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A note on a generalization of the δ-shock model

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Abstract: According to the δ-shock model, a shock-exposed system fails when an intershock time falls below a critical threshold δ. Recently, a generalization of the δ-shock model was introduced by Poursaeed (2019) under which the system fails when the intershock time falls below a threshold $\delta_1 > 0$ and also the system probably fails with probability θ if the intershock time falls in the interval $(\delta_1, \delta_2]$ for $\delta_1 < \delta_2$. In this paper, we look at this generalized model with new assumptions for intershock times. More precisely, we assume that the intershock times have a discrete distribution, and the chance of their occurrence at a critical time point is significantly high. We investigate some statistical properties of the system's lifetime, and by providing an illustrative example, we examine theoretical results numerically. Numerical results show that when the chance of intershock times occurring at a critical time point increases, the system reliability decreases significantly. Finally, the paper ends with a conclusion.

Keywords: Critical situation, δ-shock model, Intershock time, Regular situation.
Mathematics Subject Classification (2010): 62N05, 90B25, 62E15.

1 Introduction

In the real world, most systems suffer from random shocks from various sources. These random shocks often have an adverse effect on the reliability of the systems. In reliability theory, shock models are used to study the reliability behavior of systems that are subject to random shocks at random times. There are three basic types of shock models in the literature, which are extreme shock models, cumulative shock models, and run shock models (see, e.g., Anderson (1988), Gut (1990), and Mallor and Omey (2001)). The system failure scenarios under these models are as follows:

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(i) Extreme shock model: The system fails when the magnitude of a shock exceeds a critical threshold;
(ii) Cumulative shock model: The system fails when the sum of the magnitudes of shocks exceeds a critical threshold;
(iii) Run shock model: The system fails when there is a run of k shocks exceeding a critical magnitude.

In addition to the above traditional shock models, there are other shock models that have been introduced and developed in recent decades. The so-called *δ-shock model* is one of them, which has received more attention. According to the δ-shock model, the system fails when the intershock time (i.e., the time lag between two successive shocks) falls below a prefixed threshold $\delta > 0$. The δ-shock model was first introduced by Li et al. (1999), after which it was widely studied by many scientists and researchers. Below is a summary of some recent developments. Eryilmaz (2012) extended the δ-shock model using the concept of runs. Eryilmaz (2013) studied the discrete-time version of the δ-shock model, where shocks happen with accordance to a binomial process. Parvardeh and Balakrishnan (2015) introduced a mixed δ-shock model, in which, the system fails when the intershock time is smaller than a threshold δ or the magnitude of the shock is larger than another threshold δ. Wang and Peng (2017) proposed a generalized δ-shock model with two types of shocks and two different recovery times such that for a type i shock for $i = 1, 2$, the system recovery time is δ_i, and the system fails when a shock occurs while the system has not still recovered from the consequence of the previous shock. Tuncel and Eryilmaz (2018) investigated the survival function and the mean lifetime of the system failure under the δ-shock model, considering the proportional hazard rate model. Zhao et al. (2018) proposed an extension of cumulative shock and δ-shock models as a two-stage shock model with a self-healing mechanism. Entezari and Roozegar (2020) introduced a system with multiple states of failure as a mixed discrete-time δ-shock model, which is defined by combining δ-shock and extreme shock models. Lorvand et al. (2020) discussed a mixed δ-shock model for multi-state systems by assuming a renewal process of shocks, where the system fails in three specific states. Poursaeed (2021) studied the reliability analysis of an extension of the discrete time version of the δ-shock model by considering two different critical thresholds and a probable failure region. Eryilmaz and Kan (2021) introduced a particular shock model when the distributions of intershock times and magnitudes of shocks are the discrete phase-type. Lorvand and Nematollahi (2022) studied a mixed δ-shock model that is based on both intershock time and magnitude of shocks. Finkelstein and Cha (2024) revisited the classical δ-shock model and generalized it to the case of renewal processes of external shocks with arbitrary inter-arrival times and arbitrary distribution of the recovery parameter δ. Lorvand and Eryilmaz (2024) developed a new δ-shock model by considering the shock magnitude, based on which, if a shock follows a non-critical shock at time less than δ_1 or a shock follows a critical shock at time occurring less than δ_2, the system fails ($\delta_1 < \delta_2$). Farhadian and Jafari (2025) proposed a generalized version of the δ-shock model in which the critical region for intershock times is the interval $[\alpha, \delta]$ for $0 \leq \alpha < \delta$ and also introduced a critical situation for a system under the classical δ-shock model.

In this paper, we will study a generalized δ-shock model by Poursaeed (2019) under some new assumptions. In the new framework for the considered model, we assume that the intershock times have a discrete distribution, and the chance of their occurrence at a critical time point of the model is significantly high. Indeed, it is expected that increasing the chance of intershock times at a point within the critical area of the model will make the system more sensitive and increase the system's tendency to fail. Such a situation is called a *critical situation*. The idea of the critical situations was first introduced by Farhadian and Jafari (2025) for the classical δ-shock model. Our motivation for studying such a model comes from the practical aspect of real-world systems, since in many cases, the chance of intershock times occurring at a critical time point may be increased by various factors. An illustrative example will be provided after presenting the results.

The paper is organized as follows. Section 2 introduces the general framework of the model. The characteristics of intershock times are derived in Section 3. The stopping time distribution is obtained in Section 4. In Section 5, some distribution properties of the system's lifetime are investigated. An illustrative numerical example is presented in Section 6. Section 7 concludes the paper.

2 Model description

Consider a system that is subject to a sequence of external shocks that occur randomly over time. Let X_i (for $i = 1, 2, \dots$) denotes the time between the ith and $(i+1)$th shocks. Let also X denote a generic random variable of X_i's. We assume that the intershock times $X_1, X_2, \dots$ are independent and identically distributed (i.i.d.) by an arbitrary distribution with cumulative distribution function (cdf) $F(x) = 1 - \bar{F}(x) = P(X \leq x)$ (with $F(0) = 0$). Let δ_1 and δ_2 (with $0 < \delta_1 < \delta_2$) are two critical levels and suppose that the system fails if an intershock time is less than or equal to δ_1 for the first time in the sequence of shocks, and also the system probably fails with a probability θ, if the intershock time falls in the interval $(\delta_1, \delta_2]$. Otherwise, the system continues to work safely. In the case where the system failure is probable with probability θ, corresponding to intervals of the form $(\delta_1, \delta_2]$, a Bernoulli trial $Y_1, Y_2, \dots$ is considered with a success probability θ. It is assumed that Y_i's are independent of X_i's. If N_1 is the number of intershock times greater than δ_2 and N_2 is the number of intershock times that fall in the interval $(\delta_1, \delta_2]$ with Y's=0, then the number of intershock times between successive shocks until system failure is equal to $N = N_1 + N_2 + 1$. Assume that N_1 and N_2 are independent, and also assume that (N_1, N_2) is independent of X_i's. Therefore, the lifetime of the system can be defined as follows:

$$T = \sum_{i=1}^{N} X_i,$$

where the stopping random variable N is defined as

$$\{N = n\} \quad \Leftrightarrow \quad \{X_n \leq \delta_1\} \text{ or } \{\delta_1 < X_n \leq \delta_2, Y_n = 1\}.$$

The above model was first introduced by Poursaeed (2019). From here on, we look at the above model with the following two new assumptions for intershock times:

• The intershock time X take positive integer values;

• The probability that intershock time X occurs at a critical time point is considered to be significantly high.

Note that when the probability that X takes a point less than δ_1 or between $(\delta_1, \delta_2]$ increases significantly, then the reliability of the system is expected to decrease. Such a situation is a critical situation (Farhadian and Jafari, 2025). Here, we consider the system in a situation where the probability that X takes the borderline point δ_1 increases significantly. In this case, we definitely have a critical situation because the point δ_1 is in the center of two critical areas; one is a definite failure area (that is, $(0, \delta_1]$) and the other is a probable failure area (that is, $(\delta_1, \delta_2]$).

Note that in distribution theory, a significant increase in probability of a random variable at a particular point is called inflation, and a probability distribution with this property is called an inflated distribution.

3 Properties of the intershock time X

According to the descriptions in Section 2, the intershock time X takes positive integer values with a probability mass function (pmf) $P(X = x)$. We denote this pmf as $P_{reg}(X = x)$ when the probability distribution of intershock times has a regular form, that is, there is no inflation in the probability distribution. When there is inflation in the probability distribution of intershock times, we denote the pmf as $P_+(X = x)$. Thus, if X has support in $\boldsymbol{\mathcal{X}}$ and its distribution is inflated at a particular point k ($k \in \boldsymbol{\mathcal{X}}$), then the k-inflated pmf is given by

$$P_+(X = x) = \begin{cases} \alpha + (1-\alpha)P_{reg}(X = x), & \text{if } x = k, \\ (1-\alpha)P_{reg}(X = x), & \text{if } x \in \boldsymbol{\mathcal{X}} - \{k\}, \end{cases} \tag{1}$$

where $\alpha \in [0, 1]$ is an inflation parameter.

For example, if X follows a 2-inflated geometric distribution, we have $P_{reg}(X = x) = p(1-p)^{x-1}$ for $x = 1, 2, \ldots$ and $0 < p \leq 1$, thus

$$P_+(X = x) = \begin{cases} \alpha + (1-\alpha)p(1-p), & \text{if } x = 2, \\ (1-\alpha)p(1-p)^{x-1}, & \text{if } x \in \{1, 3, 4, 5, 6, \ldots\}. \end{cases} \tag{2}$$

Inflated distributions are usually studied for special distributions at particular points such as 0 and 1 (see, e.g., Rivas (2023)). In the following, we investigate some distributional properties of a general k-inflated distribution.

Theorem 3.1. *(Inflation property) Let the intershock time X follow the k-inflated distribution in* (1)*. Then $P_+(X = k) \geq P_{reg}(X = k)$.*

Proof. We have $0 \leq P_{reg}(X = x) \leq 1$ for any x in its support. On the other hand, for $\alpha = 0$, we have $P_+(X = k) = P_{reg}(X = k)$, and for $\alpha = 1$, $P_+(X = k)$ is degenerated pmf. Now, let us consider $\alpha \in (0, 1)$. Therefore, multiplying both sides of $P_{reg}(X = k) < 1$ by α and adding $-\alpha P_{reg}(X = k)$ to both sides gives

$$\alpha - \alpha P_{reg}(X = k) > 0.$$

Finally, adding both sides by $P_{reg}(X = k)$, we get $\alpha + (1-\alpha)P_{reg}(X = k) > P_{reg}(X = k)$, that is, $P_+(X = k) > P_{reg}(X = k)$. This completes the proof. □

Therefore, by Theorem 3.1, in a k-inflated distribution, the probability of occurrence of k is higher than in a distribution with regular pmf $P_{reg}(X = x)$.

In the next theorem, we obtain the cdf of a k-inflated random variable distributed by (1).

Theorem 3.2. *Let the intershock time X follow the k-inflated distribution in* (1)*. If the cdf of X in its regular mode is denoted by $F_{reg}(x)$, then the cdf of X in inflated mode is given by*

$$F_{+}(x) = \begin{cases} (1-\alpha)F_{reg}(x), & \text{if } x < k, \\ \alpha + (1-\alpha)F_{reg}(x), & \text{if } x \geq k. \end{cases}$$

Proof. Using (1), we have for $x < k$,

$$F_{+}(x) = \sum_{j=0}^{x} P_{+}(X = j) = (1-\alpha)\sum_{j=0}^{x} P_{reg}(X = j) = (1-\alpha)F_{reg}(x),$$

and if $x \geq k$, we have

$$\begin{aligned} F_{+}(x) &= \sum_{j=0}^{x} P_{+}(X = j) \\ &= (1-\alpha)\sum_{j=0}^{x} P_{reg}(X = j)) - (1-\alpha)P_{reg}(X = k) \\ &\quad + \alpha + (1-\alpha)P_{reg}(X = k) \\ &= \alpha + (1-\alpha)F_{reg}(x). \end{aligned}$$

This completes the proof. □

Theorem 3.3. *Let the intershock time X follow the k-inflated distribution in* (1)*. If the reliability function of X in regular mode is denoted by $\bar{F}_{reg}(x)$, then the reliability function of X in inflated mode is given by*

$$\bar{F}_{+}(x) = \begin{cases} \alpha + (1-\alpha)\bar{F}_{reg}(x), & \text{if } x < k, \\ (1-\alpha)\bar{F}_{reg}(x), & \text{if } x \geq k. \end{cases}$$

Proof. By using the definition of reliability function ($\bar{F}_{+}(x) = P_{+}(X > x)$) and using Theorem 3.2, the proof is straightforward. □

In the following theorem, we obtain the moments related to intershock times under k-inflated distribution in (1).

Theorem 3.4. *Let $E_{reg}[X^r]$ be the rth moment of the intershock time X in its regular mode. The rth moment of the k-inflated version of X distributed by* (1) *is*

$$E_{+}[X^r] = \alpha k^r + (1-\alpha)E_{reg}[X^r].$$

In particular, for $r = 1$, we have $E_{+}[X] = \alpha k + (1-\alpha)E_{reg}[X]$.

Proof. We have

$$
\begin{aligned}
E_+[X^r] &= \sum_x x^r P_+(X=x) \\
&= (1-\alpha)\sum_x x^r P_{reg}(X=x) - (1-\alpha)k^r P_{reg}(X=k) \\
&\quad + k^r\left(\alpha + (1-\alpha)P_{reg}(X=k)\right) \\
&= \alpha k^r + (1-\alpha)E_{reg}[X^r].
\end{aligned}
$$

The theorem is proved. □

In the following, we calculate the probability generating function (pgf) of the inflated distribution in (1).

Theorem 3.5. *If intershock time X follows the k-inflated distribution in* (1)*, then its pgf is*

$$G_X^+(z) = \alpha z^k + (1-\alpha)G_X^{reg}(z),$$

where $G_X^{reg}(z)$ is the pgf of X in its regular mode.

Proof. We have

$$
\begin{aligned}
G_X^+(z) = E_+[z^X] = \sum_x z^x P_+(X=x) &= \alpha z^k + (1-\alpha)\sum_x z^x P_{reg}(X=x) \\
&= \alpha z^k + (1-\alpha)E_{reg}[z^X] \\
&= \alpha z^k + (1-\alpha)G_X^{reg}(z).
\end{aligned}
$$

The theorem is proved. □

4 Properties of the stopping time N

In this section, we derive the pmf of random variables N_1, N_2, and N under the critical situation. The following results are a rewrite of the corresponding results from Poursaeed (2019) in terms of the inflated distribution of intershock times. Since n_1 of X's are as $X > \delta_2$, and n_2 of X's are as $(\delta_1 < X \le \delta_2, Y=0)$, and one of X's is as $X \le \delta_1$ or $(\delta_1 < X \le \delta_2, Y=1)$, therefore, the joint pmf of N_1 and N_2 is given by (for $n_1, n_2 = 0, 1, 2, \ldots$)

$$
\begin{aligned}
P(N_1=n_1, N_2=n_2) &= \binom{n_1+n_2}{n_1}\big(P_+(X>\delta_2)\big)^{n_1}\big((1-\theta)P_+(\delta_1 < X \le \delta_2)\big)^{n_2} \\
&\quad \times\big(\theta P_+(\delta_1 < X \le \delta_2) + P_+(X \le \delta_1)\big) \\
&= \binom{n_1+n_2}{n_1}\big(\bar{F}_+(\delta_2)\big)^{n_1}\Big(\left(F_+(\delta_2) - F_+(\delta_1)\right)(1-\theta)\Big)^{n_2} \\
&\quad \times\Big(\left(F_+(\delta_2) - F_+(\delta_1)\right)\theta + F_+(\delta_1)\Big). \qquad (3)
\end{aligned}
$$

To obtain $P(N_1 = n_1)$ and $P(N_2 = n_2)$ from (3), we need the following lemma.

Lemma 4.1. *(Graham et al., 1994, Page 199) Let* r *be an arbitrary real number such that* $0 < r < 1$. *Then*

$$\sum_{i=0}^{\infty} \binom{i+j}{j} r^i = \frac{1}{(1-r)^{j+1}}.$$

Now, by using (3), the pmf of N_1 is obtained as

$$\begin{aligned}
P(N_1 = n_1) &= \sum_{n_2=0}^{\infty} P(N_1 = n_1, N_2 = n_2) \\
&= \sum_{n_2=0}^{\infty} \left\{ \binom{n_1+n_2}{n_1} \left(\bar{F}_+(\delta_2)\right)^{n_1} \left((F_+(\delta_2) - F_+(\delta_1))\,(1-\theta)\right)^{n_2} \right. \\
&\quad \left. \times \left((F_+(\delta_2) - F_+(\delta_1))\,\theta + F_+(\delta_1)\right) \right\} \\
&= \left(\bar{F}_+(\delta_2)\right)^{n_1} \left((F_+(\delta_2) - F_+(\delta_1))\,\theta + F_+(\delta_1)\right) \\
&\quad \times \sum_{n_2=0}^{\infty} \binom{n_1+n_2}{n_1} \left((F_+(\delta_2) - F_+(\delta_1))\,(1-\theta)\right)^{n_2} \\
&= \frac{\left(\bar{F}_+(\delta_2)\right)^{n_1} \left((F_+(\delta_2) - F_+(\delta_1))\,\theta + F_+(\delta_1)\right)}{\left(1 - (F_+(\delta_2) - F_+(\delta_1))\,(1-\theta)\right)^{n_1+1}} \qquad \text{(by Lemma 1)} \\
&= \left(\frac{\bar{F}_+(\delta_2)}{\bar{F}_+(\delta_2) + (F_+(\delta_2) - F_+(\delta_1))\,\theta + F_+(\delta_1)} \right)^{n_1} \\
&\quad \times \frac{(F_+(\delta_2) - F_+(\delta_1))\,\theta + F_+(\delta_1)}{\bar{F}_+(\delta_2) + (F_+(\delta_2) - F_+(\delta_1))\,\theta + F_+(\delta_1)}, \quad n_1 = 0, 1, 2, \ldots, \qquad (4)
\end{aligned}$$

that is, N_1 follows the geometric distribution.

By a similar way, it can be shown that the random variable N_2 has geometric distribution with the following pmf

$$P(N_2 = n_2) = \left(\frac{(1-\theta)\,(F_+(\delta_2) - F_+(\delta_1))}{F_+(\delta_2)} \right)^{n_2} \frac{(F_+(\delta_2) - F_+(\delta_1))\,\theta + F_+(\delta_1)}{F_+(\delta_2)}, \qquad (5)$$

for $n_2 = 0, 1, 2, \ldots$.

Since $N = N_1 + N_2 + 1$, and N_1 and N_2 have geometric distribution with pmf in (4) and (5), respectively, therefore,

$$\begin{aligned}
P(N = n) \;&=\; \left(1 - \big(F_+(\delta_1) - F_+(\delta_2)\big)\theta + F_+(\delta_1)\right)^{n-1} \\
&\quad \times \left(\big(F_+(\delta_1) - F_+(\delta_2)\big)\theta + F_+(\delta_1)\right), \quad \textit{for } n = 1, 2, \ldots, \qquad (6)
\end{aligned}$$

that is, N follows the geometric distribution.

5 Properties of the system's lifetime T

In general, deriving an explicit representation of the reliability function for the considered system is difficult, or very complex if obtained. Therefore, the pgf of the system's lifetime can be useful for the calculation of the probability mass function of the system's lifetime. Below, we obtain the pgf of the system's lifetime.

Theorem 5.1. *Consider the model described in Section 2. If the distribution of X is inflated at the critical point δ_1, then the pgf of the system's lifetime is*

$$G_T(z) = \frac{\Big(\alpha + (1-\alpha)\big(\big(F_{reg}(\delta_2) - F_{reg}(\delta_1)\big)\theta + F_{reg}(\delta_1)\big)\Big)}{1 - \Big(1 - \Big(\alpha + (1-\alpha)\big(\,(F_{reg}(\delta_2) - F_{reg}(\delta_1))\,\theta + F_{reg}(\delta_1)\big)\Big)\Big)} \times \frac{\big(\alpha z^{\delta_1} + (1-\alpha)G_{reg}(z)\big)}{\big(\alpha z^{\delta_1} + (1-\alpha)G_{reg}(z)\big)}.$$

Proof. The pgf of the system's lifetime T can be calculated as follows

$$\begin{aligned} G_T(z) = E[z^T] = E\left[z^{\sum_{i=1}^N X_i}\right] &= E\left[E_+\left[z^{\sum_{i=1}^N X_i} \middle| N\right]\right] \\ &= E\left[\left(G_X^+(z)\right)^N\right] \\ &= G_N\big(G_X^+(z)\big), \end{aligned} \tag{7}$$

where $G_N(z)$ is the pgf of the random variable N.

Since N has a geometric distribution with pmf in (6), therefore, the pgf of N is obtained as

$$G_N(z) = \frac{\big(\,(F_+(\delta_2) - F_+(\delta_1))\,\theta + F_+(\delta_1)\big)z}{1 - \Big(1 - \big(\,(F_+(\delta_2) - F_+(\delta_1))\,\theta + F_+(\delta_1)\big)\Big)z}. \tag{8}$$

Hence, by using (8) in (7), we obtain

$$G_T(z) = \frac{\big(\,(F_+(\delta_2) - F_+(\delta_1))\,\theta + F_+(\delta_1)\big)G_X^+(z)}{1 - \Big(1 - \big(\,(F_+(\delta_2) - F_+(\delta_1))\,\theta + F_+(\delta_1)\big)\Big)G_X^+(z)}. \tag{9}$$

Since X is δ_1-inflated distributed, so by applying Theorem 3.5 to (9), we get the desired result. This completes the proof. □

In the next theorem, we obtain an explicit formula for the mean lifetime of the system, which defines the mean time to failure (MTTF) of the system.

Theorem 5.2. *Consider the model described in Section 2. If the distribution of X is inflated at the critical point δ_1, then the MTTF of the system is*

$$E[T] = \frac{\alpha\delta_1 + (1-\alpha)E_{reg}[X]}{\alpha + (1-\alpha)\Big(\big(F_{reg}(\delta_2) - F_{reg}(\delta_1)\big)\theta + F_{reg}(\delta_1)\Big)}.$$

Proof. The random variable N is a stopping time for $X_1, X_2, \dots$. Therefore, by using the well-known Wald's identity, the MTTF of the system can be computed as

$$E[T] = E\left[\sum_{i=1}^{N} X_i\right] = E[N]E_{+}[X]. \tag{10}$$

Since N has a geometric distribution with pmf in (6), therefore,

$$E[N] = \frac{1}{(F_{+}(\delta_2) - F_{+}(\delta_1))\,\theta + F_{+}(\delta_1)}. \tag{11}$$

Using (11) in (10), we get

$$E[T] = \frac{E_{+}[X]}{(F_{+}(\delta_2) - F_{+}(\delta_1))\,\theta + F_{+}(\delta_1)}. \tag{12}$$

Since X is δ_1-inflated distributed, so by applying Theorems 3.2 and Theorem 3.4 to (12), the desired result is obtained. This completes the proof. ☐

Remark 5.3. *Note that if we consider $\alpha = 0$, the above results investigate the system's lifetime in a regular situation corresponding to the regular distribution of intershock times.*

6 Illustrative example

Retail shop counters in stores and shopping centers are one of the most important parts in the sales cycle. They are the end stations for customers. The customers' arrival at the retail counter is a random phenomenon, and the time between the arrivals varies usually from one minute to six minutes. Minute changes are considered discrete. Suppose the arrival of customers when they reach the retail counters follows the Bernoulli process. Therefore, the interarrival time between customers follows a geometric distribution. We assume that the retail counter is a system and the customers are shocks. Accordingly, the interarrival times between customers are considered as intershock times. Now if the intershock time is less than or equal to 2 minutes, the sales clerk will have a disruption in his/her work. This is considered the system failure. If the intershock time is between 2 and 4 minutes, the sales clerk may experience a disruption in his/her work, but this is not deterministic. So, the system failure is probable with a probability θ when the intershock times are between 2 and 4 minutes. On busy days, the number of customers increases. In this way, the number of customers visiting the retail counter also increases. This will make intershock times shorter. Thus, due to the large number of customers on busy days, the intershock time may experience excessive frequency at a small point in time. We assume that the intershock times are the most frequent at point 2 minutes; that is, the probability distribution of intershock times is inflated at point 2. Obviously, this scenario follows the model described in Section 2 with $\delta_1 = 2$ and $\delta_2 = 4$.

Now, we present some computational results. Since the arrival of customers follows the Bernoulli process, the intershock times $X_1, X_2, \dots$ are i.i.d. distributed by a geometric distribution with mean $\frac{1}{p}$. That is, $P_{reg}(X_i = x) = p(1-p)^{x-1}$ for $x = 1, 2, \dots$.

On busy days, the distribution of intershock times is changed to a 2-inflated geometric distribution with inflation parameter $\alpha \in [0, 1]$ (see (2)). Using Theorems 5.1, the pmf and then the reliability function of the system's lifetime T are calculated. For $\alpha > 0$, we have an increase in the frequency of intershock times at point 2, indicating a busy day (critical situation). For $\alpha = 0$, we have the system's lifetime in regular days (regular situation). All calculations are performed using the R program. The corresponding numerical results are given in Table 1. From Table 1, it is clear that in all cases, the reliability of the system in a critical situation (when $\alpha > 0$) is significantly smaller than the reliability of the system in a regular situation (when $\alpha = 0$). Thus, the system is more stable in a regular situation. Furthermore, Figure 1 depicts the MTTF of the system versus p for some different values of θ and inflation parameter α. For $\alpha = 0$, we have the MTTF of the system in a regular situation.

Table 1: The pmf and reliability functions of the system's lifetime T.

α	p	θ	t	$P(T = t)$	$P(T = t)$ w.r.t. $\alpha = 0$	$P(T > t)$	$P(T > t)$ w.r.t. $\alpha = 0$
0.3	0.4	0.1	1	0.2139	0.2652	0.7861	0.7348
			2	0.3717	0.1948	0.4144	0.5400
			3	0.1251	0.1431	0.2893	0.3969
			4	0.1006	0.1052	0.1887	0.2917
			5	0.0600	0.0773	0.1287	0.2144
0.6	0.6	0.2	1	0.2272	0.5201	0.7728	0.4799
			2	0.6618	0.2495	0.1110	0.2304
			3	0.0532	0.1197	0.0578	0.1107
			4	0.0402	0.0574	0.0176	0.0533
			5	0.0098	0.0275	0.0078	0.0258
0.9	0.8	0.3	1	0.0797	0.7772	0.9203	0.2228
			2	0.9134	0.1687	0.0069	0.0541
			3	0.0035	0.0366	0.0034	0.0175
			4	0.0030	0.0079	0.0004	0.0096
			5	0.0001	0.0017	0.0003	0.0079

7 Conclusions

In this paper, a generalized version of the δ-shock model, first introduced by Poursaeed (2019), is studied under some new assumptions. We assumed that the intershock times are discrete and then investigated the reliability behavior of the system under the assumption that the probability that intershock times occur at a critical time point is significantly high. We obtained the probability generating function and the mean lifetime of the system under the new assumptions. With an illustrative example, we examined the theoretical results numerically. This paper can motivate further studies in this field. We recall that in the model discussed, the random variables N_1 and N_2 were assumed to be independent of the intershock times, and it was also clear that Theorem 5.1 is based on the independence of N from the intershock times. Therefore, as an idea for further work, this model can be studied based on the assumption of the dependence of these variables.

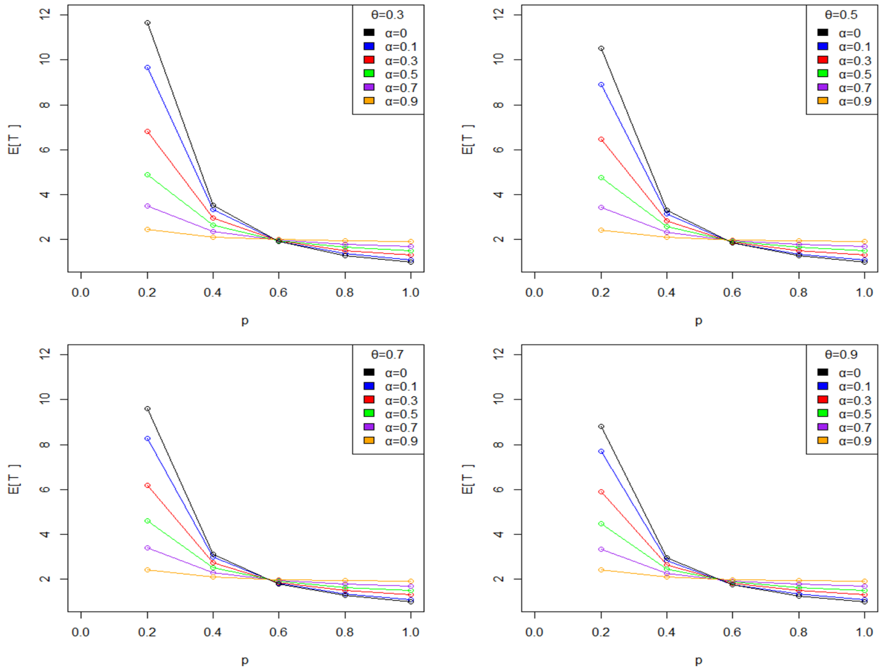

Figure 1: Plot of MTTF versus p for $\delta_1 = 2$, $\delta_2 = 4$, and some different values of α and θ.

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