Journal of Statistical Modelling: Theory and Applications
Vol. 5, No. 2, 2024, pp. 121-135
Yazd University Press 2024

JSMTA

Research Paper

Estimation parameters of Lindley distribution under type II progressive censoring with the presence of outlier data

Parviz Nasiri*, Sayeda Fatemeh Miri, Ali Shadrokh
Department of Statistics, Payam Noor University, Tehran, Iran

Received: December 09, 2024/ Revised: February 10, 2025/ Accepted: February 18, 2025

Abstract: Lindley distribution is one of the most important statistical distributions that is widely used in various fields including biology, engineering, and medicine. This distribution has worked well in modeling mortality studies, and this distribution can be used for statistical modeling of plant roots density. Recently, a number of researchers have shown that statistical distributions can be used to model or investigate the number of plant roots. One of the most important of these is the Lindley distribution, which can be used to investigate modeling data in the presence of outliers that have a Lindley or uniform distribution. In this article, we present the Lindley distribution under type II progressive censoring with the presence of outlier data, and its parameters are estimated using maximum likelihood and Bayesian methods. In the following, by applying Gibbs sampling and performing simulation, the estimators are compared with each other using the mean squared error.

Keywords: Bayesian estimation; Maximum likelihood estimation; Mean squared error; Outlier data; Progressive censoring of the second type.
Mathematics Subject Classification (2010): 65C60, 62N02.

1 Introduction

In many cases, in various fields such as reliability and other application fields, the tester cannot observe the exact failure time of the units under test due to reasons such as time constraints, lack of access to all parts, high costs, etc. In such situations, censorship methods are often used to manage this situation. Progressive censoring of the second type is one of the censoring methods in which it is possible to remove test

*Corresponding author: pnasiri@pnu.ac.ir

units at each stage. In other words, during the test, units are removed from the test. This method is a generalization of second-right type censoring. It should be noted that in the second type of censorship, the lifespan of censored units is usually longer than the lifespan of uncensored units. Also, censored units can be used as secondary data in subsequent experiments, which is economically important. As a result, progressive censoring of the second type is a flexible method that helps to manage and control the downtime of units in experiments. In past studies, many researchers have used progressive censoring of the second type in their research. This method allows them to extract the best possible information from the data with constraints such as time constraints or access to all units. This information can be obtained through statistical data analysis and advanced models to increase accuracy and reliability.

Schneider and Weissfeld (1986) presented statistical inferences on type II censored samples. Pandey et al. (1989) introduced shrinking estimators of the shape parameter of the Weibull distribution based on the second type censoring scheme. Dey and Kuo (1991) developed a new family of empirical Bayes estimators for parameter exponential distribution considering type II censored data. Dallas (1993) presented the maximum likelihood estimation of the second-type Bohr distribution parameter based on the second-type censoring scheme. Balakrishnan and Cutler (2008) presented maximum likelihood estimation of Laplace distribution parameters based on type II censored samples. Singh et al. (2005) investigated the estimation of Weibull exponential family parameters based on the second type censoring scheme. Singh and Kumar (2007) investigated the Bayesian estimation of the exponential distribution parameter under type II censoring scheme. Balakrishnan and Han (2008) obtain parameter estimates of the semi-logistic distribution under type II censored samples.

Balakrishnan and Han (2008) considered the estimation of a stress-step model from an exponential distribution based on a type II censoring scheme. Chan et al. (2008) presented a point and interval estimate on the regression model of Frein value based on type II censoring design. Balakrishnan and Han (2008) considered the estimation of a stress-step model from an exponential distribution based on a type II censoring scheme. Chan et al. (2008) have presented point and interval estimation on the regression model of outlier value based on type II censoring scheme. Madi and Raqab (2009) present Bayesian inference for the generalized exponential distribution based on progressively censored data. Iliopoulos and Balakrishnan (2011) introduced the classical statistical inference of the Laplace distribution based on type II censoring scheme. Panahi and Asadi (2011) obtained the Weibull distribution estimate based on the second type censoring scheme. Kundu and Raqab (2012) obtained a Bayesian estimate of the Weibull distribution based on a type II censoring scheme. Regarding other censored models, Singh et al. (2014); Kumar et al. (2015, 2018) and Pathak et al. (2022), respectively, in exponential distributions, exponential gamma, composite inverse Poisson distribution and investigated the Weibull-Poisson complex.

In the data analysis process, a data may have a very short or a very long life time. This data is called "outlier data". This may be due to production disruptions, equipment failure at a particular stage, or other unknown factors. In the process, if you do not consider this data in your analysis and directly analyze it with normal statistical models, it may have a destructive effect on the results. This means that it is very important to carefully check the data distribution and estimate the parameters considering the outlying data. In this case, there is a need to apply special statistical and modeling methods in order to accurately manage and analyze these outlier data.

2 Lindley distribution in the presence of data far from uniform distribution

To provide a statistical model of plant root density, Ooms and Moore (1991) showed that if plants grow in a suitable environment, root length follows an exponential distribution or Lindley distribution. Also, based on other evidence, when a plant enters the reproductive phase of growth, smaller roots grow to a depth of about 1.6 mm. Based on this, it can be said that the root density distribution has a uniform distribution. Ooms and Moore (1991) confirmed that the root density distribution has different distributions. Therefore, in examining a number of plant roots, it was concluded that some of them follow the Lindley distribution and others follow the uniform distribution. Jabbari Nooghabi (2021) showed that in the density of plant roots so that it has a Lindley distribution. Suppose we have a sample of size n that $n-k$ of them are outliers with a density function.

$$f_1(x;\theta)=\frac{x+1}{\theta^2(\theta+1)}\exp\frac{-x}{\theta},\quad x>0,\quad \theta>0, \tag{1}$$

and k (known) of them are uniform distribution with density function,

$$f_2(x;\theta,\alpha)=\frac{1}{\alpha\theta}I_{(0,\alpha\theta)}(x),\quad 0<x<\alpha\theta,\quad \alpha>0,\quad \theta>0. \tag{2}$$

According to the Dixit (1989) model, the joint density function is equal to

$$\begin{aligned}
f(x_1,x_2,\dots,x_n;\alpha,\theta) &= \prod_{i=1}^{n} f_1(x_i)\sum_{A_1,A_2,\dots,A_k}\prod_{j=1}^{k}\frac{f_2(x_{A_j})}{f_1(x_{A_j})}[C(n,k)]^{-1},\\
f(x_1,x_2,\dots,x_n;\alpha,\theta) &= \frac{\prod_{i=1}^{n}(x_i+1)\exp\left(-\frac{1}{\theta}\sum_{i=1}^{n}x_i\right)}{C(n,k)\alpha^k\theta^{2n}(\theta+1)^{n-k}}\\
&\quad\times\sum_{A_1,A_2,\dots,A_k}\frac{\exp\left(\frac{1}{\theta}\sum_{j=1}^{k}x_{A_j}\right)}{\prod_{j=1}^{k}(x_{A_j}+1)}\prod_{j=1}^{k}I(\alpha\theta-x_{A_j})\\
&= \prod_{i=1}^{n}\frac{(x_i+1)}{\theta^2(\theta+1)}\exp\frac{-x_i}{\theta}\sum_{A_1,A_2,\dots,A_k}\\
&\quad\times\prod_{j=1}^{k}\frac{\exp\left(\frac{x_{A_j}}{\theta}\right)\frac{1}{\alpha\theta}I_{0,\alpha\theta}(x_{A_j})\theta^2(\theta+1)}{(x_{A_j}+1)}[C(n,k)]^{-}1,
\end{aligned}$$

and $C(n,k)=\frac{n!}{k!(n-k)!}$ such that,

$$\sum_{A_1,A_2,\dots,A_k}=\sum_{A_1=1}^{n-k+1}\sum_{A_2=A_1+1}^{n-k+2}\cdots\sum_{A_k=A_{k-1}+1}^{n}.$$

In the special case, for $k=3$, the density function is equal to

$$\begin{aligned} f(x_1,x_2,\dots,x_n) &= \prod_{i=1}^{n} f_1(x_i) \sum_{A_1,A_2,A_3} \prod_{j=1}^{3} \frac{f_2(x_{A_j})}{f_1(x_{A_j})} [C(n,3)]^{-1} \\ &= \frac{\prod_{i=1}^{n}(x_i+1)\exp\left(-\frac{1}{\theta}\sum_{i=1}^{n} x_i\right)}{C(n,3)\alpha^3\theta^{2n}(\theta+1)^{n-3}} \\ &\quad \times \sum_{A_1,A_2,A_3} \frac{\frac{1}{\theta\sum_{j=1}^{3} x_{A_j}}}{\prod_{j=1}^{3}(x_{A_j}+1)} \prod_{j=1}^{3} I(\alpha\theta - x_{A_j}). \end{aligned}$$

According to the joint density function, the marginal density function is equal to

$$f(x) = \frac{k}{n} f_2(x) + \frac{n-k}{n} f_1(x),$$

which according to (1) and (2), the density and marginal distribution functions are respectively equal to

$$f(x;\alpha,\theta) = \frac{k}{n}\frac{1}{\alpha\theta} I_{(0,\alpha\theta)}(x) + (1-\frac{k}{n})\frac{x+1}{\theta^2(\theta+1)} \exp\left(-\frac{x}{\theta}\right), \quad x \geq 0, \alpha > 0, \theta > 0, \quad (3)$$

$$F(x;\alpha,\theta) = \frac{k}{n}\frac{x}{\alpha\theta} I_{(0,\alpha\theta)}(x) - (1-\frac{k}{n})\frac{1+\theta+x}{\theta(1+\theta)} \exp\left(-\frac{x}{\theta}\right), \quad x > 0, \alpha > 0, \theta > 0. \quad (4)$$

For more information, see Dixit and Nasiri (2001).

3 Type II progressive censoring Model

Suppose n parts are tested and if the tester decides to randomly remove R_1 out of the $n-1$ remaining test units from the test after seeing the first failure, where $R_1 = 0,\dots,n-m$ and at the time of the second failure, R_2 out of the $n-R_1-1$ remaining test units are removed from the test. crash out of the test where $R_2 = 0,\dots,n-m-R_1$ and continue to do so until at the time of the m-th failure the remaining R_m units are out of the test, so that $R_m = n-m-\sum_{i=1}^{m-1} R_i$. The values of $R_i (1 \leq i \leq m)$ are fixed and predetermined values. In addition, the failure times, which are random variables, and the ordinal statistics of the increasing censoring of the second type from the right are called and denoted by $X_{1:m:n},\dots,X_{m:m:n}$. Therefore, type two progressive censoring is displayed based on m and vector $\tilde{R} = (R_1,\dots,R_m)$. Now, to obtain the density function of the random sample $X_1,\dots,X_n$ with the increasing censored design of the second type, suppose $x_{1:m:n}^{(R_1,\dots,R_m)}, x_{2:m:n}^{(R_1,\dots,R_m)} \dots, x_{m:m:n}^{(R_1,\dots,R_m)}$, $1 \leq m \leq n$ is an progressive censoring sample of the second type, n units are taken from the distribution of $F_X(x,\alpha,\theta)$, and $\boldsymbol{X} \equiv (X_{1:m:n}^{(R_1,\dots,R_m)}, X_{2:m:n}^{(R_1,\dots,R_m)} \dots, X_{m:m:n}^{(R_1,\dots,R_m)})$ are censored. The density function of $\boldsymbol{X}$ is equal to

$$f_{X_{1:m:n},X_{2:m:n},\dots,X_{m:m:n};\alpha,\theta}(x_1,x_2,\dots,x_m) = A\prod_{i=1}^{m} f_{\alpha,\theta}(x_i)\left[1-F_{\alpha,\theta}(x_i)\right]^{R_i}, \quad (5)$$

where A is the normalizing constant is equal to

$$\begin{aligned} A &= n(n-1-R_1),(n-2-R_1-R_2),\ldots,(n-\sum_{i=1}^{(m-1)}(R_i-m+1)) \\ &= \prod_{j=1}^{m}(n-\sum_{i=1}^{j-1}R_i-j+1), \end{aligned}$$

Based on (5), the marginal distribution of $X_{i:m:n}$ for $1 \leq i \leq m$ is given by

$$f_{X_{i:m:n};\alpha,\theta}(x) = a_{i-1,n} f_{\alpha,\theta}(x) \sum_{j=1}^{i} b_{j,i,n}(1-F_{\alpha,\theta}(x))^{\gamma_{j,n}-1}, \tag{6}$$

where

$$\begin{aligned} \gamma_{j,n} &= n-j+1-\sum_{k=1}^{j-1}R_k = m-j+1+\sum_{k=j}^{m}R_k, \\ a_{i-1,n} &= \prod_{j=1}^{i}\gamma_{j,n}, \qquad i=1,\ldots,m, \\ b_{i,i,n} &= \prod_{j=1,i\neq j}\frac{1}{\gamma_{j,n}-\gamma_{j,n}}, \qquad 1\leq j\leq i\leq m. \end{aligned}$$

For more information about the progressive censoring of the second type from the right, you can refer to the articles of Balasooriya and Balakrishnan (2000); Balakrishnan and Han (2007) and Alhussain and Ahmed (2020).

4 Type II progressive censoring Lindley distribution with outliers

In this section, to obtain the model by replacing (3) and (4) in (5) and (6), the density function of $\underline{X} \equiv (x_{1:m:n}^{(R_1,\ldots,R_m)},\ldots,x_{m:m:n}^{(R_1,\ldots,R_m)})$ is given by

$$\begin{aligned} f_{X_1,\ldots,X_n}(x_1,\ldots,x_n;\alpha,\theta) = A\prod_{i=1}^{n}\Bigg(&\left(\frac{k}{n\alpha\theta}I_{0,\alpha\theta}(x_i)+\left(1-\frac{k}{n}\right)\frac{(x_i+1)}{\theta^2(\theta+1)}\exp\left(\frac{-x_i}{\theta}\right)\right) \\ &\times\Bigg(1-\frac{kx_i}{n\alpha\theta}I_{0,\alpha\theta}(x_i)+\left(1-\frac{k}{n}\right) \\ &\times\left(\frac{1+\theta+x_i}{\theta(1+\theta)}\exp\left(\frac{-x_i}{\theta}\right)\right)\Bigg)^{R_i}\Bigg). \end{aligned} \tag{7}$$

The marginal probability density function $X_{i:m:n}$, $1 \leq i \leq m$, according to (7), is given by

$$f_{X_{i:m:n;\alpha,\theta}}(x) = a_{i-1,n} f_{\alpha,\theta}(x) \sum_{j=1}^{i} b_{j,i,n}(1-F_{\alpha,\theta}(x))^{\gamma_{j,n}-1},$$

$$
\begin{aligned}
f_{X_{i:m:n;\alpha,\theta}}(x) &= a_{i-1,n}\left(\frac{k}{n\alpha\theta}I_{(0,\alpha\theta)}(x_i)+\left(1-\frac{k}{n}\right)\frac{x_i+1}{\theta^2(\theta+1)}\exp\left(\frac{-x_i}{\theta}\right)\right) \\
&\quad \times\sum_{j=1}^{i} b_{j,i,n}\Bigg(1-\frac{kx_i}{n\alpha\theta}I_{0,\alpha\theta}(x_i) \\
&\quad +\left(1-\frac{k}{n}\right)\left(\frac{1+\theta+x_i}{\theta(1+\theta)}\exp\left(\frac{-x_i}{\theta}\right)\right)\Bigg)^{\gamma_{j,n}-1}.
\end{aligned}
$$

For the special case $j=1$,

$$
\begin{aligned}
\gamma_{1,n} &= na_{i-1,n}=\gamma 1, n=n, \\
b_{1,i,n} &= \prod_{i\neq j, j=1}\frac{1}{\gamma_{i,n}-n}, \quad 1\le j\le i\le m, \\
\text{or} \qquad b_{i,n} &= \prod_{i=2}^{m}\frac{1}{\gamma_{i,n}-n}.
\end{aligned}
$$

It should be say that, the marginal probability density function of Lindley distribution under type II progressive censoring with the presence of outlier is equal to

$$
\begin{aligned}
f_{X_{i:m:n;\alpha,\theta}}(x) &= n\left(\frac{k}{n\alpha\theta}I_{(0,\alpha\theta)}(x_i)+\left(1-\frac{k}{n}\right)\frac{x_i+1}{\theta^2(\theta+1)}\exp\left(\frac{-x_i}{\theta}\right)\right) \\
&\quad \times b_{i,n}\left(1-\frac{kx_i}{n\alpha\theta}I_{0,\alpha\theta}(x_i)+\left(1-\frac{k}{n}\right)\left(\frac{1+\theta+x_i}{\theta(1+\theta)}\exp\left(\frac{-x_i}{\theta}\right)\right)\right)^{n},
\end{aligned}
$$

5 Estimation of parameters of Lindley distribution under type II progressive censoring in the presence of outlier data

In this section, the parameters of Lindley distribution under type II progressive censoring in the presence of outlier data are estimated by maximum likelihood and Bayesian methods.

5.1 Likelihood estimation

The likelihood function of the Lindley distribution under the progressive censoring of the second type with the presence of outlying data is equal to

$$
\begin{aligned}
L(\alpha,\theta|X_1,X_2,\ldots,X_n) = A\prod_{i=1}^{n}&\left(\left(\frac{k}{n\alpha\theta}I_{(0,\alpha\theta)}(x_i)+\left(1-\frac{k}{n}\right)\frac{(x_i+1)}{\theta^2(\theta+1)}\exp\left(\frac{-x_i}{\theta}\right)\right)\right. \\
&\times\Bigg(1-\frac{kx_i}{n\alpha\theta}I_{0,\alpha\theta}(x_i)+\left(1-\frac{k}{n}\right) \\
&\times\left.\left(\frac{1+\theta+x_i}{\theta(1+\theta)}\exp\left(\frac{-x_i}{\theta}\right)\right)\Bigg)_i^{R}\right).
\end{aligned}
$$

To estimate the parameters using the maximum likelihood method, if is the logarithm of the likelihood function for parameters and according to (2) it can be written as

$$l(\alpha,\theta)=\log A+\sum_{i=1}^{n}\log\left(\frac{k}{n\alpha\theta}I_{(0,\alpha\theta)}(x_i)+\left(1-\frac{k}{n}\right)\frac{x_i+1}{\theta^2(\theta+1)}\exp\left(\frac{-x_i}{\theta}\right)\right)$$
$$+\sum_{i=1}^{n}R_i\log\left(1-\frac{kx_i}{n\alpha\theta}I_{0,\alpha\theta}(x_i)+\left(1-\frac{k}{n}\right)\left(\frac{1+\theta+x_i}{\theta(1+\theta)}\exp\left(\frac{-x_i}{\theta}\right)\right)\right). \quad (8)$$

But using of (8) the parameter of can be estimated as follows

$$\hat{\alpha}\theta=x_n=\max(X_1,X_2,\ldots,X_n);\quad x_i>0,\quad \alpha>0,\quad \theta>0.$$

By Substituting of $\hat{\alpha}=\frac{x(n)}{\theta}$ in (8), $l(\theta)$ is equal to

$$l(\theta)\;=\;l\log A+\sum_{i=1}^{n}\log\left(\frac{k}{nx(n)}+\left(1-\frac{k}{n}\right)\frac{(x_i+1)}{\theta^2(\theta+1)}\exp\left(\frac{-x_i}{\theta}\right)\right)$$
$$+\sum_{i=1}^{n}\left(R_i\log 1-\frac{kx_i}{n}+\left(1-\frac{k}{n}\right)\left(\frac{1+\theta+x_i}{\theta(1+\theta)}\exp\left(\frac{-x_i}{\theta}\right)\right)\right).$$

To estimate parameter θ , the logarithm of the likelihood function can be written as

$$\frac{dl(\theta)}{d\theta}=\sum_{i=1}^{n}\left(\frac{\left(1-\frac{k}{n}\right)\exp\left(\frac{-x_i}{\theta}\right)\left(\left(\frac{2\theta(x_i+1)}{(\theta(\theta+1))^2}\right)+\frac{x_i(x_i+1)}{\theta^3(\theta+1)}\right)}{\left(\frac{k}{nx_{(n)}}+\left(1-\frac{k}{n}\right)\frac{1}{\theta^2(\theta+1)}(x_i+1)\exp\left(\frac{-x_i}{\theta}\right)\right)^2}\right)$$
$$+\sum_{i=1}^{n}R_i\left(\frac{\left(\left(1-\frac{k}{n}\right)\exp\left(\frac{-x_i}{\theta}\right)\left(\left(\frac{-x_i}{(1+\theta)^2}\right)+\left(\frac{x_i(1+\theta+x_i)}{\theta^2(1+\theta)}\right)\right)\right)}{\left(1-\frac{kx_i}{n}+\left(1-\frac{k}{n}\right)\left(\frac{1+\theta+x_i}{\theta(1+\theta)}\exp\left(\frac{-x_i}{\theta}\right)\right)\right)^2}\right)$$

$$\frac{d^2l}{d\theta^2}=\sum_{i=1}^{n}\left(\frac{2\exp\left(\frac{-x_i}{\theta}\left(\theta+x_i+1\right)\right.}{\theta^4(\theta+1)}-\frac{4\exp\left(\frac{-x_i}{\theta}\right)(x_i+1)}{\theta^3(\theta+1)}+\frac{2\exp\left(\frac{-x_i}{\theta}\right)}{\theta^2(\theta+1)}\right)$$
$$=\frac{2\sum_{i=1}^{n}\exp\left(\frac{-x_i}{\theta}\left(\theta+x_i+1\right)\right.}{\theta^4(\theta+1)}-\frac{4\sum_{i=1}^{n}\exp\left(\frac{-x_i}{\theta}\right)(x_i+1)}{\theta^3(\theta+1)}+\frac{2\sum_{i=1}^{n}\exp\left(\frac{-x_i}{\theta}\right)}{\theta^2(\theta+1)}.$$

Since the normal equation $\frac{dl(\theta)}{d\theta}$ is not an explicit function of the parameter θ and it is not possible to solve it analytically, therefore a numerical method such as the Newton-Raphson method or Algorithm EM can be used to estimate it. For this purpose, to use the Newton-Raphson method, suppose that parameter is studied. In the $(h+1)-th$ step of the algorithm iteration process, the updated parameter is

$$\theta_{i+1}=\theta_i-\frac{g(\theta_i)}{g\prime(\theta_i)},$$

where $g(\theta)=\frac{dl}{d\theta}$ and $g\prime(\theta)=\frac{d^2l}{d\theta^2}$. In which the first and second order derivatives of the likelihood function are desired, and the process of repeating the algorithm is realized until the convergence time, which is $\|\theta_{(h+1)}-\theta_{(h)}\|<\varepsilon$, and the value of the estimators can be obtained. It is also possible to discuss the parameters of the Lindley distribution under the progressive censoring of the second type with the presence of outlying data using the moment method.

5.2 Bayesian estimation

The Bayesian method is a suitable method for small sample size and sampling problems with censored samples, especially when some a priori information about the parameters of the distribution is available. In this section, a Bayesian approach is presented to estimate the parameters of the Lindley distribution under type II progressive censoring with the presence of outlier data. In Bayesian estimation, inverse gamma distributions are often used because of their flexibility as prior distributions for parameters that must have positive values. By observing the data, a posterior distribution is obtained by combining the data with the prior distribution to estimate the parameters using the obtained information. For Bayesian estimation, parameters α and θ have inverse gamma prior distributions with parameters (a_1, b_1) and (a_2, b_2) respectively.

$$\begin{aligned}\pi(\alpha, a_1, b_1) &= a_1 b_1 \alpha^{-a_1-1}\exp(-a_1\alpha^{-b_1}), \quad \alpha>0, \quad a_1>0, \quad b_1>0,\\ \pi(\alpha, a_2, b_2) &= a_2 b_2 \alpha^{-a_2-1}\exp(-a_2\alpha^{-b_2}), \quad \alpha>0, \quad a_2>0, \quad b_2>0.\end{aligned}$$

In this case, the posterior distribution can be expressed as

$$\pi(\theta,\alpha|\underline{x}) = \frac{\pi(\underline{x}|\theta,\alpha)\pi(\theta,a_1,b_1)\pi(\alpha,a_2,b_2)}{\int\int \pi(\underline{x}|\theta,\alpha)\pi(\theta,a_1,b_1)\pi(\alpha,a_2,b_2)d(\theta)d(\alpha)}, \quad x>0, \quad \alpha>0, \quad \theta>0,$$

$$\pi(\theta,\alpha|\underline{x}) = \frac{A\prod_{i=1}^{n}\left(\frac{k}{n\alpha}+\left(1-\frac{k}{n}\right)\frac{1}{\theta^2(\theta+1)}(x_i-1)\exp\left(\frac{-x_i}{\theta}\right)\left(1-\frac{kx_i}{n\alpha}+\left(1-\frac{k}{n}\right)\right.\right.}{A\int\int\prod_{i=1}^{n}\left(\frac{k}{n\alpha}+\left(1-\frac{k}{n}\right)\frac{1}{\theta^2(\theta+1)}(x_i-1)\exp\left(\frac{-x_i}{\theta}\right)\left(1-\frac{kx_i}{n\alpha}+\left(1-\frac{k}{n}\right)\right.\right.}$$

$$\frac{\left.\left.\left(\frac{1+\theta+x_i}{\theta(1+\theta)}\exp\left(\frac{-x_i}{\theta}\right)\right)^{R_r}\right)\right)a_1b_1a_2b_2\alpha^{-(a_2+1)}\exp-(a_1\alpha_{-b_1}+a_2\theta_{-b_2})}{\left.\left.\left(\frac{1+\theta+x_i}{\theta(1+\theta)}\exp\left(\frac{-x_i}{\theta}\right)\right)^{R_r}\right)\right)a_1b_1a_2b_2\alpha^{-(a_2+1)}\exp-(a_1\alpha_{-b_1}+a_2\theta_{-b_2})d\theta d\alpha}. \tag{9}$$

Since (9) is not an explicit function of the parameters, the Lindley approximation method is used. Therefore, any proportion of the integral as

$$l(x) = E[y(\alpha,\theta|x)] = \frac{\int y(\alpha,\theta)e^{L(\alpha,\theta)+G(\alpha,\theta)}d(\alpha,\theta)}{\int e^{L(\alpha,\theta)+G(\alpha,\theta)}d(\alpha,\theta)}.$$

Such that $y(\alpha,\theta)$ is a function of α and θ. Let $L(\alpha,\theta)$ is the logarithm of the maximum likelihood and $G(\alpha,\theta)$ is the logarithm of the prior distribution of α and θ. Therefore, equation $l(x)$ can be written as

$$\begin{aligned}l(x) = y(\hat\alpha,\hat\theta) + \frac{1}{2}&[(\hat y_{\theta\theta}+2\hat y_\theta\hat p_\theta)\hat\sigma_{\theta\theta}+(\hat y_{\alpha\theta}+2\hat y_\alpha\hat p_\theta)\hat\sigma_{\alpha\theta}+(\hat y_{\theta\alpha}+2\hat y_\theta\hat p_\alpha)\hat\sigma_{\theta\alpha}\\ &+(\hat y_{\alpha\alpha}+2\hat y_\alpha\hat p_\alpha)\hat\sigma_{\alpha\alpha}+(\hat y_\theta\hat\sigma_{\theta\theta}+\hat y_\alpha\hat\sigma_{\theta\alpha})(\hat L_{\theta\theta\theta}\hat\sigma_{\theta\theta}+\hat L_{\theta\alpha\theta}\hat\sigma_{\theta\alpha}+\hat L_{\alpha\theta\theta}\hat\sigma_{\alpha\theta}\\ &+\hat L_{\alpha\alpha\theta}\hat\sigma_{\alpha\alpha})+(\hat y_\theta\hat\sigma_{\alpha\theta}+\hat y_\alpha\hat\sigma_{\alpha\alpha})(\hat L_{\alpha\theta\theta}\hat\sigma_{\theta\theta}+\hat L_{\theta\alpha\alpha}\hat\sigma_{\alpha\theta}+\hat L_{\alpha\alpha\alpha}\hat\sigma_{\alpha\alpha})],\end{aligned}$$

where $\hat\alpha$ and $\hat\theta$ are the maximum likelihood estimates for α and θ, respectively, and also

$$\hat y_\theta = \frac{\partial_y(\hat\alpha,\hat\theta)}{\partial\hat\theta}, \quad \hat y_\alpha = \frac{\partial_y(\hat\alpha,\hat\theta)}{\partial\hat\alpha},$$

$$\hat{y}_{\theta\theta} = \frac{\partial_y^2(\hat{\alpha},\hat{\theta})}{\partial^2\hat{\theta}}, \quad \hat{y}_{\alpha\alpha} = \frac{\partial_y^2(\hat{\alpha},\hat{\theta})}{\partial^2\hat{\alpha}}, \quad \hat{y}_{\alpha\theta} = \hat{y}_{\theta\alpha} = \frac{\partial_y^2(\hat{\alpha},\hat{\theta})}{\partial\hat{\alpha}\partial\hat{\theta}},$$
$$\hat{p}_{\alpha} = \frac{\partial G(\hat{\alpha},\hat{\theta})}{\partial\hat{\alpha}}, \quad \hat{p}_{\theta} = \frac{\partial G(\hat{\alpha},\hat{\theta})}{\partial\hat{\theta}}, \quad \hat{L}_{\theta\theta\alpha} = \hat{L}_{\alpha\theta\theta} = \hat{L}_{\theta\alpha\theta} = \frac{\partial^3 L(\hat{\alpha},\hat{\theta})}{\partial^2\hat{\theta}\partial\hat{\alpha}},$$
$$\hat{L}_{\alpha\alpha\theta} = \hat{L}_{\theta\alpha\alpha} = \hat{L}_{\alpha\theta\alpha} = \frac{\partial^3 L(\hat{\alpha},\hat{\theta})}{\partial^2\hat{\alpha}\partial\hat{\theta}}, \quad \hat{L}_{\theta\theta\theta} = \frac{\partial^3 L(\hat{\alpha},\hat{\theta})}{\partial^3\hat{\theta}}, \quad \hat{L}_{\alpha\alpha\alpha} = \frac{\partial^3 L(\hat{\alpha},\hat{\theta})}{\partial^3\hat{\alpha}}.$$

The Bayesian estimator's parameters α and θ according to the error squared loss function are given as

$$\hat{\alpha} = E_{\alpha}(\alpha),$$
$$E_{\alpha}(\alpha) = \int_{\alpha,\theta} \theta\pi(\alpha,\theta|x)d(\alpha,\theta), y(\alpha,\theta) = \alpha.$$

6 Simulation study

In this section, to evaluate the maximum likelihood and Bayesian estimators, samples of size $n = 10(10)50$ are generated from the Lindley distribution under progressive censoring of the second type with the presence of outlying data for different values of parameters α and θ. The steps of implementing the algorithm step by step are presented below.

1. In this study, the following three designs are used for censorship.

Design one: $R_m = n - m \qquad R_1 = \ldots = R_{m-1} = 0.$

Design two: $R_2 = \ldots = R_m = 0 \qquad R_1 = n - m.$

Design three: $R_m = n - 2m + 1 \qquad R_1 = \ldots = R_{m-1} = 1.$

2. Different values of parameters are calculated using prior distributions with values of hyper parameters (a_1, b_1) and (a_2, b_2).

3. To evaluate the parameter estimation methods, the mean squared error is used, and according to the $\alpha - \bar{\hat{\alpha}} = \alpha - \frac{1}{1000}\sum_{i=1}^{1000}\hat{\alpha}_i$ and the $MSE(\hat{\alpha})\frac{1}{1000}\sum_{i=1}^{1000}(\alpha - \hat{\alpha}_i)^2$, the simulation results of estimators and for different value are given in Tables 1 to 4.

4. The simulation steps have been done with the help of software R. The number of simulations is 1000 to obtain a more stable and accurate estimate.

5. According to the simulation results, we can say

A- As the sample size increases, the MSE of all estimator's decreases.

B- By increasing the sample size for a fixed number of censors (m), the average error power of the estimator's decreases.

C- By increasing the sample size for a constant number of censors (m) for all three censoring plans, the average error power of the estimator's decreases.

D- As the number of censors (m) increases, the average error power of the estimators increases.

E- The Bayesian estimator is always efficient and has a lower average error power compared to the maximum likelihood estimator.

F- For α and θ constants, the MSE increases with the increase in the number of outlier data, which indicates the effect of the number of outlier data on the performance of the estimators.

Table 1: Estimation, Bias and MSE of estimators when $\alpha = 0.2, \theta = 2, k = 1$.

				ML method			Bayes method		
n	m	Scheme	Par	MSE	Bias	MSE	MSE	Bias	MSE
10	5	1	α	0.3890	0.1890	0.0848	0.2399	0.0399	0.0810
			θ	1.2788	-0.7212	1.3622	1.9297	-0.0703	0.3369
		2	α	0.3908	0.1908	0.0855	0.2180 5	0.0180	0.0525
			θ	1.2418	-0.7582	1.2436	1.9310	-0.0690	0.3121
		3	α	0.3861	0.1861	0.0831	0.1835	-0.0165	0.0912
			θ	1.2493	-0.75072	1.2779	1.8929	-0.1071	0.3176
20	5	1	α	0.3429	0.1429	0.0577	0.1740	-0.0261	0.0154
			θ	1.2389	-0.7610	0.7385	1.9421	-0.0579	0.1738
		2	α	0.3441	0.1441	0.0590	0.1903	-0.0097	0.0345
			θ	1.2371	-0.7630	0.7425	1.9336	-0.0664	0.1757
		3	α	1.3462	0.1462	0.0591	1.1724	-0.0276	0.0224
			θ	1.2240	-0.7760	0.7405	1.9606	-0.0394	0.1819
	10	1	α	0.4275	0.2275	0.1025	0.3471	0.1471	0.1033
			θ	1.1668	-0.8333	0.7552	1.9336	-0.0664	0.2012
		2	α	0.4268	0.2268	0.1027	0.3256	0.1256	0.1017
			θ	1.1655	-0.8345	0.7570	1.9322	-0.0678	0.1859
		3	α	0.4281	0.2281	0.1030	0.3567	0.1567	0.1093
			θ	1.1611	-0.8389	0.7625	1.9414	-0.0586	0.1953
30	5	1	α	0.2753	0.0753	0.0302	0.2302	0.0302	0.0123
			θ	1.3157	-0.6843	0.6028	1.9745	-0.0255	0.1113
		2	α	0.2729	0.0729	0.0297	0.2057	0.0057	0.0204
			θ	1.3134	-0.6843	0.6028	1.9693	-0.0307	0.1305
		3	α	0.2736	0.0736	0.0297	0.2162	0.0162	0.0290
			θ	1.3020	-0.6980	0.6151	1.9758	-0.0242	0.1156
	10	1	α	0.4228	0.2228	0.0937	0.3584	0.1584	0.0830
			θ	1.2108	-0.7892	0.6816	1.9584	-0.0416	0.1215
		2	α	0.4210	0.2210	0.0931	0.2997	0.0997	0.0836
			θ	1.2121	-0.7879	0.6787	1.9695	-0.0306	0.1280
		3	α	0.4227	0.2227	0.0940	0.3221	0.1220	0.0820
			θ	1.2051	-0.7950	0.6851	1.9532	-0.0468	0.1263
40	5	1	α	0.2210	0.0210	0.0161	0.2090	0.0090	0.0105
			θ	1.3907	-0.60913	0.5191	1.9872	-0.0128	0.0926
		2	α	0.2213	0.0213	0.0161	0.1733	-0.0267	0.0153
			θ	1.3891	-0.6109	0.6024	1.9748	-0.0252	0.1022
		3	α	0.2214	0.0214	0.0164	0.1859	-0.0141	0.0118
			θ	1.3716	-0.6284	0.5325	1.9746	-0.0254	0.0986
	10	1	α	0.3800	0.1800	0.0694	0.3532	0.1532	0.0573
			θ	1.2521	-0.7479	0.6191	1.9776	-0.0224	0.0987
		2	α	0.3835	0.1835	0.0703	0.3347	0.1347	0.0687
			θ	1.2525	-0.7475	0.6199	1.9687	-0.0313	0.0952
		3	α	0.3835	0.1835	0.0676	0.3147	0.1147	0.0528
			θ	1.2431	-0.7569	0.6280	1.9768	-0.0232	0.1000
50	5	1	α	0.2069	0.0069	0.0111	0.1988	-0.0012	0.0100
			θ	1.4521	-0.5479	0.4768	1.9926	-0.0074	0.0769
		2	α	0.2078	0.0078	0.0113	0.1946	-0.0054	0.0114
			θ	1.4586	-0.5414	0.4711	1.9849	-0.0151	0.0866
		3	α	0.2044	0.0044	0.0110	0.1946	-0.0054	0.0102
			θ	1.4438	-0.5562	0.4713	1.9894	-0.0106	0.0761
	10	1	α	0.3607	0.1607	0.0509	0.3132	0.1132	0.0432
			θ	1.2879	-0.7121	0.5743	1.9784	-0.0216	0.0773
		2	α	0.3577	0.1577	0.0486	0.3236	0.1236	0.0327
			θ	1.2924	-0.7076	0.5671	1.9747	-0.0253	0.0761
		3	α	0.3546	0.1546	0.0478	0.3403	0.1403	0.0283
			θ	1.2799	-0.7201	0.5774	1.9757	-0.0243	0.0756

Table 2: Estimation, Bias and MSE of estimators when $\alpha = 0.2, \theta = 2, k = 2$.

				ML method			Bayes method		
n	m	Scheme	Par	MSE	Bias	MSE	MSE	Bias	MSE
10	5	1	α	0.3934	0.1934	0.0861	0.2628	0.0628	0.0837
			θ	1.3380	-0.6620	1.5763	1.9333	-0.0667	0.3857
		2	α	0.4082	0.2082	0.0930	0.2690	0.0690	0.0860
			θ	1.3142	-0.6858	1.4885	1.9103	-0.0897	0.3557
		3	α	0.3808	0.1808	0.0796	0.2306	0.0306	0.0879
			θ	1.3665	-0.6335	1.5344	1.9162	-0.0838	0.3381
20	5	1	α	0.3512	0.1512	0.0626	0.2085	0.0085	0.0106
			θ	1.2389	-0.7611	0.7385	1.9541	-0.0459	0.1934
		2	α	0.3565	0.1653	0.0641	0.2196	-0.0196	0.0600
			θ	1.2415	-0.7585	0.7346	1.9360	-0.0640	0.1809
		3	α	0.3460	0.1460	0.0603	0.2030	0.0030	0.2244
			θ	1.3665	-0.6335	1.5344	1.9701	-0.0299	0.1882
	10	1	α	0.4419	0.2419	0.1119	0.3489	0.1489	0.1033
			θ	1.1623	-0.8377	0.7803	1.9618	-0.0382	0.2006
		2	α	0.4342	0.2342	0.1054	0.3428	0.1428	0.0996
			θ	1.1723	-0.8277	0.7439	1.9697	-0.0304	0.1900
		3	α	0.4352	0.2352	0.1080	0.3235	0.1234	0.1004
			θ	1.1573	-0.8427	0.7604	1.9444	-0.0556	0.1769
30	5	1	α	0.2708	0.0708	0.0295	0.2141	0.0141	0.0162
			θ	1.3108	-0.6892	0.6080	1.9708	-0.0293	0.1406
		2	α	0.2833	0.0833	0.0333	0.2226	0.0226	0.0259
			θ	1.3088	-0.6912	0.6094	1.9695	-0.0305	0.1333
		3	α	0.2674	0.0674	0.0278	0.1872	-0.0128	0.0146
			θ	1.2977	-0.7023	0.6201	1.9651	-0.0349	0.1233
	10	1	α	0.4197	0.2197	0.0907	0.3388	0.1388	0.0473
			θ	1.2081	-0.7919	0.6793	1.9691	-0.0308	0.1219
		2	α	0.4128	0.2128	0.0901	0.3074	0.1347	0.0687
			θ	1.2035	-0.7965	0.6909	1.9816	-0.0184	0.1208
		3	α	0.4211	0.2211	0.0933	0.3712	0.1712	0.0483
			θ	1.2039	-0.7962	0.6867	1.9787	-0.0213	0.1278
40	5	1	α	0.2164	0.0164	0.0151	0.2028	-0.0028	0.0137
			θ	1.3859	-0.6141	0.5230	1.9853	-0.0147	0.1013
		2	α	0.2216	0.0216	0.0165	0.2166	0.0166	0.0125
			θ	1.3842	-0.6158	0.5301	1.9680	-0.0320	0.0943
		3	α	0.2313	0.0313	0.0184	1.2017	0.0017	0.2538
			θ	1.3673	-0.6327	0.5362	1.9629	-0.0371	0.0917
	10	1	α	0.3766	0.1766	0.0651	0.3219	0.1219	0.0386
			θ	1.2477	-0.7523	0.6232	1.9580	-0.0420	0.0937
		2	α	0.3813	0.1813	0.0711	0.2927	0.0927	0.0268
			θ	1.2466	-0.7534	0.6262	1.9706	-0.0296	0.0939
		3	α	0.3803	0.1803	0.0692	0.3501	0.1501	0.0238
			θ	1.2614	-0.7386	0.6038	1.9874	-0.0126	0.0987
50	5	1	α	0.2120	0.0120	0.0120	0.2041	0.0041	0.0109
			θ	1.4476	-0.5525	0.4790	1.9962	-0.0038	0.0772
		2	α	0.2075	0.0075	0.0106	0.1907	-0.0093	0.0114
			θ	1.4541	-0.5459	0.4732	1.9930	-0.0170	0.0729
		3	α	0.2070	0.0070	0.0110	0.1946	-0.0070	0.0081
			θ	1.4395	-0.5605	0.4740	1.9886	-0.0114	0.0797
	10	1	α	0.3678	0.1677	0.0530	0.3071	0.1071	0.0288
			θ	1.3024	-0.6976	0.5566	1.9492	-0.0508	0.0768
		2	α	0.3566	0.1566	0.0474	0.2923	0.0923	0.0266
			θ	1.2845	-0.7155	0.5801	1.9660	-0.0340	0.0746
		3	α	0.3635	0.1635	0.0494	0.3540	0.1540	0.0256
			θ	1.2695	-0.7305	0.5906	1.9771	-0.0229	0.0774

Table 3: Estimation, Bias and MSE of estimators when $\alpha = 0.3, \theta = 2, k = 1$.

				ML method			Bayes method		
n	m	Scheme	Par	MSE	Bias	MSE	MSE	Bias	MSE
10	5	1	α	0.4022	0.1022	0.0528	0.3689	0.0689	0.0410
			θ	1.6316	-0.3684	2.4377	1.9016	-0.0984	0.3285
		2	α	0.4053	0.1053	0.0538	0.3581	0.0581	0.0425
			θ	1.7063	-0.2937	2.6401	1.8862	-0.1138	0.3121
		3	α	0.4144	0.1144	0.0580	0.3824	0.0824	0.0322
			θ	1.6838	-0.3162	2.4790	1.8874	-0.1126	0.3362
20	5	1	α	0.3495	0.0495	0.0357	0.3346	0.0346	0.0255
			θ	1.2500	-0.7500	0.7860	1.9509	-0.0491	0.1922
		2	α	0.3743	0.0743	0.0400	0.3402	0.0402	0.0304
			θ	1.2598	-0.7402	0.7633	1.9542	-0.0458	0.1874
		3	α	0.3663	0.0663	0.0404	0.3422	0.0422	0.0324
			θ	1.2369	-0.7631	0.7413	1.9451	-0.0549	0.1843
	10	1	α	0.4287	0.1287	0.0632	0.3751	0.0751	0.0526
			θ	1.1712	-0.8288	0.7957	1.8985	-0.1015	0.3330
		2	α	0.4278	0.1278	0.0728	0.3842	0.0842	0.0527
			θ	1.1773	-0.8227	0.7445	1.9602	-0.0398	0.1852
		3	α	0.4287	0.1287	0.0708	0.3889	0.0889	0.0603
			θ	1.1580	-0.8389	0.7625	1.9360	-0.0641	0.1952
30	5	1	α	0.2853	-0.0147	0.0227	0.3311	0.0311	0.0185
			θ	1.3171	-0.6829	0.5995	1.9695	-0.0255	0.1280
		2	α	0.2869	-0.0131	0.0246	0.3357	0.0356	0.0184
			θ	1.3110	-0.6890	0.6194	1.9747	-0.0253	0.0761
		3	α	0.2807	-0.0194	0.0218	0.3284	0.0284	0.0200
			θ	1.3077	-0.6923	0.5949	1.9532	-0.0468	0.1263
	10	1	α	0.4388	0.1388	0.0600	0.3692	0.0472	0.0438
			θ	1.2103	-0.7897	0.6752	1.9632	-0.0369	0.1742
		2	α	0.4381	0.1381	0.0597	0.3597	0.0472	0.0457
			θ	1.2065	-0.7936	0.6854	1.9406	-0.0594	0.1322
		3	α	0.4267	0.1267	0.0560	0.3552	0.0552	0.0452
			θ	1.2061	-0.7939	0.6826	1.9570	-0.0430	0.1215
40	5	1	α	0.2882	-0.0118	0.0187	0.3102	0.0102	0.0123
			θ	1.3680	-0.6320	0.5480	1.9809	-0.0191	0.0923
		2	α	0.2867	-0.0133	0.0199	0.3159	0.0159	0.0143
			θ	1.4186	-0.5815	0.4997	1.9687	-0.0313	0.0952
		3	α	0.2883	-0.0117	0.0189	0.3128	0.0128	0.0157
			θ	1.3698	-0.6302	0.5457	1.9757	-0.0243	0.0756
	10	1	α	0.3932	0.0932	0.0416	0.3482	0.0482	0.0373
			θ	1.2507	-0.7493	0.6182	1.9609	-0.0391	0.0966
		2	α	0.3946	0.0946	0.0427	0.3353	0.0353	0.0389
			θ	1.2498	-0.7502	0.6202	1.9705	-0.0296	0.0932
		3	α	0.3957	0.0957	0.0398	0.3357	0.0357	0.0528
			θ	1.2646	-0.7354	0.5986	1.9756	-0.0244	0.0227
50	5	1	α	0.2920	-0.0080	0.0126	0.3037	0.0037	0.0121
			θ	1.4559	-0.5442	0.4582	1.9816	-0.0184	0.0787
		2	α	0.2935	-0.0065	0.0121	0.3043	0.0043	0.0114
			θ	1.4503	-0.5497	0.4701	1.9849	-0.0152	0.1205
		3	α	0.2930	-0.0070	0.0136	0.3048	0.0048	0.0132
			θ	1.4612	-0.5388	0.4373	1.9857	-0.0143	0.0921
	10	1	α	0.3496	0.0496	0.0283	0.3236	0.0236	0.0202
			θ	1.3052	-0.6948	0.5520	1.8949	-0.1051	0.0775
		2	α	0.3376	0.0376	0.0242	0.3246	0.0246	0.0217
			θ	1.2877	-0.7123	0.5746	1.7807	-0.2193	0.0818
		3	α	0.3430	0.0430	0.0255	0.3235	0.3235	0.0223
			θ	1.2722	-0.7278	0.5861	1.8736	-0.1265	0.0772

Table 4: Estimation, Bias and MSE of estimators when $\alpha = 0.2, \theta = 3, k = 2$.

				ML method			Bayes method		
n	m	Scheme	Par	MSE	Bias	MSE	MSE	Bias	MSE
10	5	1	α	0.4353	0.2353	0.0960	0.2613	0.0613	0.0747
			θ	1.5070	-1.4930	2.7268	3.0702	0.0702	0.4176
		2	α	0.4377	0.2377	0.0984	0.2473	0.0473	0.0492
			θ	1.5097	-1.4903	2.7319	2.9688	-0.0312	0.4771
		3	α	0.4359	0.2359	0.0969	0.2382	0.0382	0.0330
			θ	1.6099	-1.3901	2.8018	3.0114	0.0114	0.4393
20	5	1	α	0.2784	0.0784	0.0286	0.2232	0.0232	0.0104
			θ	1.5097	-1.4903	2.7319	3.0837	0.0837	0.2319
		2	α	0.2836	0.0836	0.0306	0.2375	0.0375	0.0122
			θ	1.5679	-1.4321	2.3345	3.0108	0.0108	0.1896
		3	α	0.2769	0.0769	0.0288	0.2122	0.0122	0.0184
			θ	1.5534	-1.4466	2.3625	2.9788	-0.0212	0.1991
	10	1	α	0.4809	0.2809	0.1197	0.3349	0.1349	0.1165
			θ	1.4904	-1.5097	2.3999	2.8458	-0.1542	0.2815
		2	α	0.4893	0.2893	0.1245	0.3596	0.1596	0.1027
			θ	1.5155	-1.4848	2.3296	3.1153	0.1153	0.2028
		3	α	0.4789	0.2789	0.1187	0.3418	0.1418	0.1034
			θ	1.4939	-1.5061	2.3760	2.8712	-0.1288	0.2601
30	5	1	α	0.2369	0.0369	0.0162	0.1943	-0.0057	0.0040
			θ	1.6295	-1.3705	2.1540	2.9475	-0.0525	0.1769
		2	α	0.2370	0.0370	0.0162	0.2262	0.0262	0.0103
			θ	1.6116	-1.3884	2.1899	2.9926	-0.0075	0.1886
		3	α	0.2370	0.0370	0.0163	0.2213	0.0213	0.0128
			θ	1.6204	-1.3796	2.1440	2.9411	-0.0589	0.1725
	10	1	α	0.3851	0.1851	0.0621	0.3423	0.1423	0.0565
			θ	1.5392	-1.4608	2.2412	3.0952	0.0952	0.1958
		2	α	0.3857	0.1857	0.0614	0.2940	0.0940	0.0323
			θ	1.5351	-1.4649	2.2631	2.8737	-0.1263	0.1490
		3	α	0.3852	0.1852	0.0624	0.3562	0.1562	0.0194
			θ	1.5346	-1.4654	2.2612	3.0949	0.0949	0.1443
40	5	1	α	0.1708	-0.0292	0.0077	0.1919	-0.0081	0.0047
			θ	1.6703	-1.3297	2.0511	3.0375	0.0375	0.0856
		2	α	0.1698	-0.0302	0.0076	0.1929	-0.0071	0.0048
			θ	1.7450	-1.2550	1.9070	3.0035	0.0035	0.1065
		3	α	0.1707	-0.0293	0.0078	0.1763	-0.0237	0.0043
			θ	1.6795	-1.3205	2.0411	3.0304	0.0304	0.0887
	10	1	α	0.3023	0.1023	0.0262	0.3015	0.1015	0.0175
			θ	1.5754	-1.4246	2.1472	2.8911	-0.1089	0.1184
		2	α	0.3034	0.1034	0.0262	0.2943	0.0943	0.0183
			θ	1.5745	-1.4255	2.1499	2.9494	-0.0509	0.1060
		3	α	0.3057	0.1057	0.0270	0.2935	0.1935	0.0102
			θ	1.6317	-1.3683	2.0109	3.1122	0.1122	0.1168
50	5	1	α	0.2032	0.0032	0.0110	0.2007	0.0007	0.0105
			θ	1.7668	-1.2332	1.8190	3.0002	0.0002	0.0518
		2	α	0.2045	0.0045	0.0109	0.2010	0.0010	0.0105
			θ	1.7632	-1.2368	1.8361	2.9729	-0.0271	0.0406
		3	α	0.2031	0.0031	0.0113	0.2027	0.0027	0.0102
			θ	1.7815	-1.2185	1.7603	2.9995	-0.0005	0.0615
	10	1	α	0.2523	0.0523	0.0121	0.2436	0.0436	0.0119
			θ	1.6317	-1.3683	2.01088	3.0148	0.0148	0.0850
		2	α	0.2527	0.0527	0.0125	0.2387	0.0387	0.0117
			θ	1.6084	-1.3916	2.0712	3.0262	0.0262	0.1024
		3	α	0.2521	0.0521	0.0123	0.2326	0.0326	0.0110
			θ	1.6286	-1.3714	2.1076	2.9857	-0.0143	0.0997

References

Alhussain, Z.A. and Ahmed, E.A. (2020). Estimation of exponentiated Nadarajah-Haghighi distribution under progressively type-II censored sample with application to bladder cancer data. *Indian Journal of Pure and Applied Mathematics*, **51**:631-657.

Balakrishnan, N. and Cutler, C.D. (1996). Maximum likelihood estimation of Laplace parameters based on Type-II censored samples. *Statistical theory and applications: papers in honor of Herbert A. David*, 145–151.

Balakrishnan, N. and Han, D. (2007). Optimal progressive Type-II censoring schemes for nonparametric confidence intervals of quantiles. *Communications in Statistic-Simulation and Computation*, **36**(6):1247–1262.

Balakrishnan, N. and Han, D. (2008). Maximum likelihood estimation of Laplace parameters based on Type-II censored samples. In *Statistical Theory and Applications: Papers in Honor of Herbert A. David (Eds).*

Balasooriya, U. and Balakrishnan, N. (2000). Reliability sampling plan for log-normal distribution. *IEEE Transactions on Reliability,* **49**(2):199–203.

Chan, P.S., Ng, H.K.T., Balakrishnan, N. and Zhou, Q. (2008). Point and interval estimation for extreme-value regression model under Type II censoring. *Computational Statistics and Data Analysis,* **52**(8):4040–4058.

Dallas, R.W. (1993). Maximum likelihood estimation of Burr XII distribution parameters under Type II censoring. *Microelectronics Reliability,* **33**(9):1251–1257.

Dey, D.K. and Kuo, L. (1991). A new empirical Bayes estimator with type II censored data. *Journal Computational Statistics & Data Analysis,* **12**(3):271–279.

Dixit, U.J. (1989). Estimation of parameters of the Gamma distribution in the Presence of outliers. *Communications in Statistics-Theory and Methods,* **18**(8):3071–3085.

Dixit, U.J. and Nasiri, P.F. (2001). Estimation of parameters of the exponential distribution in the presence of outliers generated from uniform distribution. *Merton,* **59**(3-4):187–198.

Iliopoulos, G. and Balakrishnan, N. (2011). Exact likelihood inference for Laplace distribution based on Type-II censored samples. *Journal of Statistical Planning and Inference,* **141**(3):1224–1239.

Jabbari Nooghabi, M. (2021). Comparing estimation of the parameters of distribution of the root density of plants in the presence of outliers. *Environmetrics,* **32**(5):e2676.

Kumar, D., Singh, U., Singh, S.K. and Bhattacharyya, G. (2015). Bayesian Estimation of Exponentiated Gamma Parameter for Progressive Type II Censored Data with Binomial Removals. *Journal of Statistics Applications & Probability,* **4**(2):265–273.

Kumar, M., Singh, S.K. and Singh, U. (2018). Bayesian inference for Poisson-inverse exponential distribution under progressive type-II censoring with binomial removal. *International Journal of System Assurance Engineering and Management,* **9**(6):1235–1249.

Kundu, D. and Raqab, M.Z. (2012). Bayesian inference and prediction of order statistics for a Type-II censored Weibull distribution. *Journal of Statistical Planning and Inference,* **142**(2):41–47.

Madi, M.T. and Raqab, M.Z. (2009). Bayesian inference for the generalized exponential distribution based on progressively censored data. *Communications in Statistics theory and Methods,* **38**(20):16–29.

Ooms, G., and Moore, K.L. (1991). A model assay for genetic and environmental changes in the architecture of intact root systems of plants grown in vitro. *Plant Cell, Tissue and Organ Culture,* **27**:129–139.

Panahi, H. and Asadi, S. (2011). Estimation of the Weibull Distribution Based on Type-II Censored Samples. *Applied Mathematical Sciences,* **5**(52):2549–2558.

Pandey, B.N., Malik, H.J. and Srivastava, R. (1989). Shrinkage estimators for the shape parameter of Weibull distribution under Type-II censoring. *Communications in Statistics-Theory and Methods,* **18**(4):1175–1199.

Pathak, A., Kumar, M., Singh, S.K. and Singh, U. (2022). Bayesian inference: Weibull Poisson model for censored data using the expectation–maximization algorithm and its application to bladder cancer data. *Journal of Applied Statistics,* **49**(4):926–948.

Schneider, H. and Weissfeld, L. (1986). Inference based on type II censored samples. *Biometrics,* **42**(3):531–536.

Singh U. and Kumar A., (2007). Bayesian estimation of the exponential parameter under a multiply Type-II censoring scheme. *Austrian Journal of Statistics,* **36**(21):227–238.

Singh, U., Gupta, P.K. and Upadhyay, S.K. (2005). Estimation of parameters for exponentiated-Weibull family under type-II censoring scheme. *Computational Statistics & Data Analysis,* **48**(3):509–523.

Singh, S.K., Singh, U., Kumar, M. and Vishwakarma, P.K. (2014). Classical and Bayesian inference for an extension of the exponential distribution under progressive type-II censored data with binomial removals. *Journal of Statistics Applications & Probability,* **1**(3):75–86.