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Research Paper

Goodness-of-fit tests with spacings of adaptive progressively type-II censored samples from general proportional hazard rate models

Mohammad Vali Ahmadi*[1], Mahdi Doostparast[2]
[1]Department of Statistics, University of Bojnord, Bojnord, Iran
[2]Department of Statistics, Ferdowsi University of Mashhad, Mashhad, Iran

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Abstract: This article proposes two tests to assess the general proportional hazard rate models for adaptive progressively type-II censored samples. These tests are based on spacings derived from the order statistics of the adaptive progressively type-II censored sample. Under the proportional hazard rate model, the distribution of the test statistics is also derived. An extensive simulation study is performed to evaluate the power of proposed tests where the underlying distribution is exponential, Rayleigh, and Pareto. The results indicate that the proposed tests are quite powerful. In addition, we apply these tests to analyze real-world engineering reliability datasets, highlighting their practical versatility. Moreover, the point and interval estimators of the unknown parameter with the most powerful tests are derived when the adaptive progressively type-II censored sample comes from a proportional hazard rate model.

Keywords: Adaptive progressively type-II censoring; Goodness-of-fit test; Proportional hazard rate model; Uniformly most powerful test.
Mathematics Subject Classification (2010): 62N03, 62F03.

1 Introduction

Parametric statistical procedures rely on the assumption of a specific distribution for the random variable being studied. Therefore, it is crucial to test the validity of the assumed distribution using sample data before conducting any inferential procedures. The goodness-of-fit (GOF) technique, which traces its origins back to Karl Pearson's

*Corresponding author: `mv.ahmadi@ub.ac.ir`

seminal article on chi-square tests in 1900, has been extensively discussed by numerous scholars. For a comprehensive review of this topic, refer to D'Agostino and Stephens (1986).

In reliability analysis, it is often impractical to obtain complete data on failure times for all experimental units due to time and cost constraints. To address this issue, different censoring schemes have been proposed by researchers. Type-I censoring terminates the test at a predetermined time, while type-II censoring terminates the test after a specified number of failures. Epstein (1954) introduced a hybrid censoring scheme that combines Type-I and type-II censoring. Under this censoring scheme, the test stops at time $T^* = \min\{X_{n:N}, T\}$, where $X_{n:N}$ is the n-th failure time from N units put on the test, and $T > 0$ is a predetermined time. However, the above censoring schemes do not have the flexibility of allowing the removal of units during the test. To overcome this constraint, the progressive censoring scheme was developed. By combining type-II and progressive censoring, the progressive type-II censoring scheme is obtained. Under the progressive type-II censoring scheme, a certain number (N) of units are randomly selected and put on the test at time zero. The test continues until the occurrence of the n-th failure. When the i-th unit fails ($i = 1, \ldots, n-1$), R_i of the surviving units are randomly removed from the test. Finally, when the nth failure occurs, all the remaining units R_n are removed from the test. Here, n and $\boldsymbol{R} = (R_1, \ldots, R_n)$ are predetermined, and $\sum_{i=1}^{n} R_i = N-n$. The observed lifetimes of such a progressively type-II censored sample are denoted by $X_{1:n:N} < X_{2:n:N} < \cdots < X_{n:n:N}$. Progressive censoring has been extensively studied, and for more details, the readers can refer to Balakrishnan and Aggarwala (2000). Additionally, Balakrishnan and Cramer (2014) offer a comprehensive guide to the theory and methods of progressive censoring in their book.

Kundu and Joarder (2006) proposed a censoring scheme called progressive hybrid censoring, which combines aspects of the hybrid and progressive censoring schemes. In this censoring scheme, the test stops at time $T^* = \min\{X_{n:n:N}, T\}$ where $T > 0$ is a predetermined time. If $X_{n:n:N} < T$, the test stops at time $X_{n:n:N}$, and hence n failures are observed. But if $X_{J:n:N} < T < X_{J+1:n:N}$, the test stops at time T, and hence J failures are observed. Similar to the hybrid censoring scheme, the effective sample size in progressive hybrid censoring is random, and hence it may be very small. To tackle this concern, Ng et al. (2009) proposed the adaptive type-II progressive censoring scheme. Under this scheme, if $X_{n:n:N} < T$, the test concludes at time $X_{n:n:N}$. Otherwise, if $X_{J:n:N} < T < X_{J+1:n:N}$, we take the following approach: $R_{J+1} = \cdots = R_{n-1} = 0$ and $R_n = N - n - \sum_{i=1}^{J} R_i$. This formulation enables us to stop the test as soon as possible if the $(J+1)$-th failure time exceeds T for $J+1 < n$. Figures 1 and 2 provide visual representations of these scenarios. Recently, Kohansal and Haji (2023) discussed the estimation of the parameters of modified Weibull distribution on the basis of the adaptive progressively type-II censored data.

In recent years, various authors have focused on GOF tests applied to progressively censored samples. For instance, Michael and Schucany (1979) employed a transformation method to convert progressively type-II censored data into a reduced set of uncensored data. They subsequently conducted a standard GOF test for uniformity on the transformed data. Balakrishnan et al. (2002) proposed a GOF test specifically designed for the exponential model when the available sample is progressively type-

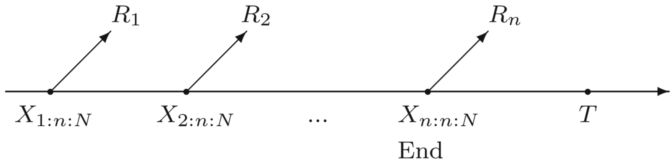

Figure 1: The test is terminated before time T ($X_{n:n:N} < T$).

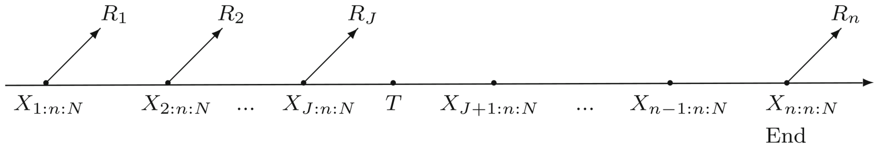

Figure 2: The test is terminated after time T ($X_{n:n:N} \geq T$).

II censored. Balakrishnan and Lin (2003) derived the exact null distribution of the test statistic proposed by Balakrishnan et al. (2002). Furthermore, Balakrishnan et al. (2004) extended their method to encompass location-scale families of distributions. More recently, Wang (2008) proposed another GOF test for the exponential distribution based on spacings derived from progressively type-II censored data. Interested readers can also refer to Zhu (2021), Doostparast (2015), Zhao et al. (2024), Mirjalili and Nadeb (2020), Qin et al. (2022) for further developments in this area.

Throughout this article, we assume that the distribution function (DF) of lifetimes satisfies the proportional hazard rate model. Let X and Y be two random variables with hazard rate functions h_F and h_G and DFs F and G, respectively. Then, according to Cox (1972), X and Y satisfy the proportional hazard rate model with the proportionality constant $\theta > 0$, if $h_G(x) = \theta\, h_F(x)$ for all x. Alternatively, this can be expressed as

$$\bar{F}(x) = \left(\bar{G}(x)\right)^{\theta}, \tag{1}$$

for all x, where $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$ are the survival functions of X and Y, respectively. This model includes several well-known lifetime distributions such as exponential, Rayleigh, Pareto, Weibull, and others. It is also a subclass of the one-parameter exponential family of distributions. Furthermore, this model is flexible enough to accommodate both monotonic and non-monotonic failure rates, even though the baseline failure rate is monotonic. For further details on proportional hazard rate models, readers may refer to Marshall and Olkin (2007). In recent years, numerous authors have explored proportional hazard rate models in the context of progressively type-II censored data. For instance, Meshkat and Dehqani (2020) proposed various predictors for the failure times of censored units in progressively censored data derived from proportional hazard rate models. Basirat (2013) examined the estimation of the stress-strength reliability parameter under progressive type-II censoring. Additionally, Basiri and Asgharzadeh (2021) investigated the determination of the optimal sample size in progressively type-II censoring, accounting for the associated experimental costs within proportional hazard rate models. For further related studies, see Chaturvedi et al. (2019), Fallah (2022), Asgharzadeh and Valiollahi (2009).

In this article, we present GOF tests for the proportional hazard rate model when the available sample is adaptive progressively type-II censored. Therefore, the remainder of the article is structured as follows. In Section 2, we outline the proposed GOF tests and derive the exact null distribution for the test statistics. Under the adaptive progressively type-II censored data coming from a proportional hazard rate model, the statistical inferences about the proportionality parameter θ are investigated in Section 3. The power of the proposed tests is then evaluated through Monte Carlo simulations in Section 4, considering null distributions such as exponential, Rayleigh, and Pareto. Section 5 is dedicated to exploring the applicability of the proposed GOF tests on various real datasets.

2 Proposed tests

Let $\mathcal{F}$ be the class of all DFs of the form $F_\theta(x) = 1 - \left(\bar{G}(x)\right)^\theta$ for all x and $\theta > 0$, where the DF G is completely known, but the proportionality constant θ is unknown. Hence

$$\mathcal{F} \equiv \mathcal{F}(G) = \left\{ F_\theta(x): \; F_\theta(x) = 1 - \left(\bar{G}(x)\right)^\theta, \; \theta > 0 \right\}. \tag{2}$$

Further, let $X_1, \ldots, X_N$ be an independent and identically distributed sample from a distribution F and $\boldsymbol{X} = (X_{1:n:N}, \ldots, X_{n:n:N})$ be the corresponding adaptive progressively type-II censored sample with progressive censoring scheme $\boldsymbol{R} = (R_1, \ldots, R_n)$ such that $X_{J:n:N} < T < X_{J+1:n:N}$. We would like to test the composite null hypothesis

$$H_0 : F \in \mathcal{F}(G), \tag{3}$$

with specified G and $\mathcal{F}$ is as in Equation (2).

Under the null hypothesis H_0 and transforming $W_{i:n:N} = -\ln\left(\bar{G}(X_{i:n:N})\right)$ for $i = 1, \ldots, n$, it can be shown that $\boldsymbol{W} = (W_{1:n:N}, \ldots, W_{n:n:N})$ is an adaptive progressively type-II censored sample coming from the exponential distribution with parameter θ. Now, we define the adaptive progressively type-II censored spacings as follows:

$$\begin{aligned}
Z_1 &= N W_{1:n:N}, \\
Z_2 &= (N - R_1 - 1)\left(W_{2:n:N} - W_{1:n:N}\right), \\
&\;\vdots \\
Z_J &= \left(N - \sum_{j=1}^{J-1} R_j - J + 1\right)\left(W_{J:n:N} - W_{J-1:n:N}\right), \\
Z_{J+1} &= \left(N - \sum_{j=1}^{J} R_j - J\right)\left(W_{J+1:n:N} - W_{J:n:N}\right), \\
&\;\vdots \\
Z_n &= \left(N - \sum_{j=1}^{J} R_j - n + 1\right)\left(W_{n:n:N} - W_{n-1:n:N}\right).
\end{aligned} \tag{4}$$

Following Theorem 6 of Cramer and Iliopoulos (2010), the spacings $Z_1, \ldots, Z_n$ in (4) are independent and identically distributed as exponential with parameter θ. Let us define

$$V_i = \frac{Z_1 + \cdots + Z_i}{Z_1 + \cdots + Z_n}, \quad i = 1, \ldots, n-1. \tag{5}$$

Based on the random sample $Z_1, \ldots, Z_n$ from the exponential distribution, Balakrishnan et al. (2002) proved that the joint distribution of $V_1, \ldots, V_{n-1}$ is equivalent to the joint distribution of the $n-1$ order statistics $U_{(1)}, \ldots, U_{(n-1)}$ generated from a random sample of size $n-1$ from the standard uniform distribution, denoted by $U_1, \ldots, U_{n-1}$. Subsequently, the distribution of test statistic

$$Y = \frac{1}{n-1} \sum_{i=1}^{n-1} \frac{Z_1 + \cdots + Z_i}{Z_1 + \cdots + Z_n} = \frac{1}{n-1} \sum_{i=1}^{n-1} V_i \stackrel{d}{=} \frac{1}{n-1} \sum_{i=1}^{n-1} U_{(i)} \stackrel{d}{=} \frac{1}{n-1} \sum_{i=1}^{n-1} U_i,$$

is similar to the distribution of the mean of $n-1$ independent and identically distributed random variables from standard uniform distribution. Furthermore, the test statistic Y can be expressed as

$$Y = \frac{\sum_{i=1}^{n-1} (n-i) Z_i}{(n-1) \sum_{i=1}^{n} Z_i}. \tag{6}$$

Thus, the numerator of Y is a linear combination of the spacings with decreasing weights, while the denominator is the sum of the spacings. When the distribution of spacings $Z_1, \ldots, Z_n$ deviates from being exponential, the spacings can either be larger or smaller than the expected values, causing the numerator of Y to become excessively large or small. Therefore, small and large values of Y lead us to the rejection of the null hypothesis H_0 in (3). To determine the critical value t_α corresponding to a significance level α under the null distribution, Balakrishnan and Lin (2003) derived the exact value of t_α such that $P(Y < t_\alpha) = \alpha$ for $3 \le n \le 20$. For $n > 20$, t_α can be approximated using the normal approximation method. Also, the power of the proposed test is

$$P\left(Y < t_{\alpha/2} \middle| H_1\right) + P\left(Y > t_{1-\alpha/2} \middle| H_1\right).$$

Using V_i's in (5), we can define another test statistic as follows

$$S = -2 \sum_{i=1}^{n-1} \log V_i = -2 \sum_{i=1}^{n-1} \log\left(\frac{Z_1 + \cdots + Z_i}{Z_1 + \cdots + Z_n}\right). \tag{7}$$

Once again, small and large values of S lead us to the rejection of the null hypothesis H_0. On the basis of the joint distribution of $V_1, \ldots, V_{n-1}$, we have

$$S = 2 \sum_{i=1}^{n-1} \left(-\log V_i\right) \stackrel{d}{=} 2 \sum_{i=1}^{n-1} \left(-\log U_{(i)}\right) \stackrel{d}{=} 2 \sum_{i=1}^{n-1} \left(-\log U_i\right).$$

Hence, the distribution of the test statistic S under the null hypothesis is chi-square with $2(n-1)$ degrees of freedom. The power function of the proposed test is

$$P\left(S < \chi^2_{\alpha/2, 2(n-1)} \middle| H_1\right) + P\left(S > \chi^2_{1-\alpha/2, 2(n-1)} \middle| H_1\right),$$

where $\chi^2_{\gamma,m}$ stands for the γ-th lower quantile of the chi-square distribution with m degrees of freedom.

In reliability and life-testing applications, the exponential, Rayleigh, and Pareto distributions are extensively used for modeling lifetime data; see, e.g., Lawless (2003). Therefore, in this article, we will specifically consider these three distributions. However, it is important to note that the tests discussed here are applicable to any distribution that belongs to the proportional hazard rate model.
1. Exponential distribution with DF $F(x) = 1 - \exp\{-\theta x\}, \ x > 0$.
2. Rayleigh distribution with DF $F(x) = 1 - \exp\{-\theta x^2\}, \ x > 0$.
3. Pareto distribution with DF $F(x) = 1 - x^{-\theta}, \ x > 1$.
Also, it should be noted that the test statistics Y and S in Equations (6) and (7), respectively, are inherently scale-invariant under the Exponential and Rayleigh distributions. However, to ensure scale invariance of Y and S under the Pareto distribution, these test statistics can be modified as

$$
\begin{aligned}
Y_\star &= \frac{1}{n-2} \sum_{i=2}^{n-1} \left(\frac{Z_2 + \cdots + Z_i}{Z_2 + \cdots + Z_n} \right), \\
S_\star &= -2 \sum_{i=2}^{n-1} \log \left(\frac{Z_2 + \cdots + Z_i}{Z_2 + \cdots + Z_n} \right),
\end{aligned}
$$

respectively. Under the null hypothesis, it is evident that the distribution of $Y_\star$ is similar to the distribution of the mean of $n-2$ independent and identically distributed random variables from standard uniform distribution. Also, the distribution of $S_\star$ under the null hypothesis follows a chi-square distribution with $2(n-2)$ degrees of freedom. Small and large values of $Y_\star$ and $S_\star$ indicate that the null hypothesis should be rejected.

3 Statistical inference

After the null hypothesis (3) is accepted, it is necessary to discuss the statistical inference about the unknown parameter θ. In order to achieve this, we develop optimal statistical methods for point and interval estimation, as well as hypothesis testing for the parameter θ. These methods are based on adaptive progressively type-II censored data from a proportional hazard rate model as defined in Equation (1).

3.1 Point estimations

Let $\boldsymbol{x} = (x_{1:n:N}, \ldots, x_{n:n:N})$ be the observed adaptive progressively type-II censored data drawn from the DF $F_\theta(\cdot)$ and the probability density function (PDF) $f_\theta(\cdot)$ and assuming $J = j$, the likelihood function (LF) of the parameter θ is

$$
L(\theta) = C \prod_{i=1}^{n} f_\theta(x_{i:n:N}) \prod_{i=1}^{j} \left[\bar{F}_\theta(x_{i:n:N})\right]^{R_i} \left[\bar{F}_\theta(x_{n:n:N})\right]^{N-n-\sum_{k=1}^{j} R_k}, \tag{8}
$$

where

$$C = N \times \begin{cases} \prod_{i=2}^{n} (N-i+1), & \text{if } j=0, \\ \prod_{i=2}^{n} \left(N-i+1-\sum_{k=1}^{\min\{i-1,j\}} R_k\right), & \text{if } j=1,2,\ldots,n. \end{cases}$$

Suppose that $F_\theta(x) = 1 - \left(\bar{G}(x)\right)^\theta$ and then with the PDF $f_\theta(x) = \theta\, g(x) \left(\bar{G}(x)\right)^{\theta-1}$, where $g(x)$ is the PDF of the DF $G(x)$, the LF of θ in (8) is reduced to

$$\begin{aligned} L(\theta) &= C\theta^n \prod_{i=1}^{n} \frac{g(x_{i:n:N})}{\bar{G}(x_{i:n:N})} \\ &\quad \times \left\{ \prod_{i=1}^{n} \bar{G}(x_{i:n:N}) \prod_{i=1}^{j} \left[\bar{G}(x_{i:n:N})\right]^{R_i} \left[\bar{G}(x_{n:n:N})\right]^{N-n-\sum_{k=1}^{j} R_k} \right\}^\theta . \end{aligned} \tag{9}$$

Hence, the logarithm of the LF is

$$\begin{aligned} \log L(\theta) &= \log C + n \log\theta + \sum_{i=1}^{n} \left(\log g(x_{i:n:N}) - \log \bar{G}(x_{i:n:N})\right) \\ &\quad + \theta \Bigg(\sum_{i=1}^{n} \log \bar{G}(x_{i:n:N}) \\ &\quad + \sum_{i=1}^{j} R_i \log \bar{G}(x_{i:n:N}) + \left(N-n-\sum_{k=1}^{j} R_k\right) \log \bar{G}(x_{n:n:N}) \Bigg). \end{aligned}$$

From $\partial \log L(\theta)/\partial\theta = 0$, we conclude that the maximum likelihood estimator for θ is

$$\hat{\theta}_{\mathrm{ML}} = \frac{n}{T_\star}, \tag{10}$$

where $T_\star = \sum_{i=1}^{n} W_{i:n:N} + \sum_{i=1}^{J} R_i W_{i:n:N} + \left(N-n-\sum_{k=1}^{J} R_k\right) W_{n:n:N}$ with $W_{i:n:N} = -\ln\left(\bar{G}(X_{i:n:N})\right)$. Notice that the PDF of $\boldsymbol{X} = (X_{1:n:N}, \ldots, X_{n:n:N})$ in (9) belongs to the exponential family of distributions. So, $T_\star$ is a complete and sufficient statistic for θ. Also, from (4), we can see that $T_\star = \sum_{i=1}^{n} Z_i$. Moreover, since $Z_1, \ldots, Z_n$ are independent and identically distributed as exponential with parameter θ, then the distribution of $T_\star$ is gamma with shape and scale parameters n and θ, respectively; that is the corresponding PDF is $f(x) = \theta^n x^{n-1} \exp\{-\theta x\}/\Gamma(n)$, where $\Gamma(n) = \int_0^\infty x^{n-1}\exp\{-\theta x\}\,\mathrm{d}x$ is the complete gamma function. Thus, we have $\mathrm{E}\big(\hat{\theta}_{\mathrm{ML}}\big) = n\theta/(n-1)$ and then the bias of $\hat{\theta}_{\mathrm{ML}}$ for estimating θ is $\mathrm{B}\big(\hat{\theta}_{\mathrm{ML}}\big) := \mathrm{E}\big(\hat{\theta}_{\mathrm{ML}}\big) - \theta = \theta/(n-1)$. Furthermore, by Corollary 1.12 on page 88 of Lehmann and Casella (1998), the uniformly minimum variance unbiased (UMVU) estimator for θ, when $n > 1$, is

$$\hat{\theta}_{\mathrm{UMVU}} = \frac{n-1}{T_\star}. \tag{11}$$

Using the distribution of $T_\star$ and after some algebraic computations, the mean squared errors (MSEs) of $\hat{\theta}_{\mathrm{ML}}$ and $\hat{\theta}_{\mathrm{UMVU}}$ are derived as

$$\mathrm{MSE}(\hat{\theta}_{\mathrm{ML}}) := \mathrm{E}(\hat{\theta}_{\mathrm{ML}} - \theta)^2 = \frac{(n+2)\theta^2}{(n-1)(n-2)},$$

$$\mathrm{MSE}(\hat{\theta}_{\mathrm{UMVU}}) = \frac{\theta^2}{n-2},$$

respectively. Thus, assuming $n > 1$, the efficiency (EFF) of $\hat{\theta}_{\mathrm{ML}}$ with respect to $\hat{\theta}_{\mathrm{UMVU}}$ is given by

$$\mathrm{EFF}(\hat{\theta}_{\mathrm{ML}}, \hat{\theta}_{\mathrm{UMVU}}) := \frac{\mathrm{MSE}(\hat{\theta}_{\mathrm{UMVU}})}{\mathrm{MSE}(\hat{\theta}_{\mathrm{ML}})} = 1 - \frac{3}{n+2} < 1.$$

Hence, for $n > 1$, $\hat{\theta}_{\mathrm{UMVU}}$ dominates $\hat{\theta}_{\mathrm{ML}}$ on the basis of the MSE criterion.

3.2 Interval estimations

From the distribution of $T_\star$, we conclude that $2\theta T_\star$ is distributed as chi-square distribution with $2n$ degrees of freedom. So, an equi-tailed confidence interval for θ at the confidence level $1-\alpha$ is

$$\left(\frac{\chi^2_{\alpha/2,2n}}{2T_\star}, \frac{\chi^2_{1-\alpha/2,2n}}{2T_\star} \right).$$

Now, we restrict ourselves to a class of intervals for θ of the form $(a/2T_\star, b/2T_\star)$ and find a member of this class with the shortest width subject to $\Pr(a < 2\theta T_\star < b) = 1-\alpha$. Using Lagrange method and after some algebraic manipulations, the optimal values for a and b are determined by numerically solving the following equations

$$\int_a^b u_{2n}(x)\,\mathrm{d}x = 1-\alpha, \quad \text{and} \quad a^{n-1}\exp\left(-\frac{a}{2}\right) = b^{n-1}\exp\left(-\frac{b}{2}\right),$$

where $u_m(x)$ is the PDF of the chi-square distribution with m degrees of freedom.

3.3 Tests of hypotheses

First, we discuss the problem of testing the null hypothesis $H_0 : \theta \le \theta_0$ against the alternative hypothesis $H_1 : \theta > \theta_0$. Since the PDF of $\boldsymbol{X} = (X_{1:n:N}, \ldots, X_{n:n:N})$ in (9) possesses the monotone likelihood ratio property in statistic $-T_\star$, then from Chapter 3 on page 61 of Lehmann and Romano (2022) the uniformly most powerful (UMP) test of size α is given by

$$\psi_{\mathrm{UMP}}(\boldsymbol{X}) = \begin{cases} 1, & \text{if } T_\star \le c, \\ 0, & \text{if } T_\star > c, \end{cases} \tag{12}$$

where for the determination of the constant c, we must solve $\mathrm{E}_{\theta_0}\big(\psi_{\mathrm{UMP}}(\boldsymbol{X})\big) = \alpha$. Since $2\theta T_\star \sim \chi^2_{2n}$, therefore we have $c = \chi^2_{\alpha,2n}/2\theta_0$. Similarly, for testing $H_0 : \theta \ge \theta_0$ versus $H_1 : \theta < \theta_0$, the UMP test of size α is given by

$$\psi_{\mathrm{UMP}}(\boldsymbol{X}) = \begin{cases} 1, & \text{if } T_\star \ge c, \\ 0, & \text{if } T_\star < c, \end{cases} \tag{13}$$

where $c = \chi^2_{1-\alpha,2n}/2\theta_0$. Comparing the UMP tests in (12) and (13), it is concluded that a UMP test for testing $H_0 : \theta = \theta_0$ against $\theta \ne \theta_0$ does not exist. Hence, for this case, we propose a UMP unbiased (UMPU) test. Using Chapter 4 on page 126 of

Lehmann and Romano (2022), the UMPU test of size α for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$ is of the form

$$\psi_{\text{UMPU}}\left(\boldsymbol{X}\right) = \begin{cases} 1, & \text{if } T_\star \leq c_1 \text{ or } \geq c_2, \\ 0, & \text{if } c_1 < T_\star < c_2, \end{cases} \tag{14}$$

where the constants c_1 and c_2 are now specified by solving $\mathrm{E}_{\theta_0}\big(\psi_{\text{UMPU}}\left(\boldsymbol{X}\right)\big) = \alpha$ and $\mathrm{E}_{\theta_0}\big(T_\star\, \psi_{\text{UMPU}}\left(\boldsymbol{X}\right)\big) = \alpha\, \mathrm{E}_{\theta_0}\left(T_\star\right)$. Since $2\theta T_\star \sim \chi^2_{2n}$, these equations are reduced to

$$\int_{C_1}^{C_2} u_{2n}(x)\,\mathrm{d}x = 1 - \alpha, \quad \text{and} \quad \int_{C_1}^{C_2} x\, u_{2n}(x)\,\mathrm{d}x = 2n(1-\alpha), \tag{15}$$

with $C_i = 2\theta_0 c_i$ for $i = 1, 2$. It is easy to show that $x\, u_{2n}(x) = 2n\, u_{2(n+1)}(x)$ for any $x > 0$ and $n \in \mathbb{N}$. Then, the second equation in (15) can be rewritten as

$$\int_{C_1}^{C_2} u_{2(n+1)}(x)\,\mathrm{d}x = 1 - \alpha.$$

Integrating by parts and using the first equation in (15), the above equation is reduced to

$$C_1^n \exp\left(-\frac{C_1}{2}\right) = C_2^n \exp\left(-\frac{C_2}{2}\right).$$

For obtaining the generalized likelihood ratio (GLR) of size α for $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$, it is necessary to derive the likelihood ratio statistic. From (9) and (10), this statistic is obtained as follows

$$\Lambda := \frac{L(\theta_0)}{L\big(\hat{\theta}_{\text{ML}}\big)} = \frac{\theta_0^n \exp\left(-\theta_0 T_\star\right)}{\hat{\theta}_{\text{ML}}^n \exp\big(-\hat{\theta}_{\text{ML}} T_\star\big)} = \left(\frac{\theta_0 T_\star}{n}\right)^n \exp\left(n - \theta_0 T_\star\right).$$

From $\Lambda \leq k$, we conclude that $T_\star^n \exp\left(-\theta_0 T_\star\right) \leq K$. Thus, the GLR test for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ is as

$$\psi_{\text{GLR}}\left(\boldsymbol{X}\right) = \begin{cases} 1, & \text{if } T_\star^n \exp\left(-\theta_0 T_\star\right) \leq K, \\ 0, & \text{if } T_\star^n \exp\left(-\theta_0 T_\star\right) > K. \end{cases} \tag{16}$$

On the basis of the distribution of $2\theta T_\star$ which is the chi-square distribution with $2n$ degrees of freedom, one can determine the constant K by solving the equation $\mathrm{E}_{\theta_0}\big(\psi_{\text{GLR}}\left(\boldsymbol{X}\right)\big) = \alpha$.

Remark 3.1. We observed that a UMP test does not exist for testing the null hypothesis $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$. This leads us to the non-existence of a uniformly most accurate (UMA) confidence interval for θ. But, using the acceptance region of the UMPU test in (14), we can show that the $100(1-\alpha)\%$ UMA unbiased (UMAU) confidence interval for θ is $(C_1/2T_\star, C_2/2T_\star)$ with

$$\int_{C_1}^{C_2} u_{2n}(x)\,\mathrm{d}x = 1 - \alpha, \quad \text{and} \quad C_1^n \exp\left(-\frac{C_1}{2}\right) = C_2^n \exp\left(-\frac{C_2}{2}\right).$$

Also, since the acceptance region of the UMP test of size α in testing $H_0 : \theta = \theta_0$ against $H_1 : \theta \geq \theta_0$ is $T_\star > \chi^2_{\alpha,2n}/2\theta_0$, then $\chi^2_{\alpha,2n}/2T_\star$ is the $100(1-\alpha)\%$ UMA lower bound for θ. Similarly, from the acceptance region of the UMP test of size α in testing $H_0 : \theta = \theta_0$ against $H_1 : \theta \leq \theta_0$, we conclude that the $100(1-\alpha)\%$ UMA upper bound for θ is $\chi^2_{1-\alpha,2n}/2T_\star$.

4 Simulation study

In this section, we evaluate the power of the proposed tests for three different null distributions: exponential, Rayleigh, and Pareto. We examine various alternative distributions with distinct hazard functions.

For the exponential null distribution, according to Noughabi (2017), the following alternative distributions are used

- Monotone decreasing hazard rate: Weibull(0.5,1) and Gamma(0.5,1),
- Monotone increasing hazard rate: Weibull(2,1) and Gamma(2,1),
- Non-Monotone hazard rate: Log-normal(0,0.5) and Log-normal(0,1.5).

Also, according to Jahanshahi et al. (2016), the following alternative distributions for the Rayleigh null distribution are considered

- Monotone decreasing hazard rate: Chi-square(1) and Weibull(0.5,1),
- Monotone increasing hazard rate: Chi-square(3) and Beta(3,1),
- Non-Monotone hazard rate: Beta(1,0.5) and Exponential(2).

Additionally, according to Saldaña-Zepeda et al. (2010), the following alternative distributions for the Pareto null distribution are investigated

- Monotone decreasing hazard rate: Weibull(0.5,1) and Gamma(0.8,1),
- Monotone increasing hazard rate: Weibull(3,1) and Gamma(3,1),
- Non-Monotone hazard rate: Log-normal(1,1) and Log-normal(5,3).

The probability density functions of these alternative distributions are presented in the table below.

alternative distribution	Probability density function
Weibull(W(α,β))	$f(x) = \frac{\alpha}{\beta^\alpha} x^{\alpha-1} \exp\{-(x/\beta)^\alpha\}$, $x > 0$, $\alpha > 0$, $\beta > 0$
Gamma(G(α,β))	$f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} \exp\{-x/\beta\}$, $x > 0$, $\alpha > 0$, $\beta > 0$
Log-normal (LN(μ,σ))	$f(x) = \frac{1}{\sqrt{2\pi}\sigma x} \exp\left\{-\left(\log(x) - \mu\right)^2 / 2\sigma^2\right\}$, $x > 0$, $\mu \in \mathbb{R}$, $\sigma > 0$
Chi-square ($\chi^2(n)$)	$f(x) = \frac{1}{2^{n/2}\Gamma(n/2)} x^{n/2-1} \exp\{-x/2\}$, $x > 0$, $n \in \mathbb{N}$
Beta (Be(α,β))	$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$, $0 < x < 1$, $\alpha > 0$, $\beta > 0$
Exponential (Exp(λ))	$f(x) = \lambda \exp\{-\lambda x\}$, $x > 0$, $\lambda > 0$

We will also compare the performance of the proposed tests against alternative tests in terms of power. Under the exponential null hypothesis, Döring and Cramer (2019) suggested a test statistic based on sample spacings and progressive type-II censored data, defined as follows

$$\delta = \sum_{i=1}^{n-1} \left| \hat{F}^\star_{n-1}(V_i) - V_i \right| + \sum_{i=1}^{n-1} \left| \lim_{t \to V_i} \hat{F}^\star_{n-1}(t) - V_i \right| - \frac{n-1}{n}, \tag{17}$$

where $\hat{F}^\star_{n-1}(t) = \frac{n-1}{n} \hat{F}_{n-1}(t)$ with $\hat{F}_{n-1}(t) = \frac{1}{n-1} \sum_{i=1}^{n-1} 1_{[0,t]}(V_j)$ as the empirical DF of V_j's, and $1_{[0,t]}(x)$ represents the indicator function. The test statistic δ provides evidence against the null distribution when it is sufficiently large. Under the Rayleigh null hypothesis, Ren and Gui (2021) proposed a test statistic based on cumulative entropy and progressive type-II censored data, given by

$$\eta = \frac{\sum_{i=0}^{n-1}(1-\tau_i)\left(\log(1-\tau_i)Q_i^{(1)} + \frac{\hat{\theta}}{3}Q_i^{(3)}\right) + \sqrt{\frac{\pi}{\hat{\theta}}}\left(\Phi\left(\sqrt{2\hat{\theta}}X_{n:n:N}\right) - \frac{1}{2}\right)}{\sum_{i=0}^{n-1}(1-\tau_i)Q_i^{(1)}} - 1, \quad (18)$$

where $\hat{\theta} = n\left(\sum_{i=1}^{n}(R_i+1)X_{i:n:N}^2\right)^{-1}$ is the maximum likelihood estimator for the parameter of the Rayleigh distribution, $\Phi(\cdot)$ is the DF of the standard normal distribution, and $Q_i^{(j)} = X_{i+1:n:N}^j - X_{i:n:N}^j$ for $j = 1, 3$. Additionally, we have $\tau_0 = 0$, $X_{0:n:N} = 0$, and

$$\tau_i = 1 - \prod_{j=n-i+1}^{n} \frac{j + R_{n-j+1} + \cdots + R_n}{j + 1 + R_{n-j+1} + \cdots + R_n}, \quad i = 1, \ldots, n-1.$$

The test statistic η provides evidence for rejecting the null distribution if its value is too large. Furthermore, under the Pareto null hypothesis, Zhang and Gui (2020) suggested a test statistic based on the cumulative hazard function and progressive type-II censored data, expressed as

$$\xi = \frac{\sum_{i=1}^{n}(S_i - \bar{S})(\hat{H}(S_i) - \bar{\hat{H}})}{\sqrt{\sum_{i=1}^{n}(S_i - \bar{S})^2}\sqrt{\sum_{i=1}^{n}(\hat{H}(S_i) - \bar{\hat{H}})^2}}, \quad (19)$$

where $\bar{S} = \frac{1}{n}\sum_{i=1}^{n} S_i$ with $S_i = \log(X_{i:n:N})$ for $i = 1, \ldots, n$, and $\bar{\hat{H}} = \frac{1}{n}\sum_{i=1}^{n}\hat{H}(S_i)$ with

$$\hat{H}(S_i) = \sum_{k=1}^{i} \frac{1}{N - \sum_{j=1}^{k-1} R_j - k + 1}, \quad i = 1, \ldots, n.$$

The estimator $\hat{H}$ is called the Nelson-Aalen estimator of the cumulative hazard function on the basis of the progressively type-II censored data. The test statistic ξ supports the rejection of the null hypothesis only for small values.

The test statistics δ, η, and ξ need to be adapted for adaptive progressively type-II censored data. To achieve this, we define the adjusted censoring scheme as follows

$$R_i' = \begin{cases} R_i\, 1_{(0,T)}(X_{i:n:N}), & \text{if } i = 1, \ldots, n-1, \\ N - n - \sum_{j=1}^{n-1} R_j', & \text{if } i = n, \end{cases}$$

where T is the threshold time for adaptive progressive censoring. To compute the power of the tests δ, η, and ξ under adaptive progressively type-II censored samples, it is essential to utilize the adjusted censoring scheme $\boldsymbol{R}' = (R_1', \ldots, R_n')$ instead of the original scheme $\boldsymbol{R} = (R_1, \ldots, R_n)$ in (17), (18), and (19).

Table 1 presents some choices for adaptive progressive censoring schemes, used in the simulation study. Also, the significance level is $\alpha = 0.05$. For each censoring scheme, we generated 10,000 sets of data to estimate the power of the proposed tests. The results of these simulations are recorded in Tables 2-4. Empirical evidences from

Tables 2-4 show the following findings:
1. The power of the proposed tests has an increasing pattern with respect to N and n under any null and alternative distributions. In other words, as $1 - n/N$ (degree of censoring) decreases, the power of the tests increases.
2. In general, we observe that the tests proposed perform very well for all null and alternative distributions considered, except for a few cases.
3. The power of tests Y and S is influenced by the censoring scheme $R_1, \ldots, R_n$, meaning that changing the censoring scheme while keeping other factors constant will impact the power of the tests. Therefore, the researcher should be careful in selecting the censoring scheme.
4. While it may seem that the effect of the threshold time T on the power of the tests is not very noticeable, the researcher should still be careful in selecting T. This careful selection not only helps increase the power of the tests but also reduces the duration of the experiment.
5. In comparing the power of the two tests Y and S, it is observed that there is no absolute superiority of either of these tests, and it depends on the null and alternative distributions. According to Table 2, when the null distribution is exponential, the power of test S is greater than Y for all alternative distributions except the distribution of LN(0,1.5). However, as seen in Table 3, based on the Rayleigh null distribution, when the alternative distribution is Be(3,1), the power of test Y is greater than S, and the opposite is true for other alternative distributions. Finally, based on Table 4 and considering the Pareto null distribution, it appears that when the alternative distribution is W(0.5,1), test S has superiority over test Y, while in the case of the alternative LN(5,3) distribution, there is no absolute superiority of either of Y and S tests, and for other alternative distributions, the superiority lies with test Y.
6. The power of tests δ, η, and ξ is influenced by both the censoring scheme $R_1, \ldots, R_n$ and the threshold time T. However, test η exhibits a significantly stronger dependence on these factors under the alternative Be(3,1) and Be(1,0.5) distributions.
7. Table 2 shows that the δ test exhibits superior performance compared to the Y and S tests exclusively when the alternative distribution is W(2,1). For other alternative distributions, the δ test generally demonstrates lower power than the Y and S tests.
8. Table 3 demonstrates that, across all alternative distributions, either the Y or the S test exhibits higher power than the η test. Furthermore, Tables 4 reveals that both the Y and S tests outperform the ξ test.

Table 1: Different choices for adaptive type-II progressive censoring scheme.

N	n	$R_1, \ldots, R_n$	T	No.
20	8	$R_1 = 12,\ R_i = 0$ for $i \neq 1$	0.25	1
			2	2
			10	3
		$R_8 = 12,\ R_i = 0$ for $i \neq 8$	0.25	4
			2	5
			10	6
		$R_1 = \cdots = R_4 = 1,\ R_5 = \cdots = R_8 = 2$	0.25	7
			2	8
			10	9
	12	$R_1 = 8,\ R_i = 0$ for $i \neq 1$	0.25	10
			2	11
			10	12
		$R_{12} = 8,\ R_i = 0$ for $i \neq 12$	0.25	13
			2	14
			10	15
		$R_1 = R_2 = 0,\ R_3 = \cdots = R_{10} = 1,\ R_{11} = R_{12} = 0$	0.25	16
			2	17
			10	18
	16	$R_1 = 4,\ R_i = 0$ for $i \neq 1$	0.25	19
			2	20
			10	21
		$R_{16} = 4,\ R_i = 0$ for $i \neq 16$	0.25	22
			2	23
			10	24
		$R_1 = \cdots = R_6 = 0,\ R_7 = \cdots = R_{10} = 1,\ R_{11} = \cdots = R_{16} = 0$	0.25	25
			2	26
			10	27
40	10	$R_1 = 30,\ R_i = 0$ for $i \neq 1$	0.25	28
			2	29
			10	30
		$R_{10} = 30,\ R_i = 0$ for $i \neq 10$	0.25	31
			2	32
			10	33
		$R_1 = \cdots = R_{10} = 3$	0.25	34
			2	35
			10	36
	20	$R_1 = 20,\ R_i = 0$ for $i \neq 1$	0.25	37
			2	38
			10	39
		$R_{20} = 20,\ R_i = 0$ for $i \neq 20$	0.25	40
			2	41
			10	42
		$R_1 = \cdots = R_{20} = 1$	0.25	43
			2	44
			10	45
	30	$R_1 = 10,\ R_i = 0$ for $i \neq 1$	0.25	46
			2	47
			10	48
		$R_{30} = 10,\ R_i = 0$ for $i \neq 30$	0.25	49
			2	50
			10	51
		$R_1 = \cdots = R_{10} = 0,\ R_{11} = \cdots = R_{20} = 1,\ R_{21} = \cdots = R_{30} = 0$	0.25	52
			2	53
			10	54

Table 2: Power estimates under the exponential distribution and $\alpha = 0.05$.

No.	W(0.5,1)			W(2,1)			LN(0,0.5)			LN(0,1.5)			G(0.5,1)			G(2,1)		
	Y	S	δ	Y	S	δ	Y	S	δ	Y	S	δ	Y	S	δ	Y	S	δ
1	0.616	0.810	0.336	0.622	0.652	0.763	0.836	0.899	0.925	0.298	0.186	0.140	0.336	0.567	0.107	0.215	0.230	0.333
2	0.622	0.806	0.333	0.696	0.705	0.813	0.933	0.965	0.972	0.300	0.184	0.140	0.330	0.556	0.106	0.3108	0.344	0.441
3	0.627	0.810	0.333	0.705	0.711	0.815	0.937	0.966	0.972	0.294	0.185	0.139	0.336	0.563	0.110	0.319	0.349	0.446
4	0.412	0.569	0.150	0.351	0.370	0.491	0.820	0.874	0.910	0.044	0.029	0.046	0.302	0.473	0.094	0.192	0.210	0.298
5	0.404	0.561	0.152	0.354	0.367	0.486	0.830	0.889	0.919	0.048	0.028	0.045	0.312	0.471	0.099	0.197	0.212	0.303
6	0.410	0.573	0.146	0.357	0.372	0.487	0.826	0.883	0.914	0.050	0.029	0.047	0.307	0.463	0.093	0.192	0.208	0.298
7	0.505	0.646	0.236	0.366	0.383	0.507	0.831	0.888	0.918	0.059	0.033	0.046	0.368	0.517	0.135	0.197	0.212	0.309
8	0.505	0.646	0.236	0.366	0.383	0.507	0.831	0.888	0.918	0.059	0.033	0.046	0.368	0.517	0.135	0.197	0.212	0.309
9	0.525	0.656	0.260	0.433	0.446	0.584	0.884	0.927	0.951	0.091	0.046	0.055	0.368	0.516	0.139	0.224	0.237	0.339
10	0.755	0.885	0.576	0.784	0.819	0.878	0.941	0.982	0.975	0.441	0.308	0.295	0.411	0.638	0.214	0.303	0.351	0.421
11	0.754	0.891	0.566	0.820	0.825	0.891	0.958	0.986	0.981	0.443	0.297	0.282	0.414	0.643	0.212	0.372	0.430	0.489
12	0.754	0.891	0.566	0.820	0.825	0.891	0.958	0.986	0.981	0.443	0.297	0.282	0.414	0.643	0.212	0.372	0.430	0.489
13	0.763	0.892	0.585	0.827	0.833	0.895	0.954	0.986	0.979	0.440	0.300	0.288	0.422	0.646	0.216	0.379	0.438	0.498
14	0.591	0.754	0.365	0.591	0.628	0.706	0.938	0.979	0.972	0.093	0.036	0.041	0.397	0.590	0.203	0.280	0.325	0.390
15	0.594	0.756	0.364	0.590	0.626	0.711	0.938	0.981	0.976	0.085	0.035	0.040	0.397	0.595	0.194	0.281	0.321	0.386
16	0.695	0.825	0.506	0.588	0.622	0.707	0.938	0.981	0.975	0.128	0.060	0.062	0.481	0.650	0.267	0.285	0.324	0.395
17	0.773	0.864	0.609	0.750	0.742	0.844	0.962	0.988	0.984	0.315	0.175	0.192	0.494	0.655	0.287	0.343	0.382	0.465
18	0.784	0.873	0.623	0.757	0.751	0.852	0.961	0.988	0.985	0.344	0.202	0.220	0.492	0.657	0.281	0.349	0.385	0.465
19	0.854	0.941	0.741	0.899	0.903	0.947	0.971	0.995	0.989	0.555	0.399	0.423	0.478	0.695	0.302	0.387	0.461	0.508
20	0.855	0.939	0.743	0.901	0.903	0.949	0.969	0.995	0.988	0.566	0.404	0.431	0.485	0.700	0.309	0.437	0.509	0.554
21	0.852	0.940	0.743	0.905	0.908	0.952	0.971	0.996	0.991	0.571	0.414	0.437	0.481	0.709	0.310	0.427	0.503	0.549
22	0.765	0.890	0.615	0.803	0.825	0.879	0.972	0.996	0.990	0.260	0.127	0.141	0.461	0.669	0.299	0.384	0.450	0.497
23	0.777	0.893	0.621	0.808	0.830	0.883	0.971	0.995	0.988	0.253	0.130	0.141	0.476	0.683	0.304	0.373	0.442	0.493
24	0.766	0.891	0.607	0.794	0.818	0.869	0.975	0.996	0.991	0.252	0.127	0.142	0.473	0.683	0.301	0.374	0.439	0.485
25	0.787	0.896	0.664	0.799	0.830	0.881	0.971	0.996	0.989	0.260	0.136	0.150	0.518	0.705	0.352	0.376	0.443	0.491
26	0.878	0.939	0.783	0.895	0.892	0.943	0.971	0.994	0.988	0.519	0.357	0.394	0.538	0.714	0.368	0.412	0.478	0.533
27	0.873	0.939	0.776	0.892	0.887	0.944	0.973	0.995	0.989	0.538	0.370	0.412	0.530	0.708	0.362	0.427	0.483	0.544
28	0.726	0.911	0.484	0.875	0.883	0.933	0.976	0.995	0.993	0.348	0.203	0.193	0.408	0.699	0.178	0.401	0.471	0.533
29	0.720	0.906	0.483	0.882	0.883	0.931	0.996	0.999	0.999	0.355	0.213	0.205	0.414	0.706	0.184	0.514	0.572	0.627
30	0.715	0.913	0.477	0.878	0.882	0.929	0.996	1.000	0.999	0.355	0.215	0.205	0.413	0.710	0.178	0.518	0.575	0.628
31	0.462	0.639	0.215	0.445	0.474	0.570	0.968	0.988	0.988	0.050	0.050	0.075	0.391	0.567	0.164	0.273	0.301	0.379
32	0.454	0.640	0.214	0.440	0.468	0.565	0.962	0.985	0.986	0.051	0.053	0.079	0.385	0.571	0.164	0.278	0.306	0.386
33	0.452	0.627	0.201	0.433	0.463	0.559	0.964	0.989	0.987	0.050	0.048	0.074	0.377	0.563	0.160	0.283	0.307	0.387
34	0.638	0.773	0.410	0.478	0.517	0.612	0.969	0.991	0.989	0.060	0.048	0.069	0.511	0.672	0.274	0.278	0.310	0.393
35	0.654	0.783	0.427	0.604	0.612	0.724	0.992	0.998	0.998	0.073	0.048	0.071	0.513	0.663	0.264	0.357	0.389	0.484
36	0.650	0.781	0.415	0.613	0.623	0.738	0.994	0.998	0.998	0.072	0.045	0.068	0.509	0.668	0.265	0.365	0.393	0.494

Continuation of Table 2.

No.	W(0.5,1)			W(2,1)			LN(0,0.5)			LN(0,1.5)			G(0.5,1)			G(2,1)		
	Y	S	δ	Y	S	δ	Y	S	δ	Y	S	δ	Y	S	δ	Y	S	δ
37	0.929	0.985	0.867	0.980	0.980	0.991	0.999	1.000	1.000	0.636	0.445	0.530	0.592	0.839	0.440	0.595	0.708	0.708
38	0.929	0.986	0.865	0.984	0.980	0.993	0.999	1.000	1.000	0.637	0.452	0.532	0.591	0.843	0.444	0.639	0.733	0.741
39	0.922	0.985	0.866	0.983	0.982	0.991	0.999	1.000	1.000	0.639	0.445	0.531	0.594	0.840	0.442	0.642	0.730	0.739
40	0.788	0.912	0.667	0.837	0.880	0.899	0.999	1.000	1.000	0.070	0.028	0.047	0.591	0.799	0.449	0.508	0.601	0.620
41	0.785	0.910	0.661	0.843	0.883	0.905	0.999	1.000	1.000	0.076	0.028	0.049	0.599	0.797	0.453	0.499	0.589	0.608
42	0.792	0.914	0.671	0.837	0.882	0.902	0.999	1.000	1.000	0.072	0.028	0.045	0.585	0.789	0.436	0.503	0.599	0.615
43	0.878	0.956	0.795	0.860	0.897	0.916	0.999	1.000	1.000	0.121	0.044	0.072	0.680	0.852	0.545	0.501	0.602	0.623
44	0.908	0.964	0.842	0.950	0.951	0.974	1.000	1.000	1.000	0.270	0.106	0.183	0.699	0.852	0.558	0.603	0.683	0.714
45	0.911	0.967	0.844	0.947	0.947	0.973	1.000	1.000	1.000	0.289	0.122	0.200	0.699	0.856	0.558	0.609	0.689	0.715
46	0.982	0.997	0.971	0.998	0.997	0.999	1.000	1.000	1.000	0.814	0.648	0.751	0.724	0.910	0.625	0.719	0.830	0.800
47	0.980	0.998	0.968	0.998	0.997	0.999	0.999	1.000	1.000	0.808	0.644	0.747	0.723	0.908	0.632	0.744	0.839	0.817
48	0.983	0.997	0.969	0.998	0.998	0.999	0.999	1.000	1.000	0.810	0.643	0.749	0.727	0.909	0.628	0.743	0.834	0.819
49	0.946	0.987	0.913	0.981	0.987	0.991	1.000	1.000	1.000	0.336	0.148	0.247	0.730	0.901	0.633	0.679	0.792	0.766
50	0.947	0.986	0.913	0.983	0.989	0.992	1.000	1.000	1.000	0.338	0.149	0.249	0.722	0.896	0.623	0.675	0.786	0.763
51	0.945	0.986	0.911	0.980	0.986	0.990	1.000	1.000	1.000	0.330	0.143	0.242	0.727	0.900	0.626	0.670	0.782	0.760
52	0.960	0.991	0.938	0.983	0.989	0.991	1.000	1.000	1.000	0.338	0.146	0.250	0.786	0.920	0.706	0.679	0.791	0.768
53	0.988	0.996	0.978	0.998	0.997	0.999	1.000	1.000	1.000	0.749	0.545	0.685	0.807	0.928	0.726	0.747	0.835	0.830
54	0.987	0.996	0.981	0.998	0.997	0.999	1.000	1.000	1.000	0.758	0.556	0.696	0.813	0.928	0.733	0.759	0.837	0.836

Table 3: Power estimates under the Rayleigh distribution and $\alpha = 0.05$.

No.	W(0.5,1)			$\chi^2(1)$			$\chi^2(3)$			Be(3,1)			Be(1,0.5)			Exp(2)		
	Y	S	η	Y	S	η	Y	S	η	Y	S	η	Y	S	η	Y	S	η
1	0.950	0.999	0.954	0.873	0.994	0.877	0.273	0.350	0.225	0.315	0.270	0.000	0.066	0.139	0.009	0.616	0.809	0.636
2	0.946	0.999	0.944	0.866	0.992	0.857	0.377	0.453	0.403	0.742	0.674	0.000	0.093	0.167	0.007	0.620	0.810	0.617
3	0.947	0.999	0.943	0.869	0.991	0.856	0.382	0.451	0.405	0.749	0.686	0.000	0.096	0.167	0.008	0.616	0.811	0.608
4	0.848	0.965	0.845	0.790	0.943	0.778	0.186	0.232	0.110	0.229	0.228	0.282	0.175	0.328	0.143	0.403	0.557	0.312
5	0.852	0.966	0.850	0.789	0.944	0.780	0.186	0.232	0.115	0.232	0.233	0.282	0.175	0.330	0.137	0.400	0.560	0.307
6	0.847	0.964	0.847	0.782	0.944	0.775	0.182	0.228	0.113	0.237	0.234	0.304	0.172	0.330	0.142	0.396	0.561	0.319
7	0.910	0.984	0.976	0.852	0.965	0.955	0.185	0.239	0.115	0.238	0.239	0.244	0.159	0.327	0.230	0.509	0.648	0.837
8	0.922	0.986	0.881	0.869	0.970	0.823	0.261	0.299	0.192	0.327	0.313	0.000	0.118	0.291	0.097	0.533	0.654	0.503
9	0.921	0.985	0.886	0.861	0.969	0.815	0.274	0.299	0.294	0.321	0.313	0.000	0.114	0.292	0.092	0.528	0.658	0.499
10	0.991	1.000	0.989	0.956	0.998	0.941	0.389	0.454	0.265	0.637	0.573	0.000	0.173	0.138	0.002	0.755	0.890	0.726
11	0.991	1.000	0.988	0.954	0.999	0.939	0.509	0.573	0.505	0.909	0.827	0.000	0.206	0.165	0.001	0.756	0.889	0.724
12	0.990	1.000	0.986	0.955	0.998	0.943	0.496	0.562	0.500	0.909	0.833	0.000	0.207	0.166	0.001	0.758	0.893	0.726
13	0.967	0.997	0.978	0.926	0.989	0.943	0.294	0.354	0.244	0.512	0.489	0.514	0.112	0.290	0.162	0.590	0.749	0.571
14	0.966	0.997	0.977	0.924	0.990	0.943	0.301	0.365	0.262	0.508	0.488	0.532	0.116	0.300	0.177	0.587	0.752	0.580
15	0.970	0.998	0.979	0.931	0.990	0.946	0.298	0.362	0.259	0.517	0.497	0.531	0.111	0.292	0.173	0.593	0.754	0.572
16	0.978	0.999	0.992	0.946	0.992	0.980	0.296	0.362	0.237	0.510	0.488	0.480	0.087	0.283	0.233	0.701	0.825	0.888
17	0.991	0.999	0.933	0.972	0.997	0.898	0.412	0.468	0.075	0.753	0.661	0.000	0.060	0.214	0.002	0.785	0.867	0.687
18	0.993	1.000	0.963	0.973	0.996	0.910	0.501	0.531	0.465	0.765	0.664	0.000	0.066	0.219	0.003	0.781	0.867	0.672
19	0.999	1.000	0.997	0.986	1.000	0.981	0.519	0.578	0.306	0.904	0.858	0.000	0.296	0.159	0.000	0.856	0.939	0.835
20	0.998	1.000	0.997	0.986	0.999	0.975	0.614	0.670	0.582	0.968	0.910	0.000	0.316	0.179	0.000	0.852	0.940	0.801
21	0.998	1.000	0.996	0.986	1.000	0.976	0.597	0.661	0.575	0.969	0.907	0.000	0.321	0.180	0.000	0.854	0.938	0.811
22	0.993	1.000	0.998	0.977	0.998	0.991	0.473	0.539	0.516	0.838	0.777	0.714	0.117	0.206	0.266	0.764	0.885	0.816
23	0.996	1.000	0.998	0.976	0.998	0.990	0.469	0.530	0.508	0.838	0.781	0.715	0.116	0.212	0.272	0.772	0.884	0.819
24	0.996	1.000	0.999	0.978	0.998	0.989	0.458	0.525	0.491	0.838	0.779	0.697	0.121	0.216	0.258	0.771	0.891	0.821
25	0.995	1.000	0.998	0.980	0.999	0.993	0.454	0.533	0.502	0.844	0.778	0.718	0.112	0.209	0.264	0.787	0.900	0.864
26	0.998	1.000	0.994	0.993	0.999	0.977	0.535	0.595	0.245	0.953	0.867	0.000	0.205	0.179	0.000	0.870	0.936	0.812
27	0.999	1.000	0.997	0.993	0.999	0.981	0.614	0.645	0.596	0.951	0.861	0.000	0.182	0.173	0.001	0.873	0.937	0.824
28	0.978	1.000	0.972	0.935	1.000	0.919	0.388	0.498	0.360	0.542	0.512	0.000	0.108	0.173	0.005	0.719	0.910	0.691
29	0.978	1.000	0.970	0.935	0.999	0.914	0.481	0.569	0.480	0.886	0.825	0.000	0.101	0.173	0.003	0.719	0.908	0.680
30	0.981	1.000	0.975	0.929	0.999	0.915	0.469	0.564	0.483	0.885	0.829	0.000	0.109	0.175	0.004	0.723	0.911	0.702
31	0.909	0.984	0.914	0.878	0.979	0.881	0.184	0.233	0.113	0.239	0.246	0.307	0.281	0.467	0.224	0.460	0.630	0.377
32	0.908	0.984	0.913	0.880	0.976	0.883	0.185	0.235	0.116	0.238	0.244	0.305	0.288	0.471	0.232	0.462	0.646	0.390
33	0.905	0.984	0.909	0.876	0.976	0.880	0.189	0.246	0.116	0.237	0.242	0.301	0.280	0.470	0.225	0.472	0.641	0.393
34	0.974	0.998	0.999	0.949	0.994	0.998	0.198	0.252	0.098	0.238	0.242	0.140	0.270	0.479	0.383	0.639	0.775	0.980
35	0.973	0.998	0.936	0.950	0.994	0.896	0.321	0.348	0.263	0.376	0.366	0.000	0.230	0.457	0.171	0.647	0.775	0.571
36	0.972	0.997	0.932	0.948	0.993	0.893	0.322	0.356	0.326	0.376	0.368	0.000	0.229	0.473	0.170	0.654	0.784	0.577

Continuation of Table 3.

No.	W(0.5,1)			$\chi^2(1)$			$\chi^2(3)$			Be(3,1)			Be(1,0.5)			Exp(2)		
	Y	S	η	Y	S	η	Y	S	η	Y	S	η	Y	S	η	Y	S	η
37	1.000	1.000	1.000	0.996	1.000	0.990	0.617	0.680	0.480	0.898	0.915	0.000	0.352	0.161	0.000	0.927	0.985	0.873
38	1.000	1.000	1.000	0.997	1.000	0.991	0.720	0.790	0.659	0.994	0.969	0.000	0.350	0.156	0.000	0.927	0.983	0.868
39	1.000	1.000	0.999	0.996	1.000	0.990	0.722	0.789	0.651	0.995	0.973	0.000	0.347	0.167	0.000	0.923	0.984	0.862
40	0.998	1.000	0.999	0.992	1.000	0.996	0.419	0.503	0.358	0.678	0.670	0.703	0.205	0.494	0.256	0.783	0.906	0.773
41	0.998	1.000	0.999	0.993	1.000	0.996	0.414	0.495	0.348	0.687	0.679	0.699	0.199	0.486	0.248	0.788	0.912	0.776
42	0.998	1.000	0.999	0.993	1.000	0.997	0.415	0.502	0.357	0.681	0.670	0.705	0.207	0.490	0.255	0.783	0.913	0.777
43	1.000	1.000	1.000	0.996	1.000	0.999	0.420	0.517	0.325	0.688	0.679	0.626	0.140	0.443	0.311	0.873	0.954	0.957
44	1.000	1.000	0.996	0.998	1.000	0.982	0.570	0.628	0.059	0.921	0.854	0.000	0.063	0.353	0.003	0.910	0.967	0.789
45	1.000	1.000	0.997	0.999	1.000	0.984	0.615	0.660	0.526	0.918	0.849	0.000	0.059	0.352	0.004	0.912	0.964	0.793
46	1.000	1.000	1.000	1.000	1.000	0.999	0.787	0.822	0.557	0.998	0.996	0.000	0.569	0.169	0.000	0.983	0.998	0.949
47	1.000	1.000	1.000	1.000	1.000	1.000	0.847	0.895	0.789	1.000	0.997	0.000	0.572	0.167	0.000	0.982	0.997	0.954
48	1.000	1.000	1.000	1.000	1.000	1.000	0.851	0.893	0.787	1.000	0.996	0.000	0.571	0.178	0.000	0.982	0.997	0.957
49	1.000	1.000	1.000	1.000	1.000	1.000	0.684	0.764	0.683	0.982	0.962	0.966	0.099	0.306	0.393	0.944	0.987	0.953
50	1.000	1.000	1.000	1.000	1.000	1.000	0.679	0.756	0.677	0.981	0.962	0.964	0.110	0.304	0.395	0.946	0.987	0.955
51	1.000	1.000	1.000	1.000	1.000	1.000	0.688	0.764	0.681	0.981	0.960	0.963	0.104	0.301	0.385	0.945	0.987	0.954
52	1.000	1.000	1.000	1.000	1.000	1.000	0.692	0.773	0.695	0.979	0.957	0.966	0.108	0.298	0.411	0.962	0.990	0.978
53	1.000	1.000	1.000	1.000	1.000	0.999	0.766	0.828	0.200	1.000	0.987	0.000	0.284	0.199	0.000	0.988	0.996	0.961
54	1.000	1.000	1.000	1.000	1.000	0.999	0.869	0.890	0.791	1.000	0.990	0.000	0.287	0.203	0.000	0.988	0.997	0.959

Table 4: Power estimates under the Pareto distribution and $\alpha = 0.05$.

No.	W(3,1)			W(0.5,1)			LN(1,1)			LN(5,3)			G(3,1)			G(0.8,1)		
	Y	S	ξ	Y	S	ξ	Y	S	ξ	Y	S	ξ	Y	S	ξ	Y	S	ξ
1	0.500	0.479	0.471	0.753	0.704	0.804	0.322	0.322	0.284	0.354	0.347	0.319	0.384	0.371	0.340	0.770	0.717	0.824
2	0.754	0.703	0.751	0.759	0.709	0.748	0.501	0.490	0.416	0.466	0.453	0.378	0.653	0.620	0.606	0.767	0.716	0.772
3	0.749	0.700	0.743	0.751	0.699	0.745	0.498	0.488	0.413	0.497	0.484	0.408	0.657	0.619	0.610	0.770	0.716	0.768
4	0.442	0.425	0.400	0.441	0.425	0.401	0.294	0.292	0.257	0.296	0.296	0.262	0.377	0.367	0.340	0.457	0.441	0.409
5	0.441	0.428	0.392	0.440	0.424	0.399	0.294	0.291	0.264	0.296	0.294	0.257	0.378	0.367	0.335	0.458	0.437	0.417
6	0.437	0.419	0.384	0.442	0.426	0.386	0.289	0.287	0.245	0.288	0.288	0.245	0.377	0.365	0.328	0.446	0.426	0.395
7	0.438	0.424	0.408	0.542	0.508	0.543	0.296	0.294	0.270	0.290	0.286	0.262	0.384	0.376	0.353	0.541	0.505	0.525
8	0.545	0.511	0.575	0.546	0.510	0.584	0.365	0.356	0.291	0.303	0.305	0.221	0.465	0.449	0.368	0.562	0.520	0.609
9	0.544	0.509	0.557	0.548	0.511	0.554	0.369	0.359	0.347	0.332	0.326	0.206	0.481	0.451	0.467	0.563	0.525	0.581
10	0.744	0.693	0.667	0.914	0.842	0.932	0.555	0.528	0.439	0.564	0.537	0.459	0.667	0.627	0.565	0.925	0.856	0.945
11	0.912	0.841	0.861	0.913	0.845	0.870	0.721	0.669	0.499	0.683	0.644	0.464	0.859	0.787	0.745	0.924	0.855	0.891
12	0.915	0.849	0.868	0.917	0.854	0.872	0.733	0.679	0.502	0.717	0.670	0.500	0.859	0.788	0.751	0.930	0.863	0.897
13	0.750	0.692	0.668	0.741	0.695	0.665	0.544	0.518	0.431	0.537	0.511	0.427	0.675	0.630	0.582	0.767	0.708	0.692
14	0.740	0.682	0.656	0.745	0.688	0.660	0.551	0.524	0.430	0.541	0.513	0.422	0.681	0.634	0.583	0.761	0.705	0.682
15	0.745	0.690	0.662	0.745	0.686	0.668	0.541	0.517	0.431	0.540	0.511	0.434	0.667	0.620	0.570	0.764	0.701	0.688
16	0.747	0.690	0.667	0.836	0.759	0.780	0.547	0.521	0.438	0.546	0.519	0.436	0.671	0.628	0.572	0.837	0.761	0.776
17	0.865	0.770	0.912	0.871	0.779	0.896	0.636	0.586	0.439	0.545	0.519	0.399	0.755	0.690	0.605	0.874	0.786	0.929
18	0.870	0.778	0.833	0.868	0.777	0.829	0.686	0.616	0.481	0.575	0.545	0.249	0.798	0.712	0.693	0.878	0.790	0.855
19	0.929	0.870	0.866	0.970	0.915	0.974	0.760	0.706	0.600	0.765	0.718	0.611	0.865	0.808	0.761	0.975	0.925	0.981
20	0.971	0.922	0.924	0.968	0.917	0.926	0.848	0.782	0.555	0.816	0.757	0.523	0.945	0.878	0.826	0.974	0.922	0.941
21	0.968	0.916	0.927	0.971	0.916	0.925	0.850	0.787	0.566	0.848	0.783	0.565	0.941	0.878	0.821	0.975	0.921	0.942
22	0.923	0.864	0.856	0.917	0.857	0.856	0.761	0.711	0.588	0.762	0.709	0.583	0.871	0.807	0.766	0.930	0.875	0.879
23	0.923	0.864	0.855	0.925	0.864	0.861	0.765	0.712	0.590	0.757	0.703	0.586	0.873	0.809	0.771	0.931	0.875	0.875
24	0.921	0.862	0.860	0.919	0.861	0.860	0.762	0.709	0.597	0.757	0.707	0.596	0.871	0.810	0.771	0.932	0.874	0.884
25	0.921	0.859	0.854	0.948	0.879	0.919	0.756	0.703	0.581	0.765	0.710	0.589	0.873	0.812	0.760	0.947	0.884	0.915
26	0.963	0.898	0.948	0.965	0.901	0.949	0.803	0.737	0.537	0.754	0.706	0.475	0.896	0.832	0.735	0.969	0.910	0.963
27	0.963	0.895	0.918	0.963	0.896	0.923	0.844	0.772	0.568	0.765	0.710	0.398	0.931	0.856	0.813	0.969	0.903	0.938
28	0.660	0.633	0.633	0.935	0.879	0.953	0.485	0.466	0.424	0.557	0.538	0.492	0.509	0.489	0.441	0.937	0.883	0.959
29	0.929	0.869	0.915	0.932	0.875	0.912	0.733	0.697	0.560	0.736	0.701	0.558	0.879	0.818	0.806	0.941	0.885	0.934
30	0.932	0.876	0.910	0.930	0.871	0.911	0.738	0.705	0.554	0.743	0.705	0.550	0.876	0.823	0.792	0.937	0.881	0.925
31	0.560	0.531	0.498	0.569	0.535	0.514	0.414	0.404	0.356	0.410	0.402	0.347	0.499	0.480	0.445	0.568	0.537	0.510
32	0.571	0.539	0.492	0.560	0.533	0.484	0.413	0.403	0.338	0.417	0.403	0.340	0.510	0.483	0.427	0.582	0.549	0.497
33	0.571	0.540	0.504	0.560	0.528	0.497	0.420	0.412	0.357	0.411	0.401	0.349	0.502	0.479	0.434	0.564	0.534	0.503
34	0.583	0.550	0.522	0.742	0.672	0.741	0.419	0.407	0.358	0.425	0.417	0.366	0.499	0.477	0.434	0.753	0.681	0.711
35	0.739	0.672	0.708	0.749	0.675	0.716	0.579	0.537	0.396	0.469	0.453	0.276	0.678	0.624	0.502	0.750	0.675	0.729
36	0.742	0.669	0.706	0.739	0.668	0.707	0.581	0.532	0.469	0.538	0.507	0.290	0.691	0.628	0.616	0.751	0.675	0.726

Continuation of Table 4.

No.	W(3,1)			W(0.5,1)			LN(1,1)			LN(5,3)			G(3,1)			G(0.8,1)		
	Y	S	ξ	Y	S	ξ	Y	S	ξ	Y	S	ξ	Y	S	ξ	Y	S	ξ
37	0.948	0.911	0.894	0.998	0.984	0.999	0.845	0.811	0.718	0.857	0.821	0.763	0.918	0.871	0.832	0.999	0.987	0.999
38	0.998	0.985	0.989	0.997	0.984	0.991	0.968	0.927	0.744	0.949	0.915	0.728	0.994	0.973	0.950	0.999	0.987	0.994
39	0.997	0.983	0.989	0.997	0.984	0.991	0.969	0.931	0.746	0.970	0.928	0.744	0.994	0.970	0.948	0.998	0.986	0.993
40	0.950	0.911	0.893	0.947	0.907	0.890	0.843	0.799	0.708	0.838	0.798	0.699	0.915	0.873	0.832	0.955	0.915	0.902
41	0.950	0.910	0.890	0.951	0.912	0.890	0.837	0.796	0.690	0.838	0.802	0.697	0.922	0.876	0.833	0.956	0.916	0.907
42	0.952	0.914	0.897	0.949	0.913	0.896	0.830	0.792	0.708	0.846	0.803	0.722	0.914	0.869	0.831	0.958	0.916	0.911
43	0.953	0.913	0.900	0.982	0.947	0.963	0.838	0.801	0.715	0.850	0.802	0.729	0.914	0.877	0.834	0.988	0.955	0.965
44	0.987	0.955	0.984	0.986	0.956	0.977	0.915	0.862	0.681	0.864	0.820	0.636	0.966	0.923	0.843	0.990	0.961	0.989
45	0.989	0.955	0.962	0.989	0.954	0.964	0.934	0.875	0.715	0.891	0.840	0.483	0.974	0.929	0.896	0.990	0.960	0.973
46	0.998	0.989	0.992	1.000	0.997	1.000	0.980	0.955	0.903	0.980	0.954	0.905	0.995	0.980	0.974	1.000	0.998	1.000
47	1.000	0.997	0.998	1.000	0.997	0.999	0.995	0.980	0.841	0.990	0.972	0.830	0.999	0.994	0.984	1.000	0.998	0.999
48	1.000	0.997	0.999	1.000	0.998	0.998	0.997	0.980	0.853	0.996	0.982	0.850	1.000	0.993	0.985	1.000	0.998	0.999
49	0.998	0.991	0.990	0.998	0.991	0.992	0.977	0.952	0.896	0.980	0.956	0.901	0.995	0.981	0.972	0.999	0.992	0.992
50	0.998	0.989	0.989	0.998	0.991	0.990	0.978	0.953	0.897	0.979	0.953	0.900	0.996	0.981	0.974	0.999	0.990	0.994
51	0.999	0.989	0.990	0.999	0.991	0.991	0.979	0.951	0.893	0.982	0.954	0.894	0.995	0.982	0.967	0.999	0.991	0.991
52	0.998	0.991	0.992	1.000	0.995	0.999	0.980	0.953	0.895	0.980	0.956	0.902	0.996	0.980	0.973	1.000	0.995	0.998
53	1.000	0.996	1.000	1.000	0.996	1.000	0.990	0.967	0.833	0.978	0.951	0.768	0.998	0.986	0.952	1.000	0.997	1.000
54	1.000	0.995	0.997	1.000	0.995	0.997	0.996	0.974	0.833	0.980	0.953	0.598	1.000	0.989	0.982	1.000	0.997	0.999

5 Illustrative examples

In this section, the GOF tests demonstrated in this article are used to address the reliability of ball bearings as an engineering challenge. The dataset, discussed by Caroni (2002), pertains to tests on the endurance of deep-groove ball bearings. Periodic inspections were conducted on the ball bearings to detect any potential issues. The number of millions of revolutions before failure for 25 ball bearings was recorded, and for our analysis, we are considering the failure times as continuous. By implementing an adaptive type-II progressive censoring scheme with $n = 10$, $T = 75$, and $\boldsymbol{R} = (0, 1, 2, \ldots, 2, 1, 1)$ on the ball bearings lifetime data, we generated the values of $\boldsymbol{X}$ and $\boldsymbol{R'}$ as shown in Table 5.

Table 5: The values of running the adaptive type-II progressive censoring scheme

i	1	2	3	4	5	6	7	8	9	10
$X_{i:n:N}$	17.88	28.92	33.00	41.52	48.48	51.96	55.56	67.80	68.64	98.64
R'_i	0	1	2	2	2	2	2	2	1	1

We observe that $J = 10$ in the data provided in Table 5. To test the null hypothesis that the censored dataset in Table 5 follows a Rayleigh distribution as predicted by theory, we computed the test statistics $Y = 0.55031$ and $S = 13.76409$. Considering a significance level of $\alpha = 0.05$, we compared the test statistics with the critical values. We found that $0.31231 = t_{0.025} < Y < t_{0.975} = 0.68769$ and $8.230746 = \chi^2_{0.025,18} < S < \chi^2_{0.975,18} = 31.52638$. As a result, the proposed tests do not reject the null hypothesis.

These results are consistent with the findings of Baratpour and Khodadadi (2012) and Jahanshahi et al. (2016). Therefore, the statistical analysis supports the conclusion that the observed data in Table 5 is in agreement with a Rayleigh distribution.

Now, we must provide the statistical inferences for the parameter θ of the Rayleigh distribution with DF $F(x) = 1 - \exp\left\{-\theta x^2\right\}$, $x > 0$. Using Equations (10) and (11), the ML and UMVU estimators of θ based on the censored dataset in Table 5 are 0.000129 and 0.000116, respectively. Also, we can show that the 95% equi-tailed confidence interval, the confidence interval with the shortest width and the UMAU confidence interval for θ are derived as $(0.000062, 0.00022)$, $(0.000055, 0.00021)$ and $(0.000064, 0.000227)$, respectively. Finally, for testing the null hypothesis $H_0 : \theta = \theta_0$ against $\theta \neq \theta_0$ with $\theta_0 = 0.0001$, using the UMPU and GLR tests at the level of $\alpha = 0.05$ in Equations (14) and (16), we observe that $c_1 < T_\star < c_2$ and $T_\star^n \exp(-\theta_0 T_\star) > K$. Hence, the null hypothesis $H_0 : \theta = \theta_0$ is accepted.

6 Conclusions

The article proposed two new tests for assessing the fit of a proportional hazard rate model, accounting for situations where only a limited portion of the random sample is accessible due to observations being influenced by adaptive progressive type-II censoring. The new tests rely on normalized spacings and involve a straightforward computational process. The exact null distribution of the test statistics has been addressed. It is obvious that the suggested tests can be used for progressively type-II censored data, conventional type-II censored data and uncensored data, all of which are subsets of adaptive progressive type-II censoring.

Clearly, the simulation study across various sample sizes and censoring schemes indicates that the new tests effectively detect deviations from the null distribution. The procedures of proposed tests have been demonstrated using real data corresponding to the proportional hazard rate model. In addition, optimal confidence intervals for the unknown parameter were developed using adaptive progressively type-II censored data from a proportional hazard rate model. These included UMA and equi-tailed confidence intervals, as well as the confidence interval with the shortest width. We also obtained UMP tests for one-sided alternative hypothesis and UMPU and GLR tests for two-sided alternative hypothesis.

There are additional noteworthy issues in this area that warrant further exploration. For instance, rather than relying on normalized spacings, we could develop goodness-of-fit tests using empirical distribution functions for adaptive progressively type-II censored data. Additionally, we are in the process of broadening the scope of the proposed tests to encompass general progressively type-II censored data, progressively first-failure censored data, and, from a broader perspective, generalized order statistics.

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