

Research Paper

Asymptotic ruin probabilities in a dependent perturbed integrated risk process with application

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Abstract: The present paper investigates two types of perturbed integrated risk models to compute the asymptotic ruin probabilities: (i) the risk model which is perturbed by log-return rate, jump process, and Brownian motion process with dependent structure between the insurance risk and investment risk when the claim sizes are pairwise strong quasi-asymptotically independent. For this model, we assume that the heavy-tailed claim sizes and return jumps are caused by the systematic factors with an arbitrarily dependent structure; (ii) the risk model in which the underlying price process is a geometric Brownian motion, and the jump diffusion process is modeled by a dependent Affine process when the claim sizes are asymptotically independent. For both dependent models, the asymptotic ruin probabilities are obtained using mathematical approaches. Moreover, some numerical studies with Monte Carlo simulation using the Farlie-Gumbel-Morgenstern copula as the joint distribution function of claim sizes and return jumps are provided to verify the performance of asymptotic results. Some of the results show that, under the framework of regular variation with dependence structure, the asymptotic finite-time ruin probability is insensitive to the claim sizes.

Keywords: Affine process; Asymptotic ruin probability; Girsanov's theorem; Heavy-tailed distribution; Perturbed integrated risk model.

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1 Introduction

Consider an insurance company which is allowed to invest its wealth into financial assets. The integrated risk process of the insurer is defined by

$$U(t) = e^{R(t)} \left(x + \int_{0-}^t e^{-R(u-)} dS(u) \right), \quad t \geq 0, \quad (1)$$

where $U(0) = x$ is the initial reserve of insurance company and $R(t)$ is the log-price process of the investment portfolio which is perturbed by a jump-diffusion process $\{R_t, t \geq 0\}$ with the following two equations:

Case 1) The jump diffusion process $\{R_t, t \geq 0\}$ is perturbed by a diffusion perturbation as

$$R_t = rt + \sigma_R W_R(t) + \sum_{i=1}^{N(t)} Y_i =: B(t) + J(t), \quad (2)$$

where r is a real log-return rate, $\sigma_R > 0$ is the volatility, the perturbation $\{W_R(t), t \geq 0\}$ is a Wiener process. The random variable $\{Y_i, i \geq 1\}$ denotes the return jumps caused by the systematic factors, and systematic factors arrive at times $\tau_i, i \geq 0$, which constitute a renewal counting process

$$N(t) = \sup \{i \in N : \tau_i \leq t\}, \quad t \geq 0.$$

Its renewal function is defined as $\lambda(t) = E[N(t)] = \sum_{i=1}^{\infty} P(\tau_i \leq t)$, by convention, $\tau_0 = 0$. In the diffusion perturbation (2), the processes $\{B(t), t \geq 0\}$ and $\{J(t), t \geq 0\}$ are the continuous and pure jump parts, respectively. It is worth noting that $\{B(t), t \geq 0\}$ is a Brownian motion and its Laplace exponent can be written as

$$\phi_B(\gamma) = \log E \left(e^{-\gamma B(1)} \right) = -r\gamma + \frac{1}{2} \sigma_R^2 \gamma^2, \quad \gamma \in (-\infty, \infty).$$

Furthermore, $S(t)$ is the surplus process of the insurance businesses and also follows a jump-diffusion process, which has the following form

$$S(t) = x + ct + \sigma_S W_S(t) - \sum_{i=1}^{N_\omega(t)} X_i, \quad (3)$$

where $c > 0$ denotes the premium rate, $\sigma_S > 0$ is the volatility, and the perturbation $\{W_S(t), t \geq 0\}$ is another Wiener process. The insurance claims $\{X_i, i \geq 1\}$ are caused by the systematic factors. Assume that the insurance claims caused by the systematic factors occur at times $\tau_i + \omega, i \geq 0$, which constitute a delayed renewal counting process

$$N_\omega(t) = \sup \{i \in N : \tau_i + \omega \leq t\}, \quad t \geq 0,$$

where $\omega \geq 0$ is a common independent delay time, the delayed renewal function of $N_\omega(t)$ is defined as $\lambda_\omega(t) = E[N_\omega(t)] = \sum_{i=1}^{\infty} P(\tau_i + \omega \leq t), t > 0$. Note that $\omega = 0$ means no claim delay, in this case, $N_\omega(t) = N(t)$. The finite-time ruin probability at time $t \geq 0$ for risk model (1) is defined as

$$\psi(x, t) = P \left(\inf_{0 \leq s \leq t} U(s) < 0 \mid U(0) = x \right).$$

Case 2) This is the Heston (1993) approach to option pricing, where he extended the Black-Scholes model given in Black and Scholes (1973), for which the underlying price process is a geometric Brownian motion, to allow “stochastic volatility”. In Heston (1993), the underlying jump diffusion process $\{Z_t = e^{R_t}, t \geq 0\}$ is modeled by Affine process as

$$dZ_t = \mu Z_t dt + \sqrt{r_t} Z_t dB_t^{(1)}, \quad t \geq 0, \quad (4)$$

where $\mu \geq 0$ is the risk-free rate, $\{B_t^{(1)}, t \geq 0\}$ is a standard Wiener process, and $\{r_t, t \geq 0\}$ is a stochastic short-rate process satisfying

$$dr_t = \zeta(l - r_t)dt + \delta\sqrt{r_t}dB_t^{(2)}, \quad t \geq 0, \quad (5)$$

with a constant starting point $r_0 > 0$. Here, $l > 0$ is the long-term mean, $\zeta > 0$ is the mean reverting speed, $\delta > 0$ is the volatility of the variance process itself, and $\{B_t^{(2)}, t \geq 0\}$ is another standard Wiener process correlated to the Wiener process $\{B_t^{(1)}, t \geq 0\}$ in (3) by

$$dB_t^{(1)} = \rho_0 B_t^{(2)} + \sqrt{1 - \rho_0^2} dB_t^{(3)}, \quad (6)$$

where $\{B_t^{(3)}, t \geq 0\}$ is a standard Wiener process independent of $\{B_t^{(2)}, t \geq 0\}$ and $\rho_0 \in [-1, 1]$ is the correlation coefficient. The negative correlation, i.e., $\rho_0 < 0$, between $\{B_t^{(1)}, t \geq 0\}$ and $\{B_t^{(2)}, t \geq 0\}$ is known as the leverage effect. In addition, we assume that the parameters in (5) satisfy the following stability condition

$$\frac{2\zeta l}{\delta^2} > 1. \quad (7)$$

This condition ensures that the variance process $\{r_t, t \geq 0\}$ remains positive, starting from positive variance r_0 , (Brigo and Mercurio, 2001). Note that the process $\{r_t, t \geq 0\}$ given in (5) is a Cox-Ingersoll-Ross process, which is investigated more carefully by Cheng and Wang (2023) to obtain the ruin probabilities.

Therefore, the risk process at time $t \geq 0$ with stochastic interest rate R_t and initial reserve $x \geq 0$ is given by

$$U_t = e^{R_t} \left(x + \int_0^t e^{-R_\theta} dR_\theta \right). \quad (8)$$

Suppose that T denotes the time to ruin of the risk process (8), i.e. $T = \inf \{t \geq 0 : U_t < 0\}$ or $T = \infty$ if $U_t \geq 0$ for all $t \geq 0$.

As similar, for the risk process (2) with jump diffusion process modeled by (4)-(7), the finite and infinite time ruin probabilities with the initial reserve $x \geq 0$ are defined by equalities:

$$\begin{aligned} \psi_A(x, t) &= P(T \leq t | U_0 = x), \quad x, t \geq 0, \\ \psi_A(x) &= P(T < \infty | U_0 = x) = P(U_t < 0 \text{ for some } t \geq 0 | U_0 = x), \end{aligned}$$

respectively. In the past, scholars have typically considered insurance risk and financial risk to be independent or have focused solely on insurance risk. For instance, Yang et

al. (2011) investigated a continuous renewal risk without interest rate, they derived the finite-time ruin probability when claim sizes are independent and inter-arrival times are negatively dependent. Wang et al. (2013) derived the asymptotic results for a risk model with a constant interest rate, where both claim sizes and inter-arrival times follow a certain dependence structure. Hao and Tang (2012) investigated a general bivariate Lévy-driven risk model, using two independent Lévy processes to describe the insurance surplus process and the log-price process of the investment. They obtained a simple and unified asymptotic formula as the initial surplus level tends to infinity. Li (2012) considered a time-dependent renewal risk model with stochastic return and obtained asymptotic formulas when claim sizes are extended regularly varying. Wang et al. (2018) obtained the finite-time ruin probability for a renewal risk model with stochastic return and Brownian perturbation when the claim sizes have a subexponential distribution. Bazyari and Roozgar (2019) computed the finite time ruin probability and structural density properties in the classical compound Poisson risk model with dependence between claim sizes and claim inter-arrival time. As the application, they studied some dependent models of the interclaim times and claim sizes. For more related papers, references can be made to Tang et al. (2010), Guo and Wang (2013), Yang et al. (2014), Li (2016) and Yang and Li (2019).

However, with the diversification of the insurance and financial industries and the increasing frequency of catastrophic events, systemic factors are seen as significant contributors to the downfall of insurance companies. Therefore, the insurance risk model should account for these systemic factors, which could lead to correlated insurance losses and fluctuations in investment returns. Natural disasters such as earthquakes, floods, or hurricanes not only cause significant payouts but also trigger financial market impacts, causing sharp fluctuations in stock, bond or real estate prices. After the 2011 Fukushima nuclear disaster in Japan, insurers faced not only massive claims but also saw related energy company stocks plummet, leading to investment losses for those holding such assets. In the event of a large-scale cyberattack, insurers may need to pay out substantial claims under cyber insurance policies. Such incidents typically have widespread impacts on financial markets, causing the stock prices of affected companies to drop. For example, as many companies increased their exposure to cyber risks during the pandemic, some insurers might have faced higher claims, while the stock market performance of related companies could also suffer, affecting the insurer's investment portfolios.

The COVID-19 pandemic drastically increased health insurance claims, straining insurers' liquidity. Simultaneously, it triggered global economic uncertainty, stock market volatility, and business disruptions. These interconnected crises exposed insurers to dual risks: rising claim payouts and investment losses.

All of the above examples prompt us to consider the connection between insurance risk and financial risk.

Recently, Guo (2022) derived an uniform asymptotic estimate for ruin probabilities in a Poisson risk model with dependent insurance risk and financial risk when the claim sizes are regularly varying. Yang et al. (2023) obtained the asymptotic finite-time ruin probability in a renewal counting process when the insurance claims and the associated random variables have different regularly varying distributions. Bazyari (2024) investigated a generalized compound renewal risk process based on the claim

amounts with dependence structures under a uniformly bounded copula function. Bazyari (2025) obtained the asymptotic exponential estimations for finite and infinite time ruin probabilities in a perturbed continuous compound Poisson risk model when the claim amounts have an arbitrary dependent structure.

Motivated by the above papers, computing the asymptotic estimation of ruin probabilities in two types of dependent perturbed continuous compound Poisson risk model is embedded in this paper. We assume that the insurance company faces both insured losses and financial failure, the insurance claims $\{X_i, i \geq 1\}$ have a common distribution F and are pairwise strong quasi-asymptotically independent (pSQAI), due to their exposure to certain common macroeconomic factors. The return jump sizes $\{Y_i, i \geq 1\}$ are nonnegative, independent and identically distributed (i.i.d.) random variables. For every $i \geq 1$, there may exist arbitrary dependence between X_i and Y_i , and between the perturbation $\{W_R(t), t \geq 0\}$ and the perturbation $\{W_S(t), t \geq 0\}$. Let G be the common distribution of

$$\tilde{X}_i = X_i e^{-Y_i}. \quad (9)$$

For simplicity, we assume that all the random sources ω , $\{N(t), t \geq 0\}$, $\{(X_i, Y_i), i \geq 1\}$ and $\{(W_R(t), W_S(t)), t \geq 0\}$ are mutually independent.

Furthermore, we also consider the risk model which the underlying price process is a geometric Brownian motion and jump diffusion process is modeled by a dependent Affine process when the claim sizes are asymptotically independent.

The rest of this paper is organized as follows: Section 2 introduces some necessary preliminaries on heavy tailed distributions. Section 3 gives the main results on the asymptotic estimation of ruin probabilities. Section 4 provides the necessary lemmas. The proof of these lemmas are given in Appendix. Section 5 presents some numerical simulations using the Farlie-Gumbel-Morgenstern (FGM) copula to verify the accuracy of our results. Moreover, the ruin probabilities are computed with the Dow Jones industrial index data. Concluding remarks are given in Section 6. The proof of theorems and Lemmas are given in Section 7.

2 Some preliminaries

Hereafter, all limit relationships hold as $x \rightarrow \infty$ unless otherwise stated. For positive functions f and g , we write $f \lesssim g$ or $f \gtrsim g$ if $\limsup f/g \leq 1$; $f \sim g$ if $\lim f/g = 1$. Furthermore, we write $f = o(g)$ if $\lim f/g = 0$; $f = O(g)$ if $\limsup f/g < \infty$; we write $f \asymp g$ if $0 < \liminf f/g < \limsup f/g < \infty$. For any real numbers a and b , we denote $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$. In addition, $I_{\{A\}}$ is the indicator function of the set A . K is a constant with different values in different places in this paper.

In this paper, we will restrict the risk distribution to some classes of heavy-tailed distributions, therefore, we recall some definitions on heavy-tailed distributions. A random variable X with a proper distribution F and $\bar{F}(x) = 1 - F(x) > 0$, for all $x \in (-\infty, \infty)$, is said to be heavy-tailed, if $Ee^{uX} = \infty$ for all $u > 0$.

For more details on the following heavy-tailed distribution classes, one can see Embrechts et al. (1997) and Foss et al. (2013).

One of the most important classes of heavy-tailed distributions is the class of dominated varying tail distributions, denoted by \mathcal{D} . A distribution F on $(-\infty, \infty)$ belongs

to \mathcal{D} , if

$$\limsup_{x \rightarrow \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty,$$

holds for any $0 < y < 1$.

Associated with \mathcal{D} is the long-tailed distribution class \mathcal{L} . A distribution F on $(-\infty, \infty)$ belongs to \mathcal{L} , if

$$\lim_{x \rightarrow \infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = 1,$$

holds for any $y \in (-\infty, \infty)$.

A distribution F on $(-\infty, \infty)$ belongs to the regularly varying distribution class $\mathcal{R}_{-\alpha}$, $\alpha \geq 0$ if

$$\lim_{x \rightarrow \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha},$$

holds for any fixed $y > 0$. One can refer to Resnick (1987) for more details about the regularly varying distribution class.

The above subclasses of heavy-tailed distributions, it is well-known that the relationship

$$\mathcal{R}_{-\alpha} \subset \mathcal{L} \cap \mathcal{D} \subset \mathcal{L},$$

holds. The lower and upper Matuszewska indices of F are defined as

$$\begin{aligned} J_F^- &= - \lim_{y \rightarrow \infty} \frac{\log \overline{F}^*(y)}{\log y}, \\ J_F^+ &= - \lim_{y \rightarrow \infty} \frac{\log \overline{F}_*(y)}{\log y}, \end{aligned}$$

respectively, where $\overline{F}_*(y) = \liminf_{x \rightarrow \infty} \overline{F}(xy)/\overline{F}(x)$ and $\overline{F}^*(y) = \limsup_{x \rightarrow \infty} \overline{F}(xy)/\overline{F}(x)$ for $y > 0$.

By Proposition 2.2.1 of Bingham et al. (1987), if $F \in \mathcal{D}$, for any $p_1 < J_F^- \leq J_F^+ < p_2$ and $C > 1$, there exists some $D > 0$ such that

$$\frac{1}{C} (y^{-p_1} \wedge y^{-p_2}) \leq \frac{\overline{F}(xy)}{\overline{F}(x)} \leq C (y^{-p_1} \vee y^{-p_2}), \tag{10}$$

holds, whenever $xy > D$ and $x > D$. From inequality (10), we get that

$$x^{-p_2} = o(\overline{F}(x)). \tag{11}$$

Furthermore, if $F \in \mathcal{R}_{-\alpha}$, $\alpha \geq 0$, then Theorem 1.5.6 of Bingham et al. (1987) tells us that, there exists some x_0 such that, for any fixed $b > 1$ and $\delta > 0$, for all $x, y > x_0$,

$$\frac{1}{b} (y^{-\alpha+\delta} \wedge y^{-\alpha-\delta}) \leq \frac{\overline{F}(xy)}{\overline{F}(x)} \leq b (y^{-\alpha+\delta} \vee y^{-\alpha-\delta}), \tag{12}$$

which implies that for any $p > \alpha$,

$$x^{-p} = o(\overline{F}(x)). \tag{13}$$

If for any $i \neq j$, the relation

$$\lim_{x_i \wedge x_j \rightarrow \infty} P(|X_i| > x_i | X_j > x_j) = 0,$$

holds, then we say that the random variables X_1, \dots, X_n are pSQAI.

Indeed, pSQAI encompasses many dependence structures such as mutually independent, pairwise FGM distributed and pairwise negatively dependent.

Furthermore, for the identically distributed random variables $\{X_i, i \geq 1\}$, and for any $i \neq j$ and $j \geq 1$, if

$$\lim_{x \rightarrow \infty} \frac{P(X_i > x, X_j > x)}{P(X_i > x)} = 0, \quad (14)$$

then we say $\{X_i, i \geq 1\}$ is asymptotically independent.

If two random variables X_1 and X_2 are pSQAI, then they are also asymptotic independent. For more details on pSQAI, one can refer to Geluk and Tang (2009), Li (2013), and Chen and Liu (2022). For convenience later, we denote

$$\begin{aligned} \Theta_i(t) &= e^{-B(\tau_i + \omega) - J((\tau_i + \omega)^-)} I_{(\tau_i + \omega \leq t)}, \\ \tilde{\Theta}_i(t) &= e^{-B(\tau_i + \omega) + Y_i - J((\tau_i + \omega)^-)} I_{(\tau_i + \omega \leq t)}. \end{aligned}$$

3 Main results

In this section, we give the following main results of this paper which are associated with the asymptotic ruin probabilities. The proof of these results are given in Appendix.

Theorem 3.1. *Consider the integrated risk model (1). Assume that the insurance claims $\{X_i, i \geq 1\}$ are pSQAI as given in (13) and the associated random variables $\{\tilde{X}_i, i \geq 1\}$ are defined in (9), which respectively have the common tail distributions satisfying $F \in \mathcal{L} \cap \mathcal{D}$ and $G \in \mathcal{L} \cap \mathcal{D}$. If $\bar{F} \asymp \bar{G}$, then for any fixed $t > 0$, the asymptotic finite time ruin probability is given by*

$$\psi(x, t) \sim \sum_{i=1}^{\infty} P(X_i \Theta_i(t) > x).$$

Remark 3.2. *From the insurance claim sizes $\{X_i, i \geq 1\}$ which are pSQAI and $\{Y_i, i \geq 1\}$ which are nonnegative random variables, $\bar{F} \asymp \bar{G}$, we derive that the associated $\{\tilde{X}_i, i \geq 1\}$ are pSQAI. On the other hand, since $\{Y_i, i \geq 1\}$ are nonnegative random variables, we get the inequality*

$$P(X_i e^{-Y_i} > x_i, X_j e^{-Y_j} > x_j) \leq P(X_i > x_i, X_j > x_j).$$

As $x_i \wedge x_j \rightarrow \infty$, combining the claim sizes $\{X_i, i \geq 1\}$ which are pSQAI and $\bar{F} \asymp \bar{G}$, we get that

$$\frac{P(X_i e^{-Y_i} > x_i, X_j e^{-Y_j} > x_j)}{P(X_j e^{-Y_j} > x_j)} \lesssim \frac{P(X_i > x_i, X_j > x_j)}{P(X_j > x_j)} \rightarrow 0,$$

which implies that $\{\tilde{X}_i, i \geq 1\}$ are also $pSQAI$. In addition, the fact that $\{Y_i, i \geq 1\}$ are nonnegative random variables, also implies that $E[e^{-pY_1}] \leq 1$ holds directly, for some $p > J_F^+$, which plays an important role in our proof. In a special case, if the insurance claims $\{X_i, i \geq 1\}$ are $pSQAI$, X and Y are independent, by Theorem 2.2 in Li (2013) we directly conclude that the associated random variables $\{\tilde{X}_i, i \geq 1\}$ are also $pSQAI$.

Corollary 3.3. *Under the conditions of Theorem 3.1, if $F \in \mathcal{R}_{-\alpha}$ and $G \in \mathcal{R}_{-\alpha}$, $\alpha \geq 0$, then the asymptotic finite time ruin probability is given by*

$$\psi(x, t) \sim \sum_{i=1}^{\infty} E \left[(\Theta_i(t) I_{\{w=0\}})^\alpha \right] \bar{F}(x) + \sum_{i=1}^{\infty} E \left[(\tilde{\Theta}_i(t) I_{\{w>0\}})^\alpha \right] \bar{G}(x).$$

Remark 3.4. *The result (3.3) can be further calculated by Lemma A.1 of Yang et al. (2023) and the final form is consistent with the result in Yang et al. (2023), though $t > 0$ is any fixed here. This also implies that asymptotics of the finite-time ruin probability are insensitive to some dependence structure between the claim sizes in the risk model (1).*

In the following Theorem, the finite time ruin probability is obtained for the risk process (8) where the jump diffusion process $\{Z_t = e^{Rt}, t \geq 0\}$ is modeled by Affine process. Note that, to compute the ruin probability we consider $\rho_0 \in [-1, 0]$, to reach the consistence with the assumption of relation (7) in Cheng and Wang (2023).

Theorem 3.5. *Consider the risk process (8) where the jump diffusion process $\{Z_t = e^{Rt}, t \geq 0\}$ is modeled by (4)-(7) with $\rho_0 \in [-1, 0]$. Suppose that the identically distributed random variables $\{X_i, i \geq 1\}$ are asymptotically independent as given in (14) with the distribution function $F(x) \in \mathcal{R}_{-\alpha}$ for some $\alpha \geq 0$, and the constant $\alpha^0 = \max\{\alpha, 2\}$ satisfies the inequality*

$$\zeta + \alpha^0 \delta \rho_0 > \delta \sqrt{\alpha^0(\alpha^0 + 1)},$$

then for any fixed $t \geq 0$, the exponential estimate for finite time ruin probability is given by

$$\psi_A(x, t) = \lambda \bar{G}(x) \int_0^t D_1(\alpha, s) \exp \{ -\alpha \mu s + r_0 D_2(\alpha, s) \} ds, \quad \text{as } x \rightarrow \infty,$$

where

$$D_1(\alpha, s) = \left(\frac{\exp \left(\frac{s(\zeta + \alpha \delta \rho_0)}{2} \right)}{\cosh \left(\frac{s\Omega(\alpha)}{2} \right) + (\zeta + \alpha \delta \rho_0) \frac{\sinh \left(\frac{s\Omega(\alpha)}{2} \right)}{\Omega(\alpha)}} \right)^{\frac{2\zeta t}{\delta^2}},$$

$$D_2(\alpha, s) = \frac{\alpha^2 + \alpha}{(\zeta + \alpha \delta \rho_0) + \Omega(\alpha) \coth \left(\frac{s\Omega(\alpha)}{2} \right)},$$

and $\Omega(\alpha) = \sqrt{(\zeta + \alpha \delta \rho_0)^2 - \delta^2(\alpha^2 + \alpha)}$. Moreover, if the constant $\alpha^0 = \max\{\alpha, 2\}$ satisfies the inequality

$$\alpha^0 \delta^2 \mu > \zeta l (\zeta + \alpha^0 \delta \rho_0) - \sqrt{(\zeta + \alpha^0 \delta \rho_0)^2 - \delta^2 \alpha^0 (\alpha^0 + 1)},$$

then the exponential estimate for infinite time ruin probability is given by

$$\psi_A(x) = \lambda \bar{G}(x) \int_0^\infty D_1(\alpha, s) \exp \{ -\alpha \mu s + r_0 D_2(\alpha, s) \} ds, \quad \text{as } x \rightarrow \infty.$$

4 Some lemmas

Before proving Theorems 3.1 and Corollary 3.1, we present some useful lemmas. The proof of these lemmas are given in Appendix. The Lemma 4.1 is from Lemma 4.1.2 given in Wang and Tang (2006).

Lemma 4.1. *Let X be a random variable with distribution function F , and θ be a positive random variable independent of X . Denote the distribution function of θX by H , then the following statements hold:*

(1) *If $F \in \mathcal{L} \cap \mathcal{D}$ and $E(\theta^\beta) < \infty$ for some $\beta > J_F^+(F)$, then $\bar{H}(x) \asymp \bar{F}(x)$ and $H \in \mathcal{L} \cap \mathcal{D}$.*

(2) *If $F \in \mathcal{R}_{-\alpha}$ and $E(\theta^\beta) < \infty$ for some $\beta > \alpha \geq 0$, then $\bar{H}(x) \sim E(\theta^\alpha)\bar{F}(x)$.*

Lemma 4.2. (1) *Under the conditions of Theorem 3.1, for any fixed $t > 0$ and any fixed $n \geq 1$, it holds that*

$$P\left(\sum_{i=1}^n X_i \Theta_i(t) > x\right) \sim \sum_{i=1}^n P(X_i \Theta_i(t) > x).$$

(2) *Under the conditions of Corollary 3.1, for any fixed $t > 0$ and any fixed $n \geq 1$, it holds that*

$$P\left(\sum_{i=1}^n X_i \Theta_i(t) > x\right) \sim \sum_{i=1}^n E[(\Theta_i(t) I_{\{w=0\}})^\alpha] \bar{F}(x) + \sum_{i=1}^n E\left[\left(\tilde{\Theta}_i(t) I_{\{w>0\}}\right)^\alpha\right] \bar{G}(x).$$

Lemma 4.3. (1) *Under the conditions in Theorem 3.1, for any fixed $\varepsilon > 0$ and $t > 0$, it holds that for all large n ,*

$$P\left(\sum_{i=n+1}^{\infty} X_i \Theta_i(t) > x\right) \leq \varepsilon \sum_{i=1}^n P(X_i \Theta_i(t) > x).$$

(2) *Under the conditions in Corollary 3.1, for any fixed $\varepsilon > 0$, $t > 0$, it holds that for all large n ,*

$$P\left(\sum_{i=n+1}^{\infty} X_i \Theta_i(t) > x\right) \leq \varepsilon \sum_{i=1}^n E[(\Theta_i(t) I_{\{w=0\}})^\alpha] \bar{F}(x) + \varepsilon \sum_{i=1}^n E\left[\left(\tilde{\Theta}_i(t) I_{\{w>0\}}\right)^\alpha\right] \bar{G}(x).$$

5 Numerical studies

In this section, we will verify the accuracy of the asymptotic estimates in Theorem 3.1 for Pareto and Weibull distributions using the Monte Carlo (MC) method. In the last example, the finite and infinite time ruin probabilities are computed using the jump diffusion process defined in (4) and (5) with the Dow Jones industrial index data.

Example 5.1. *We assume that the claim size X follows the Pareto distribution*

$$\bar{F}(x) = \left(\frac{\kappa_1}{x + \kappa_1}\right)^{\alpha_1}, \quad (15)$$

and the geometric return jump e^{-Y} also follows the Pareto distribution

$$\bar{H}(x) = \left(\frac{\kappa_2}{x + \kappa_2} \right)^{\alpha_2}, \tag{16}$$

where α_1 and α_2 are shape parameters. We mainly consider two cases, Case 1: X and e^{-Y} are independent. Case 2 : The pair (X, e^{-Y}) follows a FGM copula with the joint distribution function given by

$$\pi(x, y) = F_1(x)F_2(y) (1 + \rho\bar{F}_1(x)\bar{F}_2(y)), \quad \rho \in [-1, 1], \tag{17}$$

and the corresponding copula has the following form

$$C(u, v) = uv(1 + \rho(1 - u)(1 - v)), \quad u, v \in [0, 1], \tag{18}$$

where ρ is the dependence coefficient. For more details on copula, one can see Nelsen (2006). For convenience, we assume that $\{N(t), t \geq 0\}$ is a Poisson counting process with $E[N(t)] = \lambda t$ and $\omega = 0$. The algorithm can be described as follows:

Step 1) Simulate a random number N from a Poisson distribution with parameter λt .
 Step 2) Choose a large integer $M > 0$. Generate N random numbers from the uniform distribution on $(0, 1)$ and denote them by u_1, \dots, u_N . Set $u' = it/M$, for $i = 0, \dots, M$. Combine the above number and sort them in ascending order, writing the result as t_0, t_1, \dots, t_{N+M} , writing $s_i = t_i - t_{i-1}, i = 1, \dots, N + M$. Let τ_1, \dots, τ_N be the new locations of u_1, \dots, u_N .

Step 3) When in the independent case, we generate N numbers of X and e^{-Y} from the Pareto distributions given by (15) and (16). When in the case where (X, e^{-Y}) has a FGM copula, we generate N random pairs (u, v) by (17). From these pairs (u, v) , we generate N numbers of X by applying the inverse function $F^{-1}(u)$ and generate N numbers of e^{-Y} by applying the inverse function $H^{-1}(u)$.

Step 4) Generate $2(N+M)$ numbers of independent standard normal distributed random variables and denote them by $W_1^{(1)}, \dots, W_{N+M}^{(1)}$ and $W_1^{(2)}, \dots, W_{N+M}^{(2)}$.

Step 5) For each $n = 0, \dots, N + M$, the discounted integrated risk process is given by

$$\begin{aligned} V(t_n) &= e^{-R(t_n)}U(t_n) \\ &= x + e^{-R(t_{i-1})} \sum_{i=1}^n \left(cs_i + \sigma_S \sqrt{s_i} W_i^{(1)} \right) - \sum_{i=1}^N X_i e^{-R(\tau_{i-1})} I_{(\tau_i \leq t_n)}, \end{aligned}$$

where

$$R(t_i) = rt_i + \sigma_R \sum_{j=1}^i \sqrt{s_j} \left(hW_j^{(1)} + \sqrt{1 - h^2} W_j^{(2)} \right) - \sum_{j=1}^N \log e^{-Y_j} I_{(\tau_j \leq t_i)}.$$

Step 6) If there exists some $n \in \{0, \dots, N + M\}$ such that $V(t_n) < 0$, we write $\pi = 1$. Otherwise, we write $\pi = 0$.

Step 7) Repeat steps 1–6 N_1 times and derive results π_1, \dots, π_{N_1} . The ruin probability $\psi(x, t)$ in (3.1) can be calculated by

$$\psi_1 = \frac{\sum_{k=1}^{N_1} \pi_k}{N_1}.$$

For the independent case, the specific parameters are as follows: $\lambda = 5$, $c = 150$, $t = 1$, $r = 0.08$, $h = 0.3$, $\sigma_S = 0.2$, $\sigma_R = 0.3$, $\alpha_1 = 3.5$, $\kappa_1 = 3$, $\alpha_2 = 1.36$, $\kappa_2 = 2.5$, $M = 100$, $N_1 = 50000$. The initial wealth x changes from 500 to 2000 in steps of 15. We reduced the scale on the axis but not the actual values in our figures. The right-hand side of (3.1) can be calculated by $\psi_2 = 2n\bar{F}(x)$ using the third step in (4.3) and Lemma 4.1, and we set $n = 100$. Figure 1 shows the accuracy of the asymptotic estimate for Pareto distribution with $t = 1$, $\alpha_1 = 3.5$, $\kappa_1 = 3$, $\alpha_2 = 1.36$, $\kappa_2 = 2.5$ in the independent case. From Figure 1, we can observe that as the initial capital x increases, the values of ψ_1 and ψ_2 get closer in a certain range. This suggests that the simulations closely match the asymptotic function, thereby validating the accuracy of the asymptotic estimation in a specific context.

For the dependent case, the specific parameters are as follows: $\lambda = 5$, $c = 150$, $t = 1$, $r = 0.08$, $h = 0.3$, $\sigma_S = 0.2$, $\sigma_R = 0.3$, $\alpha_1 = 3.5$, $\kappa_1 = 1.2$, $\alpha_2 = 1.36$, $\kappa_2 = 8$, $M = 100$, $N_1 = 50000$. The initial wealth x changes from 700 to 1100 in steps of 4. Since $\bar{F}(x) \asymp \bar{G}(x)$, the right-hand side of (3.1) can be calculated by $\psi_2 = 2n\bar{F}(x)$, following the third step in (4.3) and Lemma 4.1. We set $n = 100$. When the dependence coefficient $\rho = 0.1, 0.8$, we obtain Figures 1-3, respectively. Figure 3 shows that with dependence structure, the asymptotic finite-time ruin probability is insensitive among the claim sizes. Moreover with increasing the dependent coefficient, the finite-time ruin probability, ψ_1 , is increasing.

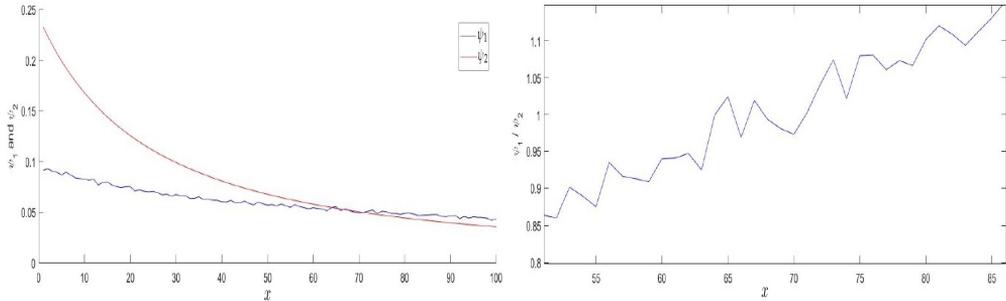


Figure 1: Accuracy of the asymptotic estimate for Pareto distribution with $t = 1$, $\alpha_1 = 3.5$, $\kappa_1 = 3$, $\alpha_2 = 1.36$, $\kappa_2 = 2.5$ in the independent case (left) and local ratio of ψ_1/ψ_2 for $N_1 = 50000$ (right).

Figure 4 shows the accuracy of the asymptotic estimate for Pareto distribution with $t = 10$, $\alpha_1 = 3.5$, $\kappa_1 = 3$, $\alpha_2 = 1.36$, $\kappa_2 = 2.5$ in the independent case. When the dependence coefficient $\rho = 0.1, 0.8$, we obtain Figure 4. The inference on these figures are similar to the Figures 1 and 3.

Example 5.2. Assume that the claim amounts $\{X_i, i = 1, 2, \dots\}$ are distributed as Weibull distribution

$$\bar{F}(x; \beta_1) = 1 - \exp(-x^{\beta_1}), \quad x \geq 0, \quad (19)$$

and the geometric return jump e^{-Y} also follows the Weibull distribution

$$\bar{H}(x, \beta_2) = 1 - \exp(-x^{\beta_2}), \quad x \geq 0, \quad (20)$$

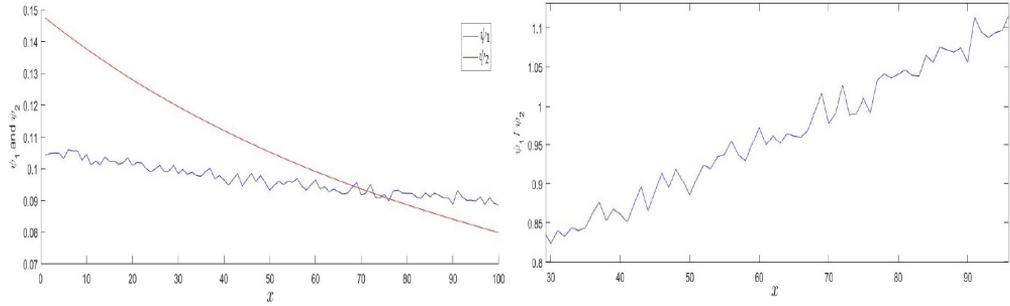


Figure 2: Accuracy of the asymptotic estimate for Pareto distribution with $t = 1$, $\alpha_1 = 3.5$, $\kappa_1 = 1.2$, $\alpha_2 = 1.36$, $\kappa_2 = 8$, $\rho = 0.1$ in the dependent case (left) and local ratio of ψ_1/ψ_2 for $N_1 = 50000$ (right).

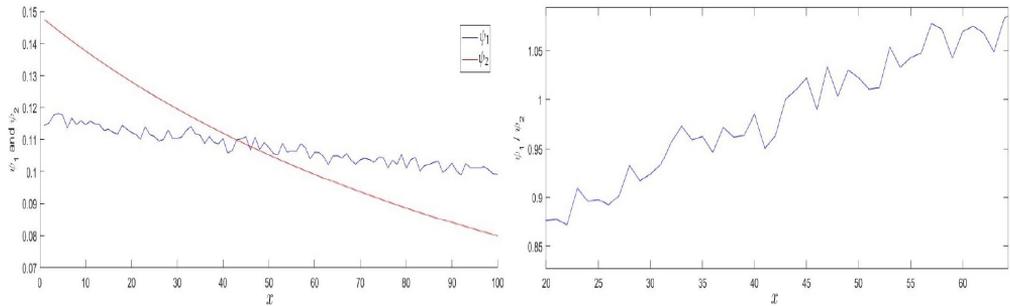


Figure 3: Accuracy of the asymptotic estimate for Pareto distribution with $t = 1$, $\alpha_1 = 3.5$, $\kappa_1 = 1.2$, $\alpha_2 = 1.36$, $\kappa_2 = 8$, $\rho = 0.8$ in the dependent case (left) and local ratio of ψ_1/ψ_2 for $N_1 = 50000$ (right).

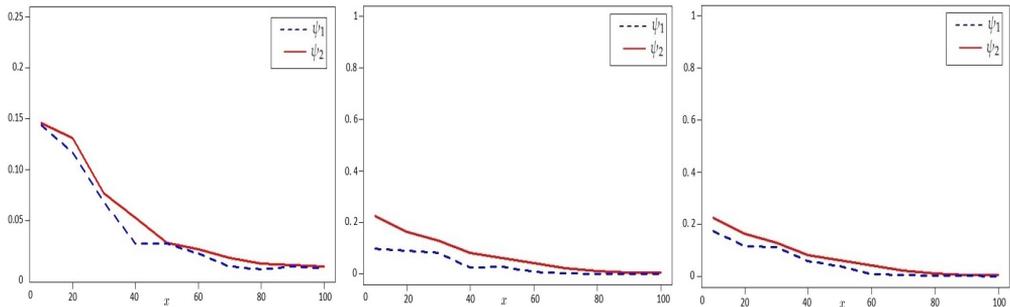


Figure 4: Accuracy of the asymptotic estimate for Pareto distribution with $t = 10$, $\alpha_1 = 3.5$, $\kappa_1 = 3$, $\alpha_2 = 1.36$, $\kappa_2 = 2.5$ (left) in the independent case, and with $t = 10$, $\alpha_1 = 3.5$, $\kappa_1 = 1.2$, $\alpha_2 = 1.36$, $\kappa_2 = 8$, $\rho = 0.1$ (middle) and $\rho = 0.8$ (right) in the dependent case.

where β_1 and β_2 are positive parameters. As similar to Example 5.1, we consider two cases, Case 1: X and e^{-Y} are independent. Case 2 : The pair (X, e^{-Y}) follows the FGM copula with the joint distribution function and corresponding copula as given in (17) and (18), respectively. Furthermore, we follow the algorithm as described

in Example 5.1. Figure 4 shows the accuracy of the asymptotic estimate for Pareto distribution with $t = 1$, $\beta_1 = 0.5$ and $\beta_2 = 1.5$ in the independent case. Figure 5 shows that with dependence structure, the asymptotic finite-time ruin probability is insensitive among the claim sizes. Moreover with increasing the dependent coefficient, the finite-time ruin probability, ψ_1 , is increasing.

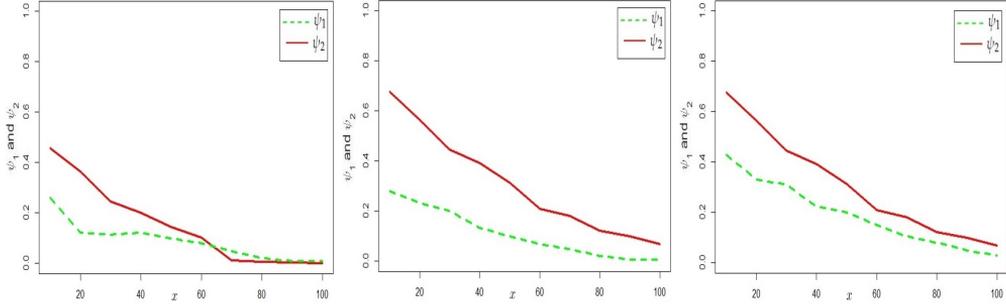


Figure 5: Accuracy of the asymptotic estimate for Weibull distribution with $t = 1$, $\beta_1 = 0.5$, $\beta_2 = 1.5$ (left) in the independent case, and with $t = 1$, $\beta_1 = 0.5$, $\beta_2 = 1.5$, $\rho = 0.1$ (middle), and with $t = 1$, $\beta_1 = 0.5$, $\beta_2 = 1.5$, $\rho = 0.8$ (right) in the dependent case.

Figure 5 shows the accuracy of the asymptotic estimate for Weibull distribution with $t = 1$, $\beta_1 = 0.5$ and $\beta_2 = 1.5$ in the independent case. Figure 5 shows that with dependence structure, the asymptotic finite-time ruin probability is insensitive among the claim sizes. As similar to Figure 4, with increasing the dependent coefficient, the finite-time ruin probability, ψ_1 , is increasing.

Example 5.3. (Ruin probabilities with the Dow Jones industrial index data). Dragulescu and Yakovenko (2002) studied the jump diffusion process $\{Z_t = e^{Rt}, t \geq 0\}$ defined in (4) and (5). They integrated the joint probability density function over variance and obtained the marginal probability distribution function of returns unconditional on variance and directly compared it with the financial data. The Dow Jones data collected for the 20-years period of 1982-2001. They determined the four (non-zero) fitting parameters of the model $\zeta = 11.35$, $\delta = 0.618$, $\mu = 0.143$ and $\xi = 0.022$, when $\rho_0 = 0$, and the result agrees very well with the Dow-Jones data. For these fitted parameters, the conditions in theorem 5.1 can be held over a certain range α when $\rho_0 = 0$. With simple computations, we have $\frac{2\zeta\xi}{\zeta^2} = 1.31$. If $\alpha^0 = \max\{\alpha, 2\} < 17.87$, then from (3.4) we have $\zeta > \delta\sqrt{\alpha^0(\alpha^0 + 1)}$, and if $\alpha^0 < 10.66$ then from (3.8) we have $\alpha^0\delta^2\mu > \zeta\xi(\zeta - \sqrt{\zeta^2 - \delta^2\alpha^0(\alpha^0 + 1)})$. Thus for $0 < \alpha < 10.66$, the conditions in Theorem 5.1 can be held. To compute the ruin probabilities, we consider two different distribution functions. For the first distribution, we suppose that the claim amounts $\{X_i, i = 1, 2, \dots\}$ form a sequence of i.i.d random variables with Pareto distribution

$$F_X(x) = 1 - \left(\frac{2}{x+2}\right)^4, \quad x > 0,$$

which $F(x) \in \mathcal{R}_{-4}$, and for the second distribution, we suppose that the claim amounts $\{X_i, i = 1, 2, \dots\}$ form a sequence of i.i.d random variables with Inverse Gamma

distribution $F_X(x) = \frac{e^{-\frac{5}{x}(x+5)}}{x}$, $x > 0$, which $F(x) \in \mathcal{R}_{-1}$. Now, using the jump diffusion process $\{Z_t = e^{\tilde{R}t}, t \geq 0\}$ defined in (4) and (5) with the Dow Jones data for $t = 50$, we obtain the finite and infinite time ruin probabilities. The results are reported in Table 1. From this table, we see that the values of ruin probabilities $\psi_A(x, t)$ and $\psi_A(x)$ for Pareto and Inverse Gamma distributions decrease as the initial reserve increases.

Table 1: Ruin probabilities for Pareto and Inverse Gamma distributions with Dow Jones industrial data.

Pareto distribution							
x	50	100	200	400	500	700	900
$\psi_A(x, t)$	34.5×10^{-2}	14.2×10^{-2}	39.5×10^{-3}	11.9×10^{-3}	57.3×10^{-4}	51.6×10^{-5}	25.4×10^{-4}
$\psi_A(x)$	39.7×10^{-2}	22.6×10^{-2}	44.2×10^{-3}	13.7×10^{-3}	64.3×10^{-4}	53.8×10^{-5}	28.7×10^{-4}
Inverse Gamma distribution							
x	50	100	200	400	500	700	900
$\psi_A(x, t)$	20.6×10^{-2}	11.2×10^{-2}	27.3×10^{-3}	8.05×10^{-3}	43.71×10^{-4}	34.60×10^{-5}	21.75×10^{-4}
$\psi_A(x)$	24.7×10^{-2}	13.4×10^{-2}	29.5×10^{-3}	9.28×10^{-3}	46.03×10^{-4}	37.02×10^{-5}	23.01×10^{-4}

6 Concluding remarks

In this work, we studied the perturbed integrated risk process of an insurance company with dependent insurance risk and investment risk. We consider two types of claim sizes: (i) pairwise strong quasi-asymptotically independent; (ii) asymptotically independent. For the first type, we assume that the claim sizes and return jumps caused by the systematic factors with arbitrarily dependent structure. The asymptotic finite-time ruin probabilities are obtained for heavy-tailed claim sizes. We verified the accuracy of the asymptotic estimates in Theorem 3.1 using the MC method, when the claim X and geometric return jump e^{-Y} followed the Pareto distributions. In our numerical examples, we mainly considered two cases to study the finite time ruin probabilities. Case 1: X and e^{-Y} are independent. Case 2 : The pair (X, e^{-Y}) follows a FGM copula. In both cases, we observed that as the initial capital increases, the values of ψ_1 and ψ_2 got closer in a certain range. For the second type, the asymptotic finite-time ruin probability derived when the jump diffusion process is modeled by Affine process and presented a real data analysis with Dow Jones industrial index data.

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Appendix

6.1 Proof of Lemma 4.2.

(1) Note that for any fixed $t > 0$ and $i \in \mathbb{N}$, on the set $\omega = 0$, $N((\omega + \tau_i)-) = N(\tau_i-) = i - 1$ since $\{N(t), t \geq 0\}$ is a renewal counting process with càdlàg paths.

Then, $\Theta_i(t)$ is a random function with related to Y_1, \dots, Y_{i-1} and irrespective of Y_i . Thus, we know that $\Theta_i(t)$ is independent of X_i . Similarly, on the set $\omega > 0$, $\tilde{\Theta}_i(t)$ is independent of \tilde{X}_i since $N((\tau_i + \omega) -) \geq i$.

Furthermore, we know that $E \left[\left(e^{-\inf_{0 \leq s \leq t} B(s) - \inf_{0 \leq s \leq t} J(s)} \right)^p \right] < \infty$ for some $p > J_F^+$ since $E[e^{-pY_1}] \leq 1$, for $p > J_F^+$. Thus, for any fixed $t > 0$ and some $p > J_F^+$, we get that

$$\begin{aligned} E \left[\left(\Theta_i(t) I_{\{\omega=0\}} \right)^p \right] &\leq E \left[\left(e^{-\inf_{0 \leq s \leq t} B(s) - \inf_{0 \leq s \leq t} J(s)} \right)^p \right] < \infty, \\ E \left[\left(\tilde{\Theta}_i(t) I_{\{\omega>0\}} \right)^p \right] &< \infty. \end{aligned}$$

Then, for any fixed $t > 0$ and any fixed $n \geq 1$, we have

$$\begin{aligned} P \left(\sum_{i=1}^n X_i \Theta_i(t) > x \right) &= P \left(\sum_{i=1}^n X_i \Theta_i(t) I_{\{\omega=0\}} > x \right) + P \left(\sum_{i=1}^n \tilde{X}_i \tilde{\Theta}_i(t) I_{\{\omega>0\}} > x \right) \\ &\sim \sum_{i=1}^n P \left(X_i \Theta_i(t) I_{\{\omega=0\}} > x \right) + \sum_{i=1}^n P \left(\tilde{X}_i \tilde{\Theta}_i(t) I_{\{\omega>0\}} > x \right) \\ &= \sum_{i=1}^n P \left(X_i \Theta_i(t) > x \right), \end{aligned} \quad (21)$$

where the second step is due to Theorem 2.3 in Li (2013).

(2) We begin with the third step of (21) and apply Lemma 4.1, for any fixed $t > 0$ and for $1 \leq i \leq n$, it holds that

$$P \left(X_i \Theta_i(t) I_{\{\omega=0\}} > x \right) \sim E \left[\left(\Theta_i(t) I_{\{\omega=0\}} \right)^\alpha \right] \bar{F}(x), \quad (22)$$

$$P \left(X_i \Theta_i(t) I_{\{\omega>0\}} > x \right) \sim E \left[\left(\tilde{\Theta}_i(t) I_{\{\omega>0\}} \right)^\alpha \right] \bar{G}(x), \quad (23)$$

therefore, this completes the proof.

6.2 7.2. Proof of Lemma 4.3.

(1) For any fixed $t > 0$ and any fixed $\gamma > 1$, choose some sufficiently large n , such that

$\sum_{i=n+1}^{\infty} i^{-\gamma} \leq 1$, then we have the inequality

$$\begin{aligned} P \left(\sum_{i=n+1}^{\infty} X_i \Theta_i(t) > x \right) &\leq P \left(\sum_{i=n+1}^{\infty} X_i \Theta_i(t) > \sum_{i=n+1}^{\infty} i^{-\gamma} x \right) \\ &\leq \sum_{i=n+1}^{\infty} P \left(X_i \Theta_i(t) > i^{-\gamma} x \right). \end{aligned} \quad (24)$$

Following the proof of Lemma 4.4 in Guo (2022), using (10) and (11), we get that for any $p > J_F^+$ and any $n \geq 1$,

$$\sum_{i=n+1}^{\infty} P \left(X_i \Theta_i(t) > i^{-\gamma} x \right)$$

$$\leq K \sum_{i=n+1}^{\infty} i^{\gamma p} \left(\bar{F}(x) P(\tau_i \leq t, \omega = 0) + \bar{G}(x) P(\tau_i + \omega \leq t, \omega > 0) \right), \quad (25)$$

where the constant $K = E \left[\left(e^{-\inf_{0 \leq s \leq t} B(s) - \inf_{0 \leq s \leq t} J(s)} \right)^p \right] > 0$. For some $p > J_F^+$, $E[e^{-pY_1}] \leq 1$, thus $E \left[\left(e^{-\inf_{0 \leq s \leq t} B(s) - \inf_{0 \leq s \leq t} J(s)} \right)^p \right] < \infty$.

For any fixed $\varepsilon > 0$, $t > 0$, some $p > J_F^+$, and sufficiently large n , we have

$$\begin{aligned} \sum_{i=n+1}^{\infty} i^{\gamma p} P(\tau_i \leq t) &= \sum_{i=n+1}^{\infty} i^{\gamma p} \int_{0^-}^t P(N(t-u) \geq i-1) P(\tau_1 \in du) \\ &\leq E \left[(N(t)+1)^{\gamma p+1} I_{\{N(t) \geq n\}} \right] P(\tau_1 \leq t) \\ &\leq \varepsilon P(\tau_1 \leq t), \end{aligned} \quad (26)$$

where we used $E \left[(N(t)+1)^{\gamma p+1} \right] < \infty$ in the last step, since $\{N(t), t \geq 0\}$ is a renewal counting process. Similarly, for any fixed $t > 0$ and sufficiently large n , the inequality

$$\begin{aligned} \sum_{i=n+1}^{\infty} i^{\gamma p} P(\tau_i + \omega \leq t, \omega > 0) &\leq \varepsilon P(\tau_1 + \omega \leq t, \omega > 0) \\ &\leq \varepsilon P(\omega > 0), \end{aligned} \quad (27)$$

holds. Combining the inequalities (26) and (27) with (25), we obtain that

$$\begin{aligned} \sum_{i=n+1}^{\infty} P(X_i \Theta_i(t) > i^{-\gamma} x) &\leq K\varepsilon \bar{F}(x) P(\omega = 0) + K\varepsilon \bar{G}(x) P(\omega > 0) \\ &\lesssim K\varepsilon \left(P(X_i \Theta_i(t) > x, \omega = 0) + P(\tilde{X}_i \tilde{\Theta}_i(t) > x, \omega > 0) \right) \\ &= K\varepsilon P(X_i \Theta_i(t) > x), \end{aligned} \quad (28)$$

where the second step is due to Lemma 4.1, then, the inequality (4.3) holds.

(2) We begin with the second step in (28), applying (22) and (23). For any fixed $t > 0$ and sufficiently large n , we obtain that

$$\begin{aligned} &\sum_{i=n+1}^{\infty} P(X_i \Theta_i(t) > i^{-\gamma} x) \\ &\lesssim K\varepsilon \left(P(X_i \Theta_i(t) > x, \omega = 0) + P(\tilde{X}_i \tilde{\Theta}_i(t) > x, \omega > 0) \right) \\ &\sim K\varepsilon E \left[(\Theta_i(t) I_{\{\omega=0\}})^\alpha \right] \bar{F}(x) + K\varepsilon E \left[(\tilde{\Theta}_i(t) I_{\{\omega>0\}})^\alpha \right] \bar{G}(x). \end{aligned} \quad (29)$$

Combining (29) with (24), then the inequality (4.4) holds directly, and this completes the proof. \square

6.3 Proof of Theorem 3.1.

Before proving Theorem 3.1, for $t \geq 0$, we set

$$p(t) = \sigma_S \sup_{0 \leq s \leq t} \left| \int_0^s e^{-R(u)} dW_S(u) \right|,$$

$$C(t) = c \int_0^t e^{-R(u)} du, t \geq 0.$$

We first deal with the lower bound of (3.1). For any $\varepsilon > 0$, any fixed $n \geq 1$ and any fixed $t > 0$, we get

$$\begin{aligned} \psi(x, t) &= P \left(\sup_{0 \leq s \leq t} \left(\sum_{i=1}^{\infty} X_i \Theta_i(s) - \int_0^s e^{-R(u)} d(cu + \sigma_S W_S(u)) \right) > x \right) \\ &\geq P \left(\sum_{i=1}^n X_i \Theta_i(t) - C(t) - p(t) > x, C(t) \leq \frac{\varepsilon x}{2}, p(t) \leq \frac{\varepsilon x}{2} \right) \\ &\geq P \left(\sum_{i=1}^n X_i \Theta_i(t) > (1 + \varepsilon)x \right) - P \left(C(t) > \frac{\varepsilon x}{2} \right) - P \left(p(t) > \frac{\varepsilon x}{2} \right) \\ &=: I_1 - I_2 - I_3. \end{aligned} \tag{30}$$

Applying Lemma 4.2 (1) with $F \in \mathcal{L} \cap \mathcal{D}$ and $G \in \mathcal{L} \cap \mathcal{D}$, we get that

$$\begin{aligned} I_1 &\sim \sum_{i=1}^n P(X_i \Theta_i(t) > (1 + \varepsilon)x) \\ &\gtrsim \sum_{i=1}^n P(X_i \Theta_i(t) > x) \\ &\sim \sum_{i=1}^{\infty} P(X_i \Theta_i(t) > x), \end{aligned}$$

where the third step is due to the fact that

$$\begin{aligned} \sum_{i=n+1}^{\infty} P(X_i \Theta_i(t) > x) &\leq \sum_{i=n+1}^{\infty} P(X_i \Theta_i(t) > i^{-\gamma} x) \\ &\leq \varepsilon \sum_{i=1}^n P(X_i \Theta_i(t) > x), \end{aligned}$$

where $\gamma > 1$ satisfies the condition in (24). Thus, we can get

$$I_1 \gtrsim \sum_{i=1}^{\infty} P(X_i \Theta_i(t) > x). \tag{31}$$

As for I_2 , for any fixed $t > 0$ and some $p > J_F^+$, we get

$$\begin{aligned} P \left(C(t) > \frac{\varepsilon x}{2} \right) &\leq \left(\frac{\varepsilon}{2} \right)^{-p} x^{-p} c^p E \left(\int_0^t e^{-R(u)} du \right)^p \\ &\leq \left(\frac{\varepsilon}{2} \right)^{-p} x^{-p} c^p t^{p-1} E \int_0^t e^{-pR(u)} du \\ &= \left(\frac{\varepsilon}{2} \right)^{-p} x^{-p} c^p t^{p-1} E \int_0^t e^{-u\phi_R(p)} du \end{aligned}$$

$$\leq Kx^{-p},$$

where the constant $K = \left(\frac{\varepsilon}{2}\right)^{-p} c^p t^p \max\{e^{t\phi_B(p)}, 1\}$, the first step is due to Chebyshev's inequality and the second step is due to Hölder's inequality.

Then, applying (11), Lemma 4.1 and the third step in (4.3), for any fixed $t > 0$ and for any $\varepsilon > 0$ we get the inequality

$$I_2 \leq \varepsilon \sum_{i=1}^n P(X_i \Theta_i(t) > x). \tag{32}$$

As for I_3 , for any fixed $\gamma > 1$ and any fixed $t > 0$, by the Burkholder-Davis-Gundy inequality and Hölder's inequality, we derive the inequality

$$\begin{aligned} I_3 &\leq \left(\frac{\varepsilon x}{2\sigma_S}\right)^{-p} E \left[\left(\sup_{0 \leq s \leq t} \left| \int_0^s e^{-R(u)} dW_S(u) \right| \right)^p \right] \\ &\leq K \left(\frac{\varepsilon x}{2\sigma_S}\right)^{-p} E \left[\left(\int_0^t e^{-2R(u)} du \right)^{\frac{p}{2}} \right]. \end{aligned}$$

For any fixed $t > 0$, if $p \leq 2$, using Hölder's inequality we get

$$E \left[\left(\int_0^t e^{-2R(u)} du \right)^{\frac{p}{2}} \right] \leq \left[E \left(\int_0^t e^{-2R(u)} du \right) \right]^{\frac{p}{2}} < \infty. \tag{33}$$

If $p > 2$, using Hölder's inequality we have

$$\begin{aligned} E \left[\left(\int_0^t e^{-2R(u)} du \right)^{\frac{p}{2}} \right] &\leq \left(\int_0^t u^{-\gamma} du \right)^{\frac{p}{2}-1} E \left(\int_0^t u^{\gamma(\frac{p}{2}-1)} e^{-pR(u)} du \right) \\ &= \left(\int_0^t u^{-\gamma} du \right)^{\frac{p}{2}-1} \int_0^t u^{\gamma(\frac{p}{2}-1)} e^{-u\phi_B(p)} E \left[E(e^{-pY_1})^{N(t)} \right] du \\ &\leq \left(\int_0^t u^{-\gamma} du \right)^{\frac{p}{2}-1} \int_0^t u^{\gamma(\frac{p}{2}-1)} e^{-u\phi_B(p)} du \\ &< \infty, \end{aligned} \tag{34}$$

where the third step is due to $E[e^{-pY_1}] \leq 1$.

Combining the inequalities (33) and (34), similar to the proof of I_2 , we derive the inequality

$$I_3 \leq \varepsilon \sum_{i=1}^n P(X_i \Theta_i(t) > x). \tag{35}$$

Combining (31), (32) and (35) with (30), we get

$$\psi(x, t) \gtrsim \sum_{i=1}^{\infty} P(X_i \Theta_i(t) > x).$$

Next, we deal with the upper bound of (3.1). For any $\varepsilon > 0$, any fixed $t > 0$ and any fixed $n \geq 1$

$$\begin{aligned}
\psi(x, t) &\leq P\left(\sum_{i=1}^{\infty} X_i \Theta_i(t) + p(t) > x\right) \\
&\leq P\left(\sum_{i=1}^n X_i \Theta_i(t) > \left(1 - \frac{\varepsilon}{2}\right)x\right) + P\left(\sum_{i=n+1}^{\infty} X_i \Theta_i(t) > \frac{\varepsilon}{2}x\right) \\
&\quad + P\left(p(t) > \frac{\varepsilon}{2}x\right) \\
&=: J_1 + J_2 + I_3.
\end{aligned} \tag{36}$$

Similarly to the proof of I_1 , for any fixed $t > 0$, we obtain that

$$J_1 \lesssim \sum_{i=1}^{\infty} P(X_i \Theta_i(t) > x). \tag{37}$$

For J_2 , applying Lemma 4.3 (1), for any fixed $t > 0$ and any fixed $n \geq 1$, we get that

$$J_2 \leq \varepsilon \sum_{i=1}^n P(X_i \Theta_i(t) > x). \tag{38}$$

Combining (35), (37) and (38) with (36), we derive that

$$\psi(x, t) \lesssim \sum_{i=1}^{\infty} P(X_i \Theta_i(t) > x),$$

and this completes the proof.

6.4 Proof of Corollary 3.1.

From (29), we know that for any $\varepsilon > 0$, any fixed $t > 0$ and any fixed $\gamma > 1$, the inequality

$$\begin{aligned}
&\sum_{i=n+1}^{\infty} P(X_i \Theta_i(t) > x) \\
&\leq \sum_{i=n+1}^{\infty} P(X_i \Theta_i(t) > i^{-\gamma}x) \\
&\leq K\varepsilon \sum_{i=1}^n E[(\Theta_i(t)I_{\{w=0\}})^\alpha] \bar{F}(x) + K\varepsilon \sum_{i=1}^n E[(\tilde{\Theta}_i(t)I_{\{w>0\}})^\alpha] \bar{G}(x),
\end{aligned} \tag{39}$$

holds. Thus, we have that

$$\sum_{i=1}^n E[(\Theta_i(t)I_{\{w=0\}})^\alpha] \bar{F}(x) + \sum_{i=1}^n E[(\tilde{\Theta}_i(t)I_{\{w>0\}})^\alpha] \bar{G}(x)$$

$$\sim \sum_{i=1}^{\infty} E [(\Theta_i(t)I_{\{w=0\}})^\alpha] \bar{F}(x) + \sum_{i=1}^{\infty} E [(\tilde{\Theta}_i(t)I_{\{w>0\}})^\alpha] \bar{G}(x). \quad (40)$$

Following the proof of Theorem 3.1 and using (12), (13), Lemma 4.2 (2), Lemma (4.3) (2), (39) and (40), the relation (3.2) holds.

6.5 Proof of Theorem 3.2.

To prove this theorem, we will apply Theorem 1 and relation (7) in Cheng and Wang (2023). To verify the conditions of Theorem 1 and relation (7) given in Cheng and Wang (2023), we need the explicit expression of $E(e^{-\xi R_t}) = E(Z_t^{-\xi})$, for fixed $\xi > 0$. Applying the Itô's formula, we can verify that the solution to the jump diffusion process (4) is

$$\begin{aligned} Z_t &= \exp \left\{ \mu t + \int_0^t \sqrt{r_t} dB_t^{(1)} - \frac{1}{2} \int_0^t r_t dt \right\} \\ &= \exp \left\{ \mu t - \frac{1}{2} \int_0^t r_t + \int_0^t \sqrt{r_t} dB_t^{(2)} + \sqrt{1 - \rho_0^2} \int_0^t \sqrt{r_t} dB_t^{(3)} \right\}. \end{aligned} \quad (41)$$

From the inequality (3.4), we take some $\xi > 0$, with $\xi > \alpha^0 = \max\{\alpha, 2\}$ such that

$$\zeta + \xi \delta \rho_0 > \delta \sqrt{\xi(\xi + 1)}. \quad (42)$$

Taking the conditional expectation of (41) with respect to the process $\{B_s^{(2)}, 0 \leq s \leq t\}$, we have

$$\begin{aligned} E(Z_t^{-\xi}) &= E(E(Z_t^{-\xi} | B_s^{(2)}, 0 \leq s \leq t)) \\ &= \exp(-\mu \xi t) E \left(\exp \left\{ \frac{\xi}{2} \int_0^t r_s ds - \xi \rho_0 \int_0^t \sqrt{r_s} dB_s^{(2)} \right\} \right. \\ &\quad \times E \left(\exp \left\{ -\xi \sqrt{1 - \rho_0^2} \int_0^t \sqrt{r_s} dB_s^{(3)} \right\} | B_s^{(2)}, 0 \leq s \leq t \right) \\ &= \exp(-\mu \xi t) E \left(\exp \left\{ \frac{\xi}{2} \int_0^t r_s ds - \xi \rho_0 \int_0^t \sqrt{r_s} dB_s^{(2)} \right\} Y(t) \right), \end{aligned} \quad (43)$$

where $Y(t)$ will obtain by applying Girsanov's theorem. From (42) and $\rho_0 \in [-1, 0]$, we have that $\zeta^2 > 2\delta^2 b_0$, where $b_0 = \frac{\xi^2(1-\rho_0^2)}{2}$, and this is consistency with the assumption of relation (7) in Cheng and Wang (2023). Therefore, by taking $a = 0$ and $b = b_0$ in (7) of Cheng and Wang (2023), we obtain

$$\begin{aligned} M_{r(t), R(t)}(0, \frac{\xi^2(1-\rho_0^2)}{2}, t) &= E \left(\exp \left\{ \frac{\xi^2(1-\rho_0^2)}{2} \int_0^t r_s ds \right\} \right) \\ &= K_1(\xi, t) \exp(r_0 K_2(\xi, t)), \end{aligned} \quad (44)$$

where

$$K_1(\xi, t) = \left(\frac{\exp\left(\frac{\zeta t}{2}\right)}{\cosh\left(\frac{t\Delta(\xi)}{2}\right) + \zeta \frac{\sinh\left(\frac{t\Delta(\xi)}{2}\right)}{\Delta(\xi)}} \right)^{\frac{2\zeta t}{\delta^2}},$$

$$K_2(\xi, t) = \frac{\xi^2(1 - \rho_0^2)}{\zeta + \Delta(\xi) \coth\left(\frac{t\Delta(\xi)}{2}\right)},$$

and $\Delta(\xi) = \sqrt{(\zeta^2 - \xi^2\delta^2(1 - \rho_0^2))}$. Therefore, for fixed ξ the equality (44) is finite, i.e.,

$$E\left(\exp\left\{\frac{\xi^2(1 - \rho_0^2)}{2} \int_0^t r_s ds\right\}\right) < \infty.$$

Applying Girsanov's theorem to $Y(t)$ in (43), we obtain

$$\begin{aligned} Y(t) &= E\left(\exp\left\{-\xi\sqrt{1 - \rho_0^2} \int_0^t \sqrt{r_s} dB_s^{(3)}\right\} | B_s^{(2)}, 0 \leq s \leq t\right) \\ &= E\left(\exp\left\{\frac{\xi^2(1 - \rho_0^2)}{2} \int_0^t r_s ds\right\}\right). \end{aligned} \quad (45)$$

Putting (45) into (43) and using the stochastic short-rate process (5), we have

$$\begin{aligned} E(Z_t^{-\xi}) &= \exp(-\mu\xi t) E\left(\exp\left\{\left(\frac{\xi^2(1 - \rho_0^2)}{2} + \frac{\xi}{2}\right) \int_0^t r_s ds - \xi\rho_0 \int_0^t \sqrt{r_s} dB_s^{(2)}\right\}\right) \\ &= \exp\left(-\mu\xi t + \frac{\xi\rho_0}{\delta}(r_0 + \zeta t)\right) \\ &\quad \times E\left(\exp\left\{\left(-\frac{\xi r_1 \rho_0}{\delta} + \left(\frac{\xi^2(1 - \rho_0^2)}{2} + \frac{\xi}{2} - \frac{\xi\zeta\rho_0}{\delta}\right) \int_0^t r_s ds\right\}\right) \\ &= \exp\left(-\mu\xi t + \frac{\xi\rho_0}{\delta}(r_0 + \zeta t)\right) \\ &\quad \times M_{r(t), R(t)}\left(-\frac{\xi\rho_0}{\delta}, \frac{\xi^2(1 - \rho_0^2)}{2} + \frac{\xi}{2} - \frac{\xi\zeta\rho_0}{\delta}, t\right). \end{aligned}$$

From the inequality (42), we see that $\zeta^2 > 2\delta^2 b_1$, where $b_1 = \frac{\xi^2(1 - \rho_0^2)}{2} + \frac{\xi}{2} - \frac{\xi\zeta\rho_0}{\delta}$. Then $M_{r(t), R(t)}(a, b, t)$ exists when $a = -\frac{\xi\rho_0}{\delta}$ and $b = b_1$. Thus by (7) of Cheng and Wang (2023), we obtain

$$E(Z_t^{-\xi}) = D_1(\xi, t) \exp(r_0 D_2(\xi, t) - \mu\xi t), \quad (46)$$

where

$$D_1(\xi, t) = \left(\frac{\exp\left(\frac{t(\zeta + \xi\delta\rho_0)}{2}\right)}{\cosh\left(\frac{t\Omega(\xi)}{2}\right) + (\zeta + \xi\delta\rho_0) \frac{\sinh\left(\frac{t\Omega(\xi)}{2}\right)}{\Omega(\xi)}}\right)^{\frac{2\zeta t}{\delta^2}}, \quad (47)$$

$$D_2(\xi, t) = \frac{\xi^2 + \xi}{(\zeta + \xi\delta\rho_0) + \Omega(\xi) \coth\left(\frac{t\Omega(\xi)}{2}\right)}, \quad (48)$$

$$\Omega(\xi) = \sqrt{(\zeta + \xi\delta\rho_0)^2 - \delta^2(\xi^2 + \xi)}. \quad (49)$$

Therefore, for fixed ξ the equality (46) is finite, i.e., $\int_0^t E(Z_s^{-\xi}) ds < \infty$. By Theorem 1 given in Cheng and Wang (2023) and relations (46)-(49), we obtain the finite time ruin probability in (3.5). Now, we prove the infinite time ruin probability in (3.9). From

(3.4)-(3.8), we take some $\xi > 0$, with $\xi > \alpha^0 = \max\{\alpha, 2\}$ such that the inequality (42) and

$$-\xi\mu + \frac{v\zeta}{\delta^2} \left((\zeta + \xi\delta\rho_0) - \sqrt{(\zeta + \xi\delta\rho_0)^2 - \delta^2\xi(\xi + 1)} \right) < 0,$$

hold. On the other hand, for the fixed ξ , from (46)-(49), as $t \rightarrow \infty$, we have the approximation

$$E(Z_t^{-\xi}) \sim D_0 \exp \left(-\mu\xi t + \frac{\xi\rho_0}{\delta} \left((r_0 + \zeta\xi t) - \sqrt{(\zeta + \xi\delta\rho_0)^2 - \delta^2\xi(\xi + 1)} \right) < 0 \right),$$

where D_0 is a positive constant. It follows that for fixed ξ and any $\xi_0 > 0$, we have $\int_0^t \max\{s^{\xi_0}, 1\} E(Z_s^{-\xi}) ds < \infty$. Thus, by Theorem 1 given in Cheng and Wang (2023) and relations (46)-(49), we derive the infinite time ruin probability in (3.9), and this completes the proof.