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Realistic economic design of Shewhart control charts in the presence of multiple independent assignable causes using Burr-XII shock model

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Abstract: This study proposes a novel framework for the realistic economic design of Shewhart control charts, addressing a critical yet overlooked limitation in traditional economic design models. Unlike classical approaches, the realistic economic design model explicitly incorporates the probability that no new assignable causes arise after the first assignable cause appears within a quality cycle an event previously assumed to be deterministic, despite its unrealistic nature in dynamic industrial environments. By integrating the flexible Burr-XII shock model, which accommodates various hazard rate behaviors (increasing, decreasing, constant, unimodal, and U-shaped), the proposed model offers a more accurate and practical tool for economic decision-making in process monitoring. The approach extends the widely cited Lorenzen and Vance (1986) framework, enabling a more comprehensive analysis of control chart performance under multiple independent assignable causes. Numerical results demonstrate that models such as that of Saadatmelli et al. (2018), based on Duncan (1971) assumptions and Burr-XII failure times, substantially underestimate the average cost per unit time. This discrepancy highlights the importance of using more realistic frameworks like our model to ensure cost-effective quality control.

Keywords: Burr-XII distribution; Hazard rate; Multiple assignable causes; Quality monitoring; Realistic economic design; Shewhart control chart.
Mathematics Subject Classification (2010): 90B25, 62N05.

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1 Introduction

The economic design of control charts has been a central topic in quality control research since Duncan (1956), who proposed a cost-based model assuming exponentially distributed failure times and the presence of a single assignable cause. Early models were based on fixed sampling intervals and simplified cost structures. Subsequent studies refined these assumptions by adopting more flexible failure time distributions and incorporating complex process behaviors. Notably, Baker (1971), Heikes et al. (1974), and Montgomery and Heikes (1976) introduced non-exponential distributions to account for varying hazard rates. McWilliams (1989) considered alternative sampling schemes, while Banerjee and Rahim (1988) employed Weibull distributions and non-uniform sampling intervals to better reflect real-world failure dynamics. Chung (1990) developed a simplified parameter estimation procedure to enhance model usability.

More recent advancements have focused on integrating uncertainties in input parameters (Safaei et al., 2015) , non-normal quality characteristics (Seif et al., 2015), and autocorrelated process data (Naderi et al., 2018, 2021). These efforts have significantly narrowed the gap between theoretical development and industrial applicability. A major milestone in this field was the introduction of the Lorenzen and Vance (1986) model, which replaced fixed cost-per-sample approaches with average run length (ARL)-based designs, thereby capturing the stochastic nature of process failures. This model has since become a foundational reference, adapted across numerous extensions involving alternative distributions and failure mechanisms. Despite these improvements, many models retain restrictive assumptions, particularly the occurrence of only one assignable cause at a time and the absence of any new causes until the first is detected. In real industrial environments, multiple independent assignable causes, such as tool wear, sensor drift, or human error, can occur simultaneously or sequentially, often impacting the process independently. Ignoring this complexity leads to underestimation of detection delay and total quality costs. Recognizing this limitation, Duncan (1971) introduced two modeling assumptions: (i) the independence of assignable cause occurrence times, and (ii) the absence of any new causes after the first failure until a correct alarm is issued. Although Duncan acknowledged the second assumption as unrealistic, its implications on cost estimation and model accuracy remained unexamined in subsequent studies. Notably, if the second assumption holds strictly in practice, it contradicts the first assumption of independence, implying dependence among cause occurrence times. Therefore, assuming independence makes the second condition only a special case among many possible process states. To address this inconsistency, Shojaei et al. (2022) introduced the realistic economic design (RED) framework, which explicitly models the probability that no further assignable cause occurs between the initial shift and its detection. Using a Weibull failure model, they demonstrated that optimal control parameters and average cost estimates differ substantially from those obtained through traditional models.

While Weibull and Gamma distributions are widely used for modeling failure times, recent literature emphasizes the enhanced flexibility of the Burr-XII distribution. Originally proposed by Burr (1942), this distribution has been employed across domains such as econometrics, hydrology, medical statistics, and quality control. Applications include variable sampling plans (Zimmer and Burr, 1963), life data modeling (Wingo, 1993), economic-statistical control charts (Chou et al., 2000), and extreme value modeling for flood prediction (Shao et al., 2004).

Burr-XII's cumulative distribution function (CDF) is

$$F(t) = 1 - (1 + (t/s)^c)^{-k}, \qquad s, k, c > 0,$$

where's represents the scale parameter, and c and k are shape parameters. The Burr-XII distribution offers significant advantages due to its ability to approximate several well-known distributions, including the exponential, gamma, Weibull, normal, and Pareto II. It features a flexible hazard rate function, capable of taking constant, increasing, decreasing, unimodal, or U-shaped forms, making it particularly suitable for modeling diverse failure mechanisms in quality and reliability engineering.

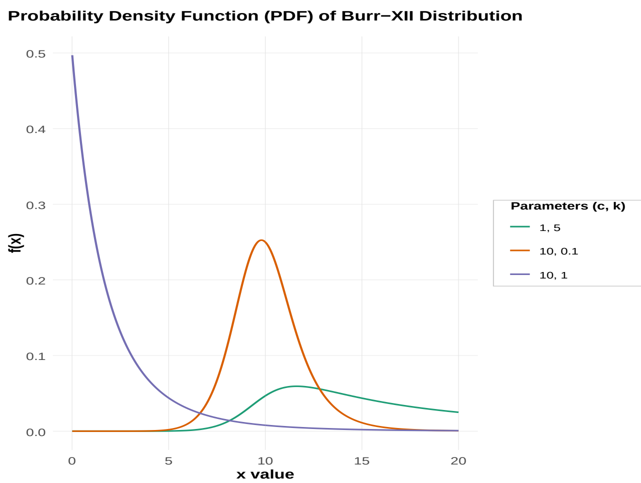
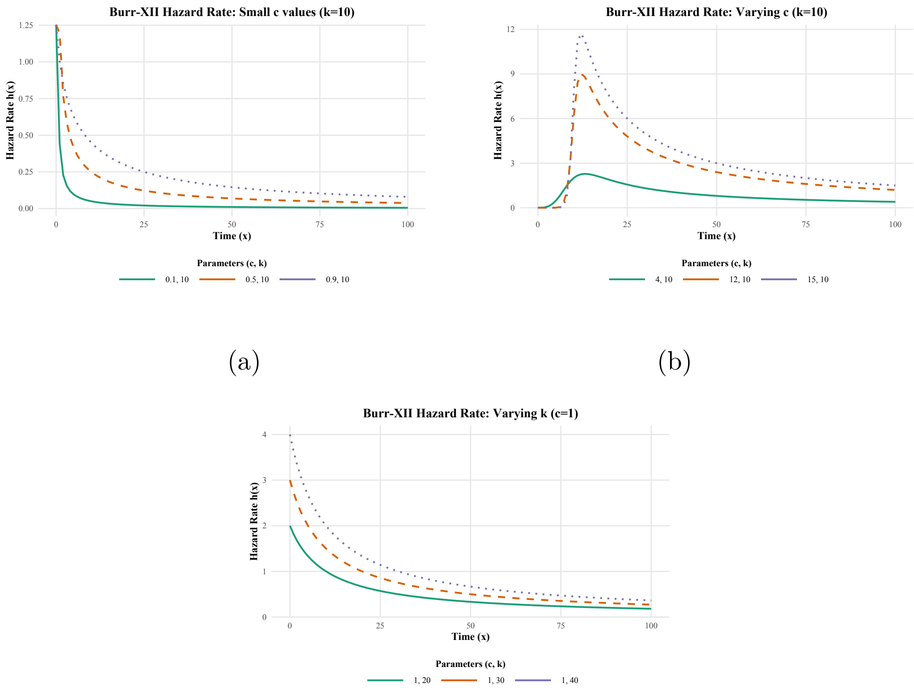


Figure 1: Burr density function for convert distributions.

Figure 1 illustrates various density shapes corresponding to different parameter settings, and Figure 2 shows how varying the Burr-XII parameters (c and k) can produce L-shaped, decreasing, or bathtub-shaped hazard rate functions. Such flexibility makes Burr-XII an excellent candidate for modeling complex, real-world failure behavior. Heydari et al. (2016) were among the first to utilize Burr-XII in control chart design, confirming its suitability for processes with non-monotonic hazard rates.

Recent studies have significantly advanced control chart methodologies, particularly in addressing complex industrial challenges. Notably, Hajiesmaeili et al. (2025) introduced adaptive Lasso charts for high-dimensional processes with dependent state sampling, reflecting the increasing complexity of modern manufacturing systems. Pour et al. (2024) applied log-logistic EWMA charts using MOPSO and VIKOR for cardiac surgery monitoring, demonstrating the relevance of advanced failure modeling in healthcare. Similarly, Lee and Chou (2024) integrated Tukey-based synthetic charts with Taguchi loss functions under log-normal assumptions. In parallel, Huang et al. (2023) proposed economic designs for p-charts considering multiple assignable causes. Healthcare applications have seen progress through Rafiei et al. (2023) risk-adjusted



(c)

Figure 2: Plots of the hazard rate function; (a) L-shape, (b) Upside-down bathtub, (c) Decreasing.

charts, demonstrating cross-industry potential. Particularly noteworthy is Salmasnia et al. (2024) integration of maintenance policies with quality control for cascade processes, offering valuable insights for extending our model. These advancements collectively underscore the growing importance of robust, economically designed control charts in modern quality management systems, while highlighting opportunities to enhance our RED model's applicability across diverse industrial contexts.

Despite these contributions, no existing study has unified RED principles, Burr-XII-based shock modeling, and multiple independent assignable causes into a single control chart framework. This paper addresses this research gap by proposing a novel economic design model for Shewhart control charts that integrates these three dimensions.

Specifically, our work builds on and differentiates itself from

- Saadatmelli et al. (2018, 2023), who optimized $\bar{X}$charts under independent assignable causes and a Burr-XII shock model, but did not incorporate RED logic.
- Shojaei et al. (2022) and Shojaei and Moghadam (2023) , who developed RED frameworks using Weibull-based shocks under multiple independent causes, but did not exploit the flexibility of Burr-XII.

To the best of our knowledge, this is the first study to extend the Lorenzen and Vance RED framework under a Burr-XII shock model while accounting for multiple independent assignable causes. The proposed model offers enhanced accuracy in cost estimation and performance analysis for industrial quality control systems.

The remainder of this article is structured as follows:
• Section 2 introduces key definitions and assumptions for RED Shewhart control charts.
• Section 3 presents the RED framework developed by Lorenzen and Vance (1986) under the Burr_XII shock model, and the proposed optimization model.
• Section 4 concludes with discussion and suggestions for future research.

2 Assumptions and notations

2.1 Assumptions

The assumptions used to design a realistic economic model in the following sections are:
1) The quality characteristic of the process follows the normal distribution with a known mean μ and a known variance σ^2 (σ^2 is constant during the process). The process is under statistical control if $\mu = \mu_0$. When the i-th assignable cause occurs, the average of the process moves from μ_0 to $\mu_0 + \delta_i \sigma^2$, in which case the process is out of statistical control. δ_i is the amount of transmission in the average due to the occurrence of the i-th assignable cause.
2) Upper control limit is $\mu_0 + L\frac{\sigma}{\sqrt{n}}$ and lower control limit is $\mu_0 - L\frac{\sigma}{\sqrt{n}}$.
3) The probability of the type I error, assuming that the quality characteristic has $N(\mu_0 - \sigma_0^2)$ distribution, is $\alpha = 2(1 - \Phi(L))$.
4) The probability of discovering a change in the average process after the i-th assignable cause occurrence is $1 - \beta_i = 1 - (\Phi(L - \delta_i\sqrt{n}) - \Phi(-L - \delta_i\sqrt{n}))$.
5) The occurrence time of the i-th assignable cause ($i = 1, 2, ..., m$) has a density function distribution of $Burr12(s, c, k_i)$;

$$f_{T_i}(t) = \frac{k_i c (t/s)^{c-1}}{s(1 + (t/s)^c)^{k_i+1}}, \qquad s, k_i, c > 0, \quad 0 \leq t < \infty.$$

6) The occurrence times of the assignable causes are independent. Therefore, $T = \min(T_1, T_2, ..., T_m)$has $Burr12(s, c, k_0)$ distribution that is $k_0 = \sum_{i=1}^{m} k_i$.
7) By selecting a random sample of size n from the production process at the time w_ν ($w_\nu = \sum_{l=1}^{v} h_l$), the process is evaluated. h_l is the l-th sampling interval and they are determined so that the probability of the process leaving the controlled state at the time when the beginning of the interval is under control remains a constant value for all sampling intervals. In other words, the following relationship must be established for $i = 1, ..., m$; $P(T < w_\nu | T > w_{\nu-1}) = P(T < w_1 | T > 0)$ then, $\bar{F}(w_\nu) = \bar{F}(h_1)\bar{F}(w_{\nu-1})$. From induction, when $v = k$; $\bar{F}(w_k) = (\bar{F}(h_1))^k$. So, $\bar{F}(w_v) = (\bar{F}(h_1))^v$, for $v = 1, 2, ...$ According to $T \sim Burr12(s, c, k_0)$, then $(1 + (w_v/s)^c)^{-k_0} = ((1 + (h_1/s)^c)^{-k_0})^v$ so,

$$h_v = w_v - w_{v-1} = s((1 + (h_1/s)^c)^v - 1)^{1/c} - s((1 + (h_1/s)^c)^{v-1} - 1)^{1/c}, \qquad v = 2, 3, ...$$

8) The time of sampling and drawing the graph is assumed to be insignificant.
9) The process is not self-correction. In other words, when the process goes out of

control, it is possible to return to the controlled state only with the intervention of the human factor.
10) The process starts from the controlled state.
11) After the occurrence of an assignable cause, another assignable causes shall not occur until a correct alarm is issued. This reception is an event that we show with I.

2.2 Notations

The cost and time parameters are as follows:
Z_0: The average time to search for false alarms,
Z_1: The average time to discover the i-th assignable cause when detected by the control chart,
Z_{2i}: The average time to repair the i-th assignable cause when discovered.
D_0: The average cost of quality per unit time for the on-control state,
D_{1i}: The average quality cost per time unit for the out-of-control state related to the i-th assignable cause,
Y: The average cost of checking false alarms when the process is under control,
D_{2i}: The cost of locating and repairing the i-th assignable cause,
a: The fixed cost of sampling,
b: The cost of each sample unit,
γ_1: Zero and one variable; If the process continues during the search time for the deviations of the cause, the value will be one and if it is stopped, the value will be zero.
γ_{2i}: Zero and one variable; If the process continues during the process modification time, when the i-th assignable cause has occurred, the value will be one and if it is stopped, the value will be zero.

In this article, after the occurrence of the i-th assignable cause, the value of change that occurs in the average process is equal to d_i, which three uniform distributions, half-normal $(1/\sqrt{2\pi})e^{-(1/2)((1/2)d_i)^2}$, and negative exponential $(1/2)e^{-(1/2)d_i}$ is considered as the prior distribution for d_i.

3 Constructing realistic Lorenzen and Vance cost model of Shewhart control charts

3.1 Calculation the ARL of the Shewhart control charts

The ARL is a key metric used to evaluate the effectiveness of control charts in statistical process control and is a key parameter in Lorenzen and Vance's economic model. It represents the average number of samples or points plotted before the first alarm occurs. For Shewhart control charts, the ARL corresponds to the mean of the geometric distribution when the process is in control ($\delta = 0$).

$$ARL_{\delta=0} = ARL_0 = E(RL) = \sum_{r=1}^{\infty} r \Pr(RL = r) = \sum_{r=1}^{\infty} r(1-\alpha)^{r-1}\,\alpha = \frac{1}{\alpha},$$

where RL represents the time it takes for the sample mean to fall out of control. This can occur in the first, second, or any subsequent logical subgroup, up to the r-th logical subgroup. If, after the occurrence of the i-th assignable cause, the process mean shifts by an amount δ_i, resulting in a new process mean of $\mu_0 + \delta_i\, \sigma_0^2$, the out-of-control ARL is expressed as

$$ARL_{\delta>0} = ARL_1 = E(RL) = \sum_{r=1}^{\infty} r \Pr(RL = r) = \sum_{r=1}^{\infty} r{\beta_i}^{r-1}\,(1-\beta_i) = \frac{1}{(1-\beta_i)}.$$

3.2 Calculating the probability of occurrence of the event I based on the Burr-XII shock model

Building upon the foundational assumptions outlined in Section 2, our economic model simplifies the computation of expected cycle time ($E(T)$) and expected cost ($E(C)$) by adopting the critical assumption that no additional assignable causes emerge during the interval between an initial assignable cause's occurrence and its subsequent detection (event I). This modeling assumption establishes a well-defined quality cycle structure that

• **Cycle Definition**: Each quality cycle initiates in the in-control state,
• **State Transition**: Progresses through an out-of-control phase following an assignable cause,
• **Termination Condition**: Concludes with complete process correction and restoration.

The mathematical foundation derives from Ross's renewal reward theorem (Ross, 2013), which establishes that the long-run average cost per unit time ($E(A)$) equals the ratio of expected cycle cost to expected cycle duration ($E(C)/E(T)$), provided both expectations remain finite.

The total cycle duration comprises four distinct temporal components:
X1: In-control duration (including false alarm investigations),
X2: Out-of-control period until proper signal generation,
X3: Diagnostic interval post-signal until assignable cause identification,
X4: Corrective action and repair time,
then,

$$E\,(T) = E\,(X_1) + E\,(X_2 + X_3 + X_4)\,. \tag{1}$$

Theorem 3.1. *Let I_i denote the occurrence of the i-th assignable cause, such that no other assignable causes occur between its occurrence and the issuance of a correct alarm. Then:*

$$P(I) = P\left(\bigcup_{i=1}^{m} I_i\right) = \sum_{i=1}^{m} P(I_i),$$

where,

$$P(I_i) = \frac{(1-\beta_i)[1-(1+(h_1/s)^c)^{-k_i}](1+(h_1/s)^c)^{-k_{-i}})}{(1-\beta_i(1+(h_1/s)^c)^{-k_{-i}})(1-(1+(h_1/s)^c)^{-k_0})}.$$

Proof. The proof is given in the Appendix □

3.2.1 The expected time of a quality cycle

To address the inherent complexity in computing the expected cycle time ($E(T)$) when considering multiple assignable causes, we employ the methodological framework developed by Shojaei et al. (2022), which decomposes $E(T)$ into two distinct components under the occurrence of Event I: 1) the in-control duration, representing the expected time the process remains in control while incorporating time allocated for false alarm investigations (corresponding to the first term in (1)), and 2) the out -of-control duration (constituting the second term in (1)), with this decomposition ultimately yielding the simplified formulation presented in (2):

$$E\left(T\right)=E\left(\sum\nolimits_{l=1}^{4}X_{l}\right)\simeq E\left(X_{1}\right)+E\left(\sum\nolimits_{l=2}^{4}X_{l}|I\right)P\left(I\right), \tag{2}$$

(Shojaei et al., 2022). Considering that the out-of-control average time is finite, establishing the above equation requires that $P(I)$ be sufficiently close to 1. Clearly, $P(I)\approx 1$ implies that the probability of more than one assignable cause occurring within each quality cycle is almost zero. The average time of the on-control state is determined by sum of the average time until the first assignable cause occurs and the average time spent investigating false alarms. According to the Lorenzen and Vance model, production may either be stopped or continue during the search for an assignable cause (γ_1), furthermore let N_{Fa} be the number of false alarms per quality cycle, then the average time of the on-control state will be $E(X_1)=E(T)+(1-\gamma_1)Z_0E(N_{Fa})$. Let N_{In} be the number of sampling in the controlled state. Therefore;$(N_{Fa}|N_{In}=v)\sim Bin(v,\alpha)$and $E(N_{Fa})=E(E(N_{Fa}|N_{In}))=E(\alpha N_{In})$. Considering that N_{In} is a non-negative random variable so, $E(N_{In})=\sum_{v=1}^{\infty}P(N_{In}\geq v)$, where $N_{In}\geq v\ \cong\ T>w_v$ then; $E(N_{In})=\sum_{v=1}^{\infty}P(T>w_v)=\sum_{v=1}^{\infty}([1+(h_1/s)^c]^{-k_0})^v=\frac{[1+(h_1/s)^c]^{-k_0}}{1-[1+(h_1/s)^c]^{-k_0}}$, so

$$E(N_{Fa})\ =\alpha\frac{(1+(h_1/s)^c)^{-k_0}}{1-(1+(h_1/s)^c)^{-k_0}}.$$

Therefore,

$$E(X_1)=sk_0\frac{\Gamma(k_0-(1/c))\Gamma(1+(1/c))}{\Gamma(k_0+1)}+(1-\gamma_1)Z_0\alpha\frac{(1+(h_1/s)^c)^{-k_0}}{1-(1+(h_1/s)^c)^{-k_0}}. \tag{3}$$

The average time of the out-of-control state $E(X_2|I)$ can be calculated as $E(X_2|I)=E(X-X_1|I)=E(X|I)-E(X_1|I)$. Considering that the process is stopped after each alarm by the control chart, then the random variables X and X_1 include the time of checking false alarms. So $X'=X-Z_0N_{Fa}$ and $X_1'=X_1-Z_0N_{Fa}$, therefore $E(X_2|I)=E(X'|I)-E({X_1}^{'}|I)$.

The analytical derivation of the out-of-control period is systematically addressed through Lemmas 3.2 to 3.4.

Lemma 3.2. *The average time from the beginning of the quality cycle to the issuance of a correct alarm (X) under the condition that only one assignable cause (I) occurs:*

$$E(X'|I)\quad=\quad\sum_{v=1}^{\infty}\sum_{i=1}^{m}w_v\frac{1}{P(I)}(1-\beta_i)\left[1-(1+(\frac{h_1}{s})^c)^{-k_i}\right]$$

$$\times \frac{\left((1+(\frac{h_1}{s})^c)^{-k_0}\right)^v - \left(\beta_i(1+(\frac{h_1}{s})^c)^{-k_{-i}}\right)^v}{(1+(\frac{h_1}{s})^c)^{-k_i} - \beta_i}$$

Proof. The proof is given in the Appendix. □

Lemma 3.3. *Let $f_T(t|I)$ be the density function of the on-control time of the process under the condition that only one assignable cause occurs until the issuance of a correct alarm, then the on-control average time under that condition (I) is equal to:*

$$E(X'_1|I) = \int_0^\infty t f_T(t|I)dt = \sum_{v=1}^{\infty} \int_{w_{v-1}}^{w_v} t f_T(t|I)dt$$

$$= \frac{1}{P(I)} \sum_{i=1}^{m} \sum_{v=1}^{\infty} (1-\beta_i) \frac{(1+(\frac{h_1}{s})^c)^{-vk_{-i}}}{1-\beta_i(1+(\frac{h_1}{s})^c)^{-k_{-i}}} \int_{w_{v-1}}^{w_v} -k_0(1+(\frac{t}{s})^c)^{-k_0-1} \frac{ct^c}{s^c} dt$$

Proof. The proof is given in the Appendix. □

Lemma 3.4. *Let $E(X_3 + X_4|I)$denotes the average time required to detect, correct, and repair the i-th assignable cause, under the condition that only one assignable cause occurs until the correct alarm is issued. Then,*

$$E(X_3 + X_4|I) = \sum_{i=1}^{m} (Z_1 + Z_{2i}) \frac{P(I_i)}{P(I)}.$$

Proof. The proof is given in the Appendix. □

Theorem 3.5. *According to 2 and considering this feature of the Lorenzen and Vance model that production can be stopped while searching for an assignable cause or not (γ_1) and this feature that production is stopped or not during the repair process (γ_2), with regard to relationships;*

$$ARL_{00} = \frac{1}{\alpha} \Rightarrow \alpha = ARL_0{}^{-1}, \quad (4)$$

$$ARL_{10} = \frac{1}{1-\beta_i} \Rightarrow (1-\beta_i) = ARL_{1i}{}^{-1} \Rightarrow \beta_i = 1 - ARL_{1i}{}^{-1}. \quad (5)$$

In the design of Shewhart control charts, the expected time of a quality cycle is given by

$$E(T) = sk_0 \frac{\Gamma(k_0-(1/c))\Gamma(1+(1/c))}{\Gamma(k_0+1)} + (1-\gamma_1)Z_0(ARL_0{}^{-1}) \frac{(1+(h_1/s)^c)^{-k_0}}{1-(1+(h_1/s)^c)^{-k_0}}$$

$$+ \sum_{v=1}^{\infty} \sum_{i=1}^{m} w_v \frac{1}{P(I)} (ARL_{1_i}^{-1})[1-(1+(\frac{h_1}{s})^c)^{-k_i}]$$

$$\times \frac{\left((1+(\frac{h_1}{s})^c)^{-k_0}\right)^v - \left((1-ARL_{1i}{}^{-1})(1+(\frac{h_1}{s})^c)^{-k_{-i}}\right)^v}{(1+(\frac{h_1}{s})^c)^{-k_i} - (1-ARL_{1i}{}^{-1})}$$

$$-\frac{1}{P(I)}\sum_{i=1}^{m}\sum_{v=1}^{\infty}(ARL_{1i}{}^{-1})\frac{\left(1+(\frac{h_1}{s})^c\right)^{-vk_{-i}}}{1-(1-ARL_{1i}{}^{-1})\left(1+(\frac{h_1}{s})^c\right)^{-k_{-i}}}$$
$$\times\int_{w_{v-1}}^{w_v}-k_0(1+(\frac{t}{s})^c)^{-k_0-1}\frac{ct^c}{s^c}dt+\sum_{i=1}^{m}[(Z_1+Z_{2i})\frac{P(I_i)}{P(I)}]$$

(According to Lemmas 3.2 to 3.4. and equation 3, the proof is clear.)

3.2.2 The expected cost of a quality cycle

The expected cost of a quality cycle consists of the following components; 1. The average cost during on-control time ($D_0E(T)$), 2. The average cost associated with searching for false alarms ($YE(N_{Fa})$), 3. The average cost during out-of-control time ($D_{1i}E(X_2|I)$), 4. The average sampling cost ($(a+bn)E(N_{TST}|I)$), where N_{TST} represents the number of samples taken until the correct alarm is issued, and 5. The average cost for location, diagnosis, and repair of the assignable cause ($\sum_{i=1}^{m}(P(I_i)/P(I))D_{2i}$).

Therefore, the total average cost in under control time and the average cost of searching for false alarms are equal to:

$$\begin{aligned} D_0E(T)+YE(N_{Fa}) &= D_0sk_0\frac{\Gamma(k_0-(1/c))\Gamma(1+(1/c))}{\Gamma(k_0+1)} \\ &\quad +Y(1-\gamma_1)\alpha\frac{(1+(h_1/s)^c)^{-k_0}}{1-(1+(h_1/s)^c)^{-k_0}}, \end{aligned} \tag{6}$$

The average out-of-control cost is equal to

$$\begin{aligned} D_{1i}E(X_2|I) &= \sum_{v=1}^{\infty}\sum_{i=1}^{m}D_{1i}w_v\frac{1}{P(I)}(1-\beta_i)[1-(1+(\frac{h_1}{s})^c)^{-k_i}] \\ &\quad\times\frac{\left((1+(\frac{h_1}{s})^c)^{-k_0}\right)^v-\left(\beta_i(1+(\frac{h_1}{s})^c)^{-k_{-i}}\right)^v}{(1+(\frac{h_1}{s})^c)^{-k_i}-\beta_i} \\ &\quad-\frac{1}{P(I)}\sum_{i=1}^{m}\sum_{v=1}^{\infty}D_{1i}(1-\beta_i)\frac{\left(1+(\frac{h_1}{s})^c\right)^{-vk_{-i}}}{1-\beta_i\left(1+(\frac{h_1}{s})^c\right)^{-k_{-i}}} \\ &\quad\times\int_{w_{v-1}}^{w_v}-k_i(1+(\frac{t}{s})^c)^{-k_i-1}\frac{ct^c}{s^c}dt+\sum_{i=1}^{m}(P(I_i)/P(I))D_{2i}. \end{aligned}$$

Lemma 3.6. *Let N_{TST} represent the total number of samples. Then,*

$$E(N_{TST}|I)=\sum_{i=1}^{m}\frac{P(I_i)}{P(I)}\left[\frac{(1+(\frac{h_1}{s})^c)^{-k_0}}{1-(1+(\frac{h_1}{s})^c)^{-k_0}}+\frac{1}{1-\beta_i(1+(\frac{h_1}{s})^c)^{-k_{-i}}}\right].$$

Proof. The proof is given in the Appendix. □

According to Lemma 3.6, the average sampling cost is equal

$$(a+bn)E(N_{TST}|I)=(a+bn)\sum_{i=1}^{m}\frac{P(I_i)}{P(I)}\left[\frac{(1+(\frac{h_1}{s})^c)^{-k_0}}{1-(1+(\frac{h_1}{s})^c)^{-k_0}}+\frac{1}{1-\beta_i(1+(\frac{h_1}{s})^c)^{-k_{-i}}}\right]. \tag{7}$$

Therefore, according to the calculation steps of E(T), the components of E(C) are as follows:

According to 6 to 7 and relations 4 and 5, the average cost of a quality cycle is given by

$$
\begin{aligned}
E(C) = & D_0 s k_0 \frac{\Gamma(k_0-(1/c))\Gamma(1+(1/c))}{\Gamma(k_0+1)} + \frac{Y}{ARL_0}\frac{\left(1+(h_1/s)^c\right)^{-k_0}}{1-\left(1+(h_1/s)^c\right)^{-k_0}} \\
& + \sum_{v=1}^{\infty}\sum_{i=1}^{m} D_{1i} w_v \frac{1}{P(I)ARL_{1i}}[1-(1+(\frac{h_1}{s})^c)^{-k_i}] \\
& \times \frac{\left((1+(\frac{h_1}{s})^c)^{-k_0}\right)^v - \left((1-ARL_{1i}{}^{-1})(1+(\frac{h_1}{s})^c)^{-k_{-i}}\right)^v}{(1+(\frac{h_1}{s})^c)^{-k_i} - (1-ARL_{1i}{}^{-1})} \\
& - \frac{1}{P(I)}\sum_{i=1}^{m}\sum_{k=1}^{\infty} D_{1i}\frac{1}{ARL_{1i}}\frac{(1+(\frac{h_1}{s})^c)^{-vk_{-i}}}{1-(1-ARL_{1i}{}^{-1})(1+(\frac{h_1}{s})^c)^{-k_{-i}}} \\
& \times \int_{w_{v-1}}^{w_v} -k_0(1+(\frac{t}{s})^c)^{-k_0-1}\frac{ct^c}{s^c}dt \\
& + \sum_{i=1}^{m} D_{1i}[(\gamma_1 Z_1+\gamma_{2i}Z_{2i})\frac{P(I_i)}{P(I)}] + \sum_{\mathrm{i=1}}^{\mathrm{m}}\frac{\mathrm{P(I_i)}}{\mathrm{P(I)}}\mathrm{D_{2i}} \\
& + (a+bn)\sum_{i=1}^{m}\frac{P(I_i)}{P(I)}\left(\left[\frac{(1+(\frac{h_1}{s})^c)^{-k_0}}{1-(1+(\frac{h_1}{s})^c)^{-k_0}} + \frac{1}{1-(1-ARL_{1i}{}^{-1})(1+(\frac{h_1}{s})^c)^{-k_{-i}}}\right]\right. \\
& \left. + (\gamma_1 Z_1+\gamma_{2i}Z_{2i})\right). \qquad (8)
\end{aligned}
$$

Considering that the object is determined the optimal values of the $\bar{\mathrm{X}}$ control chart's design parameters, in order to minimize $E(A) = E\,(C)\,/E\,(T)$ in the condition that the event I occurs with a probability close to one, the Realistic Economic Design (RED) can be formulated as follows:

$$
\text{RED Model: } \begin{cases} \text{minimize } E\,(\mathrm{A})\,, \\ \text{subject to } P\,(I) \geq p_0, \end{cases}
$$

where p_0 represents the acceptable lower limit for $P(I)$ in the selection of design parameters. To evaluate the performance of the control chart, ARL_0 and $ARL_{1overall}$ are used, such that $ARL_0 = 1/\alpha$ and $ARL_{1over\,all} = \sum_{i=1}^{m}\frac{P(I_i)}{P(I)}\frac{1}{1-\beta_i}$. In the next section, the economic design process of $\bar{\mathrm{X}}$ control chart is discussed.

3.3 Numerical analysis results

In this section, the optimal design parameters *n*, *h*, and *L* are determined by minimizing the realistic cost function. All numerical analyses were performed using *R* software, version 4.3.1, along with the latest version of the *DEoptim* package. The input parameters of the model are categorized into cost parameters (Y, D_0, D_{1i}, D_{2i}, a, b, γ_1, γ_2), time parameters (Z_0, Z_1, Z_{2i}), transfer parameters (δ_i), and Burr-XII distribution parameters (s, c, k_i). To ensure a fair comparison between our model and the model of Saadatmelli et al. (2018), the input parameters are set according to their specifications. Additionally, $\gamma_1 = \gamma_2 = 0$ is applied for consistency. The input parameters of the model are listed in the Table 1. For all assignable causes, the following values are assumed to be the same: $Y = 2000\$$, $D_0 = 210\$$, $a = 20\$$, $b = 20\$$, $Z_0 = 1.25h$ and $Z_1 = 1.25$. The parameters k_i, Z_{2i}, D_{1i}, and D_{2i} are also a function of δ_i. For details on how to determine these parameters, refer to section 5 of Saadatmelli et al. (2018). In the realistic economic design, it is assumed that p_0 takes values of 0.85, 0.90, 0.95, and 0.99. (We increase the value of p_0 to reduce the probability of multiple assignable causes occurring before issuing a correct alarm).

Table 1: The Set of input parameter values (See Saadatmelli et al. (2018) Table 3)

k_i			Z_{2i}			D^*_{2i}			D_{1i}	δ_i	i
HN_i	Un_i	NE_i	HN_i	Un_i	NE_i	HN_i	Un_i	NE_i			
10.086	6.932	11.429	2.909	2	3.293	1454	1000	1647	575	1.0	1
8.627	6.932	8.901	2.488	2	2.565	1244	1000	1283	1684	1.5	2
7.623	6.932	7.661	2.198	2	1.103	1099	1000	1103	2901	1.8	3
6.932	6.932	6.932	2.000	2	2.000	1000	1000	1000	4000	2.0	4
6.241	6.932	6.272	1.802	2	1.804	901	1000	902	5341	2.2	5
5.233	6.932	5.399	1.512	2	1.554	756	1000	777	7776	2.5	6
4.289	6.932	4.735	1.074	2	1.217	537	1000	609	12602	3.0	7

*Prior Distribution: NE; Negative-exponential; Un; Uniform, HN; Half-normal.

Tables 2 to 4 presents the optimal values of the realistic economic design parameters for $\bar{X}$ control charts in the presence of 7 independent assignable causes under the Burr-XII shock model, for varying s and c. In Table 2 s fixed and c changed, in Table 3 c fixed and s changed, and in Table 4 both c and s changed.

The analysis are conducted under both the RED model and the economic model proposed by Saadatmelli et al. (2018). Since the probability of event I is not considered in the calculation of $E(A)$ in the Saadatmelli et al. (2018) model, and to ensure a fair comparison between the two models, it is assumed that there is no restriction on the probability of event I in the RED model (i.e., the $p_0 = 0$).

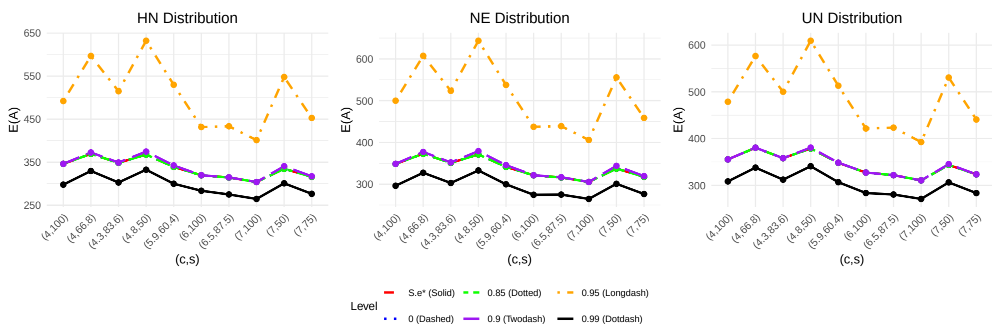


Figure 3: Impact of detection probability threshold (p_0) on expected cost ($E(A)$) across different prior distributions.

In Figure 3 the horizontal axis displays values of scale (s) and shape (c) parameters,

Table 2: Optimum design parameters for multiplicity-cause RED and Saadatmelli et al. (2018) models (economic design of $\bar{X}$ when $s = 90$ fixed and c changed).

c	PD*	p_0	n	H	L	$P(I)$	arl_0	arl_1	EA
3	HN	S.e**	4	11.35	2.49	0.922	78.28	1.29	257.58
		0	3	9.14	2.55	0.94	94.24	2	384.2
		0.85	3	9.14	2.55	0.94	94.24	2	384.2
		0.9	3	9.14	2.55	0.94	94.24	2	384.2
		0.95	2	7.5	2.44	0.952	69.11	2.63	401.78
		0.99	5	6.2	2.27	0.99	43.28	1.25	611.09
	NE	S.e	4	11.18	2.47	0.922	74.01	1.3	250.43
		0	3	9.04	2.55	0.938	93.14	2.04	387.96
		0.85	3	9.04	2.55	0.938	93.14	2.04	387.96
		0.9	3	9.04	2.55	0.938	93.14	2.04	387.96
		0.95	3	8.72	2.43	0.95	67.11	1.87	390.43
		0.99	5	6.11	2.23	0.99	39.72	1.25	623.26
	UN	S.e	3	10.63	2.56	0.922	95.53	1.37	323.07
		0	3	8.67	2.73	0.946	162.31	2	396.8
		0.85	3	8.67	2.73	0.946	162.31	2	396.8
		0.9	3	8.67	2.73	0.946	162.31	2	396.8
		0.95	3	8.21	2.81	0.95	202.28	2.13	399.52
		0.99	4	6.04	2.29	0.99	46.57	1.29	590.29
2.8	HN	S.e	3	9.01	2.53	0.915	87.67	1.49	261.18
		0	3	7.85	2.55	0.938	94.57	2	393.27
		0.85	3	7.85	2.55	0.938	94.57	2	393.27
		0.9	3	7.85	2.55	0.938	94.57	2	393.27
		0.95	3	7.03	2.57	0.952	100.09	2.05	399.63
		0.99	5	5.15	2.24	0.99	39.88	1.24	641.04
	NE	S.e	3	7.68	2.72	0.926	153.18	1.63	267.29
		0	3	7.76	2.54	0.936	92.23	2.04	397.39
		0.85	3	7.76	2.54	0.936	92.23	2.04	397.39
		0.9	3	7.76	2.54	0.936	92.23	2.04	397.39
		0.95	3	7.42	2.42	0.95	64.67	1.85	400.69
		0.99	5	5.05	2.23	0.99	39.32	1.25	654.47
	UN	S.e	3	9.16	2.56	0.922	95.53	1.37	332.26
		0	3	7.42	2.74	0.945	163.12	2	406.52
		0.85	3	7.42	2.74	0.945	163.12	2	406.52
		0.9	3	7.42	2.74	0.945	163.12	2	406.52
		0.95	3	6.71	2.88	0.95	257.88	2.3	413.96
		0.99	4	5.01	2.27	0.99	43.33	1.28	618.68
3.8	HN	S.e	5	19.97	2.34	0.905	51.85	1.17	255.62
		0	3	14.5	2.56	0.943	96.32	2.01	357.03
		0.85	3	14.5	2.56	0.943	96.32	2.01	357.03
		0.9	3	14.5	2.56	0.943	96.32	2.01	357.03
		0.95	3	13.88	2.56	0.951	96.85	2.03	358.54
		0.99	3	9.06	2.52	0.99	86.81	2.01	598.83
	NE	S.e	5	18.97	2.44	0.911	68.08	1.2	247.15
		0	3	14.37	2.56	0.941	95.83	2.07	359.84
		0.85	3	14.37	2.56	0.941	95.83	2.07	359.84
		0.9	3	14.37	2.56	0.941	95.83	2.07	359.84
		0.95	3	14.05	2.48	0.95	77.84	1.95	360.85
		0.99	5	10.73	2.26	0.99	42.84	1.26	539.56
	UN	S.e	3	16.58	2.57	0.923	98.33	1.37	300.72
		0	3	13.94	2.73	0.95	160.53	2	367.71
		0.85	3	13.94	2.73	0.95	160.53	2	367.71
		0.9	3	13.94	2.73	0.95	160.53	2	367.71
		0.95	3	13.94	2.73	0.95	160.53	2	367.71
		0.99	2	7.72	2.51	0.992	84.3	2.47	751.17

*Prior Distribution: NE: Negative-exponential; Un: Uniform, HN: Half-normal.

** S.e: Saadatmelli et al. (2018)

Continuation of Table 2.

c	PD	n	h	L	$P(I)$	arl$_0$	arl$_1$	EA
3.5	HN	4	15.36	2.5	0.92	80.52	1.3	253.45
		3	12.48	2.55	0.942	95.42	2.01	365.84
		3	12.48	2.55	0.942	95.42	2.01	365.84
		3	12.48	2.55	0.942	95.42	2.01	365.84
		3	11.94	2.54	0.95	92.21	2	367.33
		2	6.56	2.47	0.99	74.21	2.78	719.62
	NE	4	15.34	2.48	0.917	76.11	1.31	247.1
		3	12.36	2.55	0.94	94.96	2.06	368.94
		3	12.36	2.55	0.94	94.96	2.06	368.94
		3	12.36	2.55	0.94	94.96	2.06	368.94
		3	12.02	2.47	0.95	75.78	1.94	370.27
		5	8.95	2.26	0.99	42.26	1.26	565.07
	UN	3	14.37	2.57	0.923	98.33	1.37	307.94
		3	11.95	2.73	0.949	161.03	2	377.13
		3	11.95	2.73	0.949	161.03	2	377.13
		3	11.95	2.73	0.949	161.03	2	377.13
		3	11.52	2.75	0.953	169.89	2.03	378.32
		4	8.86	2.31	0.99	48.83	1.3	537.76
3.2	HN	4	13.1	2.41	0.923	62.68	1.27	255.89
		3	10.47	2.55	0.941	94.34	2	376.21
		3	10.47	2.55	0.941	94.34	2	376.21
		3	10.47	2.55	0.941	94.34	2	376.21
		3	9.92	2.55	0.95	92.86	2	378.15
		5	7.34	2.26	0.99	42.59	1.25	585.78
	NE	3	14.3	2	0.901	21.97	1.27	258.03
		3	10.36	2.55	0.939	93.93	2.05	379.67
		3	10.36	2.55	0.939	93.93	2.05	379.67
		3	10.36	2.55	0.939	93.93	2.05	379.67
		3	10.03	2.45	0.95	70.9	1.9	381.56
		5	7.2	2.26	0.99	42.47	1.26	597.18
	UN	3	12.13	2.57	0.922	98.33	1.37	316.8
		3	9.97	2.73	0.947	161.73	2	388.23
		3	9.97	2.73	0.947	161.73	2	388.23
		3	9.97	2.73	0.947	161.73	2	388.23
		3	9.4	2.73	0.956	158.54	2	390.81
		2	5.7	2.37	0.99	57.29	2.15	656.49
2.3	HN	4	5.68	2.52	0.93	85.21	1.3	264.01
		3	4.83	2.56	0.932	95.84	2	422.22
		3	4.83	2.56	0.932	95.84	2	422.22
		3	4.83	2.56	0.932	95.84	2	422.22
		3	2.73	2.53	0.98	89.71	2.01	587.35
		5	2.77	2.22	0.99	38.37	1.23	749.11
	NE	4	6.67	2.28	0.911	44.23	1.24	255.61
		3	4.79	2.53	0.93	89.35	2.01	427.6
		3	4.79	2.53	0.93	89.35	2.01	427.6
		3	4.79	2.53	0.93	89.35	2.01	427.6
		4	4.82	2.45	0.95	71.21	1.54	435.31
		5	2.72	2.2	0.99	35.99	1.24	765.87
	UN	3	5.68	2.52	0.921	85.21	1.35	264.01
		3	4.51	2.74	0.939	166.14	2	437.59
		3	4.51	2.74	0.939	166.14	2	437.59
		3	4.51	2.74	0.939	166.14	2	437.59
		3	4.51	2.54	0.95	91.91	1.72	437.28
		4	2.71	2.22	0.99	38.38	1.26	718.82

*Prior Distribution: NE: Negative-exponential; Un: Uniform, HN: Half-normal.

** S.e: Saadatmelli et al. (2018)

while the vertical axis represents corresponding $E(A)$ values. Trend lines (color-coded of green, yellow, blue, purple and black) indicate p_0 levels (0, 0.85, 0.9, 0.95, 0.99),

Table 3: Optimum design parameters for multiplicity-cause RED and Saadatmelli et al. (2018) (economic design of $\bar{X}$ when $c = 3$ fixed and s changed).

s	PD	p_0	n	h	L	$P(I)$	arl_0	arl_1	EA
55	HN	S.e	3	7.52	2.41	0.885	62.68	1.43	288.14
		0	3	6.13	2.54	0.924	91.08	1.96	416.74
		0.85	3	6.13	2.54	0.924	91.08	1.96	416.74
		0.9	3	6.13	2.54	0.924	91.08	1.96	416.74
		0.95	4	5.98	2.44	0.95	68.86	1.49	427.75
		0.99	5	3.84	2.18	0.99	34.59	1.21	782.58
	NE	S.e	4	7.79	2.5	0.887	80.52	1.31	280.1
		0	4	6.34	2.63	0.927	117.97	1.7	425.92
		0.85	4	6.34	2.63	0.927	117.97	1.7	425.92
		0.9	4	6.34	2.63	0.927	117.97	1.7	425.92
		0.95	4	5.87	2.43	0.95	66.55	1.52	434.56
		0.99	5	3.76	2.2	0.99	36.37	1.24	798.7
	UN	S.e	3	7.09	2.54	0.904	90.21	1.36	348.45
		0	3	5.89	2.61	0.937	112.33	1.79	431.02
		0.85	3	5.89	2.61	0.937	112.33	1.79	431.02
		0.9	3	5.89	2.61	0.937	112.33	1.79	431.02
		0.95	3	5.67	2.47	0.95	74.83	1.64	434.79
		0.99	4	3.76	2.2	0.99	36.25	1.25	750.77
65	HN	S.e	4	8.11	2.54	0.922	90.21	1.31	277.38
		0	3	7.03	2.79	0.909	192.7	2.44	409.82
		0.85	3	7.03	2.79	0.909	192.7	2.44	409.82
		0.9	3	7.03	2.79	0.909	192.7	2.44	409.82
		0.95	3	5.31	2.76	0.958	174.98	2.46	452.5
		0.99	5	4.52	2.21	0.99	37.35	1.23	719.28
	NE	S.e	3	8.64	2.28	0.897	44.23	1.38	273.05
		0	4	7.31	2.54	0.936	91.35	1.61	414.33
		0.85	4	7.31	2.54	0.936	91.35	1.61	414.33
		0.9	4	7.31	2.54	0.936	91.35	1.61	414.33
		0.95	4	6.85	2.47	0.95	75.22	1.55	418.76
		0.99	5	4.41	2.23	0.99	39.78	1.25	734.51
	UN	S.e	3	8.14	2.55	0.91	92.83	1.37	339.94
		0	3	6.76	2.61	0.941	113.27	1.8	419.14
		0.85	3	6.76	2.61	0.941	113.27	1.8	419.14
		0.9	3	6.76	2.61	0.941	113.27	1.8	419.14
		0.95	3	6.56	2.53	0.95	90.19	1.71	420.61
		0.99	4	4.39	2.26	0.99	42.92	1.28	691.57
70	HN	S.e	3	8.88	2.44	0.902	68.08	1.44	272.32
		0	3	7.61	2.59	0.925	106.11	2.05	401.33
		0.85	3	7.61	2.59	0.925	106.11	2.05	401.33
		0.9	3	7.61	2.59	0.925	106.11	2.05	401.33
		0.95	3	5.94	2.79	0.952	192.07	2.52	429.74
		0.99	3	3.73	2.56	0.99	98.05	2.08	830.49
	NE	S.e	4	9.91	2.41	0.924	62.68	1.11	265.97
		0	3	6.99	2.6	0.936	107.89	2.13	406.79
		0.85	3	6.99	2.6	0.936	107.89	2.13	406.79
		0.9	3	6.99	2.6	0.936	107.89	2.13	406.79
		0.95	4	7.34	2.49	0.95	79.05	1.57	412.18
		0.99	5	4.78	2.21	0.99	37.22	1.24	707.39
	UN	S.e	3	8.65	2.55	0.913	92.83	1.37	336.21
		0	3	7.18	2.61	0.943	113.68	1.81	413.86
		0.85	3	7.18	2.61	0.943	113.68	1.81	413.86
		0.9	3	7.18	2.61	0.943	113.68	1.81	413.86
		0.95	3	7.04	2.55	0.95	93.19	1.73	414.68
		0.99	4	4.76	2.23	0.99	39.82	1.26	666.79

Continuation of Table 3.

s	PD	N	h	L	$P(I)$	arl_0	arl_1	EA
75	HN	3	9.44	2.29	0.915	45.41	1.37	273.53
		3	7.88	2.56	0.934	95.68	2	396.19
		3	7.88	2.56	0.934	95.68	2	396.19
		3	7.88	2.56	0.934	95.68	2	396.19
		2	6.46	2.4	0.95	61.88	2.52	416.31
		5	5.21	2.22	0.99	38.32	1.23	669.38
	NE	3	9.87	2.14	0.909	30.9	1.32	262.48
		3	7.79	2.54	0.932	91.69	2.03	400.48
		3	7.79	2.54	0.932	91.69	2.03	400.48
		3	7.79	2.54	0.932	91.69	2.03	400.48
		4	7.83	2.51	0.95	82.93	1.59	406.26
		5	5.08	2.25	0.99	41.7	1.26	683.32
	UN	3	9.15	2.55	0.916	92.83	1.37	332.78
		3	7.59	2.62	0.945	114.13	1.81	408.96
		3	7.59	2.62	0.945	114.13	1.81	408.96
		3	7.59	2.62	0.945	114.13	1.81	408.96
		3	7.49	2.57	0.95	99.21	1.76	409.36
		4	5.08	2.25	0.99	41.37	1.27	644.77
80	HN	5	11.18	2.4	0.917	60.99	1.18	265.47
		3	8.3	2.55	0.936	94.97	2	391.93
		3	8.3	2.55	0.936	94.97	2	391.93
		3	8.3	2.55	0.936	94.97	2	391.93
		3	7.46	2.57	0.951	100.94	2.05	398.66
		5	5.53	2.25	0.99	41.27	1.24	647.9
	NE	4	10.24	2.41	0.919	62.68	1.28	257.4
		3	8.22	2.54	0.934	92.57	2.04	396.02
		3	8.22	2.54	0.934	92.57	2.04	396.02
		3	8.22	2.54	0.934	92.57	2.04	396.02
		3	7.75	2.43	0.95	67.25	1.87	400.36
		5	5.43	2.24	0.99	40.26	1.25	661.34
	UN	3	9.65	2.56	0.918	95.53	1.37	329.6
		3	8	2.62	0.947	114.75	1.82	404.38
		3	8	2.62	0.947	114.75	1.82	404.38
		3	8	2.62	0.947	114.75	1.82	404.38
		3	7.93	2.59	0.95	105.51	1.78	404.53
		4	5.42	2.25	0.99	41.23	1.27	624.83
95	HN	4	11.47	2.51	0.949	60.99	1.18	255.17
		3	9.56	2.55	0.941	93.8	1.99	380.7
		3	9.56	2.55	0.941	93.8	1.99	380.7
		3	9.56	2.55	0.941	93.8	1.99	380.7
		3	9.26	2.48	0.95	77.44	1.9	381.95
		5	6.54	2.27	0.99	43.45	1.25	594.88
	NE	3	11.93	2.33	0.908	50.48	1.41	248.51
		3	9.45	2.55	0.939	93.39	2.05	384.31
		3	9.45	2.55	0.939	93.39	2.05	384.31
		3	9.45	2.55	0.939	93.39	2.05	384.31
		3	9.14	2.45	0.95	71.27	1.9	386.14
		5	6.45	2.24	0.99	40.5	1.25	606.69
	UN	3	11.12	2.57	0.924	98.33	1.37	321.34
		3	9.21	2.62	0.951	115.55	1.83	392.34
		3	9.21	2.62	0.951	115.55	1.83	392.34
		3	9.21	2.62	0.951	115.55	1.83	392.34
		3	9.21	2.62	0.951	115.55	1.83	392.34
		4	6.36	2.31	0.99	48.78	1.3	575.31

Table 4: Optimum design parameters for multiplicity-cause RED and Saadatmelli et al. (2018) models (economic design of $\bar{X}$ when c and s changed).

c	s	PD	p_0	n	h	L	$P(I)$	arl_0	arl_1	EA
6	100	HN	S.e	3	28.35	2.67	0.937	131.83	1.57	283.49
			0	3	27.8	2.57	0.948	99.7	2.04	319.52
			0.9	3	27.8	2.57	0.948	99.7	2.04	319.52
			0.9	3	27.8	2.57	0.948	99.7	2.04	319.52
			1	3	27.73	2.55	0.95	93.54	2	319.56
			1	5	23.17	2.3	0.99	47.46	1.26	431.55
		NE	S.e	4	30.55	2.54	0.935	90.21	1.33	274.57
			0	3	27.61	2.57	0.946	99.39	2.09	321.28
			0.9	3	27.61	2.57	0.946	99.39	2.09	321.28
			0.9	3	27.61	2.57	0.946	99.39	2.09	321.28
			1	3	27.42	2.54	0.95	91.48	2.05	321.41
			1	5	22.85	2.33	0.99	50.98	1.29	437.41
		UN	S.e	4	29.43	2.67	0.951	131.83	1.25	283.49
			0	3	27.23	2.64	0.957	123	1.87	327.31
			0.9	3	27.23	2.64	0.957	123	1.87	327.31
			0.9	3	27.23	2.64	0.957	123	1.87	327.31
			1	3	27.23	2.64	0.957	123	1.87	327.31
			1	4	22.87	2.32	0.99	49.25	1.3	421.59
7	50	HN	S.e	4	20.3	2.59	0.923	104.19	1.33	300.8
			0	3	19.4	2.57	0.925	98.71	2	334.64
			0.9	3	19.4	2.57	0.925	98.71	2	334.64
			0.9	3	19.4	2.57	0.925	98.71	2	334.64
			1	4	19.23	2.48	0.95	77.52	1.53	340.47
			1	5	15.84	2.3	0.99	47.78	1.26	547.82
		NE	S.e	4	20.34	2.58	0.918	101.21	1.34	300.85
			0	3	19.3	2.56	0.923	95.9	2.04	337.31
			0.9	3	19.3	2.56	0.923	95.9	2.04	337.31
			0.9	3	19.3	2.56	0.923	95.9	2.04	337.31
			1	4	19.09	2.46	0.95	72.97	1.55	343.84
			1	5	15.77	2.25	0.99	41.8	1.26	555.8
		UN	S.e	4	20.3	2.65	0.93	124.23	1.25	306.3
			0	3	19.08	2.64	0.938	121.1	1.83	343.55
			0.9	3	19.08	2.64	0.938	121.1	1.83	343.55
			0.9	3	19.08	2.64	0.938	121.1	1.83	343.55
			1	3	18.73	2.53	0.95	88.47	1.71	345.33
			1	5	15.99	2.34	0.99	52.72	1.2	530.59
4	100	HN	S.e	4	18.92	2.63	0.942	117.11	1.34	297.74
			0	3	17.35	2.56	0.947	97.11	2.02	345.96
			0.9	3	17.35	2.56	0.947	97.11	2.02	345.96
			0.9	3	17.35	2.56	0.947	97.11	2.02	345.96
			1	3	17.23	2.53	0.95	88.73	1.97	346.09
			1	5	13.42	2.28	0.99	45.12	1.25	491.99
		NE	S.e	4	19.24	2.56	0.938	95.53	1.33	296.17
			0	3	17.19	2.56	0.945	96.77	2.08	348.4
			0.9	3	17.19	2.56	0.945	96.77	2.08	348.4
			0.9	3	17.19	2.56	0.945	96.77	2.08	348.4
			1	3	17.03	2.5	0.95	82.64	1.99	348.72
			1	5	13.25	2.27	0.99	43.22	1.27	499.96
		UN	S.e	4	18.52	2.66	0.952	127.97	1.25	308.35
			0	3	16.87	2.63	0.956	119.49	1.85	355.43
			0.9	3	16.87	2.63	0.956	119.49	1.85	355.43
			0.9	3	16.87	2.63	0.956	119.49	1.85	355.43
			1	3	16.87	2.63	0.956	119.49	1.85	355.43
			1	4	13.1	2.34	0.99	52.45	1.32	478.83

Continuation of Table 4.

c	s	PD	n	h	L	$P(I)$	arl_0	arl_1	EA
4.8	50HN		4	13.59	2.57	0.912	98.33	1.32	332.44
			3	12.51	2.55	0.926	94.98	1.98	367.61
			3	12.51	2.55	0.926	94.98	1.98	367.61
			3	12.51	2.55	0.926	94.98	1.98	367.61
			4	12.38	2.48	0.95	76.72	1.53	374.58
			5	9.37	2.25	0.99	41.92	1.24	632.13
		NE	4	13.88	2.55	0.908	92.83	1.33	332.84
			3	12.41	2.55	0.924	94.68	2.04	371.18
			3	12.41	2.55	0.924	94.68	2.04	371.18
			3	12.41	2.55	0.924	94.68	2.04	371.18
			4	12.21	2.47	0.95	76.1	1.56	379.08
			5	9.27	2.24	0.99	40.27	1.25	643.56
		UN	4	13.22	2.64	0.934	120.61	1.25	340.79
			3	12.2	2.63	0.939	118.04	1.82	378.76
			3	12.2	2.63	0.939	118.04	1.82	378.76
			3	12.2	2.63	0.939	118.04	1.82	378.76
			3	11.93	2.52	0.95	85.84	1.69	380.69
			4	9.25	2.26	0.99	42.25	1.28	609.28
4.3	83.6	HN	4	19.73	2.54	0.932	113.72	1.54	302.74
			3	16.9	2.56	0.942	97.16	2.02	348.39
			3	16.9	2.56	0.942	97.16	2.02	348.39
			3	16.9	2.56	0.942	97.16	2.02	348.39
			3	16.66	2.48	0.95	77.21	1.89	349.17
			5	13.05	2.29	0.99	46.26	1.26	514.9
		NE	3	17.39	2.62	0.937	127.97	1.37	302.79
			3	16.76	2.56	0.94	96.76	2.07	350.97
			3	16.76	2.56	0.94	96.76	2.07	350.97
			3	16.76	2.56	0.94	96.76	2.07	350.97
			3	16.37	2.49	0.95	78.51	1.96	352.14
			5	12.96	2.24	0.99	40.05	1.25	523.56
		UN	4	17.98	2.66	0.948	127.97	1.25	312.16
			3	16.47	2.63	0.952	119.56	1.84	358.1
			3	16.47	2.63	0.952	119.56	1.84	358.1
			3	16.47	2.63	0.952	119.56	1.84	358.1
			3	16.47	2.63	0.952	119.56	1.84	358.1
			4	12.78	2.34	0.99	52.21	1.31	500.23
7	75	HN	4	30.3	2.55	0.927	104.19	1.33	276.42
			3	28.14	2.57	0.938	100.12	2.03	315.88
			3	28.14	2.57	0.938	100.12	2.03	315.88
			3	28.14	2.57	0.938	100.12	2.03	315.88
			3	27.69	2.47	0.95	75.88	1.89	317.12
			5	23.76	2.31	0.99	48.48	1.27	452.75
		NE	4	30.19	2.59	0.923	104.19	1.35	276.42
			3	27.98	2.57	0.937	99.74	2.08	317.73
			3	27.98	2.57	0.937	99.74	2.08	317.73
			3	27.98	2.57	0.937	99.74	2.08	317.73
			3	27.47	2.46	0.95	72.87	1.91	319.36
			5	23.62	2.28	0.99	44.36	1.27	458.73
		UN	4	29.5	2.67	0.942	131.83	1.25	283.39
			3	27.68	2.64	0.949	123.55	1.86	323.48
			3	27.68	2.64	0.949	123.55	1.86	323.48
			3	27.68	2.64	0.949	123.55	1.86	323.48
			3	27.65	2.64	0.95	122	1.85	323.48
			4	23.53	2.31	0.99	49.06	1.3	440.79

Continuation of Table 4.

c	s	PD	p_0	n	h	L	$P(I)$	arl_0	arl_1	EA
7	100	HN	S.e	4	39.73	2.55	0.935	92.83	1.31	264.51
			0	3	36.64	2.57	0.947	101.17	2.05	303.77
			0.9	3	36.64	2.57	0.947	101.17	2.05	303.77
			0.9	3	36.64	2.57	0.947	101.17	2.05	303.77
			1	3	36.54	2.54	0.95	91.71	1.99	303.85
			1	5	31.66	2.32	0.99	49.39	1.27	401.19
		NE	S.e	3	37.52	2.65	0.93	124.23	1.58	264.62
			0	3	36.44	2.57	0.945	100.82	2.1	305.21
			0.9	3	36.44	2.57	0.945	100.82	2.1	305.21
			0.9	3	36.44	2.57	0.945	100.82	2.1	305.21
			1	3	36.21	2.53	0.95	90	2.04	305.38
			1	5	31.4	2.31	0.99	48.5	1.29	405.86
		UN	S.e	4	38.48	2.67	0.949	131.83	1.25	270.89
			0	3	36.04	2.65	0.956	124.94	1.87	310.52
			0.9	3	36.04	2.65	0.956	124.94	1.87	310.52
			0.9	3	36.04	2.65	0.956	124.94	1.87	310.52
			1	3	36.04	2.65	0.956	124.94	1.87	310.52
			1	4	31.32	2.34	0.99	51.89	1.31	392.55
4	66.8	HN	S.e	3	12.61	2.62	0.922	87.67	1.31	329.5
			0	3	12.25	2.56	0.935	95.86	2	369.21
			0.9	3	12.25	2.56	0.935	95.86	2	369.21
			0.9	3	12.25	2.56	0.935	95.86	2	369.21
			1	3	11.79	2.44	0.95	68.61	1.83	372.53
			1	5	8.98	2.27	0.99	44.13	1.25	596.75
		NE	S.e	4	14.04	2.53	0.917	87.67	1.32	327.34
			0	3	11.93	2.58	0.936	102.17	2.1	372.85
			0.9	3	11.93	2.58	0.936	102.17	2.1	372.85
			0.9	3	11.93	2.58	0.936	102.17	2.1	372.85
			1	4	12.23	2.52	0.95	87.33	1.61	377.19
			1	5	8.87	2.25	0.99	41.79	1.26	607.46
		UN	S.e	4	13.05	2.65	0.943	124.23	1.25	337.89
			0	3	11.93	2.63	0.946	117.48	1.83	380.36
			0.9	3	11.93	2.63	0.946	117.48	1.83	380.36
			0.9	3	11.93	2.63	0.946	117.48	1.83	380.36
			1	3	11.83	2.58	0.95	104.03	1.78	380.59
			1	4	8.82	2.29	0.99	46.33	1.29	576.58
5.9	60.4	HN	S.e	4	20.68	2.49	0.999	78.28	1.29	299.85
			0	3	19.17	2.56	0.932	98.13	2.01	338.88
			0.9	3	19.17	2.56	0.932	98.13	2.01	338.88
			0.9	3	19.17	2.56	0.932	98.13	2.01	338.88
			1	3	18.62	2.42	0.95	65.06	1.81	342.34
			1	5	15.39	2.3	0.99	47.31	1.26	529.79
		NE	S.e	4	20.5	2.56	0.918	124.23	1.37	299.7
			0	3	19.04	2.56	0.931	97.6	2.06	341.44
			0.9	3	19.04	2.56	0.931	97.6	2.06	341.44
			0.9	3	19.04	2.56	0.931	97.6	2.06	341.44
			1	3	18.48	2.39	0.95	60.93	1.82	345.59
			1	5	15.28	2.27	0.99	43.11	1.26	537.73
		UN	S.e	4	20.19	2.65	0.938	124.23	1.25	306.73
			0	3	18.79	2.64	0.944	121.02	1.84	348.06
			0.9	3	18.79	2.64	0.944	121.02	1.84	348.06
			0.9	3	18.79	2.64	0.944	121.02	1.84	348.06
			1	3	18.6	2.59	0.95	104.31	1.78	348.48
			1	4	15.27	2.27	0.99	43.88	1.28	513.12

Continuation of Table 4.

c	s	PD	n	h	L	$P(I)$	arl_0	arl_1	EA
6.5	87.5	HN	3	30.47	2.65	0.932	124.23	1.56	274.91
			3	29.85	2.57	0.944	100.27	2.04	314.31
			3	29.85	2.57	0.944	100.27	2.04	314.31
			3	29.85	2.57	0.944	100.27	2.04	314.31
			3	29.59	2.51	0.95	84.16	1.94	314.69
			5	25.22	2.31	0.99	49.02	1.27	433.38
		NE	3	30	2.66	0.933	127.97	1.59	275.01
			3	29.67	2.57	0.942	99.89	2.09	316.03
			3	29.67	2.57	0.942	99.89	2.09	316.03
			3	29.67	2.57	0.942	99.89	2.09	316.03
			3	29.34	2.5	0.95	80.62	1.97	316.63
			5	24.98	2.31	0.99	48.94	1.29	438.93
		UN	4	31.4	2.67	0.947	131.83	1.25	280.37
			3	29.32	2.64	0.953	123.8	1.86	321.79
			3	29.32	2.64	0.953	123.8	1.86	321.79
			3	29.32	2.64	0.953	123.8	1.86	321.79
			3	29.32	2.64	0.953	123.8	1.86	321.79
			5	25.39	2.39	0.99	60.78	1.22	423.41

with the red line representing Saadatmelli et al. (2018) model. Panel (HN Distribution) shows results for the half-normal distribution, (NE Distribution) for the negative exponential distribution, and (UN Distribution) for the uniform prior distribution. The graph demonstrates a consistent positive relationship between p_0 and $E(A)$ across all distributions.

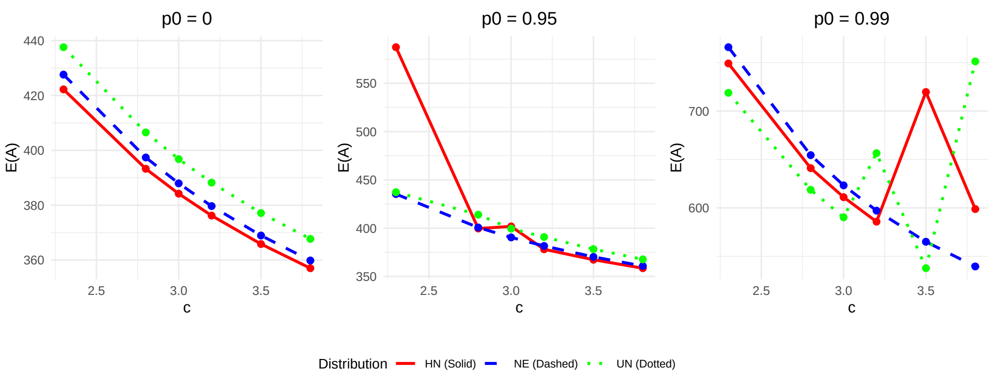


Figure 4: Effect of shape parameter (c) on expected cost ($E(A)$) across different prior distributions.

In Figure 4 The horizontal axis represents values of the shape parameter (c), while the vertical axis displays corresponding expected cost ($E(A)$) values. Trend lines are coded by line style: solid line for half-normal distribution, dashed line for negative exponential distribution, and dotted line for uniform prior distribution. Panel ($p_0 = 0$) presents results for $p_0 = 0$, ($p_0 = 0.95$) for $p_0 = 0.95$, and ($p_0 = 0.99$) for $p_0 = 0.99$. The graph demonstrates an inverse relationship between parameter c and $E(A)$), where increasing values of c correspond to decreasing $E(A)$ values across all distributions.

In Figure 5 The horizontal axis represents values of the shape parameter (c), while the vertical axis displays corresponding expected cost ($E(A)$) values. Trend lines are

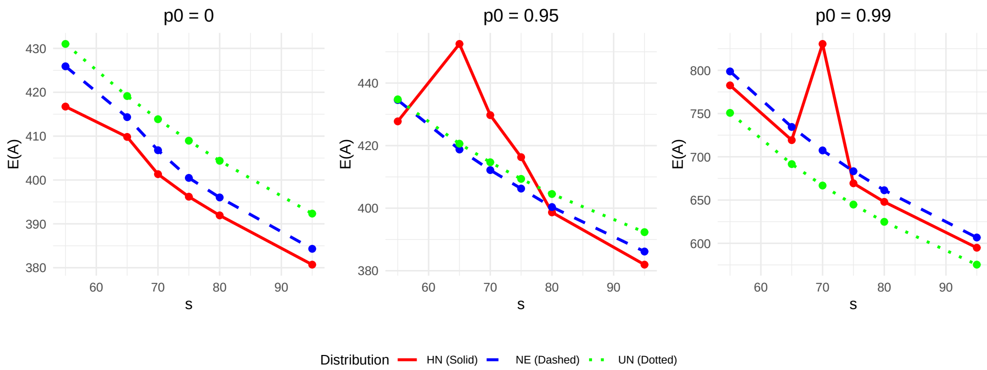


Figure 5: Effect of scale parameter (c) on expected cost ($E(A)$)) across different prior distributions.

coded by line style: solid line for half-normal distribution, dashed line for negative exponential distribution, and dotted line for uniform prior distribution. Panel ($p_0 = 0$) presents results for $p_0 = 0$, ($p_0 = 0.95$) for $p_0 = 0.95$, and ($p_0 = 0.99$) for $p_0 = 0.99$. The graph demonstrates an inverse relationship between parameter c and $E(A)$, where increasing values of s correspond to decreasing $E(A)$ values across all distributions.

Based on the analysis of Tables 2 to 4 and Figure 3 derived from Table 4, it is evident that the Realistic Economic Design (RED) Model consistently yields higher values of $E(A)$ (expected cost per unit time) compared to the model proposed by Saadatmelli et al. (2018) across all three prior distributions (NE, Un, and HN) and varying values of c and s. For example, under the HN distribution with $c = 3$ and $s = 90$, the $E(A)$ value in Saadatmelli et al. (2018) model is 253.45$, whereas the RED model which incorporates the occurrence of event I, results in significantly higher $E(A)$ value of 365.84$. This discrepancy underscores a critical limitation in Saadatmelli et al. (2018) approach: while they assumed the occurrence of event I, they failed to include it in their $E(A)$ calculations, leading to a substantial underestimation of costs.

Furthermore, the analysis reveals that as p_0 (the acceptable lower limit for $P(I)$) increases from 0.85 to 0.99, the $E(A)$ value rises across all three prior distributions (NE, HN, and Un). This trend highlights the importance of accurately accounting for event I in cost calculations, as its exclusion can lead to significant deviations in economic design outcomes. These findings emphasize the necessity of adopting the RED Model for more accurate and reliable economic design of control charts, particularly in scenarios involving multiple assignable causes and varying process conditions.

The analysis of Figures 4 and 5, derived from Tables 2 and 3, reveals important trends in the behavior of $E(A)$ (expected cost per unit time) and the optimal parameters of control charts. As c increases while s remains fixed, and as s increases while c remains fixed, the value of $E(A)$ generally decreases across all three distributions (NE, HN, and Un). Additionally, for constant values of c and s, as p_0 increases from 0.85 to 0.99, the optimal sample size (n) shows an upward trend, while the sampling interval (h) and control limit multiplier (L) exhibit a downward trend.

4 Conclusion

For many years, the research community has dedicated significant effort to designing effective quality control charts. This paper introduces a Realistic Economic Design (RED) Model for Shewhart control charts using the Burr-XII shock model to address scenarios involving multiple independent assignable causes. By extending Lorenzen and Vance's framework, the proposed model provides a more accurate and practical approach to economic design. A numerical example demonstrates the application of the solution method, revealing that Saadatmelli et al. (2018) model significantly underestimate the average cost per unit time of the quality cycle. The numerical results demonstrated that when employing the Saadatmelli et al. (2018) economic model for determining the design parameters of $\bar{X}$ control charts, the estimated average cost per unit time of the quality cycle would be underestimated by 33.84% to 173.58% compared to the actual value. These findings highlight the importance of adopting the RED Model for more reliable and accurate economic design in industrial applications. Future research should explore the application of the RED Model to parametric, non-parametric, and adaptive control charts under the Burr-XII shock model. Furthermore, the incorporation of control variables (covariates) into the regression control chart method presents a promising direction for future studies, which could further enhance the model's flexibility and applicability in various industrial contexts.

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Appendix

4.1 Proof of Theorem 3.1

To outline the proof steps, the following terms are defined

$$
\begin{aligned}
k_0 &= \sum_{i=1}^{m} k_i, \qquad k_{-i} = k_0 - k_i, \\
\mathbf{T}_{-i} &= (T_1, T_2, ..., T_{i-1}, T_{i+1}, ..., T_m), \\
\{\mathbf{T}_{-i} > X\} &= \{T_1 > X, T_2 > X, ..., T_{i-1} > X, T_{i+1} > X, ..., T_m > X\}, \\
I_i &= \{T_i \leq X, \mathbf{T}_{-i} > X\}, \\
I_i \cap I_j &= \{T_i \leq X, \mathbf{T}_{-i} > X\} \cap \{T_j \leq X, \mathbf{T}_{-j} > X\} \\
&= \{T_i \leq X, \mathbf{T}_{-j} > X, T_j \leq X, \mathbf{T}_{-i} > X\} = \phi, \qquad i < j.
\end{aligned}
$$

Therefore, for $i < j$, $I_i \cap I_j$ is null.

Clearly, according to (8), the occurrence of the i-th assignable cause, such that no other assignable causes occur between its occurrence and the issuance of a correct alarm (I_i) is equivalent to the event $\{T_i \le X, T_{-i} > X\}$, therefore, using the law of total probability

$$
\begin{aligned}
P(I_i) &= P(T_i \le X, \mathbf{T}_{-i} > X) = \sum_{v=1}^{\infty} P(T_i \le X, \mathbf{T}_{-i} > X, X = w_v) \\
&= \sum_{v=1}^{\infty} \sum_{l=1}^{v} P(w_{l-1} < T_i \le w_l,\, \mathbf{T}_{-i} > w_v,\, X = w_v) \\
&= \sum_{v=1}^{\infty} \sum_{l=1}^{v} P(X = w_v | w_{l-1} < T_i \le w_l,\, \mathbf{T}_{-i} > w_v) P(w_{l-1} < T_i \le w_l,\, \mathbf{T}_{-i} > w_v).
\end{aligned}
$$

Based on the definition of the variables ($T_i \sim Burr12(s, c, k_i)$ and $(1 + (w_l/s)^c)^{-k_i} = (1 + (h_1/s)^c)^{-vk_i}$) and the underlying assumptions

$$
P(X = w_v | w_{l-1} < T_i \le w_l,\, T_{-i} > w_v) = \beta_i^{v-l}(1 - \beta_i), \tag{9}
$$

$$
\begin{aligned}
P(w_{l-1} < T_i \le w_l,\, \mathbf{T}_{-i} > w_v) &= P(w_{l-1} < T_i \le w_l) P(\mathbf{T}_{-i} > w_v) \\
&= [(1 + (w_{l-1}/s)^c)^{-k_i} - (1 + (w_l/s)^c)^{-k_i}] \\
&\quad \times (1 + (w_v/s)^c)^{-k_{-i}} \\
&= [(1 + (h_1/s)^c)^{-k_i(l-1)} - (1 + (h_1/s)^c)^{-k_i l}] \\
&\quad \times (1 + (h_1/s)^c)^{-k_{-i} v}.
\end{aligned} \tag{10}
$$

From (9) and (10) and Geometric series:

$$
\begin{aligned}
P(I_i) &= \sum_{v=1}^{\infty} \sum_{l=1}^{v} \beta_i^{v-l}(1 - \beta_i)[(1 + (h_1/s)^c)^{-k_i(l-1)} - (1 + (h_1/s)^c)^{-k_i l}](1 + (h_1/s)^c)^{-k_{-i} v} \\
&= \sum_{v=1}^{\infty} (1 - \beta_i)(1 + (h_1/s)^c)^{-k_{-i} v}[(1 + (h_1/s)^c)^{k_i} - 1] \\
&\quad \times \frac{\beta_i^v (1 + (h_1/s)^c)^{-k_i} - (1 + (h_1/s)^c)^{-k_i(v+1)}}{\beta_i - (1 + (h_1/s)^c)^{-k_i}} \\
&= \frac{(1 - \beta_i)[(1 + (h_1/s)^c)^{k_i} - 1](1 + (h_1/s)^c)^{-k_i}}{\beta_i - (1 + (h_1/s)^c)^{-k_i}} \\
&\quad \times \left[\frac{\beta_i (1 + (h_1/s)^c)^{-k_{-i}})}{1 - \beta_i (1 + (h_1/s)^c)^{-k_{-i}})} - \frac{(1 + (h_1/s)^c)^{-k_0}}{1 - (1 + (h_1/s)^c)^{-k_0}} \right] \\
&= \frac{(1 - \beta_i)[1 - (1 + (h_1/s)^c)^{-k_i}](1 + (h_1/s)^c)^{-k_{-i}})}{(1 - \beta_i (1 + (h_1/s)^c)^{-k_{-i}})(1 - (1 + (h_1/s)^c)^{-k_0})}.
\end{aligned}
$$

Clearly $I = \bigcup_{i=1}^{m} I_i$ and since for each $i \ne j$ we have $I_i \cap I_j = \emptyset$, it follows that:$P(I) = \sum_{i=1}^{m} P(I_i)$.

4.2 Proof of Lemma 3.2

$$\begin{aligned}
E(X'|I) &= \sum_{v=1}^{\infty} w_v P(X = w_v|I) = \sum_{v=1}^{\infty} w_v \sum_{i=1}^{m} \frac{P(X = w_v, I_i)}{P(I)} \\
&= \sum_{v=1}^{\infty}\sum_{i=1}^{m} w_v \frac{1}{P(I)} \underbrace{P(X = w_v, T_i \le w_v, \vec{T}_{-i} > w_v)}_{:=D}, \\
D &= P(X = w_v, T_i \le w_v, \vec{T}_{-i} > w_v) \\
&= \sum_{l=1}^{v} P(X = w_v, w_{l-1} < T_i \le w_l, \vec{T}_{-i} > w_v) \\
&= \sum_{l=1}^{v} P(X = w_v | w_{l-1} < T_i \le w_l, \vec{T}_{-i} > w_v) P(w_{l-1} < T_i \le w_l, \vec{T}_{-i} > w_v) \\
&= (1-\beta_i)[1-(1+(\frac{h_1}{s})^c)^{-k_i}] \frac{\left(1+(\frac{h_1}{s})^c\right)^{-vk_0} - \left(\beta_i(1+(\frac{h_1}{s})^c)^{-k_{-i}}\right)^v}{\left(1+(\frac{h_1}{s})^c\right)^{-k_i} - \beta_i}.
\end{aligned}$$

4.3 Proof of Lemma 3.3

$$E(X_1'|I) = \int_0^{\infty} t f_T(t|I)dt = \sum_{v=1}^{\infty} \int_{w_{v-1}}^{w_v} t f_T(t|I)dt,$$

where

$$\begin{aligned}
f_T(t|I)dt &\approx \frac{P(t < T \le t+dt, I)}{P(I)} \\
&= \sum_{l=1}^{\infty} \frac{P(t < T \le t+dt, I, X = w_l)}{P(I)} \\
&= \sum_{l=v}^{\infty} \frac{P(t < T \le t+dt, I, X = w_l)}{P(I)}.
\end{aligned}$$

According to $\{I_i\} \equiv \{T_i < X, \vec{T}_{-i} > X\}$

$$\begin{aligned}
\frac{P(t < T \le t+dt, I)}{P(I)} &= \sum_{l=v}^{\infty}\sum_{i=1}^{m} \frac{P(t < T_i \le t+dt, T_i < w_l, \vec{T}_{-i} > w_l, X = w_l)}{P(I)} \\
&= \sum_{i=1}^{m} \frac{1}{P(I)} \sum_{l=v}^{\infty} P(X = w_l | t < T_i \le t+dt, \vec{T}_{-i} > w_l) \\
&\quad \times P(t < T_i \le t+dt, \vec{T}_{-i} > w_l) \\
&= \sum_{i=1}^{m} \frac{1}{P(I)} (1-\beta_i) P(t < T_i \le t+dt) \\
&\quad \times \sum_{l=v}^{\infty} \beta_i^{(l-v)} \left(1+(\frac{h_1}{s})^c\right)^{-lk_{-i}}.
\end{aligned}$$

While $dt \to 0$

$$\begin{aligned} f_T(t|I) &= \sum_{i=1}^{m} \frac{1}{P(I)}(1-\beta_i) f_{T_i}(t) \frac{\left(1+(\frac{h_1}{s})^c\right)^{-vk_{-i}}}{1-\beta_i\left(1+(\frac{h_1}{s})^c\right)^{-k_{-i}}} \\ &= \sum_{i=1}^{m} \frac{1}{P(I)}(1-\beta_i)\left[k_i(1+(\frac{t}{s})^c)^{-k_i-1}\frac{ct^{c-1}}{s^c}\right] \frac{\left(1+(\frac{h_1}{s})^c\right)^{-vk_{-i}}}{1-\beta_i\left(1+(\frac{h_1}{s})^c\right)^{-k_{-i}}}. \end{aligned}$$

So,

$$E(X_1^{'}|I) = \frac{1}{P(I)}\sum_{i=1}^{m}\sum_{v=1}^{\infty}(1-\beta_i)\frac{\left(1+(\frac{h_1}{s})^c\right)^{-vk_{-i}}}{1-\beta_i\left(1+(\frac{h_1}{s})^c\right)^{-k_{-i}}}\int_{w_{v-1}}^{w_v} k_i(1+(\frac{t}{s})^c)^{-k_i-1}\frac{ct^c}{s^c}dt.$$

The integral $\int_{w_{v-1}}^{w_v} k_i(1+(\frac{t}{s})^c)^{-k_i-1}\frac{ct^c}{s^c}dt$ is finite. Considering that in the distribution of Burr-XII, that $s, k_i, c > 0$ and $0 \le t < \infty$

$$\int_{w_{v-1}}^{w_v} k_i(1+(\frac{t}{s})^c)^{-k_i-1}\frac{ct^c}{s^c}dt \le \sum_{v=1}^{\infty}\int_{w_{v-1}}^{w_v} k_i(1+(\frac{t}{s})^c)^{-k_i-1}\frac{ct^{c-1}}{s^c}dt = E(t) \le \infty.$$

4.4 Proof of Lemma 3.4

$$\begin{aligned} E(X_4|I) &= \sum_{i=1}^{m} Z_{2i} P(X_4 = Z_{2i}|I) = \sum_{i=1}^{m} Z_{2i}\frac{P(X_4 = Z_{2i}, I)}{P(I)} \\ &= \sum_{i=1}^{m}\sum_{j=1}^{\infty} Z_{2i}\frac{P(T_i \le w_j, \vec{T}_{-i} > w_j, X = w_j)}{P(I)} = \sum_{i=1}^{m} Z_{2i}\frac{P(I_i)}{P(I)}. \end{aligned}$$

4.5 Proof of Lemma 3.6

$$\begin{aligned} \sum_{v=1}^{\infty} v\frac{P(N_{TST} = v, I)}{P(I)} &= \sum_{v=1}^{\infty}\sum_{i=1}^{m}\frac{v}{P(I)}P(X = w_v, T_i < w_v, \vec{T}_{-i} > w_v) \\ &= \sum_{v=1}^{\infty}\sum_{i=1}^{m}\frac{v}{P(I)}\sum_{l=1}^{v}P(X = w_v | w_{l-1} < T_i < w_l, \vec{T}_{-i} > w_v) \\ &\quad \times P(w_{l-1} < T_i < w_l, \vec{T}_{-i} > w_v) \\ &= \sum_{i=1}^{m}\frac{(1-\beta_i)[1-\left(1+(\frac{h_1}{s})^c\right)^{-k_i}]}{P(I)(\beta_i - \left(1+(\frac{h_1}{s})^c\right)^{-k_i})} \\ &\quad \times \sum_{v=1}^{\infty} v[\beta_i^v(1+(\frac{h_1}{s})^c)^{-vk_{-i}} - (1+(\frac{h_1}{s})^c)^{-vk_0}] \\ &= \sum_{i=1}^{m}\frac{P(I_i)}{P(I)}\left[\frac{\left(1+(\frac{h_1}{s})^c\right)^{-k_0}}{1-\left(1+(\frac{h_1}{s})^c\right)^{-k_0}} + \frac{1}{1-\beta_i\left(1+(\frac{h_1}{s})^c\right)^{-k_{-i}}}\right]. \end{aligned}$$