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Bayesian, E-Bayesian and hierarchical Bayesian estimations for optimization of traffic intensity in the M/M/m/K queue based on fuzzy index

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Abstract: This paper focuses on the M/M/m/K queuing model, where inter-arrival and service times follow exponential distributions. We discuss the fuzzy average degree of customer satisfaction and evaluate traffic intensity based on a fuzzy index, using Bayesian, E-Bayesian, and hierarchical Bayesian methods, applying the general entropy loss function. Additionally, the maximum likelihood estimation method is utilized for estimation. To compare the performances of the proposed estimation methods, a Monte Carlo simulation is conducted. Evaluation criteria, such as the cost function and the average customer satisfaction index, are used to select the most appropriate estimation method for the present paper. Finally, a numerical example is provided to determine the most suitable estimator.

Keywords: Average customer satisfaction index; Cost function; M/M/m/K queuing model; Traffic intensity; Fuzzy index.
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1 Introduction

The problem of estimation concerning the parameters of the queuing model, such as the arrival rate, service rate and traffic intensity is an important problem having wide practical applications in real situations. The Bayesian method, as an alternative to the classical method, is in statistical inference. In the Bayesian inference, the performance of the estimator depends on the prior distribution and also on the loss function used.

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The parameters of a prior distribution called hyper-parameters. Dey (2008) obtained Bayesian estimation of the traffic intensity in M/M/1 queue and queue characteristics under quadratic loss function. Cruz et al. (2017) have Bayesian estimated of traffic intensity based on queue length in a multi-server M/M/s queue. Golzade Gervi et al. (2019) compared record ranked set sampling scheme with inverse sampling scheme in empirical Bayesian estimations and predictions of exponential distribution. Das and Pardhan (2024) considered maximum likelihood and Bayesian estimation of traffic intensity for a bulkservice queue with batch-size-dependent service mechanism.

E-Bayesian and hierarchical Bayesian methods in estimation theory have studied by many author. Ando (2007) studied Bayesian predictive information criterion for the evaluation of hierarchical Bayesian and empirical Bayes models. Reyad et al. (2017) obtained E-Bayesian and Hierarchical Bayesian estimations based on Dual generalized order statistics from the inverse weibull model. Yousefzadeh (2017) used E-Bayesian and hierarchical Bayesian estimations for the system reliability parameter based on asymmetric loss function. Kizilaslan (2017) studied the E-Bayesian and hierarchical Bayesian estimations for the proportional reversed hazard rate model based on record values. Fayyaz Heidari et al. (2022) have computed E-Bayesian and hierarchical Bayesian estimation of rayleigh distribution parameter with type-II censoring from imprecise data. Recently, Shi et al. (2024) studied the E-Bayesian and hierarchical Bayesian estimations for the reliability analysis of Kumaraswamy generalized distribution based on upper record values. The problem of E-Bayesian and hierarchical Bayes estimation methods in estimation theory have been studied by several authors, see (Richard, 2011; Han, 2009; Wang et al., 2012; Han, 2017).

Some authors have used fuzzy sets in theory estimation. Akbari and Rezaei (2007) studied a new method for estimating fuzzy spot for uniformly minimum variance. Jenab and Rashidi (2009) considered fuzzy Bayesian condition monitoring model based on exponential distribution. Pak et al. (2014) conducted wide studies on inferential procedures for lifetime distributions based on fuzzy lifetime data. Also see, Pak et al. (2013). Gholizadeh et al. (2016) has considered fuzzy E-Bayesian and hierarchical Bayesian estimations on the Kumaraswamy distribution using censoring Data. Yaghoobzadeh (2019) studied E-Bayesian and H-Bayesian of Gompertz distribution under type II censoring based on fuzzy data.

The Bayesian estimation methods in the field of queueing theory have attracted the attention of many researchers. Chowdhury and Mukherjee (2013) obtained estimation of traffic intensity based on queue length in a single M/M/1 queue. Estimation of waiting time distribution in an M/M/1 queue was studied by Chowdhury and Mukherjee (2011). Ren and Wang (2012) investigated Bayes estimation of traffic intensity in M/M/1 queue under a precautionary loss function. Nonparametric estimation of service time characteristics in infinite-server queues with nonstationary poisson input are derived by Goldenshluger and Koops (2019). Singh and Acharya (2019) studied the equivalence between Bayes and the maximum likelihood estimator in M/M/1 queue. Schweer and Wichelhaus (2020) introduced non-parametric estimation of the service time distribution in discrete time queuing model. Moreover, Chandrasekhar et al. (2021) studied classical and Bayes estimation in the M/D/1 queuing system. Makhdoom (2023) derived a new optimum statistical estimation of the traffic intensity parameter for the M/M/1/K queuing model based on fuzzy and non-fuzzy criteria.

Hendi et al. (2024) have used E-Bayesian and hierarchical Bayesian estimation for traffic intensity in the M/M/1 queueing system. Also, Makhdoom and Yaghoobzadeh Shahrastan (2024) considered improving the quality of M/M/m/K queueing systems using system cost function optimization. Recently, Makhdoom and Yaghoobzadeh Shahrastani (2024) focused on Bayesian estimation in the M/M/1 queueing model under a type II censoring scheme based on fuzzy data.

Queueing systems with finite capacity are important concepts and factors such as system cost and customer satisfaction are considered evaluation criteria for a queuing system. Efforts are made to minimize system cost and maximize customer satisfaction in each queuing system. The aim of the present paper is the selection of the best estimate for the traffic intensity parameter of a queuing model based on fuzzy index, incorporating a factor called the average customer satisfaction index in choosing the most appropriate estimate.

The remaining of this paper is organized as follows. In section 2, first M/M/m/K queuing model and its cost function are introduced, next Bayes, E-Bayesian and hierarchical Bayesian estimations are defined. Section 3 contains the maximum likelihood estimation (MLE), Bayes estimation, E-Bayesian and hierarchical Bayesian estimations of traffic intensity in M/M/m/K queuing model. Section 4, introduces the results and compares of the proposed estimation methods for traffic intensity using Monte Carlo simulation and a numerical example. Section 5 concludes.

2 Mathematical concepts

In the current section, first we define E-Bayesian, hierarchical Bayesian, fuzzy set probability function and average degree of customer satisfaction average degree of customer satisfaction. Next, we introduce the M/M/m/K queuing model, evaluation criteria and their cost function.

Definition 2.1. *(Han, 1997) Suppose $\pi(b_1, b_2)$ is the joint prior distribution of hyper-parameters of b_1 and b_2 and $\hat{\theta}^B$ is Bayes estimator of θ. Then the E-Bayesian of θ, $\hat{\theta}^{EB}$, is defined as follows*

$$\begin{aligned}\hat{\theta}^{EB} &= E_{\pi(b_1,b_2)}(\hat{\theta}^B)\\ &= \int_{\Lambda_1}\int_{\Lambda_2} \hat{\theta}^B \pi(b_1, b_2) db_1 db_2, \quad b_1 \in \Lambda_1, b_2 \in \Lambda_2.\end{aligned}$$

Definition 2.2. *(Han, 1997) If $\pi(\theta|\lambda)$ and $\pi^{'}(\lambda)$ are respectively the corresponding prior distributions to parameter θ and hyper-parameter λ. Then the hierarchical prior distribution of parameter θ is obtained by*

$$\pi^{''}(\theta) = \int_{\Lambda} \pi(\theta|\lambda)\pi^{'}(\lambda)d\lambda, \quad \lambda \in \Lambda.$$

Definition 2.3. *(Zadeh, 1968) If $\mu_{\widetilde{A}}(\omega)$ is membership function of $\widetilde{A}$, for each $\omega \in \Omega$. Then the probability function of $\widetilde{A}$ can be defined as*

$$P(\widetilde{A}) = \sum_{\omega\in\Omega} \mu_{\widetilde{A}}(\omega) P_\omega, \quad \mu_{\widetilde{A}}(\omega) : \Omega \to [0, 1],$$

where (Ω, F, P) *is a probability space.* Ω *stands for the sample space,* F *represents the sigma algebra on* Ω*, and* P *stands for probability measure, then* $\widetilde{A}$ *shows the fuzzy event in* Ω*.*

2.1 M/M/m/K queuing model

Queueing model has applications in various fields, particularly in estimating queue parameters such as arrival rate, service rate, and traffic intensity. In this section, we give a brief description of the M/M/m/K queuing model. The M/M/m/K queuing model has m servers with a service rate of μ, independent of the number of customers in the system and the rate of customers visiting the system is equal to λ, independent of the system status and μ. Also, the time between visits and customer service time is exponentially distributed with parameters λ and μ. In this model, the exit rate is different from the service rate. If the number of customers in the system is less than m, the exit rate is $n\mu$, otherwise it is $m\mu$. Therefore, the exit rate of system is defined as

$$\mu_n = \begin{cases} n\mu, & n < m, \\ m\mu, & m \leq n \leq K, \\ 0, & n > K. \end{cases}$$

In each queuing system, if at a certain moment the number of customers in the system is n, then the time takes for the number of people to reach $(n+1)$ is considered random variable with exponential distribution with parameter λ_n. So, the login rate to the system (λ_n) is

$$\lambda_n = \begin{cases} \lambda, & n < K, \\ 0, & n \geq K. \end{cases}$$

According Allen (1990) and assuming $r = \frac{\lambda}{\mu}$, the distribution of the number of customers in the system is as follows

$$P_n = \begin{cases} \frac{r^n}{n!} P_0, & 1 \leq n \leq m-1, \\ \frac{r^n}{m! m^{n-m}} P_0, & m \leq n \leq K, \end{cases}$$

where by assumption $\rho = \frac{r}{m}$, we have

$$P_0 = \begin{cases} (\sum_{n=0}^{m} \frac{r^n}{n!} + \frac{r^m (\rho - \rho^{K-m+1})}{m!(1-\rho)})^{-1}, & \rho \neq 1, \\ (\sum_{n=0}^{m} \frac{r^n}{n!} + \frac{r^m (K-m)}{m!})^{-1}, & \rho = 1. \end{cases} \tag{1}$$

The average number of customers in the queue, denoted $L_q = \sum_{n=m}^{K} (n-m) P_n$ is given by

$$L_q = \begin{cases} \rho \frac{r^m}{m!} [1 - \rho^{K-m+1} - (K-m+1)\rho^{K-m}(1-\rho)] P_0, & \rho \neq 1, \\ \frac{r^m (K-m)(K-m+1)}{2m!} P_0, & \rho = 1, \end{cases} \tag{2}$$

and the average number of customers in the system, denoted $L = \sum_{n=0}^{K} n P_n$, is as follows

$$L = \sum_{n=0}^{m-1} n P_n + \sum_{n=m}^{K} n P_n = L_q + m - P_0 \sum_{n=0}^{m-1} \frac{r^n (m-n)}{n!}. \tag{3}$$

In queuing theory, the mean time that a customer remains in the system (W) and the mean time a customer waits in the queue (W_q) are obtained by

$$W_q = \frac{L_q}{\bar{\lambda}}, \quad W = \frac{L}{\bar{\lambda}}, \tag{4}$$

where (assuming $m \leq K$), the entry rate into the system is $\bar{\lambda} = \lambda(1 - P_K)$.

2.2 Cost function of M/M/m/K queuing model

In each queuing system, it is important to reduce the queue length and customer waiting time and also increase customer satisfaction. For this purpose, the number of service providers should be increased and this requires cost. In the present paper, the cost function is considered as follows

$$\begin{aligned} C_m(\rho) &= C_1(L - L_q) + C_2(m - L + L_q) + C_3 m + C_4(\lambda - \bar{\lambda}) + C_5 L_q + C_6(L - L_q) \\ &= (C_1 + C_6 - C_2)(L - L_q) + (C_2 + C_3)m + C_4(\lambda - \bar{\lambda}) + C_5 L_q, \end{aligned} \tag{5}$$

where $C_1(L-L_q)$ stands for the operating cost of service providers that provide service, $C_2(m - L + L_q)$ denotes the cost of maintaining unemployed servants, $C_3 m$ represents the investment cost of service providers per unit of time, $C_4(\lambda - \bar{\lambda})$ stands the cost of losing customers, $C_5 L_q$ shows the cost of wasting customers time in the queue, and $C_6(L - L_q)$ represents the cost of wasting customers time while receiving the service. Therefore, by using (1) to (3) and $\lambda - \bar{\lambda} = \frac{\lambda r^K}{m! m^{K-m}} P_0$, the cost function defined in (5) is rewritten as

$$\begin{aligned} C_m(\rho) &= (C_1 + C_6 - C_2)(m - P_0 \sum_{n=0}^{m-1} \frac{r^n(m-n)}{n!}) + (C_2 + C_3)m + \frac{C_4 \lambda r^K}{m! m^{K-m}} P_0 \\ &\quad + \frac{C_5 r^m[\rho + (K-m)\rho^{K-m+2} - (K-m+1)\rho^{K-m+1}]}{m!(1-\rho)^2} P_0, \quad \rho \neq 1. \end{aligned} \tag{6}$$

2.3 Average degree of customer satisfaction (ADCS)

Considering the importance of customer satisfaction, we want to examine this criteria in M/M/m/K queuing model. Pardo and De la Fuente (2008) introduced degree of customer satisfaction by observing the length of the queue at the moment of logging into the system. If the customer faces short, medium and long queues at the moment of arrival, degree of customer satisfaction will be high (a_1), medium (a_2) and low (a_3), respectively, where ($a_1 \geq a_2 \geq a_3$). Due to the relativity of the concepts above, the specified queues are considered as fuzzy sets in the following, respectively

$$\begin{aligned} &\textit{short queue} = \widetilde{A} = \{(0, \mu_{\widetilde{A}}(0)), (1, \mu_{\widetilde{A}}(1)), \dots, (K, \mu_{\widetilde{A}}(K))\}, \\ &\textit{medium queue} = \widetilde{B} = \{(0, \mu_{\widetilde{B}}(0)), (1, \mu_{\widetilde{B}}(1)), \dots, (K, \mu_{\widetilde{B}}(K))\}, \\ &\textit{long queue} = \widetilde{C} = \{(0, \mu_{\widetilde{C}}(0)), (1, \mu_{\widetilde{C}}(1)), \dots, (K, \mu_{\widetilde{C}}(K))\}, \end{aligned}$$

where $\mu_{\widetilde{A}}$, $\mu_{\widetilde{B}}$ and $\mu_{\widetilde{C}}$ are membership function of the fuzzy sets $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$, respectively. According to Dubois and Prade (1980), we have $\mu_{\widetilde{A}}(i) + \mu_{\widetilde{B}}(i) + \mu_{\widetilde{C}}(i) =$

$1\ for\ i = 1, 2, \ldots, K$. Also, $\mu_{\widetilde{A}}(i)$ represents the degree of membership of the fuzzy set $\widetilde{A}$ when there are i customers in the queue. Therefore, according to Definition 2.3, the probability of a customer logging into the system with short, medium and long queues are as the following, respectively

$$\pi(\widetilde{A}) = \sum_{n=0}^{K} \mu_{\widetilde{A}}(n) P_n, \quad \pi(\widetilde{B}) = \sum_{n=0}^{K} \mu_{\widetilde{B}}(n) P_n, \quad \pi(\widetilde{C}) = \sum_{n=0}^{K} \mu_{\widetilde{C}}(n) P_n.$$

Consequently, the ADCS is obtained as

$$ADCS = a_1 \pi(\widetilde{A}) + a_2 \pi(\widetilde{B}) + a_3 \pi(\widetilde{C}). \tag{7}$$

3 MLE, Bayes, E-Bayes and hierarchical Bayes estimations

This section deals with the problem of estimation the traffic intensity ($\rho = \frac{\lambda}{m\mu}$) parameter in M/M/m/K queuing system using the MLE, Bayes, E-Bayesian and hierarchical Bayesian estimations based on the general entropy loss function. The Bayesian method is one of several methods used to estimate the parameters of statistical distributions. Choosing appropriate prior distributions for the parameter space is crucial for reducing the error of the Bayesian estimator. The loss function we considered for Bayes estimation is the general entropy loss function of the form

$$L(\hat{\theta}, \theta) = q[(\frac{\hat{\theta}}{\theta})^p - p \ln(\frac{\hat{\theta}}{\theta}) - 1], \qquad p \neq 0, \quad q > 0. \tag{8}$$

It was proposed by Calabria and Pulcini (1996) and its minimum occurs at $\hat{\theta} = \theta$. Because the value of q does not play any role on the optimization of the loss function, so without loss of generality we assume $q = 1$. The Bayes point estimator for θ under general entropy loss function is of the form

$$\hat{\theta}^{GB} = [E(\theta^{-p}|\boldsymbol{X})]^{-\frac{1}{p}},$$

provided that expectation exist and is finite. The proper choice for p is a challenging task for an analyst because it reflects the asymmetry of the loss function in a practical situation.

Remark 3.1. *When $p = -1$, the Bayes point estimator for the θ coincide with the Bayes point estimators under the symmetric squared error loss function of the form $(\hat{\theta} - \theta)^2$. The corresponding Bayes point estimator of θ is given by $E(\theta|\boldsymbol{X})$.*

Remark 3.2. *In (8), if we replace $\ln(\frac{\hat{\theta}}{\theta})$ by $(\hat{\theta} - \theta)$, we get the linear exponential (Linex) loss function of the form $q[exp\ (p(\hat{\theta} - \theta)) - p(\hat{\theta} - \theta) - 1]$. The Bayes point estimator of θ under the asymmetric linex loss function is given by, $-\frac{1}{p} \ln E(e^{-p\theta}|\boldsymbol{X})$.*

In the following, we will present the MLE, Bayes estimation, E-Bayesian and Hierarchical Bayesian estimations of the traffic intensity ($\rho = \frac{\lambda}{m\mu}$) parameter in M/M/m/K queuing system. We start with the MLE.

3.1 MLE

Let $V_1, \ldots, V_{n_1}$ be times between two consecutive arrivals of size n_1 from exponential distribution with parameter λ and the pdf $f(v, \lambda)$ given by $f(v, \lambda) = \lambda e^{-\lambda v}, v > 0, \lambda > 0$, and also $U_1, \ldots, U_{n_2}$, be service times of size n_2 from exponential distribution with parameter μ and the pdf $g(u, \mu)$ given by $g(u, \mu) = \mu e^{-\mu u}, u > 0, \mu > 0$. Also V_i's and U_i's are independent random variables. Now, let $\boldsymbol{X} = \{U_1, \ldots, U_{n_2}, V_1, \ldots, V_{n_1}\}$, and also $T_1 = \sum_{i=1}^{n_1} V_i$ and $T_2 = \sum_{i=1}^{n_2} U_i$. It can be shown that T_1 follows from gamma distribution with the shape parameter n_1 and λ. Similarly T_2 follows from gamma distribution with the shape parameter n_2 and μ. As discussed before, T_1 and T_2 are independent random variables. So, it can be obtained a closed-form expression for the ML estimation of the parameters λ and μ as $\hat{\lambda} = \frac{n_1}{T_1}$ and $\hat{\mu} = \frac{n_2}{T_2}$, respectively. Using the invariance property of ML estimators, one can immediately obtain the ML of ρ as

$$\hat{\rho}^{ML} = \frac{n_1}{mn_2} \frac{T_2}{T_1}. \tag{9}$$

3.2 Bayesian estimation

Because λ is non negative, a natural choice for the prior of λ would be to assume that its density is of the following form

$$\pi(\lambda|a, b) = \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda}. \tag{10}$$

The hyper-parameters $a(> 0)$ and $b(> 0)$ are chosen to reflect the prior knowledge about λ, $\Gamma(.)$ denotes the complete gamma function. Similarly, we can use the following informative gamma prior of μ with pdf

$$\pi(\mu|r, c) = \frac{c^r}{\Gamma(r)} \mu^{r-1} e^{-c\mu}, r > 0, c > 0. \tag{11}$$

Based on (10) and (11), and after some algebraic computations, we obtain

$$\pi(\lambda, \mu|\boldsymbol{X}) = \frac{\Gamma(n_1 + a)\Gamma(n_2 + r)}{(T_1 + b)^{T_1+a}(T_2 + c)^{T_2+r}} \lambda^{n_1+a-1} \mu^{n_2+r-1} e^{-\lambda(T_1+b)-\mu(T_2+c)}.$$

With assumption $\phi = \frac{\Gamma(a+n_1-p)\Gamma(r+n_2+p)}{\Gamma(a+n_1)\Gamma(r+n_2)}$, under general entropy loss function, the Bayesian estimator of ρ will be equal to $\{E(\rho^{-p}|\boldsymbol{X})\}^{-\frac{1}{p}}$ or equivalently

$$\hat{\rho}^B = \frac{1}{m} (\phi(T_1))^{-\frac{1}{p}} \frac{c + T_2}{b + T_1}. \tag{12}$$

3.3 E-Bayesian estimation

According to Han (1997), in (10), a and b are considered so that $\pi(\lambda|a, b)$ decreases with respect to λ. The derivative of $\pi(\lambda|a, b)$ with respect to λ is attained as

$$\frac{d\pi(\lambda|a, b)}{d\lambda} = \frac{b^a \lambda^{a-2} e^{-b\lambda}}{\Gamma(a)} \left((a - 1) - b\lambda\right).$$

It follows that $b > 0$ and $0 < a \leq 1$. By increasing b, Bayesian estimation of λ decreases. As a result, the hyper-parameter b should be bounded from above, that is $0 < b < c_1$, where c_1 is constant. On the other hand, the appropriate distribution for b is uniform distribution (see Han (1997)). So, in this paper, we consider uniform distribution $U(0, c_1)$ as a prior distribution for b, $\pi(b)$. By considering $a = 1$, (10) can be written as follows

$$\pi(\lambda|b) = be^{-b\lambda}, \quad \lambda > 0, b > 0, \tag{13}$$

similarly, by using uniform distribution $U(0, c_2)$ as a prior distribution for constant c, $\pi(c)$, and also assumption $r = 1$, (11) is given by

$$\pi(\mu|c) = ce^{-c\mu}, \quad , \mu > 0, c > 0. \tag{14}$$

According to the Definition 2.1 and (12) to (14), the E-Bayesian estimation of ρ will be derived as

$$\hat{\rho}^{EB} = \frac{1}{c_1 c_2}\int_0^{c_2}\int_0^{c_1} \hat{\rho}^B \pi(b, c) db dc = \frac{(\phi(T_1))^{-\frac{1}{p}} \left[(c_2 + T_1)^2 - T_1^2\right]}{2mc_1c_2} \log\left(\frac{c_1 + T_2}{T_2}\right). \tag{15}$$

3.4 Hierarchical Bayesian estimation

Bayesian inference is sensitive to the choice of hyper-parameters for informative priors, so care must be taken in the selection of values. In the Bayesian estimation, if the hyper-parameters are unknown, they can be estimated by using the hierarchical method. Hierarchical Bayesian modeling is a statistical model that estimates the parameters of the posterior distribution using the Bayesian method. The sub models combine to form the hierarchical model, and Bayes theorem is used to integrate them with the observed data and account for all the uncertainty that is present. The result of this integration is it allows calculation of the posterior distribution of the prior, providing an updated probability estimate. Now, By using (13) and (14) and Definition 2.2, the hierarchical prior distributions of λ and μ are considered, respectively

$$\pi(\lambda) = \int_0^{c_1} \pi(\lambda|b)\pi(b) db = \frac{1 - (1 + c_1\lambda)e^{-c_1\lambda}}{c_1\lambda^2}, \tag{16}$$

$$\pi(\mu) = \int_0^{c_2} \pi(\mu|c)\pi(c) dc = \frac{1 - (1 + c_2\mu)e^{-c_2\mu}}{c_2\mu^2}. \tag{17}$$

Thus, according to (16) and (17), the hierarchical posterior distribution of λ and μ is derived as

$$\pi^{**}(\lambda, \mu|\boldsymbol{X}) = \frac{\lambda^{a+n_1-3}\mu^{r+n_2-3}e^{-\lambda(b+T_1)-\mu(c+T_2)}S(\lambda, \mu)}{\int_0^\infty\int_0^\infty \lambda^{a+n_1-3}\mu^{r+n_2-3}e^{-\lambda(b+T_1)-\mu(c+T_2)}S(\lambda, \mu) d\lambda d\mu}, \tag{18}$$

where

$$S(\lambda, \mu) = \left(1 - c_1\lambda e^{-c_1\lambda} - e^{-c_1\lambda}\right)\left(1 - c_2\mu e^{-c_2\mu} - e^{-c_2\mu}\right).$$

From (18), the hierarchical Bayesian estimation of ρ under the general entropy loss function can be written as follows

$$\hat{\rho}^{HB} = \frac{1}{m}\left\{\frac{\int_0^\infty\int_0^\infty \lambda^{a+n_1-(p+3)}\mu^{r+n_2+p-3}e^{-\lambda(b+T_1)-\mu(c+T_2)}S(\lambda, \mu) d\lambda d\mu}{\int_0^\infty\int_0^\infty \lambda^{a+n_1-3}\mu^{r+n_2-3}e^{-\lambda(b+T_1)-\mu(c+T_2)}S(\lambda, \mu) d\lambda d\mu}\right\}^{-\frac{1}{p}}. \tag{19}$$

4 Simulation study and numerical illustration

In this section, we analyze Monte Carlo simulation for illustrative purposes and a numerical example is performed to compare performance of the methods described in the preceding sections.

4.1 Simulation study

In this section, we conduct simulation study to compare the performance of MLE, Bayesian, E-Bayesian, and hierarchical Bayesian estimates of ρ. Then, we calculate the cost function values for these estimates via the Monte Carlo simulation and determine the ADCS. We use the following algorithm:
Algorithm
Step 1. Generate a random sample of size n_1(=50), say $(V_1, \ldots, V_{n_1})$ and a random sample of size n_2(=40), say $(U_1, \ldots, U_{n_2})$, from an exponential distributions with parameters $\lambda = 3$ and $\mu = 5$, respectively, and the $\hat{\rho}^{ML}$ is obtained.
Step 2. For given values of $a = 3, b = 4, c = 5, r = 3, p = 2.5$, $c_1 = 6$ and $c_2 = 7$, compute the MLE, Bayes, E-Bayesian and hierarchical Bayesian estimations of ρ, as given in (9), (12), (15) and (19), respectively.
Step 3. For, $C_1 = 400$, $C_2 = 300$, $C_3 = 1000$, $C_4 = 250$, $C_5 = 200$ and $C_6 = 150$, calculate the cost function of estimations of ρ by using (6).
Step 4. Repeat Steps 1 to 3 for 5000 times and use the average of estimates obtained in Step 3, $\hat{\rho}^{ML}$, $\hat{\rho}^{B}$, $\hat{\rho}^{EB}$ and $\hat{\rho}^{HB}$, as the final values.

In this simulation study, we use $a_1 = 1$, $a_2 = 0.6$ and $a_3 = 0.35$. Also, we consider the fuzzy sets $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$ as follows

$$\widetilde{A} = \{(0, 0.6), (1, 0.5), (2, 0.7), (3, 0.8), (4, 0.9), (5, 0.2), (6, 0), (7, 0.4), (8, 0.8)\},$$
$$\widetilde{B} = \{(0, 0.3), (1, 0.3), (2, 0.2), (3, 0.1), (4, 0), (5, 0.7), (6, 1), (7, 0.5), (8, 0.1)\},$$
$$\widetilde{C} = \{(0, 0.1), (1, 0.2), (2, 0.1), (3, 0.1), (4, 0.1), (5, 0.1), (6, 0), (7, 0.1), (8, 0.1)\}.$$

The corresponding simulation results are reported in Table 1. It can observed that the cost function increases as m increases. Moreover, the cost function values under E-Bayesian method are smaller than those obtained under the other methods.

Table 1: Bayes, E- Bayesian and hierarchical Bayesian estimations of ρ and the cost function values based on them for $m = 1(1)8$.

m	$\hat{\rho}^{B}$	$\hat{\rho}^{EB}$	$\hat{\rho}^{HB}$	$\hat{\rho}^{ML}$	$C_m(\hat{\rho}^{B})$	$C_m(\hat{\rho}^{EB})$	$C_m(\hat{\rho}^{HB})$	$C_m(\hat{\rho}^{ML})$
$m = 1$	0.7822	0.7349	0.8007	0.7063	1521	1503	1522	1589
$m = 2$	0.3886	0.3641	0.4061	0.2736	2798	2785	2807	2839
$m = 3$	0.2597	0.2433	0.2657	0.1412	4096	4083	4100	4106
$m = 4$	0.1968	0.1849	0.2001	0.1662	5399	5385	5400	5466
$m = 5$	0.1605	0.1510	0.1617	0.1391	6701	6689	6702	6773
$m = 6$	0.1313	0.1234	0.1356	0.1601	7997	7985	8003	8040
$m = 7$	0.1127	0.1058	0.1158	0.0861	9297	9285	9302	9350
$m = 8$	0.1006	0.0947	0.1012	0.0679	10601	10589	10602	10635

We obtained the distribution of the number of customers in M/M/m/8 queuing system for $m = 1(1)8$. It is clear that the distribution of the number of customers (P_n), is decreasing with respect to n when m is kept fixed. The results are reported in Tables 2.We normalized the ADCS and cost function for the proposed estimation methods. The normalization of ADCS is shown as ${A_N}^B$, ${A_N}^{EB}$, ${A_N}^{HB}$ and ${A_N}^{ML}$ for Bayes, E-Bayes, hierarchical Bayes and ML estimations, respectively. Also, ${C_N}^B$, ${C_N}^{EB}$, ${C_N}^{HB}$ and ${C_N}^{ML}$ represents the normalized cost function. The results of normalization are reported in Table 4.

Table 2: The distribution of the number of customers in the different systems of in M/M/m/8 (stable state) for m=1(1)8.

P_n	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$
				$\hat{\rho}^B$				
P_0	0.2446	0.4404	0.4568	0.4549	0.4482	0.4548	0.4543	0.4472
P_1	0.1913	0.3423	0.3559	0.3581	0.3597	0.3583	0.3584	0.3598
P_2	0.1497	0.1330	0.1386	0.1409	0.1443	0.1411	0.1414	0.1448
P_3	0.1171	0.0517	0.0360	0.0369	0.0368	0.0371	0.0372	0.0389
P_4	0.0916	0.0201	0.0094	0.0073	0.0077	0.00729	0.00733	0.00781
P_5	0.0716	0.0078	0.0024	0.0014	0.0012	0.00112	0.00115	0.00125
P_6	0.0560	0.0030	0.00063	0.00028	0.00020	0.00015	0.00015	0.00017
P_7	0.0438	0.0012	0.00016	5.5×10^{-5}	3.2×10^{-5}	1.9×10^{-5}	1.7×10^{-5}	3.94×10^{-5}
P_8	0.0343	0.0005	4.3×10^{-5}	1.1×10^{-5}	5.1×10^{-6}	2.6×10^{-6}	1.93×10^{-6}	1.95×10^{-6}
				$\hat{\rho}^{EB}$				
P_0	0.2828	0.4662	0.4803	0.4771	0.4699	0.4769	0.4768	0.4688
P_1	0.2076	0.3395	0.3506	0.3529	0.3548	0.3531	0.3537	0.3552
P_2	0.1527	0.1236	0.1279	0.1305	0.1339	0.1307	0.1308	0.1345
P_3	0.1122	0.0450	0.0311	0.0322	0.0337	0.0323	0.0324	0.0339
P_4	0.0825	0.0164	0.0076	0.0059	0.0064	0.0059	0.0058	0.0064
P_5	0.0606	0.0059	0.0018	0.0011	0.00096	0.00088	0.00089	0.00097
P_6	0.0445	0.0022	0.00044	0.00020	0.00014	0.00011	0.00010	0.00012
P_7	0.0327	0.0008	0.00011	3.7×10^{-5}	2.2×10^{-5}	1.4×10^{-5}	1.2×10^{-5}	1.3×10^{-5}
P_8	0.0241	0.0003	2.7×10^{-5}	6.9×10^{-6}	3.3×10^{-6}	1.7×10^{-6}	1.22×10^{-6}	1.3×10^{-6}
				$\hat{\rho}^{HB}$				
P_0	0.2305	0.4226	0.4485	0.4489	0.4455	0.4433	0.4446	0.4450
P_1	0.1845	0.3432	0.3575	0.3593	0.3602	0.3606	0.3604	0.3603
P_2	0.1478	0.1394	0.1425	0.1438	0.1456	0.1467	0.1461	0.1458
P_3	0.1183	0.0566	0.0379	0.0384	0.0392	0.0398	0.0395	0.0394
P_4	0.0947	0.0229	0.0101	0.0076	0.0078	0.0080	0.0079	0.0077
P_5	0.0759	0.0093	0.0027	0.0015	0.0012	0.0013	0.0014	0.0011
P_6	0.0607	0.0038	0.00071	0.00030	0.00020	0.00017	0.00018	0.00016
P_7	0.0486	0.0015	0.00018	6.2×10^{-5}	3.3×10^{-5}	2.4×10^{-5}	2.01×10^{-5}	2.02×10^{-5}
P_8	0.0389	0.0006	5.01×10^{-5}	1.2×10^{-5}	5.4×10^{-6}	3.3×10^{-6}	2.3×10^{-6}	2.04×10^{-6}
				$\hat{\rho}^{ML}$				
P_0	0.2201	0.4017	0.4193	0.4136	0.4312	0.3133	0.3125	0.4152
P_1	0.1746	0.3116	0.3218	0.3214	0.3519	0.2417	0.2411	0.3319
P_2	0.1379	0.1278	0.1397	0.1386	0.1357	0.1377	0.1298	0.1279
P_3	0.1082	0.0951	0.0345	0.0321	0.0312	0.0298	0.0265	0.0196
P_4	0.0875	0.0211	0.0100	0.0071	0.0069	0.0072	0.0068	0.0059
P_5	0.0621	0.0087	0.0025	0.0013	0.0010	0.0011	0.0012	0.0009
P_6	0.0312	0.0029	0.00069	0.00029	0.00018	0.00014	0.00015	0.00012
P_7	0.0217	0.0011	0.00015	5.12×10^{-5}	2.2×10^{-5}	1.4×10^{-5}	1.2×10^{-5}	1.4×10^{-5}
P_8	0.0118	0.0004	4.17×10^{-5}	1.1×10^{-5}	4.5×10^{-6}	2.4×10^{-6}	3.4×10^{-6}	2.1×10^{-6}

The P_F sets and then the ADCS under the proposed estimation methods for $m = 1(1)8$ are also reported in Table 3.

In order to determine the appropriate estimator, based on the normalization of cus-

Table 3: The values of P_F and $ADCS$ under the methods of estimating ρ.

Method	P_F	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$
	$\pi(\widetilde{A})$	0.5826	0.5903	0.5869	0.5870	0.5864	0.5885	0.5874	0.5879
B	$\pi(\widetilde{B})$	0.3039	0.2757	0.2775	0.2771	0.2759	0.2774	0.2768	0.2760
	$\pi(\widetilde{C})$	0.1135	0.1339	0.1355	0.1358	0.1357	0.1360	0.1359	0.1361
	$ADCS$	0.8047	0.8026	0.8008	0.8007	0.7994	0.8025	0.8010	0.8011
	$\pi(\widetilde{A})$	0.5894	0.5885	0.5852	0.5854	0.5878	0.5871	0.5858	0.5861
EB	$\pi(\widetilde{B})$	0.2949	0.2777	0.2797	0.2793	0.2794	0.2846	0.2793	0.2783
	$\pi(\widetilde{C})$	0.1165	0.1337	0.1350	0.1353	0.1362	0.1361	0.1354	0.1355
	$ADCS$	0.8071	0.8019	0.8002	0.8003	0.8031	0.8055	0.8008	0.8005
	$\pi(\widetilde{A})$	0.5496	0.5916	0.5877	0.5875	0.5879	0.5883	0.5882	0.5878
HB	$\pi(\widetilde{B})$	0.3079	0.2744	0.2768	0.2764	0.2758	0.2756	0.2759	0.2757
	$\pi(\widetilde{C})$	0.1124	0.1339	0.1357	0.1359	0.1357	0.1361	0.1360	0.1358
	$ADCS$	0.7737	0.8031	0.8013	0.8009	0.8010	0.8014	0.8015	0.8008
	$\pi(\widetilde{A})$	0.5316	0.5712	0.5710	0.5719	0.5782	0.5695	0.5689	0.5691
ML	$\pi(\widetilde{B})$	0.2976	0.2645	0.2639	0.2682	0.2691	0.2698	0.2692	0.2689
	$\pi(\widetilde{C})$	0.1112	0.1239	0.1245	0.1251	0.1293	0.1349	0.1312	0.1309
	$ADCS$	0.7491	0.7732	0.7729	0.7766	0.7850	0.7786	0.7763	0.7762

Table 4: Normalization of average values of customer satisfaction degree and cost function.

m	${A_N}^B$	${A_N}^{EB}$	${A_N}^{HB}$	${A_N}^{ML}$	${C_N}^B$	${C_N}^{EB}$	${C_N}^{HB}$	${C_N}^{ML}$
$m=1$	0.9770	1	0.9586	1	0.9882	1	0.9875	1
$m=2$	0.9994	0.9985	1	0.3874	0.9954	1	0.9922	0.5597
$m=3$	0.9993	0.9986	1	0.1999	0.9968	1	0.9959	0.3869
$m=4$	0.9997	0.9992	1	0.2354	0.9974	1	0.9972	0.2907
$m=5$	0.9954	1	0.9973	0.1969	0.9982	1	0.9981	0.2346
$m=6$	0.9963	1	0.9949	0.2267	0.9985	1	0.9978	0.1976
$m=7$	0.9992	0.9912	1	0.1219	0.9987	1	0.9982	0.1699
$m=8$	1	0.9993	0.9994	0.0961	0.9989	1	0.9988	0.1494

tomer satisfaction and cost function, an appropriate criteria for comparing estimators is defined as $AE = wA_N + (1-w)C_N$ where $0 < w < 1$. The acronyms A_N and C_N stand for the normalized average degree of customer satisfaction and system cost, respectively. AE index is a linear weighted combination of the normalized system cost and ADCS used to determine the estimator of traffic intensity parameter. It is obvious that, by increasing the degree of satisfaction and decreasing the system cost, the value of AE increases. Therefore, an estimator with a larger AE index is considered more suitable. For $w = 0.6$ and using the results of Table 3, the value of AE under Bayes estimation, E-Bayes, hierarchical Bayesian and ML estimations, denoted by AE^B, AE^{EB}, AE^{HB} and AE^{ML}, is calculated and the results of the selected estimator are presented in Table 5.

From Table 5, it is observed that, for $m = 7$, the hierarchical Bayesian estimation of ρ is better than the other three estimators (The bolded). In other cases, E-Bayes estimation of ρ is better. For instant, the results for $m = 5, 7$ and 8 are plotted in Figure 1. The optimal values of ρ and $r = \frac{\lambda}{\mu}$ are also shown in Table 6 . That's mean, system cost is minimum and the ADCS is the most.

Table 5: The values of AE and selection of estimator.

	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$\mathbf{m=7}$	$m=8$
AE^B	0.9815	0.9978	0.9983	0.9987	0.9965	0.9972	**0.9990**	0.9996
AE^{EB}	1	0.9991	0.9992	0.9995	1	1	**0.9946**	0.9997
AE^{HB}	0.9702	0.9969	0.9984	0.9989	0.9976	0.9961	**0.9993**	0.9995
AE^{ML}	0.8639	0.4908	0.3121	0.2686	0.3134	0.2092	**0.1507**	0.1281
Selected estimator	EB	EB	EB	EB	EB	EB	**HB**	EB

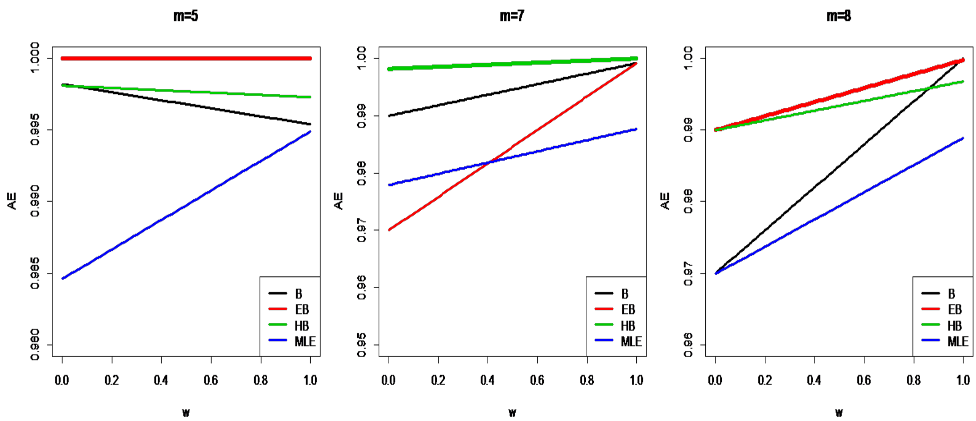

Figure 1: The plot of AE versus w for $m = 5, 7, 8$.

Table 6: Optimal values of ρ and r for $m = 1(1)8$ in M/M/m/8 model.

	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$\mathbf{m=7}$	$m=8$
$\hat{\rho}$	0.7349	0.3641	0.2433	0.1849	0.1510	0.1234	**0.1158**	0.0947
$\hat{r}$	0.7349	0.7282	0.7299	0.7396	0.7550	0.7404	**0.8106**	0.7576

4.2 Numerical illustration

In order to illustrate the results obtained in the preceding section, we intend to present the analysis of a numerical example. Therefore, we consider the city's traffic department, only has the capacity to park six cars at a time. Cars are entered randomly based on a Poisson distribution, and on average, one car is entered every two hours for technical inspection (if any). This office has four parking lots. The data are given in Table 7.

Table 7: Extracted data (in hour) of the city's traffic department for technical inspection.

Time intervals between inputs	4.189	3.281	1.365	2.245	1.371	3.109
	1.315	2.711	3.579	0.374	1.762	.360
Time intervals between services	0.852	2.154	1.178	3.121	4.228	2.621
	0.997	1.193	0.2794	2.146	1.393	3.503

Here, the time of the technical inspection is a random variable with an exponential distribution and a mean of ten hours. This is a M/M/m/K queuing system with

$\lambda = 0.5$, $\mu = 0.1$, $m = 4$, $K = 6$, $r = \frac{\lambda}{\mu} = 5$ and $\rho = \frac{\lambda}{m\mu} = 1.25(> 1)$. From (1) we get $P_0 = (\sum_{n=0}^{4} \frac{5^n}{n!} + \frac{5^4(1.25-1.25^{6-4+1})}{4!(1-1.25)})^{-1} = 0.0072$. This indicates that the probability of the system being empty is 0.0072. By using (2), the average number of cars waiting in the technical inspection queue is equal to $L_q = 0.822$. Thus $P_K = 0.2935$, $\lambda P_K = 0.1467$, $\bar{\lambda} = \lambda(1 - P_K) = 0.3532$, $L = 4.3538$. Also, substituting L_q, L and $\bar{\lambda}$ in (4), yields $W_q = 2.3268$ and $W = 12.3268$. We consider the city's traffic department, only has the capacity to park one car at a time i.e. $m = 1$. Based on the assumed values for parameters, various estimates for ρ and $C(\rho)$ have been calculated in Table 8. The values of P_n for different estimates of ρ are reported in Table 9. Now assuming $a_1 = 1$, $a_2 = 0.6$, $a_3 = 0.35$ and considering the fuzzy sets $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$

$$\widetilde{A} = \{(0, 0.6), (1, 0.5), (2, 0.7), (3, 0.8), (4, 0.9), (5, 0.2), (6, 0)\},$$
$$\widetilde{B} = \{(0, 0.3), (1, 0.3), (2, 0.2), (3, 0.1), (4, 0), (5, 0.7), (6, 1)\},$$
$$\widetilde{C} = \{(0, 0.1), (1, 0.2), (2, 0.1), (3, 0.1), (4, 0.1), (5, 0.1), (6, 0)\},$$

also, using (7) and the results in Table 12, we have computed the values of ADCS. The results are presented in Table 10.

Table 8: Bayes, E-Bayesian and hierarchical Bayesian estimations of ρ and $C(\rho)$ based on numerical illustration.

	$\hat{\rho}^B$	$\hat{\rho}^{EB}$	$\hat{\rho}^{HB}$	$\hat{\rho}^{ML}$
Estimations	2.0831	2.0358	2.1018	2.0076
$C(\hat{\rho})$	0.1787	0.1542	0.1962	0.0637

Table 9: The values of P_n based on numerical illustration for different estimates of ρ.

P_n	$\hat{\rho}^B$	$\hat{\rho}^{EB}$	$\hat{\rho}^{HB}$	$\hat{\rho}^{ML}$
P_0	0.10157	0.47085	0.20812	0.11084
P_1	0.09901	0.32121	0.18988	0.10704
P_2	0.09766	0.21144	0.18172	0.10337
P_3	0.09651	0.15236	0.17409	0.09981
P_4	0.09422	0.13425	0.16701	0.09634
P_5	0.08931	0.12017	0.16043	0.09299
P_6	0.08672	0.11291	0.15432	0.08974

Table 10: The values of $ADCS$ based on numerical illustration for different estimates of ρ.

	$\hat{\rho}^B$	$\hat{\rho}^{EB}$	$\hat{\rho}^{HB}$	$\hat{\rho}^{ML}$
$\pi(\widetilde{A})$	0.3587	0.8579	0.6687	0.3775
$\pi(\widetilde{B})$	0.2386	0.4922	0.4398	0.2508
$\pi(\widetilde{C})$	0.0677	0.1731	0.1271	0.0716
$ADCS$	0.5256	1.2138	0.9770	0.5530

5 Conclusions

In this paper, we have considered the M/M/m/K queuing model where inter-arrival and service times follow exponential distributions with parameters λ and μ, respectively.

We derived the ML, Bayes, E-Bayesian and hierarchical Bayesian estimations for the traffic intensity under the general entropy loss function. A criterion for evaluating estimates, denoted by AE, based on the normalized system cost and average degree of customer satisfaction was defined. A better estimator has larger AE value. Monte Carlo simulation results suggest that, the hierarchical Bayes estimation for $m = 7$ outperforms other estimates. But, when $m = 1, 2, \ldots, 6, 8$, E-Bayes estimation of the traffic intensity has a better performance than other estimators. Next, our study shows that, the cost function increases as m increases. Moreover, the cost function values under E-Bayesian method are smaller than those obtained under the other methods. Also, the distribution of the number of customers are decreasing with respect to n when m is kept fixed. Finally, the results in the numerical example section confirm the results of the simulation study section. The proposed topic in this manuscript, can be considered for the M/M/m/K queuing model based on fuzzy and non-fuzzy criteria, for other methods. Work in sensitivity analysis to hyper-parameters (informative and uninformative), comparison of empirical Bayesian and MLE methods with Bayesian method is currently under progress and we hope to report the results in future soon.

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