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Research Paper

Bayesian D-optimal design for regression-based stopping time of a system under the δ-shock model

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Abstract: According to the δ-shock model, the system fails when the inter-arrival time between two successive shocks falls below a given critical threshold, δ. Therefore, the system's failure depends on both the probabilistic behavior of the inter-arrival times and the threshold δ. As a result, the factors affecting these two variables also influence the system's stopping time. In this paper, we consider a regression model with an explanatory factor for the system's stopping time and then apply the Bayesian D-optimal criterion to obtain optimal designs. Assuming a uniform prior, the obtained Bayesian optimal designs all have a general form corresponding to the design space.

Keywords: Bayesian D-optimal design, δ-shock model, Fisher information matrix, Intershock time, Stopping time.
Mathematics Subject Classification (2010): 62K05, 62N05, 90B25.

1 Introduction

Most engineering systems are often exposed to random shocks from external sources. Shock models are usually used to study the survival of systems exposed to shock. One of the most widely used shock models is the *δ-shock model*, which has received much attention in applied probability and reliability engineering in recent years. Under the δ-shock model, the system failure occurs when the inter-arrival time between two successive shocks (or intershock time) is less than a critical threshold $\delta > 0$. Many studies have been done on the δ-shock model, for some of them, see e.g., Eryilmaz (2012), Eryilmaz and Bayramoglu (2014), Parvardeh and Balakrishnan (2015), Tuncel and Eryilmaz (2018), Lorvand et al. (2020), Poursaeed (2020), Entezari (2021), and

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Farhadian and Jafari (2024). The traditional studies that have been done in δ-shock modeling mostly focus on finding the reliability characteristics of the system's lifetime for various extensions and generalizations of the δ-shock model.

Among recent developments, studies that have considered the system stopping time as a regression model and have revealed the role of factors affecting the stopping time by implementing optimal design are not found in the relevant literature. This motivated us to write about this subject in this paper. Therefore, this paper establishes a novel approach in which the system stopping time will be considered as a regression model based on an influencing factor, then optimal design will be applied to identify the optimal points for estimating the regression parameters. In fact, the random behavior of shocks requires consideration of various factors that can contribute to this random behavior, and therefore, the optimal design approach can play a useful role in the optimal analysis of shock behavior and the optimal performance of systems exposed to shocks. The topic of optimal design was widely developed, so that nowadays studying and modeling data without using optimal designs is practically not cost-effective.

In the context of experimental design, the concept of optimal design refers to a specific category of experimental designs that are classified based on certain statistical criteria. Usually, in model-based optimal designs, the inferential aim is to estimate the parameters of the model so that estimators with minimum variance are of interest. The optimization process is completely dependent on the considered optimality criteria, which are usually defined in terms of the information matrix. In linear models, the information matrix and then optimality criteria do not depend on the unknown parameters of the model. So, reaching an optimal design does not have significant computational complexity. However, in general linear and non-linear models, since unknown parameters usually appear in the entries of the information matrix, so the optimality criteria depend on these parameters. Thus deriving an optimal design from the optimization problem may involve computational complexities. For a more advanced study on this topic, see Atkinson et al. (2007). Our focus is on the optimization problem involving the non-linear model.

The paper is organized as follows. The description of the δ-shock model is derived in Section 2. Section 3 gives a brief introduction to optimal design. The statement of the optimal design issue on the δ-shock model is presented in Section 4. Section 5 concludes the paper.

2 The δ-shock model

Assume that a system is subject to a sequence of external random shocks. Let U_i be the ith intershock time, i.e., the time lag between the ith and $(i+1)$th shocks for $i = 1, 2, \ldots$, and let also $U_1, U_2, \ldots$ are independent and identically distributed (i.i.d.) by an arbitrary continuous distribution with the cumulative distribution function (cdf) $F(u)$. Under the well-known δ-shock model, the performance of the system is such that if $U_n \leq \delta$ for a given threshold $\delta > 0$, the system fails. Otherwise, the system continues to work without any performance problems under the influence of shocks. Thus, the lifetime of the system is $T_\delta = \sum_{i=1}^{N} U_i$, where the stopping random variable N is defined as follows

$$\{N = n\} \text{ iff } \{U_1 > \delta, U_2 > \delta, \dots, U_{n-1} > \delta, U_n \leq \delta\},$$

for $n \in \mathbb{N} = \{1, 2, \dots\}$. Clearly, the probability mass function of N is

$$\Pr(N = n) = F(\delta)(1 - F(\delta))^{n-1}, \qquad n \in \mathbb{N}. \tag{1}$$

Besides, the reliability probability function of the system's lifetime T_δ is given by the following formula (see Eryilmaz and Bayramoglu (2014)):

$$\Pr(T_\delta > t) = \sum_{n=1}^{\infty} (1 - F(\delta))^{n-1} \int_0^\delta \Pr\left(S^*_{n-1} > t - u\right) dF(u),$$

where S^*_n is the nth arrival time of a renewal process whose inter-arrival times have the cdf,

$$F^*(u) = \frac{F(u) - F(\delta)}{1 - F(\delta)}, \quad for\ u > \delta.$$

Furthermore, the mean lifetime of the system, that is, the system's mean time to failure (MTTF) is $E(T_\delta) = \frac{E(U)}{F(\delta)}$.

Recent years have seen a significant development in the field of delta shock modeling. Most of these developments have focused on providing new generalizations and extensions. For some extension and generalization of the δ-shock model, see, for example, Wang and Peng (2017), Poursaeed (2021), Goyal et al. (2022), and Farhadian and Jafari (2025). In fact, the simplicity and practical aspects of delta shock modeling make it attractive for use in applied sciences. This model is used in reliability engineering, queuing systems, system safety, and management of systems. As an example in engineering, suppose that the components of an electrical device are prone to overheating due to electrical shocks such as high voltage or low current. It is clear that if the time interval between two consecutive electric shocks is large, the components heated by the previous shock have enough time to cool down spontaneously. However, if the time interval between two consecutive electric shocks is small (smaller than a threshold δ), then the components do not have enough time to cool down, and, in addition, their temperature increases due to the next shock. This may result in the burning of the components and then damaging the electrical device. Obviously, the performance of an electrical device in such a scenario can be described and investigated using the δ-shock model.

3 Optimal experimental design issue

In the context of experimental design, the common problem is to find optimal points (based on a certain criterion) to estimate the unknown parameters of a linear or nonlinear model, in which the response variable Y and the explanatory variable x (that varies in a compact space $\boldsymbol{\chi} \subseteq \mathbb{R}$) are related with the equation $E[Y|x] = \eta(x, \boldsymbol{\theta})$, where $\boldsymbol{\theta} \in \boldsymbol{\Theta} \subseteq \mathbb{R}^m$ (for $m \geq 1$) is the unknown parameter vector and η is a given function. In the case where the scalar response variable Y is distributed as a member of the exponential family, the Fisher information matrix for the parameter $\boldsymbol{\theta}$ at the point x is given by

$$\mathcal{I}(x,\boldsymbol{\theta}) = \frac{1}{Var(Y|x)}\left(\frac{\partial}{\partial\boldsymbol{\theta}}\eta(x,\boldsymbol{\theta})\right)\left(\frac{\partial}{\partial\boldsymbol{\theta}}\eta(x,\boldsymbol{\theta})\right)^T \in \mathbb{R}^{m\times m}. \tag{2}$$

An approximate design ξ for this model is a probability measure on the design space $\boldsymbol{\chi}$ with finite support $x_1, x_2, \ldots, x_\tau$ and weights $w_1, w_2, \ldots, w_\tau$ assigned to x_i's. This τ-point design is usually denoted by

$$\xi = \begin{Bmatrix} x_1 & x_2 & \ldots & x_\tau \\ w_1 & w_2 & \ldots & w_\tau \end{Bmatrix} \in \boldsymbol{\Xi}, \tag{3}$$

where $\boldsymbol{\Xi} = \{\xi | 0 \leq w_i \leq 1; \sum_{i=1}^{\tau} w_i = 1, x \in \boldsymbol{\chi}\}$ (see, e.g., Kiefer (1974)).

Note that for a nonlinear model with m parameters, the number of points in the optimal design satisfies the inequality $m \leq \tau \leq 1 + \frac{m(m+1)}{2}$ (see, e.g., Silvey (1980)). The information matrix of the design in Eq. (3) is defined as follows (see, e.g., Atkinson et al. (2007)):

$$\mathcal{M}(\xi,\boldsymbol{\theta}) = \sum_{i=1}^{\tau} w_i \mathcal{I}(x_i,\boldsymbol{\theta}). \tag{4}$$

The optimal design problem is finding $\xi^* = \begin{Bmatrix} x_1^* & x_2^* & \cdots & x_\tau^* \\ w_1^* & w_2^* & \ldots & w_\tau^* \end{Bmatrix}$, which maximizes a function $\psi(.)$ (which is defined by a proper criterion) of the information matrix $\mathcal{M}(\xi,\boldsymbol{\theta})$, i.e.,

$$\xi^* = \arg\max_{\xi\in\boldsymbol{\Xi}} \psi\big(\mathcal{M}(\xi,\boldsymbol{\theta})\big).$$

There are different criteria for this optimization. One of the most common such criteria is *D-optimality*, that is, the criterion $\psi(\mathcal{M}(\xi,\boldsymbol{\theta})) = \log\left(\det(\mathcal{M}(\xi,\boldsymbol{\theta}))\right)$ for when $\mathcal{M}(\xi,\boldsymbol{\theta})$ is non-singular (see, e.g., Silvey (1980) and Pukelsheim (1993)). From Eqs. (2) and (4), it can be observed that when the model is nonlinear, the information matrix $\mathcal{M}(\xi,\boldsymbol{\theta})$ depends on the unknown parameters $\boldsymbol{\theta}$. In this case, to implement a design, the role of the parameter in the information matrix must be solved first. A typical traditional method is the local optimal method, which is methodologically simple, however, it has been criticized by numerous authors because if the unknown parameters are not properly replaced by guess values, the resulting optimal designs can be very inefficient in real model settings. A more robust approach is to use a prior distribution instead of guess values for the unknown parameters, which is the concept of the Bayesian optimal design. One of the most common criteria in the Bayesian optimality is *D-optimality* (see, e.g., Berger (1985) and Chaloner and Larntz (1989)). A design is called Bayesian D-optimal with respect to a given prior π on $\boldsymbol{\theta}$, if it maximizes the function

$$\Psi_\pi(\xi) = E[\psi(\xi,\boldsymbol{\theta})] = \int_{\boldsymbol{\theta}} \psi(\xi,\boldsymbol{\theta})\pi(\boldsymbol{\theta})d\boldsymbol{\theta}, \tag{5}$$

where $\psi(\xi,\boldsymbol{\theta}) = \ln\left(det\left(\mathcal{M}(\xi,\boldsymbol{\theta})\right)\right)$ is the D-optimality criterion.

Obviously, if the prior distribution π changes, the design that maximizes (5) may also change. This allows the researcher to consider different prior distributions corresponding to different information of the unknown parameter and hence obtain various designs. The process of selecting a prior is based on pre-experimental and subjective knowledge. However, in some circumstances, it is difficult for the experimenter to specify a prior on the parameter space $\boldsymbol{\theta}$. To address this challenge, researchers proposed various solutions; for example, in some cases, an unknown prior distribution for the parameter $\boldsymbol{\theta}$ can be considered, and in some cases, the nonparametric Bayesian approach can be used; in some cases, the local optimal design approach is suggested. Some references are Abdollahi et al. (2023), Abdollahi et al. (2025), Dette and Neugebauer (1997), Dette et al. (2006), Goudarzi et al. (2019), and Parsa Maram and Jafari (2016).

4 Optimal design for the δ-shock model

In this section, we introduce a regression model for the stopping time of a system subjected to random shocks under the δ-shock model and then investigate D-optimal designs for this model. The main purpose of using a regression model is to show that shock behavior depends on factors that we may be able to control and achieve optimal values for, which helps ensure the health and optimal performance of systems and increase their efficiency.

4.1 The model

According to the δ-shock model, the number of random shocks until the system failure has a geometric distribution with pmf $\Pr(N = n) = F(\delta)\big(1 - F(\delta)\big)^{n-1}$, where $n \in \mathbb{N}$ and $F(u)$ is the cdf of intershock times (see (1)). So, it is clear that the system stopping time depends on both the probabilistic behavior of the intershock times and the critical threshold δ. Therefore, the effect of an explanatory factor on these two variables also affects the system stopping time. Thus, the parameter $F(\delta)$ can be described in terms of a regression model given an explanatory variable x. The choice of regression model can be made based on the conditions under which the system operates. Here, we consider a scenario in which increasing the value of x reduces the mean of the system stopping time. Such a scenario describes a situation where a factor, by becoming larger or more intense, leads to a reduction in the time interval between successive shocks. For example, the following model can satisfy this condition:

$$F(\delta|x) = \frac{1}{1 + e^{-\theta x}}, \tag{6}$$

where $x \in \boldsymbol{\chi}$ and $\theta > 0$ is the unknown regression coefficient.

It is clear that model (6) preserves the property $0 < F(\delta) \leq 1$. Besides, since N is a geometric random variable with the pmf in (1), so $E(N) = \frac{1}{F(\delta)}$. Therefore,

$$E(N|x) = \frac{1}{F(\delta|x)} = 1 + e^{-\theta x} = \eta(x, \theta). \tag{7}$$

Obviously, increasing the value of x leads to decreasing $E(N|x)$.

In the upcoming subsection, optimal designs are obtained for the model in (7).

4.2 Design of experiments

In the following, we investigate the D-optimal design for the regression model in (7). It should be noted that this model is of the exponential type, and exponential models have been widely studied in the literature. However, we outline the steps of the method to arrive at a general form for a one-point optimal design. To this end, we first need to get the Fisher information matrix for the model (7). For the sake of simplicity (and without loss of generality), we assume that the system stopping time occurs randomly with variance 1, that is, $Var(N|x) = 1$. This assumption actually represents a framework in which the variance is known, meaning that if we change the known variance, it does not affect the optimal points and consequently the optimal design. However, it is important to note that in a framework where the variance is unknown, the optimal points and consequently the optimal design will be different depending on the parametric expression of the variance. Accordingly, using (2), we have

$$\mathcal{I}(x, \theta) = x^2 e^{-2\theta x}.$$

The Fisher information matrix for the one-point design $\xi = \begin{Bmatrix} x \\ 1 \end{Bmatrix}$ is obtained from (4) as follows

$$\mathcal{M}(\xi, \theta) = x^2 e^{-2\theta x}. \tag{8}$$

Hence, the D-optimal criterion (i.e., $\psi(\mathcal{M}(\xi, \theta)) = \log(\det(\mathcal{M}(\xi, \theta)))$) is

$$\psi(\mathcal{M}(\xi, \theta)) = \log\left(x^2 e^{-2\theta x}\right) = 2\log(x) - 2\theta x. \tag{9}$$

It can be seen from (9) that the D-optimal criterion depends on θ. Thus, the parameter θ affects the optimization process. Therefore, we must use the Bayesian D-optimal criterion to neutralize the effect of θ on the optimization process. By using (5), the Bayesian D-optimal criterion with respect to a given prior $\pi(\theta)$ on $\boldsymbol{\theta}$ is

$$\Psi_\pi(\xi) = \int_{\boldsymbol{\theta}} (2\log(x) - 2\theta x)\, \pi(\theta) d\theta.$$

Considering the uniform prior distribution $U(\alpha, \beta)$ (with $\alpha \geq 0$ and $\beta > \alpha$) for θ, it becomes

$$\Psi_\pi(\xi) = 2\log(x) - (\alpha + \beta)x.$$

Since a one-point Bayesian D-optimal design is a design $\xi^* = \begin{Bmatrix} x^* \\ 1 \end{Bmatrix}$ that satisfies $\xi^* = \arg\max_{\xi \in \boldsymbol{\Xi}} \Psi_\pi(\xi)$, therefore

$$\xi^* = \arg\max_{\xi \in \boldsymbol{\Xi}} \left(2\log(x) - (\alpha + \beta)x\right). \tag{10}$$

Finding $\xi^* = \begin{Bmatrix} x^* \\ 1 \end{Bmatrix}$ in equation (10) requires simple calculations. Indeed, the maximum value of $2\log(x) - (\alpha + \beta)x$ over $[0, \infty)$ is found by identifying the root of its derivative, where that root makes the second derivative negative. By performing calculations, we find that the point $x = \frac{2}{\alpha+\beta}$ maximizes $2\log(x) - (\alpha + \beta)x$ on the interval $[0, \infty)$. Therefore, we conclude that under the uniform prior distribution $U(\alpha, \beta)$ with $\alpha \geq 0$ and $\beta > \alpha$, if the design space is considered to be the interval $[0, \infty)$, the Bayesian

D-optimal point is equal to $x^* = \frac{2}{\alpha+\beta}$. Note that the expected value of the prior distribution is equal to $\frac{\alpha+\beta}{2}$. Consequently, the optimal point becomes smaller as the expected value of the prior distribution increases. In addition, if the design space is a non-negative interval with finite bounds, the D-optimal point is determined by whether the upper and lower bounds are smaller and larger than $\frac{2}{\alpha+\beta}$, respectively. Accordingly, it is possible to provide a general formulation for the Bayesian D-optimal design for some different choices of design space. By considering the design space $\boldsymbol{\chi} = [a, b] \subseteq [0, \infty)$, the results of the optimization process are summarized in Table 1.

Table 1: Bayesian D-optimal design for model (7).

Prior distribution	Design space	$\xi^* = \begin{Bmatrix} x^* \\ 1 \end{Bmatrix}$
$U(\alpha, \beta)$, $\alpha \geq 0, \beta > \alpha$	$[a, b]$, $b \leq \frac{2}{\alpha+\beta}$	$\begin{Bmatrix} b \\ 1 \end{Bmatrix}$
	$[a, b]$, $a \geq \frac{2}{\alpha+\beta}$	$\begin{Bmatrix} a \\ 1 \end{Bmatrix}$
	$[a, b]$, $a < \frac{2}{\alpha+\beta} < b$	$\begin{Bmatrix} \frac{2}{\alpha+\beta} \\ 1 \end{Bmatrix}$

To see how the rules in Table 1 work in practice, let's take the case where the design space is $[0, 1.5]$ and $\theta \sim U(0, 1)$. Here, $\alpha = 0$ and $\beta = 1$. First, we find the ideal optimal point, which is $x^* = \frac{2}{0+1} = 2$. Since the upper bound of our design space, $b = 1.5$, is less than this ideal point, we turn to the first rule in the table. Just as it guides, the design 'snaps' to this boundary, giving us the final optimal design of $\xi^* = \begin{Bmatrix} 1.5 \\ 1 \end{Bmatrix}$.

4.3 The D-efficiency

The D-efficiency of a τ-point design $\xi = \begin{Bmatrix} x_1 & x_2 & \dots & x_\tau \\ w_1 & w_2 & \dots & w_\tau \end{Bmatrix}$ with respect to the τ-point Bayesian D-optimal design $\xi^* = \begin{Bmatrix} x_1^* & x_2^* & \dots & x_\tau^* \\ w_1^* & w_2^* & \dots & w_\tau^* \end{Bmatrix}$ for a model with m parameters is given by

$$D_{eff}(\xi^*, \xi) = \left(\frac{\det(\mathcal{M}(\xi, \boldsymbol{\theta}_{true}))}{\det(\mathcal{M}(\xi^*, \boldsymbol{\theta}_{true})} \right)^{\frac{1}{m}},$$

where $\boldsymbol{\theta}_{true}$ is the true parameter values.

For the model in (7), we have $m = 1$. We also have $\mathcal{M}(\xi, \theta) = x^2 e^{-2\theta x}$ (see (8)), meaning that $\det \mathcal{M}(\xi, \theta) = x^2 e^{-2\theta x}$. Thus,

$$D_{eff}(\xi^*, \xi) = \frac{x^2 e^{-2x\theta_{true}}}{x^{*2} e^{-2x^*\theta_{true}}} = \left(\frac{x}{x^*} \right)^2 e^{2(x^*-x)\theta_{true}}. \tag{11}$$

Accordingly, for each Bayesian D-optimal design in Table 1, the D-efficiency of a design $\xi = \begin{Bmatrix} x \\ 1 \end{Bmatrix}$ with respect to the Bayesian D-optimal design $\xi^* = \begin{Bmatrix} x^* \\ 1 \end{Bmatrix}$ is obtained as in Table 2.

Figure 1 depicts the plot of the D-efficiency (11) considering the design space $[0, \infty]$, different uniform prior distributions, and some values of θ_{true}. It can be seen that the

Table 2: The D-efficiencies for the Bayesian D-optimal designs in Table 1.

Prior distribution	Design space	$D_{eff}(\xi^*, \xi)$
$U(\alpha, \beta)$, $\alpha \geq 0, \beta > \alpha$	$[a, b]$, $b \leq \frac{2}{\alpha+\beta}$	$(\frac{x}{b})^2 e^{2(b-x)\theta_{true}}$
	$[a, b]$, $a \geq \frac{2}{\alpha+\beta}$	$(\frac{x}{a})^2 e^{2(a-x)\theta_{true}}$
	$[a, b]$, $a < \frac{2}{\alpha+\beta} < b$	$(\frac{\alpha+\beta}{2} x)^2 e^{2(\frac{2}{\alpha+\beta} - x)\theta_{true}}$

D-efficiency increases as the expected value of the prior distribution increases. In addition, increasing the value of θ_{true} leads to a decrease in the growth rate of the D-efficiency curve. Therefore, choosing a uniform prior with a large expected value can have a significant impact on increasing the efficiency of the optimal design.

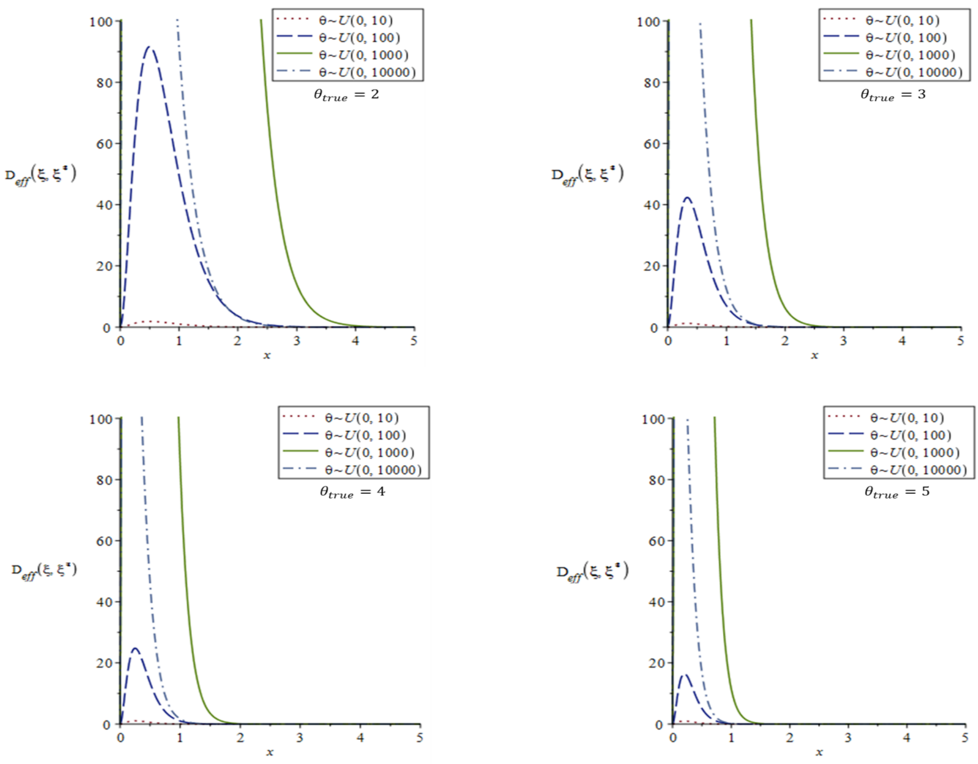

Figure 1: Plot of D-efficiency in (11) considering the design space $[0, \infty]$.

5 Conclusions and future directions

In this paper, the optimal experimental design for the δ-shock model is discussed. We considered the mean of the system stopping time under the classical δ-shock model as a regression model and investigated the Bayesian D-optimal design for this regression model. The regression model was chosen to represent a situation in which increasing the value of an explanatory variable leads to a decrease in the time interval between successive shocks, which leads to a decrease in the mean of the system stopping time. The purpose of establishing the optimal design problem in δ-shock modeling is to give importance to the impact of possible secondary factors on system failure. This is a new approach to shock modeling, and the advantages of this approach include better understanding of secondary factors in the shock environment, more detailed study of the lifetime of a system subjected to random shocks, and finding optimal points for estimating the parameters of the model describing the system. For future study, the discussed approach can be developed to various extensions and generalizations of the δ-shock model as well as other shock models. Additionally, designs with more points can be explored, such as two- and three-point designs. Moreover, since in this article we assumed that the variance of the regression model is known, and this itself can be a limitation, in more advanced studies the variance can be considered unknown.

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