Journal of Statistical Modelling: Theory and Applications
Vol. 5, No. 2, 2024, pp. 77-95
Yazd University Press 2024

JSMTA

Research Paper

Improved parameters estimation in the multicollinear Poisson regression model based on Stein-Liu estimators

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Received: March 06, 2024/ Revised: July 23, 2024/ Accepted: February 22, 2024

Abstract: This paper addressed parameter estimation in the Poisson regression model in the presence of multicollinearity when it is surmised that the parameter vector is restricted to a linear subspace. To improve the efficiency of parameter estimation, we proposed the Stein-Liu and positive Stein-Liu strategies. The proposed estimators' asymptotic distributional biases and variances were derived, and their variances were compared. The performance of the proposed estimators was investigated through an extensive Monte Carlo simulation study. The suggested estimators were also applied to data from Swedish football. The results confirmed that the performances of our estimators were superior to the unrestricted Liu estimator. As an important result, the Stein-Liu estimators uniformly perform better than the unrestricted Liu estimator.

Keywords: Monte Carlo simulation; Multicollinearity problem; Positive Stein-Liu estimator; Relative efficiency; Stein-Liu estimator.
Mathematics Subject Classification (2010): 62F10, 62F12.

1 Introduction

Many scientific fields such as economics, engineering, public health, insurance, and epidemiology research may comprise count data. The Poisson regression model is appropriate to analyze count data with the same mean and variance. The great source for the Poisson model is the textbook Cameron and Trivedi (2013). The multicollinearity problem occurs in modeling count data when there are some highly correlated predictor variables. This problem leads to the enhancement of the variance of the maximum likelihood estimator (MLE) of the parameters vector, therefore, the interpretations

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based on this estimator are not correct. The ridge estimator introduced by Hoerl and Kennard (1970) is a solution to this problem. The ridge estimator has been applied by Lukman et al. (2023) and Roozbeh et al. (2024) among several others in regression models. Another popular method is the Liu estimator proposed by Liu (1993). This estimator is a linear function of shrinkage parameter, hence has more advantages over the ridge estimator. Several authors have applied the Liu methods in different regression models such as Akdeniz et al. (2022), Arashi et al. (2022), Gelman and Golam Kibria (2020), Qasim et al. (2020), Algamal and Asar (2018), Asar (2016), and Månsson et al. (2016) among many others.

In the context of a regression model, the MLE is commonly used for parameters estimation. However, using uncertain prior information or simply prior information in the estimation process may improve the performance of the estimators. Therefore, the estimator based on the prior information called the restricted estimator performs better than the estimator with no prior information well known unrestricted estimator (or MLE). The prior information is incorporated in the model via a linear restriction on the parameters vector as follows (Ahmed, 2014)

$$\boldsymbol{R}\boldsymbol{\beta} = \boldsymbol{r}, \tag{1}$$

where $\boldsymbol{R}$ is a known matrix of order $p_2 \times p$ and $\boldsymbol{r}$ is a $p_2 \times 1$ vector with constant elements. To improve the estimation of parameters, Stein (1956) introduced Stein and positive Stein estimators by combing the unrestricted and restricted estimators. These estimators perform superiorly better than the maximum likelihood estimator in some parts of the parameter space. We can refer to two great textbooks of Ahmed (2014) and Saleh (2006) in the context of Stein estimators. In recent years, the Stein strategies have been applied in several regression models to improve estimation strategies by various authors. For example, see Zandi et al. (2021, 2023), Al-Momani and Arashi (2024), Plessis et al. (2023) Yuzbasi et al. (2020), and Arashi and Roozbeh (2019) among many others. Recently some authors proposed Stein-Liu estimators to improve estimation of the parameters in the zero-inflated negative binomial model (Zandi et al., 2024), the beta regression model (Arabi Belaghi et al., 2022), and the elliptical linear regression model (Arashi et al., 2014). Also Gelman and Golam Kibria (2020) consider both the unrestricted and restricted Liu estimators in the Poisson regression model with correlated predictor variables. In order to improve this study, our main motivation is to propose improved estimation strategies for estimation of the parameters in the Poisson regression model under a linear restriction on the parameters vector. We also assume that there is near collinearity between the predictor variables. Under these assumptions, we propose the Stein-Liu and positive Stein-Liu estimators. We derive the asymptotic distributional biases (ADBs) and asymptotic distributional variances (ADVs) of the proposed estimators and compare their asymptotic variances. We conduct a Monte Carlo simulation study and apply a real application to compare the benefit of the performance of our estimators with respect to the unrestricted Liu estimator. The results demonstrate the advantage of our estimators in different parts of the parameter space. Based on our results, the performance of the Stein-Liu and positive Stein-Liu estimators are uniformly better than the unrestricted Liu estimator.

The following is how this paper is organized: The Poisson regression model and proposed estimators are discussed in Section 2. In Section 3, it is determined how to obtain the ADBs and ADVs to probe the theoretical properties of the proposed estimators. The asymptotic variances of the proposed estimators are compared in Section 4. In Section 5, a Monte Carlo simulation study is conducted as a numerical comparison to investigate and confirm the theoretical results of the suggested estimators. A real data application to illustrate our findings is analyzed in Section 6. Concluding remarks are presented in Section 7.

2 Model specifications and estimators

The Poisson regression model, unrestricted Liu, and restricted Liu estimators, and Stein-Liu estimators are discussed in this section.

2.1 The Poisson regression model and Liu estimators

Let $Y_1, \ldots, Y_n$ be independent random variables from the Poisson distribution and $y_1, \ldots, y_n$ be the corresponding observations having the probability mass function as follows

$$P_{Y_i}(y_i; \mu_i) = \frac{\mu_i^{y_i} e^{-\mu_i}}{y_i!}, \qquad y_i = 0, 1, 2, \ldots,$$

where $\log(\mu_i) = \boldsymbol{x}_i'\boldsymbol{\beta}$ is the link function of the Poisson regression model where, $\boldsymbol{x}_i = (x_{i1}, \ldots, x_{ip})'$ is the ith row of design matrix $\boldsymbol{X}$ of order $n \times p$ with p predictor variables and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)'$ is the unknown regression parameter vector. The mean and variance in the Poisson regression model are the same as $\boldsymbol{E}(Y_i) = \boldsymbol{V}(Y_i) = \mu_i$. The log-likelihood function of the Poisson model is given as follows

$$\mathcal{L}(\boldsymbol{\beta}; y_i) = \sum_{i=1}^{n} \left\{ y_i\, \boldsymbol{x}_i'\boldsymbol{\beta} - e^{\boldsymbol{x}_i'\boldsymbol{\beta}} - \ln(y_i!) \right\}. \tag{2}$$

The maximum likelihood estimator (or unrestricted estimator) of the parameter vector $\boldsymbol{\beta}$ in this model can be obtained utilizing the iterative weighted least squares (IWLS) algorithm to maximize the log-likelihood function (2) as follows

$$\hat{\boldsymbol{\beta}}_{MLE} = \left(\boldsymbol{X}'\hat{\boldsymbol{\mathcal{W}}}\boldsymbol{X}\right)^{-1}(\boldsymbol{X}'\hat{\boldsymbol{\mathcal{W}}}\boldsymbol{z}),$$

where $\hat{\boldsymbol{\mathcal{W}}} = \text{diag}(\hat{\mu}_1, \hat{\mu}_2, \ldots, \hat{\mu}_n)$, $\hat{\mu}_i = e^{\boldsymbol{x}_i'\hat{\boldsymbol{\beta}}_{MLE}}$, and $z_i = \boldsymbol{x}_i'\hat{\boldsymbol{\beta}}_{MLE} + \frac{y_i - \hat{\mu}_i}{\hat{\mu}_i}$ is the ith element of the vector $\boldsymbol{z}$, for $i = 1, 2, \ldots, n$. The MLE is commonly used to estimate the parameter vector $\boldsymbol{\beta}$ when the predictor variables are independent. However, this estimator is sensitive to the multicollinearity problem. In this situation, the Liu estimator introduced by Liu (1993) is widely applied to solve this problem. The Liu estimator (or unrestricted Liu (UL)) is defined by Gelman and Golam Kibria (2020) in the Poisson regression model as follows

$$\hat{\boldsymbol{\beta}}_{UL} = \left(\boldsymbol{X}'\hat{\boldsymbol{\mathcal{W}}}\boldsymbol{X} + \boldsymbol{I}_p\right)^{-1}\left(\boldsymbol{X}'\hat{\boldsymbol{\mathcal{W}}}\boldsymbol{X} + d\,\boldsymbol{I}_p\right)\hat{\boldsymbol{\beta}}_{MLE},$$

where $\boldsymbol{I}_p$ is an identity matrix of order p and $0 \leq d \leq 1$ is the shrinkage parameter. Following Gelman and Golam Kibria (2020), the MSE function of the UL estimator is defined as

$$MSE(\hat{\boldsymbol{\beta}}_{UL}) = \sum_{j=1}^{p} \frac{(\lambda_j + d)^2}{\lambda_j(\lambda_j + 1)^2} + (d-1)^2 \sum_{j=1}^{p} \frac{\alpha_j^2}{(\lambda_j + 1)^2}, \tag{3}$$

where λ_j is the jth eigenvalue of matrix $\boldsymbol{C} = \boldsymbol{X}'\hat{\boldsymbol{\mathcal{W}}}\boldsymbol{X}$ and α_j is the jth element of $\boldsymbol{G}'\boldsymbol{\beta}$, where $\boldsymbol{G}$ is a $p \times p$ matrix whose its jth column is the corresponding eigenvector of λ_j where, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$. Based on the work of Månsson (2013), we estimate the shrinkage parameter d as follows

$$\hat{d} = \max\Big(0, \text{median}\Big\{\frac{\hat{\alpha}_j^2 - 1}{\lambda_j^{-1} + \hat{\alpha}_j^2}\Big\}\Big),$$

where $\hat{\alpha} = (\hat{\alpha}_1, \ldots, \hat{\alpha}_p)' = \boldsymbol{G}'\hat{\boldsymbol{\beta}}_{MLE}$ and the expression $\frac{\hat{\alpha}_j^2 - 1}{\lambda_j^{-1} + \hat{\alpha}_j^2}$ is obtained by taking the first derivative of (3) with respect to d and setting it to zero.

Under the linear restriction (1), the restricted estimator or restricted maximum likelihood estimator of $\boldsymbol{\beta}$ is more efficient than the unrestricted estimator which is defined by Gelman and Golam Kibria (2020) in the Poisson regression model as follows

$$\hat{\boldsymbol{\beta}}_R = \hat{\boldsymbol{\beta}}_{MLE} - \boldsymbol{C}^{-1}\boldsymbol{R}'\big(\boldsymbol{R}\boldsymbol{C}^{-1}\boldsymbol{R}'\big)^{-1}(\boldsymbol{R}\hat{\boldsymbol{\beta}}_{MLE} - \boldsymbol{r}),$$

Indeed, the restricted estimator is obtained by maximizing the log-likelihood function (2) with respect to $\boldsymbol{\beta}$ under the linear restriction (1). In the presence of multicollinearity, $\boldsymbol{C}$ may be a singular matrix, thus $\hat{\boldsymbol{\beta}}_R$ is not suitable because of having high variance. One solution is using restricted Liu (RL) estimator as follows (Gelman and Golam Kibria, 2020)

$$\hat{\boldsymbol{\beta}}_{RL} = \boldsymbol{A}\hat{\boldsymbol{\beta}}_{MLE} - \boldsymbol{A}\boldsymbol{C}^{-1}\boldsymbol{R}'\big(\boldsymbol{R}\boldsymbol{C}^{-1}\boldsymbol{R}'\big)^{-1}(\boldsymbol{R}\hat{\boldsymbol{\beta}}_{MLE} - \boldsymbol{r}),$$

where $\boldsymbol{A} = \big(\boldsymbol{X}'\hat{\boldsymbol{\mathcal{W}}}\boldsymbol{X} + \boldsymbol{I}_p\big)^{-1}\big(\boldsymbol{X}'\hat{\boldsymbol{\mathcal{W}}}\boldsymbol{X} + d\,\boldsymbol{I}_p\big)$. The RL estimator has been applied by Zandi et al. (2024), Gelman and Golam Kibria (2020), and Wu and Asar (2017), and Månsson et al. (2016).

2.2 Improved estimators

The Stein and positive Stein estimators first have been introduced by Stein (1956) by combining the unrestricted and restricted estimators when the predictor variables are independent and there is the prior information as a linear restriction. Our goal in this paper is to improve the estimation process in the Poisson regression model in the presence of multicollinearity and under the linear restriction (1). Parallel to the idea of Zandi et al. (2024), Arabi Belaghi et al. (2022), and Arashi et al. (2014), we propose the Stein-Liu (SL) estimator utilizing $\hat{\boldsymbol{\beta}}_{UL}$ and $\hat{\boldsymbol{\beta}}_{RL}$ as follows

$$\hat{\boldsymbol{\beta}}_{SL} = \hat{\boldsymbol{\beta}}_{RL} + \Big(1 - \frac{p_2 - 2}{T_n}\Big)\Big(\hat{\boldsymbol{\beta}}_{UL} - \hat{\boldsymbol{\beta}}_{RL}\Big), \qquad p_2 = 3, 4, 5, \ldots,$$

here T_n denotes the likelihood ratio test statistic for testing the validity of the prior information as the following hypotheses

$$\mathcal{H}_0 \,:\, \boldsymbol{R}\boldsymbol{\beta} = \boldsymbol{r} \quad vs. \quad \mathcal{H}_1 \,:\, \boldsymbol{R}\boldsymbol{\beta} \neq \boldsymbol{r}.$$

Therefore, T_n is defined as follows

$$T_n = 2\{\mathcal{L}(\hat{\boldsymbol{\beta}}_{UL}; y_i) - \mathcal{L}(\hat{\boldsymbol{\beta}}_{RL}; y_i)\},$$

where $\mathcal{L}(\hat{\boldsymbol{\beta}}_{UL}; y_i)$ and $\mathcal{L}(\hat{\boldsymbol{\beta}}_{RL}; y_i)$ are, respectively, the log-likelihood function (2) values for the unrestricted and restricted Liu estimators. The test statistic T_n under $\mathcal{H}_0$ has the asymptotically Chi-squared distribution with p_2 degree of freedom when $n \to \infty$. To remedy the Stein-Liu estimator's over-shrinking issue, an adjusted version of this estimator is called the positive Stein-Liu (PSL) estimator that eliminates the negative values of weighted function $1 - \frac{p_2-2}{T_n}$ and is defined as follows

$$\hat{\boldsymbol{\beta}}_{PSL} = \hat{\boldsymbol{\beta}}_{RL} + \max\Big\{0, 1 - \frac{p_2-2}{T_n}\Big\}\Big(\hat{\boldsymbol{\beta}}_{UL} - \hat{\boldsymbol{\beta}}_{RL}\Big), \qquad p_2 = 3, 4, 5, \ldots.$$

For $\big(\boldsymbol{X}'\hat{\boldsymbol{\mathcal{W}}}\boldsymbol{X} + \boldsymbol{I}_p\big)^{-1}\big(\boldsymbol{X}'\hat{\boldsymbol{\mathcal{W}}}\boldsymbol{X} + d\,\boldsymbol{I}_p\big) = \boldsymbol{I}_p$, the Stein-Liu and positive Stein-Liu estimators reduce to the Stein and positive Stein estimators, respectively.

3 Asymptotic properties

Now, the ADB and the ADV of the $\hat{\boldsymbol{\beta}}_{UL}$, $\hat{\boldsymbol{\beta}}_{RL}$, $\hat{\boldsymbol{\beta}}_{SL}$, and $\hat{\boldsymbol{\beta}}_{PSL}$ are theoretically discussed under the sequence of local alternatives as

$$\mathcal{K}_{(n)} \,:\, \boldsymbol{R\beta} = \boldsymbol{r} + \frac{\boldsymbol{\delta}}{\sqrt{n}}, \tag{4}$$

where $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_{p_2})' \in R^{p_2}$ is a $p_2 \times 1$ known vector. Let $\hat{\boldsymbol{\beta}}_\diamond$ be any of the proposed estimators of $\boldsymbol{\beta}$, the ADB of $\hat{\boldsymbol{\beta}}_\diamond$ is defined as follows

$$ADB(\hat{\boldsymbol{\beta}}_\diamond) = \lim_{n\to\infty} E\Big(\sqrt{n}(\hat{\boldsymbol{\beta}}_\diamond - \boldsymbol{\beta})\Big).$$

Also, the ADV of $\hat{\boldsymbol{\beta}}_\diamond$ is defined as:

$$ADV(\hat{\boldsymbol{\beta}}_\diamond) = \lim_{n\to\infty} E\Big(\sqrt{n}(\hat{\boldsymbol{\beta}}_\diamond - \boldsymbol{\beta})\,\sqrt{n}(\hat{\boldsymbol{\beta}}_\diamond - \boldsymbol{\beta})'\Big).$$

In the following theorem, the asymptotic distributional biases of the proposed estimators are obtained.

Theorem 3.1. *Under the sequence of local alternatives in* (4) *and the usual regularity conditions of the MLE, as $n \to \infty$, the ADBs of the proposed estimators are*

$$\begin{aligned}
ADB(\hat{\boldsymbol{\beta}}_{UL}) &= (\boldsymbol{A} - \mathbf{I}_p)\,\boldsymbol{\beta}, \\
ADB(\hat{\boldsymbol{\beta}}_{RL}) &= (\boldsymbol{I}_p - \boldsymbol{\mathcal{J}}\,\mathbf{R})(\boldsymbol{A} - \boldsymbol{I}_p)\,\boldsymbol{\beta} - \boldsymbol{\mathcal{J}}\,\boldsymbol{\delta}, \\
ADB(\hat{\boldsymbol{\beta}}_{SL}) &= ADB(\hat{\boldsymbol{\beta}}_{UL}) - (p_2-2)\,\boldsymbol{\mathcal{J}}\,\left[\boldsymbol{R}\,(\boldsymbol{A} - \boldsymbol{I}_p)\,\boldsymbol{\beta} + \boldsymbol{\delta}\right] E\Big[\frac{1}{\chi^2_{p_2+2}(\Delta^*)}\Big], \\
ADB(\hat{\boldsymbol{\beta}}_{PSL}) &= ADB(\hat{\boldsymbol{\beta}}_{SL}) - \,\boldsymbol{\mathcal{J}}\,[\boldsymbol{R}(\boldsymbol{A} - \boldsymbol{I}_p)\,\boldsymbol{\beta} + \boldsymbol{\delta}]\Big\{\boldsymbol{\Psi}_{p_2+2}(\chi^2_{p_2,\alpha}; \Delta^*)
\end{aligned}$$

$$+(p_2-2)\,E\Big[\frac{I(\chi^2_{p_2+2}(\Delta^*)<p_2-2)}{\chi^2_{p_2+2}(\Delta^*)}\Big]\Big\},$$

where $\boldsymbol{\mathcal{J}}=\boldsymbol{C}^{-1}\boldsymbol{R}'(\boldsymbol{R}\boldsymbol{C}^{-1}\boldsymbol{R}')^{-1}$, $\boldsymbol{\Psi}_v(.;\Delta^*)$ *is the cumulative distribution function of the* $\chi^2_v(\Delta^*)$ *distribution, and* $\Delta^*=\boldsymbol{\delta}'(\boldsymbol{R}\boldsymbol{C}^{-1}\boldsymbol{R}')^{-1}\boldsymbol{\delta}$ *is the non-centrality parameter.*

Proof. See Appendix. □

The asymptotic distributional variances of the proposed estimators are presented n the following theorem.

Theorem 3.2. *Under the local alternatives in* (4) *and the usual regularity conditions of the MLE, as* $n\to\infty$, *the ADVs of the estimators are*

$$\begin{aligned}
ADV(\hat{\boldsymbol{\beta}}_{UL}) &= \boldsymbol{A}\boldsymbol{C}^{-1}\boldsymbol{A}' + \Big[(\boldsymbol{A}-\mathbf{I}_p)\,\boldsymbol{\beta}\Big]\Big[(\boldsymbol{A}-\mathbf{I}_p)\,\boldsymbol{\beta}\Big]',\\
ADV(\hat{\boldsymbol{\beta}}_{RL}) &= \boldsymbol{A}\,\boldsymbol{C}^{-1}\,\boldsymbol{A}' - \boldsymbol{\mathcal{J}}\,\boldsymbol{R}\,\boldsymbol{A}\,\boldsymbol{C}^{-1}\,\boldsymbol{A}'\\
&\quad+\Big[(\mathbf{I}_p-\boldsymbol{\mathcal{J}}\,\boldsymbol{R})\,(\boldsymbol{A}-\boldsymbol{I}_p)\,\boldsymbol{\beta}-\boldsymbol{\mathcal{J}}\,\boldsymbol{\delta}\Big]\Big[(\mathbf{I}_p-\boldsymbol{\mathcal{J}}\,\boldsymbol{R})\,(\boldsymbol{A}-\boldsymbol{I}_p)\,\boldsymbol{\beta}-\boldsymbol{\mathcal{J}}\,\boldsymbol{\delta}\Big]',\\
ADV(\hat{\boldsymbol{\beta}}_{SL}) &= ADV(\hat{\boldsymbol{\beta}}_{UL}) - 2\,(p_2-2)\,\Bigg(\Big[(\boldsymbol{I}_p-\boldsymbol{\mathcal{J}}\,\boldsymbol{R})\,(\boldsymbol{A}-\boldsymbol{I}_p)\,\boldsymbol{\beta}-\boldsymbol{\mathcal{J}}\,\boldsymbol{\delta}\Big]\\
&\quad\times\big[\boldsymbol{\mathcal{J}}\,\boldsymbol{R}[\boldsymbol{A}-\boldsymbol{I}_p]\,\boldsymbol{\beta}+\boldsymbol{\mathcal{J}}\,\boldsymbol{\delta}\big]\,E\Big[\frac{1}{\chi^2_{p_2+2}(\Delta^*)}\Big]\Bigg)\\
&\quad+(p_2-2)(p_2-4)\boldsymbol{\mathcal{J}}\boldsymbol{R}\boldsymbol{A}\boldsymbol{C}^{-1}\boldsymbol{A}'\Bigg(E\Big[\frac{1}{(\chi^2_{p_2+2}(\Delta^*))^2}\Big]-E\Big[\frac{1}{\chi^2_{p_2+2}(\Delta^*)}\Big]\Bigg)\\
&\quad+(p_2-2)\,(p_2-4)\,[\boldsymbol{\mathcal{J}}\,\boldsymbol{R}\,(\boldsymbol{A}-\boldsymbol{I}_p)\,\boldsymbol{\beta}+\boldsymbol{\mathcal{J}}\,\boldsymbol{\delta}]\,[\boldsymbol{\mathcal{J}}\,\boldsymbol{R}\,(\boldsymbol{A}-\boldsymbol{I}_p)\,\boldsymbol{\beta}+\boldsymbol{\mathcal{J}}\,\boldsymbol{\delta}]'\\
&\quad\times\Bigg(E\Big[\frac{1}{(\chi^2_{p_2+4}(\Delta^*))^2}\Big]-E\Big[\frac{1}{\chi^2_{p_2+4}(\Delta^*)}\Big]\Bigg),\\
ADV(\hat{\boldsymbol{\beta}}_{PSL}) &= ADV(\hat{\boldsymbol{\beta}}_{SL}) - 2\,\Bigg(\{(\boldsymbol{I}_p-\boldsymbol{\mathcal{J}}\,\boldsymbol{R})\,(\boldsymbol{A}-\boldsymbol{I}_p)\,\boldsymbol{\beta}-\boldsymbol{\mathcal{J}}\,\boldsymbol{\delta}\}\,[\boldsymbol{\mathcal{J}}\,\boldsymbol{R}\,(\boldsymbol{A}-\boldsymbol{I}_p)\,\boldsymbol{\beta}]\\
&\quad\times E\Big[\Big(1-\frac{p_2-2}{\chi^2_{p_2+2}(\Delta^*)}\Big)\,I(\chi^2_{p_2+2}(\Delta^*)<p_2-2)\Big]\Bigg)\\
&\quad-\Bigg(\boldsymbol{\mathcal{J}}\,\boldsymbol{R}\,\boldsymbol{A}\,\boldsymbol{C}^{-1}\,\boldsymbol{A}'\,E\Big[\Big(1-\frac{p_2-2}{\chi^2_{p_2+2}(\Delta^*)}\Big)^2\,I(\chi^2_{p_2+2}(\Delta^*)<p_2-2)\Big]\\
&\quad+[\boldsymbol{\mathcal{J}}\,\boldsymbol{R}\,(\boldsymbol{A}-\boldsymbol{I}_p)\,\boldsymbol{\beta}+\boldsymbol{\mathcal{J}}\,\boldsymbol{\delta}]\,[\boldsymbol{\mathcal{J}}\,\boldsymbol{R}\,(\boldsymbol{A}-\boldsymbol{I}_p)\,\boldsymbol{\beta}+\boldsymbol{\mathcal{J}}\,\boldsymbol{\delta}]'\\
&\quad\times E\Big[\Big(1-\frac{p_2-2}{\chi^2_{p_2+4}(\Delta^*)}\Big)^2\,I(\chi^2_{p_2+4}(\Delta^*)<p_2-2)\Big]\Bigg).
\end{aligned}$$

Proof. See Appendix. □

4 Comparison the asymptotic variances of the proposed estimators

We now compare the ADVs of the proposed estimators following Arabi Belaghi et al. (2022). The following definition, which is helpful for this comparison, is presented first.

Definition 4.1. *Let $\mathcal{B}$ be the parameter space of $\boldsymbol{\beta}$. If $\hat{\boldsymbol{\beta}}_*$ and $\hat{\boldsymbol{\beta}}_{**}$ be two estimators of $\boldsymbol{\beta}$, such that $ADV(\hat{\boldsymbol{\beta}}_\star) \leq ADV(\hat{\boldsymbol{\beta}}_{\star\star})$ for all values of $\boldsymbol{\beta} \in \mathcal{B}$, with strict inequality for at least one $\boldsymbol{\beta}$, we say that $\hat{\boldsymbol{\beta}}_*$ dominates $\hat{\boldsymbol{\beta}}_{**}$.*

Comparing $\hat{\boldsymbol{\beta}}_{RL}$ and $\hat{\boldsymbol{\beta}}_{UL}$

$\hat{\boldsymbol{\beta}}_{RL}$ dominates $\hat{\boldsymbol{\beta}}_{UL}$ if

$$\begin{aligned} ADV(\hat{\boldsymbol{\beta}}_{RL}) - ADV(\hat{\boldsymbol{\beta}}_{UL}) = & \boldsymbol{\mathcal{J}RAC}^{-1}\boldsymbol{A}' - \Big[\big\{(\boldsymbol{A}-\boldsymbol{I}_p)\boldsymbol{\beta}\big\}\big\{\boldsymbol{\mathcal{J}R}(\boldsymbol{A}-\boldsymbol{I}_p)\boldsymbol{\beta}+\boldsymbol{\mathcal{J}\delta}\big\}'\Big] \\ & \times\Big[\big\{\boldsymbol{\mathcal{J}R}(\boldsymbol{A}-\boldsymbol{I}_p)\boldsymbol{\beta}+\boldsymbol{\mathcal{J}\delta}\big\}\big\{(\boldsymbol{A}-\boldsymbol{I}_p)\boldsymbol{\beta}\big\}'\Big] \\ & \times\Big(\boldsymbol{\mathcal{J}R}(\boldsymbol{A}-\boldsymbol{I}_p)\boldsymbol{\beta}+\boldsymbol{\mathcal{J}\delta}\Big)\Big(\boldsymbol{\mathcal{J}R}(\boldsymbol{A}-\boldsymbol{I}_p)\boldsymbol{\beta}+\boldsymbol{\mathcal{J}\delta}\Big)' \\ & \leq 0, \qquad \forall\ \Delta^* \in (0,\infty). \end{aligned}$$

Comparing $\hat{\boldsymbol{\beta}}_{SL}$ and $\hat{\boldsymbol{\beta}}_{UL}$

$\hat{\boldsymbol{\beta}}_{SL}$ dominates $\hat{\boldsymbol{\beta}}_{UL}$ if

$$\begin{aligned} ADV(\hat{\boldsymbol{\beta}}_{SL}) - ADV(\hat{\boldsymbol{\beta}}_{UL}) = & -2\,(p_2-2)\Bigg(\big[(\boldsymbol{I}_p-\boldsymbol{\mathcal{J}}\,\boldsymbol{R})\,(\boldsymbol{A}-\boldsymbol{I}_p)\,\boldsymbol{\beta}-\boldsymbol{\mathcal{J}}\,\boldsymbol{\delta}\big] \\ & \times\big[\boldsymbol{\mathcal{J}}\,\boldsymbol{R}[\boldsymbol{A}-\boldsymbol{I}_p]\,\boldsymbol{\beta}+\boldsymbol{\mathcal{J}}\,\boldsymbol{\delta}\big]\,E\Big[\frac{1}{\chi^2_{p_2+2}(\Delta^*)}\Big]\Bigg) \\ & +(p_2-2)\,(p_2-4)\,\boldsymbol{\mathcal{J}}\,\boldsymbol{R}\,\boldsymbol{A}\,\boldsymbol{C}^{-1}\,\boldsymbol{A}'\Bigg(E\Big[\frac{1}{(\chi^2_{p_2+2}(\Delta^*))^2}\Big] \\ & -E\Big[\frac{1}{\chi^2_{p_2+2}(\Delta^*)}\Big]\Bigg) + (p_2-2)(p_2-4) \\ & \times[\boldsymbol{\mathcal{J}R}(\boldsymbol{A}-\boldsymbol{I}_p)\boldsymbol{\beta}+\boldsymbol{\mathcal{J}\delta}][\boldsymbol{\mathcal{J}}\,\boldsymbol{R}\,(\boldsymbol{A}-\boldsymbol{I}_p)\,\boldsymbol{\beta}+\boldsymbol{\mathcal{J}}\,\boldsymbol{\delta}]' \\ & \times\Bigg(E\Big[\frac{1}{(\chi^2_{p_2+4}(\Delta^*))^2}\Big]-E\Big[\frac{1}{\chi^2_{p_2+4}(\Delta^*)}\Big]\Bigg) \\ & \leq 0, \qquad \forall\ \Delta^* \in (0,\infty). \end{aligned}$$

Comparing $\hat{\boldsymbol{\beta}}_{PSL}$ and $\hat{\boldsymbol{\beta}}_{UL}$

$\hat{\boldsymbol{\beta}}_{PSL}$ dominates $\hat{\boldsymbol{\beta}}_{UL}$ if

$$\begin{aligned} ADV(\hat{\boldsymbol{\beta}}_{PSL}) - ADV(\hat{\boldsymbol{\beta}}_{UL}) = & -2(p_2-2)\Bigg(\big[(\boldsymbol{I}_p-\boldsymbol{\mathcal{J}R})(\boldsymbol{A}-\boldsymbol{I}_p)\boldsymbol{\beta}-\boldsymbol{\mathcal{J}\delta}\big] \\ & \times\big[\boldsymbol{\mathcal{J}R}[\boldsymbol{A}-\boldsymbol{I}_p]\boldsymbol{\beta}+\boldsymbol{\mathcal{J}\delta}\big]E\Big[\frac{1}{\chi^2_{p_2+2}(\Delta^*)}\Big]\Bigg) \\ & +(p_2-2)(p_2-4)\boldsymbol{\mathcal{J}RAC}^{-1}\boldsymbol{A}'\Bigg(E\Big[\frac{1}{(\chi^2_{p_2+2}(\Delta^*))^2}\Big]-E\Big[\frac{1}{\chi^2_{p_2+2}(\Delta^*)}\Big]\Bigg) \\ & +(p_2-2)(p_2-4)[\boldsymbol{\mathcal{J}R}(\boldsymbol{A}-\boldsymbol{I}_p)\boldsymbol{\beta}+\boldsymbol{\mathcal{J}\delta}][\boldsymbol{\mathcal{J}R}(\boldsymbol{A}-\boldsymbol{I}_p)\boldsymbol{\beta}+\boldsymbol{\mathcal{J}\delta}]' \\ & \times\Bigg(E\Big[\frac{1}{(\chi^2_{p_2+4}(\Delta^*))^2}\Big]-E\Big[\frac{1}{\chi^2_{p_2+4}(\Delta^*)}\Big]\Bigg) \\ & -2\Bigg(\{(\boldsymbol{I}_p-\boldsymbol{\mathcal{J}R})(\boldsymbol{A}-\boldsymbol{I}_p)\boldsymbol{\beta}-\boldsymbol{\mathcal{J}\delta}\}[\boldsymbol{\mathcal{J}R}(\boldsymbol{A}-\boldsymbol{I}_p)\boldsymbol{\beta}] \end{aligned}$$

$$\times E\Big[\Big(1-\frac{p_2-2}{\chi^2_{p_2+2}(\Delta^*)}\Big)I(\chi^2_{p_2+2}(\Delta^*)<p_2-2)\Big]\Big)$$
$$-\Big(\boldsymbol{\mathcal{J}RAC}^{-1}\boldsymbol{A}'E\Big[\Big(1-\frac{p_2-2}{\chi^2_{p_2+2}(\Delta^*)}\Big)^2 I(\chi^2_{p_2+2}(\Delta^*)<p_2-2)\Big]$$
$$+[\boldsymbol{\mathcal{J}R}(\boldsymbol{A}-\boldsymbol{I}_p)\boldsymbol{\beta}+\boldsymbol{\mathcal{J}\delta}][\boldsymbol{\mathcal{J}R}(\boldsymbol{A}-\boldsymbol{I}_p)\boldsymbol{\beta}+\boldsymbol{\mathcal{J}\delta}]'$$
$$\times E\Big[\Big(1-\frac{p_2-2}{\chi^2_{p_2+4}(\Delta^*)}\Big)^2 I(\chi^2_{p_2+4}(\Delta^*)<p_2-2)\Big]\Big)\leq 0,\quad \forall\Delta^*\in(0,\infty).$$

Comparing $\hat{\boldsymbol{\beta}}_{PSL}$ and $\hat{\boldsymbol{\beta}}_{SL}$

$\hat{\boldsymbol{\beta}}_{PSL}$ dominates $\hat{\boldsymbol{\beta}}_{SL}$ if

$$ADV(\hat{\boldsymbol{\beta}}_{PSL})-ADV(\hat{\boldsymbol{\beta}}_{SL})=-2\Big(\{(\boldsymbol{I}_p-\boldsymbol{\mathcal{J}}\,\boldsymbol{R})\,(\boldsymbol{A}-\boldsymbol{I}_p)\,\boldsymbol{\beta}-\boldsymbol{\mathcal{J}}\,\boldsymbol{\delta}\}\,[\boldsymbol{\mathcal{J}}\,\boldsymbol{R}\,(\boldsymbol{A}-\boldsymbol{I}_p)\,\boldsymbol{\beta}]$$
$$\times E\Big[\Big(1-\frac{p_2-2}{\chi^2_{p_2+2}(\Delta^*)}\Big)\,I(\chi^2_{p_2+2}(\Delta^*)<p_2-2)\Big]\Big)$$
$$-\Big(\boldsymbol{\mathcal{J}}\,\boldsymbol{R}\,\boldsymbol{A}\,\boldsymbol{C}^{-1}\,\boldsymbol{A}'\,E\Big[\Big(1-\frac{p_2-2}{\chi^2_{p_2+2}(\Delta^*)}\Big)^2\,I(\chi^2_{p_2+2}(\Delta^*)<p_2-2)\Big]$$
$$+[\boldsymbol{\mathcal{J}}\,\boldsymbol{R}\,(\boldsymbol{A}-\boldsymbol{I}_p)\,\boldsymbol{\beta}+\boldsymbol{\mathcal{J}}\,\boldsymbol{\delta}]\,[\boldsymbol{\mathcal{J}}\,\boldsymbol{R}\,(\boldsymbol{A}-\boldsymbol{I}_p)\,\boldsymbol{\beta}+\boldsymbol{\mathcal{J}}\,\boldsymbol{\delta}]'$$
$$\times E\Big[\Big(1-\frac{p_2-2}{\chi^2_{p_2+4}(\Delta^*)}\Big)^2\,I(\chi^2_{p_2+4}(\Delta^*)<p_2-2)\Big]\Big)\leq 0,\quad \forall\,\Delta^*\in(0,\infty).$$

5 Simulation study

We do a Monte Carlo simulation study to evaluate and compare the performance of the suggested estimators to the unrestricted Liu estimator using the simulated relative efficiency (SRE) criteria, which is defined as follows

$$SRE(\hat{\boldsymbol{\beta}}_{UL},\hat{\boldsymbol{\beta}}_*)=\frac{SMSE(\hat{\boldsymbol{\beta}}_{UL})}{SMSE(\hat{\boldsymbol{\beta}}_*)},$$

where $\hat{\boldsymbol{\beta}}_*$ is any one of the $\hat{\boldsymbol{\beta}}_{RL}$, $\hat{\boldsymbol{\beta}}_{SL}$, and $\hat{\boldsymbol{\beta}}_{PSL}$, and $SMSE(\hat{\boldsymbol{\beta}}_*)$ is the simulated mean squared error of $\hat{\boldsymbol{\beta}}_*$ and is defined

$$SMSE(\hat{\boldsymbol{\beta}}_*)=\frac{\sum_{t=1}^{2000}(\hat{\boldsymbol{\beta}}_*-\boldsymbol{\beta})'_t(\hat{\boldsymbol{\beta}}_*-\boldsymbol{\beta})_t}{2000}.$$

It's clear that, for the value of SRE is greater than one, $\hat{\boldsymbol{\beta}}_*$ dominates the unrestricted Liu estimator.

We generate the correlated predictor variables in the Poisson model as follows (Zandi et al., 2024)

$$x_{ij}=\sqrt{1-\rho^2}u_{ij}+\rho\,u_{ip},\quad i=1,2,\ldots,n,\quad j=1,2,\ldots,p,$$

where x_{ij}s are elements of $\boldsymbol{X}$, ρ represents the correlation level between the predictor variables and u_{ij}s are generated independently from $N(0,1)$. Hence, the response variable Y_i is generated from the Poisson distribution with the mean parameter as follows

$$\mu_i = e^{\boldsymbol{x}'_i \boldsymbol{\beta}}. \tag{5}$$

We consider a special case of the linear restriction (1), where $\boldsymbol{R} = [\mathbf{0}_{p_1 \times p_2}, \boldsymbol{I}_{p_2 \times p_2}]$ such that $\mathbf{0}$ is a zeros matrix and $\boldsymbol{I}$ is an identity matrix and $\boldsymbol{r} = \mathbf{0}$. In this case, the parameter vector $\boldsymbol{\beta}$ can be partitioned as $\boldsymbol{\beta} = (\boldsymbol{\beta}'_1, \boldsymbol{\beta}'_2)'$. Therefore, the linear restriction reduces to $\boldsymbol{\beta}_2 = \mathbf{0}$, which means that some of the predictors do not have a significant effect on the response variable. So, $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ are $p_1 \times 1$ and $p_2 \times 1$ sub-vectors contain the parameters related to the significant (active) and non significant (inactive) predictors, respectively. We set $\boldsymbol{\beta}_1 = (0.09, 0.27, 0.34)'$ and $\boldsymbol{\beta}_2 = (\mathbf{0}_{p_2})'$ in (5) to generate Y_i. For performance comparison of our proposed estimators to the unrestricted Liu estimator in different parts of the parameter space, we define a distance between the proposed sub-model and the simulated model as $\Delta = \|\boldsymbol{\beta} - \boldsymbol{\beta}^0\|^2$, in which $\boldsymbol{\beta}^0 = (\boldsymbol{\beta}'_1, \mathbf{0}_{p_2})$ is the true parameter and $\|.\|$ is the Euclidean norm. The simulation is conducted in R statistical software for different values of $\Delta = \{0, 0.4, 0.6, 0.8, 1, 2, 3, 4\}$, the number of active parameter $p_1 = 3$, the number of inactive parameters $p_2 = 3, 5, 7$, two sample sizes $n = 35, 50$, and three values of the correlation level between the predictor variables $\rho = 0.90, 0.93, 0.95$. The simulations are replicated 2000 times for each case to accurate the simulation results. The SREs of the proposed estimators are reported in Tables 1-3 and Figures 1-2. Our findings are summarized as follows

1. The SREs of all estimators increase as the number of inactive predictors p_2 increases.
2. The simulated relative efficiency of all estimators increases by increasing the correlation ρ between the predictor variables.
3. The SRE of all estimators decreases as the sample size n increases.
4. The restricted Liu estimator outperforms better than the Stein-Liu estimators at $\Delta = 0$. However, as Δ moves away from zero, the performance of this estimator sharply decreases, so that for $\Delta > 1$, its SRE becomes unbounded.
5. As we would expect, the positive Stein-Liu estimator dominates the Stein-Liu estimator at $\Delta = 0$ and near it in any situation. These two estimators perform equally well and better than the restricted Liu estimator for $\Delta \geq 2$.
6. An important result is that the SREs of the Stein-Liu and positive Stein-Liu estimators are greater than one for all values of Δ. It means that these two estimators uniformly outperform the unrestricted Liu estimator.

6 Empirical application

To evaluate the performance of our proposed estimators, we apply Swedish football data during 2021 by considering the performance of Swedish football teams in the top Swedish league (Allsvenskan). This dataset includes $n = 242$ observations and contains six predictor variables as the pinnacle home win odds (x_1), pinnacle away win odds (x_2), oddsportal maximum home win (x_3), oddsportal maximum away win (x_4), average oddsportal home win (x_5) and average oddsportal away win (x_6). The response variable is the number of full-time away team goals (FTATG) (Qasim et al. (2020)). Based on the Chi-square goodness of fit test in Qasim et al. (2020), the test statistic value is $\chi^2 = 1.7359$ and $p - value = 0.8843$. Therefore, the obtained p-value confirms that at a 0.05 significance level, the response variable follows a Poisson distribution.

Table 1: *SREs* of the proposed estimators with respect to $\hat{\boldsymbol{\beta}}_{UL}$ for $p_2 = 3$.

		$n = 35$			$n = 50$		
ρ	Δ	RL	SL	PSL	RL	SL	PSL
0.90	0.0	4.946	1.456	1.627	3.139	1.198	1.490
	0.4	3.125	1.374	1.439	2.001	1.167	1.319
	0.6	2.115	1.259	1.303	1.313	1.172	1.197
	0.8	1.453	1.196	1.201	0.879	1.111	1.119
	1.0	1.035	1.136	1.137	0.614	1.078	1.078
	2.0	0.303	1.037	1.037	0.173	1.019	1.019
	3.0	0.139	1.017	1.017	0.078	1.008	1.008
	4.0	0.079	1.010	1.010	0.044	1.005	1.005
0.93	0.0	5.627	1.532	1.661	3.365	1.288	1.493
	0.4	3.946	1.450	1.512	2.469	1.285	1.383
	0.6	2.836	1.350	1.385	1.756	1.213	1.269
	0.8	2.030	1.264	1.275	1.238	1.156	1.175
	1.0	1.485	1.192	1.195	0.894	1.109	1.115
	2.0	0.456	1.053	1.053	0.264	1.029	1.029
	3.0	0.211	1.024	1.024	0.121	1.012	1.012
	4.0	0.121	1.014	1.014	0.068	1.007	1.007
0.95	0.0	5.764	1.464	1.595	3.534	1.386	1.507
	0.4	4.418	1.417	1.501	2.853	1.295	1.433
	0.6	3.381	1.343	1.408	2.178	1.245	1.335
	0.8	2.540	1.223	1.317	1.619	1.176	1.243
	1.0	1.922	1.222	1.241	1.210	1.158	1.172
	2.0	0.632	1.070	1.070	0.382	1.044	1.044
	3.0	0.298	1.031	1.031	0.177	1.019	1.019
	4.0	0.171	1.018	1.018	0.101	1.010	1.010

Table 2: *SREs* of the proposed estimators with respect to $\hat{\boldsymbol{\beta}}_{UL}$ for $p_2 = 5$.

		$n = 35$			$n = 50$		
ρ	Δ	RL	SL	PSL	RL	SL	PSL
0.90	0.0	5.661	2.025	2.450	5.035	1.871	2.440
	0.4	3.374	1.787	1.960	2.684	1.715	1.829
	0.6	2.312	1.588	1.650	1.667	1.433	1.510
	0.8	1.614	1.425	1.437	1.086	1.310	1.322
	1.0	1.166	1.301	1.303	0.749	1.216	1.216
	2.0	0.356	1.075	1.075	0.208	1.057	1.057
	3.0	0.166	1.030	1.030	0.094	1.026	1.026
	4.0	0.095	1.015	1.015	0.053	1.014	1.014
0.93	0.0	5.889	2.028	2.461	5.282	1.803	2.465
	0.4	3.911	1.892	2.076	3.274	1.833	1.977
	0.6	2.845	1.641	1.786	2.194	1.577	1.662
	0.8	2.073	1.525	1.560	1.498	1.409	1.443
	1.0	1.542	1.393	1.401	1.063	1.302	1.307
	2.0	0.498	1.107	1.107	0.310	1.084	1.084
	3.0	0.235	1.044	1.044	0.142	1.038	1.038
	4.0	0.136	1.022	1.022	0.081	1.021	1.021
0.95	0.0	6.098	2.071	2.340	5.518	1.987	2.457
	0.4	4.379	1.860	2.046	3.844	1.956	2.105
	0.6	3.354	1.717	1.819	2.762	1.702	1.820
	0.8	2.544	1.578	1.622	1.978	1.495	1.586
	1.0	1.949	1.457	1.472	1.448	1.401	1.420
	2.0	0.672	1.148	1.148	0.447	1.118	1.118
	3.0	0.323	1.065	1.065	0.207	1.052	1.052
	4.0	0.188	1.035	1.035	0.118	1.029	1.029

Table 3: $SREs$ of the proposed estimators with respect to $\hat{\boldsymbol{\beta}}_{UL}$ for $p_2 = 7$.

		$n=35$			$n=50$		
ρ	Δ	RL	SL	PSL	RL	SL	PSL
0.90	0.0	8.802	2.783	3.203	6.337	2.661	3.269
	0.4	5.482	2.470	2.730	3.680	2.202	2.414
	0.6	3.751	2.156	2.331	2.359	1.860	1.920
	0.8	2.605	1.910	1.992	1.563	1.596	1.609
	1.0	1.871	1.714	1.714	1.088	1.423	1.424
	2.0	0.560	1.238	1.238	0.306	1.120	1.120
	3.0	0.259	1.111	1.111	0.139	1.056	1.056
	4.0	0.148	1.064	1.064	0.079	1.033	1.033
0.93	0.0	9.044	2.756	3.091	7.187	2.770	3.305
	0.4	6.285	2.550	2.822	4.680	2.385	2.621
	0.6	4.593	2.326	2.509	3.188	2.052	2.149
	0.8	3.342	2.096	2.201	2.196	1.773	1.803
	1.0	2.478	1.894	1.944	1.565	1.567	1.573
	2.0	0.788	1.333	1.333	0.458	1.167	1.167
	3.0	0.369	1.156	1.156	0.210	1.077	1.077
	4.0	0.212	1.089	1.089	0.119	1.044	1.044
0.95	0.0	9.157	2.638	2.878	7.285	2.779	3.216
	0.4	6.822	2.490	2.694	5.228	2.466	2.709
	0.6	5.274	2.315	2.474	3.804	2.178	2.299
	0.8	4.018	2.121	2.238	2.745	1.899	1.960
	1.0	3.083	1.944	2.020	2.019	1.691	1.712
	2.0	1.058	1.405	1.406	0.626	1.223	1.223
	3.0	0.507	1.203	1.203	0.291	1.105	1.105
	4.0	0.294	1.119	1.119	0.166	1.061	1.061

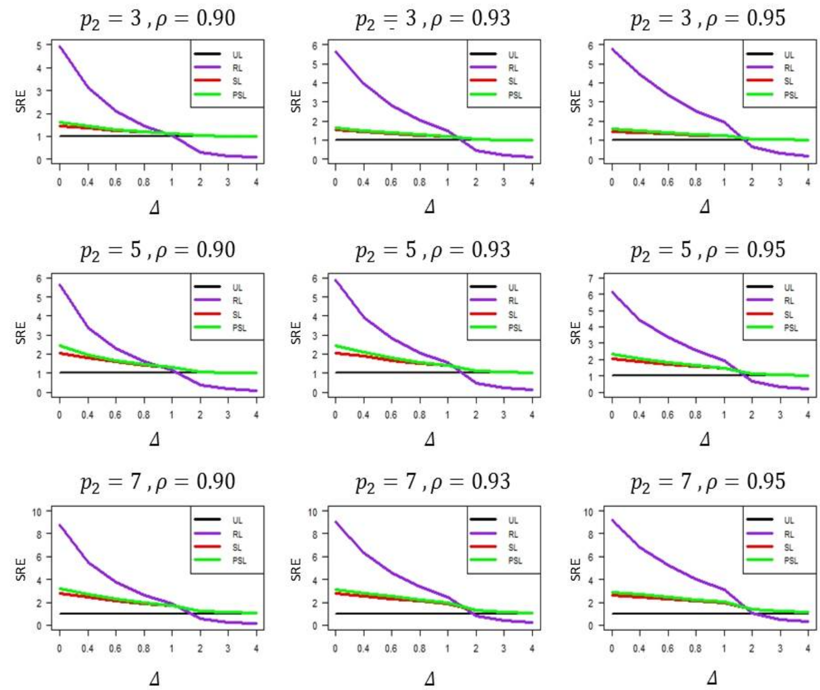

Figure 1: SREs of the proposed estimators with respect to the $\hat{\boldsymbol{\beta}}_{UL}$ for $n = 35$.

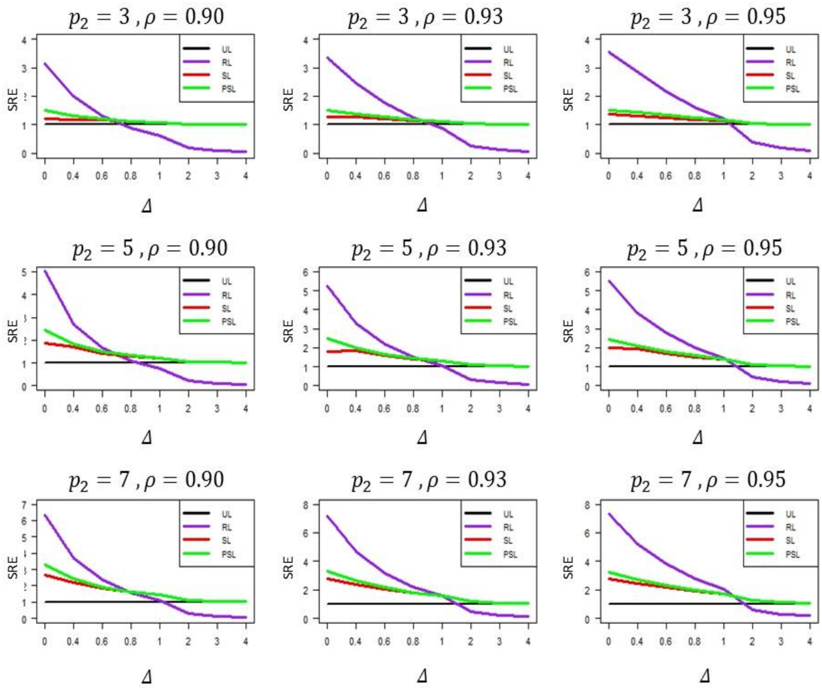

Figure 2: SREs of the proposed estimators with respect to the $\hat{\boldsymbol{\beta}}_{UL}$ for $n = 50$.

The summary statistics of the data is reported in Table 4. The correlation matrix of the predictor variables in Table 5 shows that there are high correlations between all predictors. Based on the Akaike information criterion (AIC) and Bayesian information criterion (BIC), x_2, x_3 and x_6 are the active predictors. Thus, the linear restriction (1) is written as follows

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence, $\beta_1 = \beta_4 = \beta_5 = 0$ are the non-significant parameters and so $p_1 = 3$, $p_2 = 3$ and $p = 6$. We set $\boldsymbol{\beta}^0 = (0, -0.2740, -1.3879, 0, 0, 0.4684)'$ in the candidate sub model. We chose $M = 80$ observations and used bootstrap sampling to replace them 2000 times from the original data in order to compare the performance of the different estimators. The results for the significant parameters $\hat{\beta}_2, \hat{\beta}_3$ and $\hat{\beta}_6$ at $\Delta = 0$ are reported in Table 6 that completely agree with the theoretical and numerical results.

7 Conclusions

In this paper, we improved parameters estimation in the Poisson regression model with correlated predictor variables under linear restriction on the parameters vector

Table 4: Summary statistics of Swedish football data.

Variables	Min	Q_1	Median	Mean	Q_3	Max
y	0.000	0.000	1.000	1.145	2.000	5.000
x_1	1.150	1.735	2.295	2.686	3.145	9.280
x_2	1.340	2.442	3.310	4.280	5.040	23.210
x_3	1.180	1.790	2.380	2.781	3.223	10.210
x_4	1.370	2.462	3.420	4.436	5.152	23.350
x_5	1.150	1.702	2.250	2.604	3.060	9.240
x_6	1.300	2.342	3.185	4.016	4.770	18.490

Table 5: Correlation matrix of Swedish football data.

Variables	x_1	x_2	x_3	x_4	x_5	x_6
x_1	1.000					
x_2	−0.610	1.000				
x_3	0.995	−0.600	1.000			
x_4	−0.605	0.997	−0.595	1.000		
x_5	0.999	−0.609	−0.609	0.998	1.000	
x_6	−0.632	0.997	−0.622	0.997	−0.631	1.000

Table 6: Estimates, standard errors (in parentheses), and SREs of the active coefficients in Swedish football data with respect to the unrestricted Liu estimator.

	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_6$	SRE
UL	0.345 (2.488)	0.028 (3.200)	−0.516 (2.530)	1.000
RL	0.454 (1.866)	0.101 (0.152)	−0.497 (1.893)	9.738
SL	0.366 (2.191)	0.053 (2.430)	−0.500 (2.268)	1.649
PSL	0.372 (2.196)	0.061 (2.245)	−0.503 (2.263)	1.751

using the Stein-Liu and positive Stein-Liu estimators. We developed the ADBs and ADVs of suggested estimators and compared their variances. We used a Monte Carlo simulation study and a real dataset to investigate the performance of the proposed estimators. The results revealed that the Stein-Liu and positive Stein-Liu estimators uniformly outperform the unrestricted Liu estimator. The restricted Liu estimator performed better than the Stein-Liu estimators at $\Delta = 0$ (i.e. when candidate submodel was corrected), however the performance of the Stein-Liu estimators was better than the restricted Liu estimator when Δ moved away from zero.

Acknowledgement

The authors are thankful to the reviewers for the insightful comments and suggestions that have resulted in a much improved version of this paper.

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Appendix

Proof of Theorem 3.1

We present the following lemmas which are useful for proof of the Theorems 3.1 and 3.2.

Lemma 7.1. *Let $\boldsymbol{y}$ be a p_2-dimensional random vector distributed as $\mathcal{N}_{p_2}(\boldsymbol{\mu_y}, \boldsymbol{\Sigma_y})$. Then, for any measurable function φ, we have*

$$\begin{aligned} E[\boldsymbol{y}\,\varphi(\boldsymbol{y}'\boldsymbol{y})] &= \boldsymbol{\mu_y}\, E[\varphi(\chi^2_{p_2+2}(\Delta^*))], \\ E[\boldsymbol{y}\boldsymbol{y}'\,\varphi(\boldsymbol{y}'\boldsymbol{y})] &= \boldsymbol{\Sigma_y}\, E[\varphi(\chi^2_{p_2+2}(\Delta^*))] + \boldsymbol{\mu}'_{\boldsymbol{y}}\,\boldsymbol{\mu_y} E[\varphi(\chi^2_{p_2+4}(\Delta^*))], \end{aligned} \tag{6}$$

where Δ^* is the non-centrality parameter.

Proof. See Judge and Bock (1978). □

Lemma 7.2. *Under the sequence of local alternatives $\{\mathcal{K}_{(n)}\}$ in (4) and the usual regularity conditions of MLE, as $n \to \infty$*

$$\begin{aligned} \boldsymbol{\mathcal{P}}_n &= \sqrt{n}(\hat{\boldsymbol{\beta}}_{UL} - \boldsymbol{\beta}) \xrightarrow{D} \boldsymbol{\mathcal{P}} \sim \mathcal{N}_p\Big(\boldsymbol{A} - \mathbf{I}_p\,,\, \boldsymbol{A}\boldsymbol{C}^{-1}\boldsymbol{A}'\Big), \\ \boldsymbol{\mathcal{Q}}_n &= \sqrt{n}(\hat{\boldsymbol{\beta}}_{RL} - \boldsymbol{\beta}) \\ &\xrightarrow{D} \boldsymbol{\mathcal{Q}} \sim \mathcal{N}_p\Big((\mathbf{I}_p - \boldsymbol{\mathcal{J}}\boldsymbol{R})(\boldsymbol{A} - \mathbf{I}_P)\boldsymbol{\beta} - \boldsymbol{\mathcal{J}}\boldsymbol{\delta}\,, \boldsymbol{A}\,\boldsymbol{C}^{-1}\boldsymbol{A}' - \boldsymbol{\mathcal{J}}\mathbf{R}\,\boldsymbol{A}\,\boldsymbol{C}^{-1}\boldsymbol{A}'\Big), \\ \boldsymbol{\mathcal{S}}_n &= \sqrt{n}(\hat{\boldsymbol{\beta}}_{UL} - \hat{\boldsymbol{\beta}}_{RL}) \xrightarrow{D} \boldsymbol{\mathcal{S}} \sim \mathcal{N}_p\Big(\boldsymbol{\mathcal{J}}[\mathbf{R}\,(\boldsymbol{A} - \mathbf{I}_p)\boldsymbol{\beta} + \boldsymbol{\delta}]\,,\, \boldsymbol{\mathcal{J}}\mathbf{R}\boldsymbol{A}\,\boldsymbol{C}^{-1}\boldsymbol{A}'\Big), \end{aligned}$$

$$\begin{pmatrix} \boldsymbol{\mathcal{P}}_n \\ \boldsymbol{\mathcal{Q}}_n \\ \boldsymbol{\mathcal{S}}_n \end{pmatrix} \xrightarrow{D} \begin{pmatrix} \boldsymbol{\mathcal{P}} \\ \boldsymbol{\mathcal{Q}} \\ \boldsymbol{\mathcal{S}} \end{pmatrix} \sim \mathcal{N}_{3k}\left[\begin{pmatrix} \mathbf{0} \\ (\boldsymbol{I}_p - \boldsymbol{\mathcal{J}}\mathbf{R})\,(\boldsymbol{A} - \mathbf{I}_p)\,\boldsymbol{\beta} - \boldsymbol{\mathcal{J}}\boldsymbol{\delta} \\ \boldsymbol{\mathcal{J}}\boldsymbol{R}(\boldsymbol{A} - \mathbf{I}_P)\,\boldsymbol{\beta} + \boldsymbol{\mathcal{J}}\boldsymbol{\delta} \end{pmatrix}, \begin{pmatrix} \boldsymbol{A}\,\boldsymbol{C}^{-1}\boldsymbol{A}' & \boldsymbol{A}\,\boldsymbol{C}^{-1}\boldsymbol{A}' - \boldsymbol{\mathcal{J}}\mathbf{R}\,\boldsymbol{A}\,\boldsymbol{C}^{-1}\boldsymbol{A}' & \boldsymbol{\mathcal{J}}\mathbf{R}\boldsymbol{A}\,\boldsymbol{C}^{-1}\boldsymbol{A}' \\ & \boldsymbol{A}\,\boldsymbol{C}^{-1}\boldsymbol{A}' - \boldsymbol{\mathcal{J}}\mathbf{R}\,\boldsymbol{A}\,\boldsymbol{C}^{-1}\boldsymbol{A}' & \mathbf{0} \\ & & \boldsymbol{\mathcal{J}}\mathbf{R}\boldsymbol{A}\,\boldsymbol{C}^{-1}\boldsymbol{A}' \end{pmatrix} \right],$$

where $\boldsymbol{\mathcal{J}} = \boldsymbol{C}^{-1}\boldsymbol{R}'(\boldsymbol{R}\,\boldsymbol{C}^{-1}\boldsymbol{R}')^{-1}$.

Now, we proof Theorem 3.1 using Lemma 7.2 as follows

$$\begin{aligned} ADB(\hat{\boldsymbol{\beta}}_{UL}) &= \lim_{n\to\infty} E\left[\sqrt{n}\left(\hat{\boldsymbol{\beta}}_{UL} - \boldsymbol{\beta}\right)\right] = E[\boldsymbol{\mathcal{P}}] = (\boldsymbol{A} - \boldsymbol{I}_p)\,\boldsymbol{\beta}, \\ ADB(\hat{\boldsymbol{\beta}}_{RL}) &= \lim_{n\to\infty} E\left[\sqrt{n}\left(\hat{\boldsymbol{\beta}}_{RL} - \boldsymbol{\beta}\right)\right] = E[\boldsymbol{\mathcal{Q}}] = (\boldsymbol{I}_p - \boldsymbol{\mathcal{J}}\,\mathbf{R})(\boldsymbol{A} - \boldsymbol{I}_p)\,\boldsymbol{\beta} - \boldsymbol{\mathcal{J}}\,\boldsymbol{\delta}. \end{aligned}$$

For $\hat{\boldsymbol{\beta}}_{SL}$, we have

$$\begin{aligned} ADB(\hat{\boldsymbol{\beta}}_{SL}) &= \lim_{n\to\infty} E\left[\sqrt{n}\left(\hat{\boldsymbol{\beta}}_{SL} - \boldsymbol{\beta}\right)\right] \\ &= \lim_{n\to\infty} \left\{ E\left[\sqrt{n}\left(\hat{\boldsymbol{\beta}}_{UL} - \boldsymbol{\beta}\right)\right] - (p_2 - 2)\, E\left[T_n^{-1}\sqrt{n}\left(\hat{\boldsymbol{\beta}}_{UL} - \hat{\boldsymbol{\beta}}_{RL}\right)\right] \right\} \\ &= E[\boldsymbol{\mathcal{P}}] - (p_2 - 2)\, E[T_n^{-1}\,\boldsymbol{\mathcal{S}}] \end{aligned}$$

$$= \quad ADB(\hat{\boldsymbol{\beta}}_{UL}) - (p_2-2)\,\boldsymbol{\mathcal{J}}\;[\boldsymbol{R}\,(\boldsymbol{A}-\boldsymbol{I}_p)\,\boldsymbol{\beta}+\boldsymbol{\delta}]\,E\Big[\frac{1}{\chi^2_{p_2+2}(\Delta^*)}\Big].$$

Finally,

$$\begin{aligned}
ADB(\hat{\boldsymbol{\beta}}_{PSL}) &= \lim_{n\to\infty} E\Big[\sqrt{n}\Big(\hat{\boldsymbol{\beta}}_{PSL}-\boldsymbol{\beta}\Big)\Big]\\
&= \lim_{n\to\infty} E\Big[\sqrt{n}\Big(\hat{\boldsymbol{\beta}}_{SL}-\boldsymbol{\beta}\Big) - (1-(p_2-2)T_n^{-1})\\
&\quad \times\sqrt{n}\Big((\hat{\boldsymbol{\beta}}_{UL}-\hat{\boldsymbol{\beta}}_{RL})I(T_n<p_2-2)\Big)\Big]\\
&= ADB(\hat{\boldsymbol{\beta}}_{SL})\\
&\quad -\lim_{n\to\infty} E\Big[\sqrt{n}\Big(\hat{\boldsymbol{\beta}}_{UL}-\hat{\boldsymbol{\beta}}_{RL}\Big)(1-(p_2-2)T_n^{-1})I(T_n<p_2-2)\Big]\\
&= ADB(\hat{\boldsymbol{\beta}}_{SL}) - E[\boldsymbol{\mathcal{S}}(1-(p_2-2)T_n^{-1})I(T_n<p_2-2)]\\
&= ADB(\hat{\boldsymbol{\beta}}_{SL}) - E[\boldsymbol{\mathcal{S}}]\Big\{\boldsymbol{\Psi}_{p_2+2}(\chi^2_{p_2,\alpha};\Delta^*) + (p_2-2)E\Big[\frac{I(T_n<p_2-2)}{\chi^2_{p_2+2}(\Delta^*)}\Big]\Big\}\\
&= ADB(\hat{\boldsymbol{\beta}}_{SL}) - \boldsymbol{\mathcal{J}}[\boldsymbol{R}(\boldsymbol{A}-\boldsymbol{I}_p)\boldsymbol{\beta}+\boldsymbol{\delta}]\\
&\quad \Big\{\boldsymbol{\Psi}_{p_2+2}(\chi^2_{p_2,\alpha};\Delta^*) + (p_2-2)E\Big[\frac{I(\chi^2_{p_2+2}(\Delta^*)<p_2-2)}{\chi^2_{p_2+2}(\Delta^*)}\Big]\Big\}.
\end{aligned}$$

Proof of Theorem 3.2

Here, we compute the asymptotic distributional variances of the proposed estimators

$$\begin{aligned}
ADV(\hat{\boldsymbol{\beta}}_{UL}) &= \lim_{n\to\infty} E\Big(\sqrt{n}(\hat{\boldsymbol{\beta}}_{UL}-\boldsymbol{\beta})\sqrt{n}(\hat{\boldsymbol{\beta}}_{UL}-\boldsymbol{\beta})'\Big)\\
&= \lim_{n\to\infty} E(\boldsymbol{\mathcal{P}}_n\,\boldsymbol{\mathcal{P}}_n{}')\\
&= E(\boldsymbol{\mathcal{P}}\,\boldsymbol{\mathcal{P}}')\\
&= Var(\boldsymbol{\mathcal{P}}) + E(\boldsymbol{\mathcal{P}})\,E(\boldsymbol{\mathcal{P}}')\\
&= \boldsymbol{A}\boldsymbol{C}^{-1}\boldsymbol{A}' + \Big[(\boldsymbol{A}-\mathbf{I}_p)\,\boldsymbol{\beta}\Big]\,\Big[(\boldsymbol{A}-\mathbf{I}_p)\,\boldsymbol{\beta}\Big]',\\
ADV(\hat{\boldsymbol{\beta}}_{RL}) &= \lim_{n\to\infty} E\Big(\sqrt{n}(\hat{\boldsymbol{\beta}}_{RL}-\boldsymbol{\beta})\,\sqrt{n}(\hat{\boldsymbol{\beta}}_{RL}-\boldsymbol{\beta})'\Big)\\
&= \lim_{n\to\infty} E(\boldsymbol{\mathcal{Q}}_n\,\boldsymbol{\mathcal{Q}}_n{}')\\
&= E(\boldsymbol{\mathcal{Q}}\,\boldsymbol{\mathcal{Q}}')\\
&= \boldsymbol{A}\,\boldsymbol{C}^{-1}\,\boldsymbol{A}' - \boldsymbol{\mathcal{J}}\,\mathbf{R}\,\boldsymbol{A}\,\boldsymbol{C}^{-1}\,\boldsymbol{A}'\\
&\quad + \Big[(\mathbf{I}_p-\boldsymbol{\mathcal{J}}\,\mathbf{R})\,(\boldsymbol{A}-\boldsymbol{I}_p)\,\boldsymbol{\beta}-\boldsymbol{\mathcal{J}}\,\boldsymbol{\delta}\Big]\,\Big[(\mathbf{I}_p-\boldsymbol{\mathcal{J}}\,\mathbf{R})\,(\boldsymbol{A}-\boldsymbol{I}_p)\,\boldsymbol{\beta}-\boldsymbol{\mathcal{J}}\,\boldsymbol{\delta}\Big]'.
\end{aligned}$$

The asymptotic distributional variance of $\hat{\boldsymbol{\beta}}_{SL}$ can be obtained as follows

$$\begin{aligned}
\mathcal{V}(\hat{\boldsymbol{\beta}}_{SL}) &= \lim_{n\to\infty} E\Big(\sqrt{n}(\hat{\boldsymbol{\beta}}_{SL}-\boldsymbol{\beta})\,\sqrt{n}(\hat{\boldsymbol{\beta}}_{SL}-\boldsymbol{\beta})'\Big)\\
&= \lim_{n\to\infty} E\Big[\sqrt{n}\Big(\hat{\boldsymbol{\beta}}_{UL} + \{1-(p_2-2\}T_n^{-1})\,(\hat{\boldsymbol{\beta}}_{UL}-\hat{\boldsymbol{\beta}}_{UL})-\boldsymbol{\beta}\Big)\\
&= \lim_{n\to\infty} E[(\boldsymbol{\mathcal{P}}_n-(p_2-2)\,T_n^{-1}\,\boldsymbol{\mathcal{S}}_n)\,(\boldsymbol{\mathcal{P}}_n-(p_2-2)\,T_n^{-1}\,\boldsymbol{\mathcal{S}}_n)']
\end{aligned}$$

$$= E[(\boldsymbol{\mathcal{P}}\,\boldsymbol{\mathcal{P}}'] - 2(p_2-2)\underbrace{E[\boldsymbol{\mathcal{P}}\,\boldsymbol{\mathcal{S}}'\,T_n^{-1}]}_{m_1} + (p_2-2)^2\underbrace{E[\boldsymbol{\mathcal{S}}\,\boldsymbol{\mathcal{S}}'\,\boldsymbol{T}_n^{-2}]}_{m_2},$$

we can write m_1 as follows

$$\begin{aligned} m_1 &= E[\boldsymbol{\mathcal{P}}\,\boldsymbol{\mathcal{S}}'\,T_n^{-1}] \\ &= E[\boldsymbol{\mathcal{S}}\,\boldsymbol{\mathcal{S}}'\,T_n^{-1}] + \big[(\boldsymbol{I}_p - \boldsymbol{\mathcal{J}}\,\boldsymbol{R})\,(\boldsymbol{A}-\boldsymbol{I}_p)\,\boldsymbol{\beta} - \boldsymbol{\mathcal{J}}\,\boldsymbol{\delta}\big]\,E[\boldsymbol{\mathcal{S}}\,T_n^{-1}] \\ &= Var[\boldsymbol{\mathcal{S}}]\,E\Big[\frac{1}{\chi^2_{p_2+2}(\Delta^*)}\Big] + E[\boldsymbol{\mathcal{S}}]\,E[\boldsymbol{\mathcal{P}}']\,E\Big[\frac{1}{\chi^2_{p_2+4}(\Delta^*)}\Big] \\ &\quad + \big[(\boldsymbol{I}_p - \boldsymbol{\mathcal{J}}\,\boldsymbol{R})\,(\boldsymbol{A}-\boldsymbol{I}_p)\,\boldsymbol{\beta} - \boldsymbol{\mathcal{J}}\,\boldsymbol{\delta}\big]\,E[\boldsymbol{\mathcal{S}}\,T_n^{-1}], \end{aligned}$$

and by using (6), m_2 becomes

$$m_2 = E[\boldsymbol{\mathcal{S}}\,\boldsymbol{\mathcal{S}}'\,T_n^{-2}] = Var[\boldsymbol{\mathcal{S}}]\,E\Big[\frac{1}{(\chi^2_{p_2+2}(\Delta^*))^2}\Big] + E[\boldsymbol{\mathcal{S}}]\,E[\boldsymbol{\mathcal{P}}']\,E\Big[\frac{1}{(\chi^2_{p_2+4}(\Delta^*))^2}\Big],$$

Therefore,

$$\begin{aligned} ADV(\hat{\boldsymbol{\beta}}_{SL}) &= ADV(\hat{\boldsymbol{\beta}}_{UL}) - 2(p_2-2)\Bigg(\big[(\boldsymbol{I}_p - \boldsymbol{\mathcal{J}}\boldsymbol{R})(\boldsymbol{A}-\boldsymbol{I}_p)\boldsymbol{\beta} - \boldsymbol{\mathcal{J}}\boldsymbol{\delta}\big] \\ &\quad \times\big[\boldsymbol{\mathcal{J}}\boldsymbol{R}[\boldsymbol{A}-\boldsymbol{I}_p]\boldsymbol{\beta} + \boldsymbol{\mathcal{J}}\boldsymbol{\delta}\big]E\Big[\frac{1}{\chi^2_{p_2+2}(\Delta^*)}\Big]\Bigg) + (p_2-2)(p_2-4) \\ &\quad \times\boldsymbol{\mathcal{J}}\boldsymbol{R}\boldsymbol{A}\boldsymbol{C}^{-1}\boldsymbol{A}'\Bigg(E\Big[\frac{1}{(\chi^2_{p_2+2}(\Delta^*))^2}\Big] - E\Big[\frac{1}{\chi^2_{p_2+2}(\Delta^*)}\Big]\Bigg) \\ &\quad + (p_2-2)(p_2-4)[\boldsymbol{\mathcal{J}}\boldsymbol{R}(\boldsymbol{A}-\boldsymbol{I}_p)\boldsymbol{\beta} + \boldsymbol{\mathcal{J}}\boldsymbol{\delta}][\boldsymbol{\mathcal{J}}\boldsymbol{R}(\boldsymbol{A}-\boldsymbol{I}_p)\boldsymbol{\beta} + \boldsymbol{\mathcal{J}}\boldsymbol{\delta}]' \\ &\quad \times\Bigg(E\Big[\frac{1}{(\chi^2_{p_2+4}(\Delta^*))^2}\Big] - E\Big[\frac{1}{\chi^2_{p_2+4}(\Delta^*)}\Big]\Bigg). \end{aligned}$$

Finally, we can write $ADV(\hat{\boldsymbol{\beta}}_{PSL})$ as follows

$$\begin{aligned} ADV(\hat{\boldsymbol{\beta}}_{PSL}) &= \lim_{n\to\infty} E\Big(\sqrt{n}(\hat{\boldsymbol{\beta}}_{PSL} - \boldsymbol{\beta})\sqrt{n}(\hat{\boldsymbol{\beta}}_{PSL} - \boldsymbol{\beta})'\Big) \\ &= \lim_{n\to\infty} E\Big[\sqrt{n}\Big(\hat{\boldsymbol{\beta}}_{SL} - (1-(p_2-2)T_n^{-1})I(T_n < p_2-2)(\hat{\boldsymbol{\beta}}_{UL} - \hat{\boldsymbol{\beta}}_{RL}) - \boldsymbol{\beta}\Big) \\ &\quad \times\sqrt{n}\Big(\hat{\boldsymbol{\beta}}_{SL} - (1-(p_2-2)T_n^{-1})I(T_n < p_2-2)(\hat{\boldsymbol{\beta}}^{UL} - \hat{\boldsymbol{\beta}}^{RL}) - \boldsymbol{\beta}\Big)'\Big] \\ &= ADV(\hat{\boldsymbol{\beta}}_{SL}) - 2\underbrace{E[\boldsymbol{\mathcal{S}}\boldsymbol{\mathcal{Q}}'(1-(p_2-2)T_n^{-1})I(T_n < p_2-2)]}_{m_3} \\ &\quad - \underbrace{E[\boldsymbol{\mathcal{S}}\boldsymbol{\mathcal{S}}'(1-(p_2-2)T_n^{-1})^2 I(T_n < p_2-2)]}_{m_4}, \end{aligned}$$

Now, we obtain m_3 as follows

$$\begin{aligned} m_3 &= E[\boldsymbol{\mathcal{S}}\boldsymbol{\mathcal{Q}}'(1-(p_2-2)T_n^{-1})I(T_n < p_2-2)] \\ &= E[\boldsymbol{\mathcal{S}}E\{\boldsymbol{\mathcal{Q}}'(1-(p_2-2)T_n^{-1})I(T_n < p_2-2)|\boldsymbol{\mathcal{S}}\}] \\ &= E[\boldsymbol{\mathcal{S}}\{(\boldsymbol{I}_p - \boldsymbol{\mathcal{J}}\boldsymbol{R})(\boldsymbol{A}-\boldsymbol{I}_p)\boldsymbol{\beta} - \boldsymbol{\mathcal{J}}\boldsymbol{\delta}\}' \times (1-(p_2-2)T_n^{-1})I(T_n < p_2-2)] \\ &= \{(\boldsymbol{I}_p - \boldsymbol{\mathcal{J}}\boldsymbol{R})(\boldsymbol{A}-\boldsymbol{I}_p)\boldsymbol{\beta} - \boldsymbol{\mathcal{J}}\boldsymbol{\delta}\}E[\boldsymbol{\mathcal{S}}(1-(p_2-2)T_n^{-1})I(T_n < p_2-2)] \\ &= \{(\boldsymbol{I}_p - \boldsymbol{\mathcal{J}}\boldsymbol{R})(\boldsymbol{A}-\boldsymbol{I}_p)\boldsymbol{\beta} - \boldsymbol{\mathcal{J}}\boldsymbol{\delta}\}E[\boldsymbol{\mathcal{S}}]E\Big[\Big(1-\frac{p_2-2}{\chi^2_{p_2+2}(\Delta^*)}\Big)I(\chi^2_{p_2+2}(\Delta^*) < p_2-2)\Big] \end{aligned}$$

$$
\begin{aligned}
&= \{(\boldsymbol{I}_p - \boldsymbol{\mathcal{J}}\boldsymbol{R})(\boldsymbol{A} - \boldsymbol{I}_p)\boldsymbol{\beta} - \boldsymbol{\mathcal{J}}\boldsymbol{\delta}\}[\boldsymbol{\mathcal{J}}\boldsymbol{R}(\boldsymbol{A} - \boldsymbol{I}_p)\boldsymbol{\beta} - \boldsymbol{\mathcal{J}}\boldsymbol{\delta} + \boldsymbol{\mathcal{J}}\boldsymbol{\delta}] \\
&\quad \times E\Big[\Big(1 - \frac{p_2 - 2}{\chi^2_{p_2+2}(\Delta^*)}\Big) I(\chi^2_{p_2+2}(\Delta^*) < p_2 - 2)\Big]
\end{aligned}
$$

Based on Equation (6), m_4 becomes

$$
\begin{aligned}
m_4 &= E[\boldsymbol{\mathcal{S}}\boldsymbol{\mathcal{S}}'(1 - (p_2 - 2)T_n^{-1})^2 I(T_n < p_2 - 2)] \\
&= Var(\boldsymbol{\mathcal{S}}) E\Big[\Big(1 - \frac{p_2 - 2}{\chi^2_{p_2+2}(\Delta^*)}\Big)^2 I(\chi^2_{p_2+2}(\Delta^*) < p_2 - 2)\Big] \\
&\quad + E(\boldsymbol{\mathcal{S}}) E(\boldsymbol{\mathcal{S}}) E\Big[\Big(1 - \frac{p_2 - 2}{\chi^2_{p_2+4}(\Delta^*)}\Big)^2 I(\chi^2_{p_2+4}(\Delta^*) < p_2 - 2)\Big] \\
&= \boldsymbol{\mathcal{J}}\boldsymbol{R}\boldsymbol{A}\boldsymbol{C}^{-1}\boldsymbol{A}' E\Big[\Big(1 - \frac{p_2 - 2}{\chi^2_{p_2+2}(\Delta^*)}\Big)^2 I(\chi^2_{p_2+2}(\Delta^*) < p_2 - 2)\Big] \\
&\quad + [\boldsymbol{\mathcal{J}}\boldsymbol{R}(\boldsymbol{A} - \boldsymbol{I}_p)\boldsymbol{\beta} + \boldsymbol{\mathcal{J}}\boldsymbol{\delta}][\boldsymbol{\mathcal{J}}\boldsymbol{R}(\boldsymbol{A} - \boldsymbol{I}_p)\boldsymbol{\beta} + \boldsymbol{\mathcal{J}}\boldsymbol{\delta}]' \\
&\quad \times E\Big[\Big(1 - \frac{p_2 - 2}{\chi^2_{p_2+4}(\Delta^*)}\Big)^2 I(\chi^2_{p_2+4}(\Delta^*) < p_2 - 2)\Big]
\end{aligned}
$$

Therefore, $ADV(\hat{\boldsymbol{\beta}}_{PSL})$ becomes

$$
\begin{aligned}
\mathcal{V}(\hat{\boldsymbol{\beta}}_{PSL}) &= ADV(\hat{\boldsymbol{\beta}}_{SL}) \\
&\quad - 2\Big(\{(\boldsymbol{I}_p - \boldsymbol{\mathcal{J}}\boldsymbol{R})(\boldsymbol{A} - \boldsymbol{I}_p)\boldsymbol{\beta} - \boldsymbol{\mathcal{J}}\boldsymbol{\delta}\}[\boldsymbol{\mathcal{J}}\boldsymbol{R}(\boldsymbol{A} - \boldsymbol{I}_p)\boldsymbol{\beta} - \boldsymbol{\mathcal{J}}\boldsymbol{\delta} + \boldsymbol{\mathcal{J}}\boldsymbol{\delta}] \\
&\quad \times E\Big[\Big(1 - \frac{p_2 - 2}{\chi^2_{p_2+2}(\Delta^*)}\Big) I(\chi^2_{p_2+2}(\Delta^*) < p_2 - 2)\Big]\Big) \\
&\quad - \Big(\boldsymbol{\mathcal{J}}\boldsymbol{R}\boldsymbol{A}\boldsymbol{C}^{-1}\boldsymbol{A}' E\Big[\Big(1 - \frac{p_2 - 2}{\chi^2_{p_2+2}(\Delta^*)}\Big)^2 I(\chi^2_{p_2+2}(\Delta^*) < p_2 - 2)\Big] \\
&\quad + [\boldsymbol{\mathcal{J}}\boldsymbol{R}(\boldsymbol{A} - \boldsymbol{I}_p)\boldsymbol{\beta} + \boldsymbol{\mathcal{J}}\boldsymbol{\delta}][\boldsymbol{\mathcal{J}}\boldsymbol{R}(\boldsymbol{A} - \boldsymbol{I}_p)\boldsymbol{\beta} + \boldsymbol{\mathcal{J}}\boldsymbol{\delta}]' \\
&\quad \times E\Big[\Big(1 - \frac{p_2 - 2}{\chi^2_{p_2+4}(\Delta^*)}\Big)^2 I(\chi^2_{p_2+4}(\Delta^*) < p_2 - 2)\Big]\Big).
\end{aligned}
$$