 0.4066 | 0.9659 |
| | $(\{7\}, F^{-1}(0.7))$ | 0.3783 | 0.9184 | 0.3500 | 0.9671 | 0.3444 | 0.9532 |
| | $(\{8\}, F^{-1}(0.7))$ | 0.3797 | 0.9198 | 0.3510 | 0.9687 | 0.3453 | 0.9562 |
| $(\{4\}, F^{-1}(0.5))$ | $(\{5\}, F^{-1}(0.7))$ | 0.4719 | 0.8969 | 0.4280 | 0.9811 | 0.4182 | 0.9680 |
| | $(\{6\}, F^{-1}(0.7))$ | 0.4876 | 0.8859 | 0.4358 | 0.9785 | 0.4259 | 0.9645 |
| | $(\{7\}, F^{-1}(0.7))$ | 0.4089 | 0.9153 | 0.3757 | 0.9733 | 0.3696 | 0.9567 |
| | $(\{8\}, F^{-1}(0.7))$ | 0.4098 | 0.9100 | 0.3766 | 0.9705 | 0.3703 | 0.9535 |

# 5 Data analysis

In this section, the analysis of a pair of real data sets is presented for illustrative purposes. The data sets show the breaking strengths of jute fiber at two different gauge lengths. These two data sets were used by Xia et al. (2009). Let $X$ and $Y$

Table 5: The average lengths and coverage probabilities of the asymptotic confidence interval and the Bayesian and HPD credible intervals of $R$ when $\theta_1 = 1$, $\theta_2 = 2$, $\alpha_1 = 0.5$, $\beta_1 = 1.5$, $\alpha_2 = 2$, $\beta_2 = 2$.

| $(\boldsymbol{r}_1, t_0)$ | $(\boldsymbol{r}_2, t_0')$ | AC | | Bayesian | | HPD | |
|---|---|---|---|---|---|---|---|
| | | AL | CP | AL | CP | AL | CP |
| $(\{1\}, F^{-1}(0.4))$ | $(\{5\}, F^{-1}(0.6))$ | 0.5135 | 0.8757 | 0.4766 | 0.9923 | 0.4669 | 0.9836 |
| | $(\{6\}, F^{-1}(0.6))$ | 0.5403 | 0.8824 | 0.4918 | 0.9901 | 0.4819 | 0.9792 |
| | $(\{7\}, F^{-1}(0.6))$ | 0.4634 | 0.8993 | 0.4199 | 0.9849 | 0.4128 | 0.9712 |
| | $(\{8\}, F^{-1}(0.6))$ | 0.4662 | 0.8981 | 0.4228 | 0.9828 | 0.4157 | 0.9683 |
| $(\{2\}, F^{-1}(0.4))$ | $(\{5\}, F^{-1}(0.6))$ | 0.5632 | 0.8630 | 0.5196 | 0.9914 | 0.5024 | 0.9745 |
| | $(\{6\}, F^{-1}(0.6))$ | 0.5846 | 0.8644 | 0.5346 | 0.9900 | 0.5174 | 0.9720 |
| | $(\{7\}, F^{-1}(0.6))$ | 0.5355 | 0.8767 | 0.4677 | 0.9777 | 0.4535 | 0.9433 |
| | $(\{8\}, F^{-1}(0.6))$ | 0.5336 | 0.8764 | 0.4680 | 0.9752 | 0.4533 | 0.9381 |
| $(\{3\}, F^{-1}(0.4))$ | $(\{5\}, F^{-1}(0.6))$ | 0.4491 | 0.8861 | 0.4267 | 0.9782 | 0.4184 | 0.9702 |
| | $(\{6\}, F^{-1}(0.6))$ | 0.4797 | 0.8818 | 0.4431 | 0.9671 | 0.4346 | 0.9595 |
| | $(\{7\}, F^{-1}(0.6))$ | 0.3803 | 0.9104 | 0.3608 | 0.9751 | 0.3553 | 0.9637 |
| | $(\{8\}, F^{-1}(0.6))$ | 0.3828 | 0.9065 | 0.3632 | 0.9726 | 0.3575 | 0.9603 |
| $(\{4\}, F^{-1}(0.4))$ | $(\{5\}, F^{-1}(0.6))$ | 0.4814 | 0.8928 | 0.4550 | 0.9856 | 0.4462 | 0.9789 |
| | $(\{6\}, F^{-1}(0.6))$ | 0.5072 | 0.8829 | 0.4696 | 0.9806 | 0.4604 | 0.9714 |
| | $(\{7\}, F^{-1}(0.6))$ | 0.4191 | 0.9140 | 0.3934 | 0.9797 | 0.3872 | 0.9682 |
| | $(\{8\}, F^{-1}(0.6))$ | 0.4218 | 0.9165 | 0.3956 | 0.9803 | 0.3893 | 0.9690 |
| $(\{1\}, F^{-1}(0.5))$ | $(\{5\}, F^{-1}(0.7))$ | 0.5136 | 0.8796 | 0.4761 | 0.9934 | 0.4669 | 0.9844 |
| | $(\{6\}, F^{-1}(0.7))$ | 0.5285 | 0.8854 | 0.4842 | 0.9908 | 0.4749 | 0.9823 |
| | $(\{7\}, F^{-1}(0.7))$ | 0.4617 | 0.9017 | 0.4198 | 0.9861 | 0.4130 | 0.9737 |
| | $(\{8\}, F^{-1}(0.7))$ | 0.4616 | 0.9026 | 0.4197 | 0.9874 | 0.4130 | 0.9734 |
| $(\{2\}, F^{-1}(0.5))$ | $(\{5\}, F^{-1}(0.7))$ | 0.5495 | 0.8623 | 0.5087 | 0.9901 | 0.4938 | 0.9712 |
| | $(\{6\}, F^{-1}(0.7))$ | 0.5610 | 0.8657 | 0.5156 | 0.9881 | 0.5006 | 0.9694 |
| | $(\{7\}, F^{-1}(0.7))$ | 0.5174 | 0.8847 | 0.4558 | 0.9729 | 0.4439 | 0.9426 |
| | $(\{8\}, F^{-1}(0.7))$ | 0.5172 | 0.8754 | 0.4561 | 0.9762 | 0.4441 | 0.9463 |
| $(\{3\}, F^{-1}(0.5))$ | $(\{5\}, F^{-1}(0.7))$ | 0.4478 | 0.8951 | 0.4262 | 0.9795 | 0.4181 | 0.9733 |
| | $(\{6\}, F^{-1}(0.7))$ | 0.4650 | 0.8891 | 0.4346 | 0.9722 | 0.4265 | 0.9658 |
| | $(\{7\}, F^{-1}(0.7))$ | 0.3774 | 0.9195 | 0.3600 | 0.9799 | 0.3546 | 0.9719 |
| | $(\{8\}, F^{-1}(0.7))$ | 0.3797 | 0.9188 | 0.3618 | 0.9782 | 0.3562 | 0.9700 |
| $(\{4\}, F^{-1}(0.5))$ | $(\{5\}, F^{-1}(0.7))$ | 0.4708 | 0.8925 | 0.4477 | 0.9846 | 0.4389 | 0.9778 |
| | $(\{6\}, F^{-1}(0.7))$ | 0.4861 | 0.8776 | 0.4559 | 0.9792 | 0.4470 | 0.9698 |
| | $(\{7\}, F^{-1}(0.7))$ | 0.4086 | 0.9177 | 0.3855 | 0.9795 | 0.3794 | 0.9689 |
| | $(\{8\}, F^{-1}(0.7))$ | 0.4092 | 0.9145 | 0.3864 | 0.9787 | 0.3803 | 0.9675 |

denote breaking strength of jute fiber of gauge length 10 mm and breaking strength of jute fiber of gauge length 20 mm, respectively. These data sets have been used in many studies related to the stress-strength model; we refer to Mirjalili et al. (2016), Nadeb et al. (2019), Bhattacharya and Aslam (2020), Yazgan et al. (2022), Chacko et al. (2023), Pasha-Zanoosi, H. (2023), Sarhan and Tolba (2023), Abdelwahab et al. (2024), Garg et al. (2024), Saini et al. (2024). We apply the Kolmogorov-Smirnov test for each data

Table 6: Data Set 1 (Breaking strength of jute fiber of gauge length 10 mm).

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 693.73 | 704.66 | 323.83 | 778.17 | 123.06 | 637.66 | 383.43 | 151.48 | 108.94 | 50.16 |
| 671.49 | 183.16 | 257.44 | 727.23 | 291.27 | 101.15 | 376.42 | 163.40 | 141.38 | 700.74 |
| 262.90 | 353.24 | 422.11 | 43.93 | 590.48 | 212.13 | 303.90 | 506.60 | 530.55 | 177.25 |

Table 7: Data Set 2 (Breaking strength of jute fiber of gauge length 20 mm).

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 71.46 | 419.02 | 284.64 | 585.57 | 456.60 | 113.85 | 187.85 | 688.16 | 662.66 | 45.58 |
| 578.62 | 756.70 | 594.29 | 166.49 | 99.72 | 707.36 | 765.14 | 187.13 | 145.96 | 350.70 |
| 547.44 | 116.99 | 375.81 | 581.60 | 119.86 | 48.01 | 200.16 | 36.75 | 244.53 | 83.55 |

set separately to fit the model. It is observed that for the Data Set 1, the Kolmogorov-Smirnov statistic is 0.1224 with $p$-value=0.7141 when $X \sim \text{PHR}(e^{-(x-36)},\ 0.0030)$, and for the Data Set 2, the Kolmogorov-Smirnov statistic is 0.1466 with $p$-value=0.4934 when $Y \sim \text{PHR}(e^{-(y-36)},\ 0.0033)$. Thus, based on the complete data sets, we have $\hat{R} = 0.4803$.

For illustrative the purposes, we consider two different Type-I progressive hybrid censoring schemes.
**Scheme 1:**

$$\begin{aligned} \boldsymbol{r}_1 &= (2,\ 1,\ 2,\ 1,\ 2,\ 1,\ 2,\ 1,\ 2,\ 1,\ 2,\ 1), \quad t_0 = 250, \\ \boldsymbol{r}_2 &= (2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2), \quad t_0' = 300. \end{aligned}$$

By applying these censoring schemes on the complete data, we obtained

$$\begin{aligned} \boldsymbol{x} &= (43.93,\ 101.15,\ 108.94,\ 123.06,\ 141.38,\ 151.48,\ 163.40,\ 212.13), \\ \boldsymbol{y} &= (36.75,\ 45.58,\ 48.01,\ 83.55,\ 113.85,\ 166.49,\ 200.16,\ 244.53). \end{aligned}$$

In this case, we have $\hat{R} = 0.4712$ and for $\alpha_1 = 0.9, \beta_1 = 1.1,\ \alpha_2 = 2.1,\ \beta_2 = 1.2$ we obtain $\hat{R}_{\text{sq}} = 0.4413$, $\hat{R}_{\text{wsq}} = 0.4467$, $\hat{R}_{\text{St}} = 0.4107$ and $\hat{R}_{0-1} = 0.4329$. Also, the 0.95% asymptotic, Bayesian and HPD confidence intervals are (0.2621, 0.6803), (0.2464, 0.6621) and (0.2363, 0.6420), respectively.
**Scheme 2:**

$$\begin{aligned} \boldsymbol{r}_1 &= \boldsymbol{r}_2 = (1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1), \\ t_0 &= t_0' = 400. \end{aligned}$$

By applying these censoring schemes, we observed

$$\begin{aligned} \boldsymbol{x} &= (43.93,\ 50.16,\ 101.15,\ 108.94,\ 123.06,\ 141.38,\ 163.40,\ 177.25,\ 183.16,\ 212.13, \\ &\quad 257.44,\ 323.83,\ 353.24,\ 383.43), \\ \boldsymbol{y} &= (36.75,\ 45.58,\ 48.01,\ 71.46,\ 99.72,\ 116.99,\ 119.86,\ 145.96,\ 166.49,\ 187.13, \\ &\quad 244.53,\ 284.64,\ 375.81). \end{aligned}$$

In this case, we have $\hat{R} = 0.4886$ and for $\alpha_1 = 0.9, \beta_1 = 1.1, \alpha_2 = 2.1, \beta_2 = 1.2$ we obtain $\hat{R}_{\text{sq}} = 0.4678$, $\hat{R}_{\text{wsq}} = 0.4698$, $\hat{R}_{\text{St}} = 0.4696$ and $\hat{R}_{0-1} = 0.4650$. Also, the 0.95% asymptotic, Bayesian and HPD credible intervals are (0.3098, 0.6675), (0.2893, 0.6509) and (0.2945, 0.6547), respectively.

## 6 Conclusion

In this paper, we considered the estimation of $R = P(X < Y)$ based on Type-I progressively hybrid censored samples, when $X$ and $Y$ are belonging to the PHR models. We got the maximum likelihood and some Bayes estimators of $R$ under some losses. Also, we presented the asymptotic confidence interval and the Bayesian intervals of $R$. The performances of MLEs, Bayes estimators and the proposed intervals are evaluated via simulation. The results of simulation for considered cases show that the Bayes estimators have the least MSE when we consider the weighted squared error loss function with the weight $r(1-r)$. Also, the Bayesian and HPD credible intervals are better than the asymptotic confidence interval in terms of average length and coverage probability. Also, it is observed that for point estimation and constructing the confidence intervals, it is better to apply Type-II censoring schemes. Finally, we consider a pair of real data sets and computed the MLEs, Bayes estimators, asymptotic confidence interval, and Bayesian and HPD credible intervals under two different Type-I progressive hybrid censoring schemes.

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