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Prediction of missing order statistics for generalized extreme value distribution

Isaac Almasi, Nabaz Esmailzadeh*
Department of Statistics, Faculty of Basic Science, Razi University, Kermanshah, Iran

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Abstract: The prediction of missing order statistics for the Generalized Extreme Value distribution is investigated, with a focus on the Fréchet, Gumbel, and Weibull subtypes governed by the shape parameter γ. This paper first establishes a necessary and sufficient condition for the existence of conditional moments of order statistics based on the domain of γ. We then derive predictors using three distinct methods: the best unbiased predictor, the conditional median predictor, and the conditional average predictor. A comprehensive simulation study reveals that the optimal method is distribution-dependent. For the Gumbel distribution, the best unbiased predictor provides superior accuracy and robustness. For the Fréchet distribution, the best unbiased predictor is unequivocally superior, while the conditional average predictor and the conditional median predictor demonstrate significant bias and instability. For the Weibull distribution, the choice is critically sensitive to γ, best unbiased predictor is preferred for milder shapes, whereas the conditional average predictor is more robust for stronger shapes. These findings provide a clear, evidence-based framework for selecting the appropriate prediction method in extreme value analysis. To demonstrate the performance of the proposed methods, a numerical example utilizing a real-world data set is provided.

Keywords: Extreme value distribution; Extreme value index; Missing order statistics; Prediction.
Mathematics Subject Classification (2010): 62N01, 62G32, 62G30.

*Corresponding author: n.esmailzadeh@razi.ac.ir

1 Introduction

The family of extreme value distributions, including the Fréchet, Gumbel, and Weibull types, offers a versatile framework for modeling skewness and heavy-tailed behavior in statistical applications. The Fréchet, Gumbel, and Weibull distribution functions can be expressed within a one-parameter family of distributions via the Von Mises-Jenkinson parameterization (Ahsanullah, 2016; Jenkinson, 1955; Kotz and Nadarajah, 2000). Notably, the generalized extreme value (GEV) distribution has a distribution function defined by

$$G_\gamma(x) = \begin{cases} \exp(-(1+\gamma x)^{-1/\gamma}), & \gamma \neq 0, 1+\gamma x > 0, \\ \exp(-e^{-x}), & \gamma = 0. \end{cases} \tag{1}$$

The parameter γ, known as extreme value index (EVI), determines the shape of the GEV distribution. If $\gamma > 0$, G_γ represents a heavy-tailed distribution and is said to be in the domain of attraction of the Fréchet distribution. If $\gamma = 0$, the Gumbel domain of attraction represents the simplest case for inference. Lastly, if $\gamma < 0$, then $x < -1/\gamma$, the distribution function is said to be in the domain of attraction of the Weibull distribution. This domain includes short-tailed distributions with a finite right endpoint.

The reality of data collection often involves incomplete observations due to censoring, a phenomenon in which the precise value of an observation remains unknown. Order statistics, which represent the ordered values within a sample, are a powerful analytical tool for such data. The versatility of order statistics in analyzing extreme-value distributions with censoring is evident in its widespread application across diverse fields. In reliability analysis, order statistics are fundamental for estimating and predicting component failures of lifetime distributions. Analysis of failure time often involves censoring and making orders statistics indispensable for accurate estimation. In hydrology, order statistics are used to model extreme rainfall events, design flood defenses, and assess the risk of flooding (Liu et al., 2023; Belzile, 2019). Long-term rainfall data are often incomplete because of missing observations or historical record limitations, necessitating the use of methods that effectively handle censoring. Financial applications involve modeling extreme market fluctuations, estimating the value at risk, and managing financial risk (Belzile, 2019). Financial data often exhibit heavy tails and may contain censored observations, requiring the use of robust methods based on order statistics. Environmental science uses order statistics to analyze air pollution levels, assess environmental risks, and predict extreme weather events (Belzile, 2019). Environmental data are often censored due to the limitations of monitoring equipment or the infrequent occurrence of extreme events. Medical research applies order statistics to model survival times, analyze censored data in clinical trials, and assess treatment effectiveness (Ballenberger et al., 2012). Clinical trial data are frequently censored due to patients dropping out of the study or the study ending before all patients experience the event of interest.

Let $X_1, \ldots, X_n$ be the random variables of GEV with a parameter γ, and let $Y_1 \leq Y_2 \leq \ldots \leq Y_n$ be the corresponding order statistics. We assume that some of the order statistics values are lost (or censored), that is we only observed the dataset $\boldsymbol{Y} = (Y_1, \ldots, Y_m)$ that $0 \leq m \leq n$. The prediction of censored order statistics and the

interpolation of missing order statistics are problems of fundamental interest. Several methods have been proposed for interpolating order statistics. Razmkhah et al. (2010) compared missing-order statistics interpolation (prediction or reconstruction) methods. Khatib et al. (2013) developed Bayesian interpolation assuming that the scale parameter has a conjugate prior and that the underlying distribution is exponential. The best unbiased prediction of missing order statistics of a stable distribution based on the conditional expected value has been studied in Almasi et al. (2017), Almasi et al. (2018).

Previous studies on the prediction of order statistics have primarily focused on distributions with finite moments of all orders, such as the exponential distributions with finite variance (Razmkhah et al., 2010; Khatib et al., 2013). In such settings, the existence of conditional moments for prediction is guaranteed, and no fundamental limitations arise from the distribution's tail properties. In contrast, the GEV distribution, by its nature as a limit law for maxima, features a shape parameter γ that governs its tail behavior and directly determines the finiteness of its moments. This introduces a nontrivial theoretical constraint: predictors based on moments, such as the best unbiased predictor (BUP) or those evaluated via Mean Squared Error (MSE), are only valid when the requisite conditional moments exist. Therefore, extending prediction methodology to the GEV family requires not only new derivations but also a rigorous analysis of moment existence conditions, a fundamental challenge absent in the previously studied light-tailed or stable frameworks.

The GEV distribution family forms the focus of this study. This study addresses a nontrivial theoretical problem concerning the prediction of missing order statistics, an area previously studied primarily within exponential and stable distributions. The core mathematical contribution of this study is the extension of the BUP approach to the generalized extreme value family. We present detailed derivations of the BUP, Conditional Median Predictor (CMP), and Conditional Average Predictor (CAP), supported by properties of order statistics and conditional distributions. The theoretical foundations are strengthened by rigorous results on the conditions for the existence of conditional moments under the GEV model, with proofs concerning moment existence and tail behavior that are consistent with established extreme value theory.

This study addresses the issue of predicting missing order statistics in the generalized extreme value distribution. BUP is used to generate imputed values for missing orders. We derive the BUP for Fréchet, Gumbel, and Weibull. Additionally, we examine CMP, which has been explored by several authors, including Raqab and Nagaraja (1995), Raqab et al. (2007). We also consider the CAP studied by Razmkhah et al. (2010) and Asgharzadeh et al. (2012).

The performance of the derived predictor in the three methods is also examined through a comprehensive simulation for various extreme value indexes, sample sizes, and the number of missing values.

The remainder of this paper is structured as follows: Section 2 introduces the necessary and sufficient conditions for the existence of conditional moments of order statistics of GEV random variables. In Section 3, using the results obtained in Section 2, we attempt to predict the missed order statistics. Section 4 examines the performance of the methods mentioned in Section 3 through comprehensive simulation. A numerical example utilizing real-world dataset is provided in Section 5. Given the number of

abbreviations used, a summary of all abbreviated terms is provided in the following table for convenience:

Term	Abbreviation
Generalized Extreme Value	GEV
Extreme Value Index	EVI
Best Unbiased Predictor	BUP
Conditional Median Predictor	CMP
Conditional Average Predictor	CAP
Mean Square Error	MSE

2 Existence of conditional moments of order statistics

Let $Y_i, i = 1, \ldots, n$ be order statistics from a GEV distribution with a common probability density function and cumulative distribution function, denoted as $g_\gamma(x)$ and $G_\gamma(x)$, respectively.

For an Extreme random variable $X \sim G_\gamma(x)$ with $\gamma \in R$, if $\gamma > 0$, moments $E|X|^k$ of order $k > 1/\gamma$ are not finite (De Haan and Ferreira, 2006). Formally, heavy-tailed distributions are those whose tail probability, $1 - G_{\gamma>0}(x)$, is larger than that of an exponential distribution, (Neves and Greetham, 2020). Thus, noting that if $\gamma > 0$,

$$\lim_{\lambda\to\infty} \lambda^{1/\gamma} P(X > \lambda) = \gamma^{-1/\gamma}.$$

Lemma 2.1. *Let $X_1, \ldots, X_n$ be random variables of $G_\gamma(x)$ with the extreme value index, $\gamma > 0$. Let $Y_1 \le Y_2 \le \cdots \le Y_n$ be the corresponding order statistics, then*

$$\lim_{\lambda\to\infty} \lambda^{(n-t+1)/\gamma} P(Y_t > \lambda) = \binom{n}{n-t+1} \gamma^{-(n-t+1)/\gamma}. \tag{2}$$

Proof. It can be show that

$$P(Y_t > \lambda) = \sum_{i=n-t+1}^{n} (-1)^{i-(n-t+1)} \binom{i-1}{n-t} \sum_{1\le h_1<\cdots<h_i\le n} P(X_{h_1} > \lambda, \ldots, X_{h_i} > \lambda).$$

Refer to Samorodnitsky (1986) for a proof.

From independence and regularly varying property (1), we have

$$\begin{aligned} \sum_{1\le h_1<\cdots<h_i\le n} P(X_{h_1} > \lambda, \ldots, X_{h_i} > \lambda) &= \sum_{1\le h_1<\cdots<h_i\le n} \prod_{j=1}^{i} P(X_{h_j} > \lambda) \\ &= \binom{n}{i} (P(X > \lambda))^i. \end{aligned}$$

Therefore,

$$P(Y_t > \lambda) = \sum_{i=n-t+1}^{n} (-1)^{i-(n-t+1)} \binom{i-1}{n-t} \binom{n}{i} (P(X > \lambda))^i.$$

So,

$$\lim_{\lambda\to\infty}\lambda^{(n-t+1)/\gamma}P(Y_t>\lambda) = \sum_{i=n-t+1}^{n}\{(-1)^{i-(n-t+1)}\binom{i-1}{n-t}\binom{n}{i} \times \lim_{\lambda\to\infty}\lambda^{(n-t+1)/\gamma}(P(X>\lambda))^i\}.$$

Thus

$$\lim_{\lambda\to\infty}\lambda^{(n-t+1)/\gamma}P(Y_t>\lambda) = \begin{cases}\binom{n}{n-t+1}\gamma^{-(n-t+1)/\gamma}, & i=n-t+1,\\ 0, & i>n-t+1.\end{cases}$$

□

Suppose that we have observed the data set $\boldsymbol{Y}=(Y_1,\ldots,Y_m)$, where $m<n$. A necessary and sufficient condition for the existence of order statistics moments is introduced in the following lemma.

Lemma 2.2. *Let k be a non-negative number, Y_m and Y_t, $m,t\in\{1,\ldots,n\}$, $m<t$ be order statistics from GEV with the extreme value index, $\gamma>0$. Then $E(Y_t^k|y_1,\ldots,y_m)$ is finite if and only if $t<n+1-k\gamma$.*

Proof. Using the Markov property of order statistics (Theorem 2.4.3, Arnold et al. (1992)), it is enough to show that $E(|Y_t|^k|y_m)<+\infty$. Now, we have

$$\begin{aligned}E(|Y_t|^k|y_m) &= \int_0^{+\infty}P(|Y_t|^k>\lambda|y_m)\mathrm{d}\lambda\\ &= k\int_0^{\epsilon^{1/k}}u^{k-1}P(|Y_t|>u|y_m)\mathrm{d}u + k\int_{\epsilon^{1/k}}^{+\infty}u^{k-1}P(|Y_t|>u|y_m)\mathrm{d}u,\end{aligned}$$

where $\epsilon>0$. The first term is finite. Using (2) and Theorem 2.4.1 in Balakrishnan and Cohen (1991), second term is finite if and only if

$$\int_{\epsilon^{1/k}}^{+\infty}u^{k-1}u^{-(n-t+1)/\gamma}\mathrm{d}u<+\infty,$$

which is finite if and only if $t<n+1-\gamma k$. □

3 Prediction in generalized extreme value distribution

In this section, we describe the three methods for predicting missed order statistics. The predictions for the Fréchet, Gumbel, and Weibull distributions are derived in the following subsections.

CMP approach has been studied by several authors, including Raqab and Nagaraja (1995) and Raqab et al. (2007). We denote the CMP by $\hat{Y}_{t,\mathrm{CMP}}$. We say that $\hat{Y}_{t,\mathrm{CMP}}$ is CMP of Y_t if the condition

$$P(Y_t\le\hat{Y}_{t,\mathrm{CMP}}|Y_1,\ldots,Y_m)=P(Y_t\ge\hat{Y}_{t,\mathrm{CMP}}|Y_1,\ldots,Y_m),$$

is satisfied. The CMP can be computed as follows,

$$\hat{Y}_{t,\mathrm{CMP}} = G_\gamma^{-1}\{G_\gamma(Y_m) + \mathrm{med}(V)(1 - G_\gamma(Y_m))\}, \tag{3}$$

where $V \sim \mathrm{Beta}(t-m, n+1-t)$ and $\mathrm{med}(V)$ represents the median of V.

The CAP of Y_t has been studied by Razmkhah et al. (2010) and Asgharzadeh et al. (2012). In (3), they use the average value instead of median. The CAP is defined and computed as follows,

$$\hat{Y}_{t,\mathrm{CAP}} = G_\gamma^{-1}\{\frac{t-m}{n-m+1} + \frac{n-t+1}{n-m+1}G_\gamma(y_m))\}. \tag{4}$$

It is logical to consider $E(Y_t|y_1,\ldots,y_m)$ as a predictor of Y_t having observed $y_1,\ldots,y_m$. Let us recall that the conditional density function of y_t given y_m takes the following form

$$g_\gamma(y_t|y_m) = \frac{(1-G_\gamma(y_m))^{m-n}}{\mathrm{Beta}(n_1+1, n_2+1)}\Big(G_\gamma(y_t) - G_\gamma(y_m)\Big)^{n_1}\Big(1-G_\gamma(y_t)\Big)^{n_2} g_\gamma(y_t), \tag{5}$$

for $y_t \geq y_m$, where $n_1 = t-m-1$, $n_2 = n-t$. This is the density of the (tm)-th order statistic from a sample of size nm drawn from the conditional distribution $X|X > y_m$.

By the change of variable $V = \frac{G_\gamma(Y_t)-G_\gamma(y_m)}{1-G_\gamma(y_m)}$ and using (5), the BUP of t'th order statistic $(t = m+1,\ldots,n)$, if it exists, is defined and computed as follows,

$$\begin{aligned}\hat{Y}_t &= E(Y_t|y_1,\ldots,y_m)\\ &= E\Big(G_\gamma^{-1}\Big(G_\gamma(y_m) + V(1-G_\gamma(y_m))\Big)\Big),\end{aligned} \tag{6}$$

where $V \sim \mathrm{Beta}(n_1+1, n_2+1)$,

$$\hat{Y}_{t,\mathrm{BUP}} = \frac{(1-G_\gamma(y_m))^{m-n}}{\mathrm{Beta}(n_1+1,n_2+1)}\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\Bigg\{\binom{n_1}{i}\binom{n_2}{j}(-1)^{n_1+j-i}G_\gamma(y_m)^{n_1-i} \times \int_{G_\gamma(y_m)}^{1} G_\gamma^{-1}(u)u^{i+j}du\Bigg\}. \tag{7}$$

3.1 Prediction in Gumbel distribution

By applying the well–known formula $(1+\gamma x)^{1/\gamma} \to \exp(x)$ as $\gamma \to 0$, the Gumbel distribution function obtained as follows

$$G_{\gamma=0}(x) = \exp(-e^{-x}), \qquad x \in R.$$

Using (6), the BUP of the tth order statistic is as follows,

$$\hat{Y}_{t,\mathrm{BUP}} = y_m - E(\log[1 - e^{y_m}\log[V(e^{e^{-y_m}} - 1) + 1]]),$$

where $V \sim \mathrm{Beta}(t-m, n+1-t)$.

Furthermore, using (7), the BUP of the tth order statistic are as follows,

$$\hat{Y}_{t,\mathrm{BUP}} = \frac{(1-G_0(y_m))^{m-n}}{\mathrm{Beta}(n_1+1,n_2+1)} \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} \binom{n_1}{i}\binom{n_2}{j}(-1)^{n_1+j-i+1} G_0(y_m)^{n_1-i} \times \int_{G_0(y_m)}^{1} \log\Big(-\log(u)\Big) u^{i+j} du,$$

where the last part of above equation is

$$\frac{\mathrm{Exp}(-y_m-(i+j+1)e^{-e^{-y_m}}) - \gamma_0 - \mathrm{E}_1((i+j+1)e^{-e^{-y_m}}) - \log(i+j+1))}{1+i+j},$$

where $E_n(z) = -\int_1^\infty \frac{e^{-zt}}{t^n} dt$, gives the exponential integral function.

By (3), the CMP is computed as follows,

$$\hat{Y}_{t,\mathrm{CMP}} = y_m - \log\Big(1 - e^{y_m}\log\Big(e^{e^{-y_m}}\frac{t-m-\frac{1}{3}}{n-m+\frac{1}{3}} + \frac{n-t+\frac{2}{3}}{n-m+\frac{1}{3}}\Big)\Big).$$

By (4), the CAP is defined and computed as follows,

$$\hat{Y}_{t,\mathrm{CAP}} = y_m - \log\Big(1 - e^{y_m}\log\Big(e^{e^{-y_m}}\frac{t-m}{n-m+1} + \frac{n-t+1}{n-m+1}\Big)\Big).$$

3.2 Prediction in Fréchet distribution

By applying the reparameterization $\alpha = 1/\gamma$ and the change of variable $z = 1+\gamma x$ in (1), if $\gamma > 0$, then the random variable Z follows a standard Fréchet distribution , and its cumulative distribution function is

$$G_\alpha(z) = \exp(-z^{-\alpha}), \qquad z \geq 0, \qquad \alpha > 0,$$

where α is the unknown shape parameter.

Using (7), the BUP of the tth order statistic are as follows,

$$\hat{Y}_{t,\mathrm{BUP}} = \frac{(1-G_\alpha(y_m))^{m-n}}{\mathrm{Beta}(t-m,n-t+1)} \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} \binom{n_1}{i}\binom{n_2}{j}(-1)^{n_1-i+j} G_\alpha(y_m)^{n_1-i} \times \int_{G_\alpha(y_m)}^{1} \Big(-\log(u)\Big)^{-1/\alpha} u^{i+j} du,$$

where the last part of above equation if $\alpha \in N$ is

$$(1+i+j)^{1/\alpha-1}\Gamma\Big(1-1/\alpha\Big) - \mathrm{E}_{1/\alpha}\Big((1+i+j)y_m^{-\alpha}\Big) y_m^{-\alpha+1}$$

where $E_n(z) = -\int_1^\infty \frac{e^{-zt}}{t^n} dt$.

By (3), the CMP is defined and computed as follows,

$$\hat{Y}_{t,\mathrm{CMP}} = \Big(y_m^{-\alpha} - \log\Big(\frac{t-m-\frac{1}{3}}{n-m+\frac{1}{3}} e^{y_m^{-\alpha}} + \frac{n-t+\frac{2}{3}}{n-m+\frac{1}{3}}\Big)\Big)^{-1/\alpha}.$$

By (4), the CAP is defined and computed as follows,

$$\hat{Y}_{t,\mathrm{CAP}} = \Big(y_m^{-\alpha} - \log\Big(\frac{t-m}{n-m+1} e^{y_m^{-\alpha}} + \frac{n-t+1}{n-m+1}\Big)\Big)^{-1/\alpha}.$$

3.3 Prediction in Weibull distribution

By applying the reparameterization $\alpha = -1/\gamma$ and the change of variable $z = -1 - \gamma x$ in (1), if $\gamma < 0$, then the random variable Z follows a standard Weibull distribution and its cumulative distribution function is

$$G_\alpha(x) = \exp(-(-z)^\alpha), \qquad z \le 0, \qquad \alpha > 0,$$

where α is the unknown shape parameter.

Using (7), the BUP interpolators of the tth order statistic are as follows,

$$\hat{Y}_{t,\mathrm{BUP}} = \frac{(1 - G_{-\alpha}(y_m))^{m-n}}{\mathrm{Beta}(t-m, n-t+1)} \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \binom{n_1}{i} \binom{n_2}{j} (-1)^{n_1+j-i+1} G_{-\alpha}(y_m)^{n_1-i} \times \int_{G_{-\alpha}(y_m)}^{1} \Big(-\log(u) \Big)^{1/\alpha} u^{i+j} du,$$

where the last part of above equation if $\alpha \in N$ is

$$(1+i+j)^{-1/\alpha-1} \Big(\Gamma(1 + a/\alpha) - \Gamma(1 + a/\alpha, (i+j+1)(-ym)^\alpha \Big),$$

where $E_n(z) = -\int_1^\infty \frac{e^{-zt}}{t^n} dt$, where the principal value of the integral is taken.

By (3), the CMP is defined and computed as follows,

$$\hat{Y}_{t,\mathrm{CMP}} = -\Big((-y_m)^\alpha - \log\Big(\frac{t - m - \frac{1}{3}}{n - m + \frac{1}{3}} e^{(-y_m)^\alpha} + \frac{n - t + \frac{2}{3}}{n - m + \frac{1}{3}} \Big) \Big)^{1/\alpha}.$$

By (4), the CAP is defined and computed as follows,

$$\hat{Y}_{t,\mathrm{CAP}} = -\Big((-y_m)^\alpha - \log\Big(\frac{t - m}{n - m + 1} e^{(-y_m)^\alpha} + \frac{n - t + 1}{n - m + 1} \Big) \Big)^{1/\alpha}.$$

4 Simulation study

In this section, we examine the performance of the three methods mentioned in Section 3 through comprehensive simulation. We consider various settings for the sample size n, the number of available samples m, and the parameter γ. The sample sizes we consider are n =10, 20, 50 and 100, with the corresponding values of $m = 7, 16, 45$ and 90, respectively. For each setting and distribution, we repeat the simulation 1000 times and calculate the mean and MSE of the predictions. The results are presented in Tables 1-5.

The scatter-plots of predictions versus true observations are presented in Figures 1, 2 and 3 for the Gumbel, Fréchet and Weibull distributions,respectively, where $n = 20$ and $m = 16$ in the Appendix. We have not included them for other settings to conserve space and because of the similarity in the structure of the graphs. Following, we present the results in details for three distributions.

Table 1: Means and MSEs for Gumbel distribution.

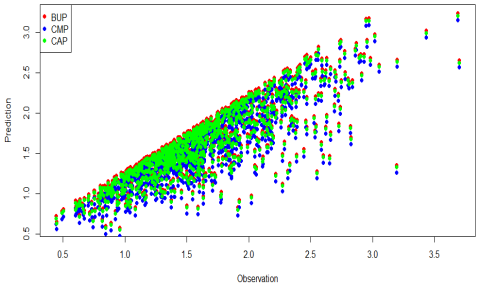
	$n = 10$, $m = 7$			$n = 20$, $m = 16$			$n = 50$, $m = 45$			$n = 100$, $m = 90$	
	Mean	MSE		Mean	MSE		Mean	MSE		Mean	MSE
y_8	1.276		y_{17}	1.679		y_{46}	2.364		y_{91}	2.301	
$\hat{Y}_{BUP}$	1.277	0.133		1.672	0.077		2.373	0.189		2.303	0.026
$\hat{Y}_{CMP}$	1.156	0.146		1.583	0.0862		2.287	0.054		2.267	0.012
$\hat{Y}_{CAP}$	1.234	0.133		1.646	0.078		2.338	0.049		2.298	0.011
y_9	1.824		y_{18}	2.048		y_{47}	2.657		y_{92}	2.392	
$\hat{Y}_{BUP}$	1.824	0.399		2.033	0.198		2.638	0.132		2.397	0.102
$\hat{Y}_{CMP}$	1.686	0.418		1.934	0.211		2.566	0.129		2.359	0.0249
$\hat{Y}_{CAP}$	1.686	0.418		1.962	0.205		2.598	0.124		2.385	0.024
y_{10}	2.940		y_{19}	2.581		y_{48}	3.007		y_{93}	2.483	
$\hat{Y}_{BUP}$	2.897	1.495		2.544	0.472		2.993	0.252		2.460	0.205
$\hat{Y}_{CMP}$	2.685	1.559		2.419	0.497		2.897	0.2446		2.436	0.049
$\hat{Y}_{CAP}$	2.452	1.732		2.377	0.512		2.896	0.244		2.457	0.047
			y_{20}	3.631		y_{49}	3.510		y_{94}	2.643	
$\hat{Y}_{BUP}$				3.572	1.440		3.493	0.469		2.641	0.260
$\hat{Y}_{CMP}$				3.367	1.507		3.378	0.477		2.586	0.065
$\hat{Y}_{CAP}$				3.099	1.720		3.312	0.499		2.600	0.064
						y_{50}	4.479		y_{95}	2.764	
$\hat{Y}_{BUP}$							4.493	1.636		2.754	0.324
$\hat{Y}_{CMP}$							4.272	1.489		2.729	0.0836
$\hat{Y}_{CAP}$							3.982	1.694		2.735	0.083
									y_{96}	2.960	
$\hat{Y}_{BUP}$										2.965	0.309
$\hat{Y}_{CMP}$										2.917	0.136
$\hat{Y}_{CAP}$										2.910	0.137
									y_{97}	3.240	
$\hat{Y}_{BUP}$										3.264	0.380
$\hat{Y}_{CMP}$										3.193	0.185
$\hat{Y}_{CAP}$										3.168	0.188
									y_{98}	3.639	
$\hat{Y}_{BUP}$										3.622	0.399
$\hat{Y}_{CMP}$										3.558	0.272
$\hat{Y}_{CAP}$										3.502	0.284
									y_{99}	4.096	
$\hat{Y}_{BUP}$										4.126	0.496
$\hat{Y}_{CMP}$										4.047	0.474
$\hat{Y}_{CAP}$										3.926	0.500
									y_{100}	4.954	
$\hat{Y}_{BUP}$										5.166	0.849
$\hat{Y}_{CMP}$										4.977	0.792
$\hat{Y}_{CAP}$										4.633	0.895

4.1 Simulation results for Gumbel

The results of 1000 times simulations are presented in table 1 for Gumbel distribution. The scatter-plots of predictions versus true observations are also presented in Figure

Table 2: Means and MSEs for Fréchet distribution.

	$n=10$				$n=20$			
	m =7,$\gamma=0.25$		m =7,$\gamma=0.5$		m =16,$\gamma=0.25$		m =16,$\gamma=0.5$	
	Mean	MSE	Mean	MSE	Mean	MSE	Mean	MSE
y_8	1.383		1.986		1.516		2.384	
$\hat{Y}_{BUP}$	1.386	0.020	1.996	0.169	1.518	0.012	2.388	0.122
$\hat{Y}_{CMP}$	1.338	0.022	1.844	0.189	1.481	0.014	2.262	0.137
$\hat{Y}_{CAP}$	1.364	0.021	1.915	0.174	1.504	0.013	2.333	0.124
y_9	1.615		2.526		1.668		3.447	
$\hat{Y}_{BUP}$	1.615	0.081	2.732	0.502	1.679	0.035	2.877	0.352
$\hat{Y}_{CMP}$	1.539	0.087	2.401	0.471	1.628	0.037	2.665	0.375
$\hat{Y}_{CAP}$	1.539	0.087	2.400	0.471	1.639	0.0363	2.702	0.364
y_{10}	2.069		4.678		1.971		3.447	
$\hat{Y}_{BUP}$	2.181	0.330	5.285	1.493	1.952	0.140	3.810	0.918
$\hat{Y}_{CMP}$	1.963	0.327	3.673	2.334	1.863	0.151	3.350	0.777
$\hat{Y}_{CAP}$	1.852	0.363	3.269	3.391	1.844	0.156	3.281	0.793
					y_{20} 2.413		y_{20} 7.016	
$\hat{Y}_{BUP}$					2.548	0.449	7.494	1.535
$\hat{Y}_{CMP}$					2.295	0.444	5.211	4.801
$\hat{Y}_{CAP}$					2.146	0.501	4.557	7.730



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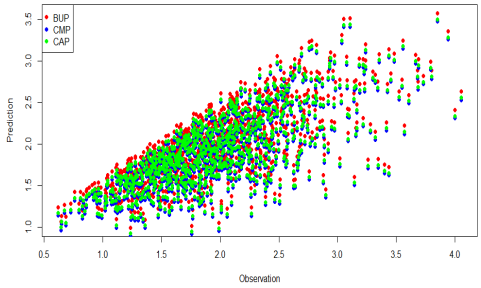

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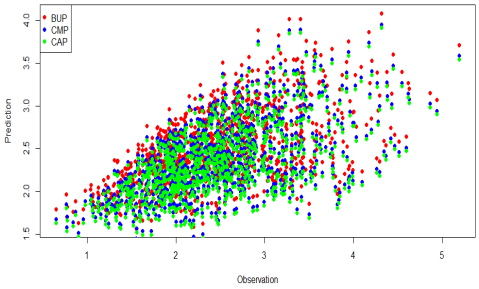

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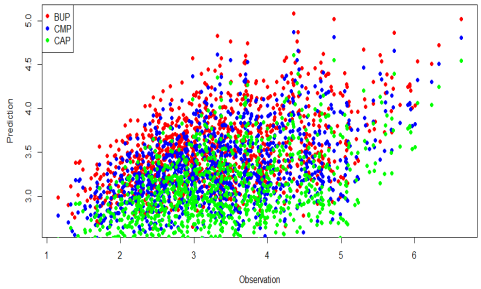


$t=20, \gamma=0$

Figure 1: Plots of predictions vs. observations for Gumbel distribution, $n=20, m=16$.

Table 3: Means and MSEs for Fréchet distribution.

	$n=50$						$n=100$			
	$m=45,\gamma=0.25$		$m=45,\gamma=0.5$		$m=90,\gamma=0.25$		$m=90,\gamma=0.5$	
	Mean	MSE	Mean	MSE	Mean	MSE	Mean	MSE
y_{46}	1.807		y_{46} 3.336		y_{91} 1.783		y_{91} 3.095	
$\hat{Y}_{BUP}$	1.801	0.009	3.329	0.135	1.782	0.021	3.081	0.152
$\hat{Y}_{CMP}$	1.767	0.011	3.197	0.154	1.768	0.002	3.044	0.034
$\hat{Y}_{CAP}$	1.790	0.009	3.281	0.138	1.782	0.002	3.090	0.032
y_{47}	1.931		y_{47} 3.689		y_{92} 1.822		y_{92} 3.212	
$\hat{Y}_{BUP}$	1.934	0.025	3.805	0.287	1.812	0.116	3.210	0.291
$\hat{Y}_{CMP}$	1.891	0.026	3.609	0.279	1.801	0.006	3.131	0.088
$\hat{Y}_{CAP}$	1.906	0.025	3.667	0.273	1.813	0.006	3.172	0.083
y_{48}	2.094		y_{48} 4.370		y_{93} 1.872		y_{93} 3.286	
$\hat{Y}_{BUP}$	2.105	0.073	4.625	0.715	1.867	0.174	3.313	0.423
$\hat{Y}_{CMP}$	2.044	0.075	4.287	0.651	1.839	0.014	3.232	0.111
$\hat{Y}_{CAP}$	2.042	0.074	4.287	0.651	1.849	0.014	3.266	0.109
y_{49}	2.439		y_{49} 5.623		y_{94} 1.922		y_{94} 3.508	
$\hat{Y}_{BUP}$	2.446	0.185	6.021	1.268	1.919	0.268	3.485	0.522
$\hat{Y}_{CMP}$	2.337	0.196	5.295	1.253	1.902	0.015	3.425	0.176
$\hat{Y}_{CAP}$	2.299	0.205	5.124	1.409	1.909	0.015	3.449	0.173
y_{50}	3.015		y_{50} 11.735		y_{95} 2.014		y_{95} 3.764	
$\hat{Y}_{BUP}$	3.241	0.601	12.046	1.347	2.010	0.261	3.815	0.584
$\hat{Y}_{CMP}$	2.921	0.554	8.377	13.171	1.975	0.028	3.675	0.308
$\hat{Y}_{CAP}$	2.716	0.632	7.245	22.472	1.978	0.028	3.685	0.306
					y_{96} 2.134		y_{96} 4.115	
$\hat{Y}_{BUP}$					2.140	0.331	4.238	0.752
$\hat{Y}_{CMP}$					2.075	0.051	4.082	0.412
$\hat{Y}_{CAP}$					2.072	0.051	4.068	0.413
					y_{97} 2.258		y_{97} 4.718	
$\hat{Y}_{BUP}$					2.268	0.256	4.875	0.925
$\hat{Y}_{CMP}$					2.222	0.065	4.667	0.719
$\hat{Y}_{CAP}$					2.208	0.067	4.609	0.729
					y_{89} 2.509		y_{98} 5.624	
$\hat{Y}_{BUP}$					2.493	0.280	5.952	1.141
$\hat{Y}_{CMP}$					2.427	0.146	5.581	1.168
$\hat{Y}_{CAP}$					2.393	0.153	5.426	1.208
					y_{99} 2.878		y_{99} 5.756	
$\hat{Y}_{BUP}$					2.905	0.340	6.036	1.087
$\hat{Y}_{CMP}$					2.763	0.245	5.634	1.362
$\hat{Y}_{CAP}$					2.681	0.271	5.478	1.430
					y_{100} 3.557		y_{100} 16.950	
$\hat{Y}_{BUP}$					3.865	0.785	17.167	1.372
$\hat{Y}_{CMP}$					3.498	0.681	12.019	27.998
$\hat{Y}_{CAP}$					3.209	0.795	10.118	50.621

1. Across all population sizes, from the small case of $n = 10, m = 7$ to the large case of $n = 100, m = 90$, BUP's mean estimate is invariably the closest to the true population value y_k. This establishes it as a virtually unbiased estimator.

Table 4: Means and MSEs for Weibull distribution.

	$n=10$						$n=20$			
	$m=7,\gamma=-0.25$		$m=7,\gamma=-0.5$		$m=16,\gamma=-0.25$		$m=16,\gamma=-0.5$	
	Mean	MSE		Mean	MSE		Mean	MSE		Mean	MSE
y_8	-0.556		y_8	-0.738		y_{17}	-0.666		y_{17}	-0.451	
$\hat{Y}_{BUP}$	-0.553	0.008		-0.736	0.004		-0.668	0.002		-0.459	0.003
$\hat{Y}_{CMP}$	-0.578	0.008		-0.755	0.004		-0.682	0.002		-0.471	0.004
$\hat{Y}_{CAP}$	-0.556	0.008		-0.741	0.004		-0.671	0.002		-0.459	0.003
y_9	-0.442		y_9	-0.654		y_{18}	-0.608		y_{18}	-0.377	
$\hat{Y}_{BUP}$	-0.430	0.014		-0.645	0.009		-0.612	0.004		-0.383	0.006
$\hat{Y}_{CMP}$	-0.439	0.015		-0.659	0.009		-0.624	0.004		-0.394	0.006
$\hat{Y}_{CAP}$	-0.439	0.014		-0.659	0.009		-0.620	0.004		-0.388	0.006
y_{10}	-0.288		y_{10}	-0.516		y_{19}	-0.533		y_{19}	-0.294	
$\hat{Y}_{BUP}$	-0.279	0.020		-0.509	0.019		-0.541	0.008		-0.302	0.008
$\hat{Y}_{CMP}$	-0.269	0.021		-0.516	0.019		-0.551	0.008		-0.306	0.008
$\hat{Y}_{CAP}$	-0.302	0.020		-0.547	0.019		-0.556	0.008		-0.312	0.009
						y_{20}	-0.423		y_{20}	-0.193	
$\hat{Y}_{BUP}$							-0.429	0.013		-0.199	0.009
$\hat{Y}_{CMP}$							-0.434	0.013		-0.191	0.009
$\hat{Y}_{CAP}$							-0.468	0.014		-0.218	0.009

In terms of MSE, BUP is consistently strong, often having the lowest error, especially in smaller samples. In larger samples, while its MSE is sometimes slightly exceeded by CAP's, it remains highly competitive, confirming its overall robustness and reliability as the top-performing method.

The CAP method emerges as a precise but slightly biased challenger, particularly in large samples. Its defining characteristic is a tendency to shrink predictions towards the center, leading to a consistent underestimation of larger true values. This is evident in the $n = 100$ scenario, where for high values like y_{99} and y_{100}, CAP's estimates are systematically lower than the true values and those of BUP. However, this trade-off appears to benefit its precision. For the large population $n = 100, m = 90$, CAP achieves the lowest MSE for the majority of the missing values, such as from y_{92} to y_{98}. This indicates that while its predictions may be slightly off on average, they exhibit less variability from one simulation to the next. Therefore, CAP is an excellent choice when working with large samples and the primary objective is to minimize the total squared prediction error, even if it comes at the cost of a small, consistent bias.

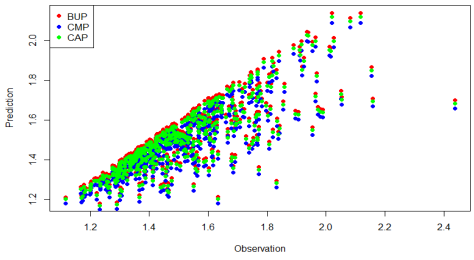
The CMP method is unequivocally the weakest performer in this comparison. It displays the highest level of bias, consistently providing the lowest mean estimate for almost every missing value across all scenarios, which signifies a severe and systematic underestimation problem. Consequently, its MSE is generally the highest among the three methods. There are very few cases, primarily in the $n = 100$ scenario, where its MSE is not the worst, but it never emerges as the best. Based on these results, the CMP method is not recommended for this imputation task, as it fails to provide accurate or precise predictions.

In summary, the hierarchy of methods is clear. BUP is the best overall method, offering superior accuracy and robust performance. CAP is a strong specialized alternative for large samples where minimizing variance is the paramount concern, accepting

Table 5: Means and MSEs for Weibull distribution.

	$n=50$				$n=100$			
	$m=45,\gamma=-0.25$		$m=45,\gamma=-0.5$		$m=90,\gamma=-0.25$		$m=90,\gamma=-0.5$	
	Mean	MSE	Mean	MSE	Mean	MSE	Mean	MSE
y_{46}	-0.560		-0.318		y_{91} -0.564		y_{91} -0.314	
$\hat{Y}_{BUP}$	-0.561	0.001	-0.312	0.006	-0.563	0.002	-0.314	0.001
$\hat{Y}_{CMP}$	-0.570	0.001	-0.329	0.001	-0.569		-0.319	
$\hat{Y}_{CAP}$	-0.563	0.001	-0.320	0.001	-0.565		-0.315	
y_{47}	-0.526		-0.282		y_{92} -0.548		y_{92} -0.299	
$\hat{Y}_{BUP}$	-0.527	0.004	-0.309	0.078	-0.548	0.006	-0.293	0.019
$\hat{Y}_{CMP}$	-0.535	0.002	-0.289	0.002	-0.553	0.001	-0.302	
$\hat{Y}_{CAP}$	-0.531	0.002	-0.285	0.002	-0.550		-0.298	
y_{48}	-0.487		-0.243		y_{93} -0.533		y_{93} -0.282	
$\hat{Y}_{BUP}$	-0.485	0.007	-0.194	0.142	-0.535	0.023	-0.297	0.037
$\hat{Y}_{CMP}$	-0.493	0.003	-0.245	0.003	-0.536	0.001	-0.284	0.001
$\hat{Y}_{CAP}$	-0.493	0.003	-0.245	0.003	-0.534	0.001	-0.281	0.001
y_{49}	-0.429		-0.192		y_{94} -0.515		y_{94} -0.264	
$\hat{Y}_{BUP}$	-0.430	0.007	-0.216	0.069	-0.509	0.041	-0.243	0.036
$\hat{Y}_{CMP}$	-0.437	0.005	-0.193	0.004	-0.518	0.001	-0.265	0.001
$\hat{Y}_{CAP}$	-0.444	0.006	-0.199	0.004	-0.516	0.001	-0.263	0.001
y_{50}	-0.343		-0.128		y_{95} -0.494		y_{95} -0.245	
$\hat{Y}_{BUP}$	-0.343	0.009	-0.121	0.008	-0.501	0.035	-0.256	0.042
$\hat{Y}_{CMP}$	-0.346	0.009	-0.121	0.004	-0.497	0.001	-0.244	0.001
$\hat{Y}_{CAP}$	-0.372	0.009	-0.140	0.004	-0.496	0.001	-0.244	0.001
					y_{96} -0.469		y_{96} -0.222	
$\hat{Y}_{BUP}$					-0.465	0.036	-0.196	0.033
$\hat{Y}_{CMP}$					-0.472	0.002	-0.221	0.001
$\hat{Y}_{CAP}$					-0.473	0.002	-0.222	0.001
					y_{97} -0.441		y_{97} -0.196	
$\hat{Y}_{BUP}$					-0.440	0.034	-0.204	0.036
$\hat{Y}_{CMP}$					-0.444	0.002	-0.195	0.002
$\hat{Y}_{CAP}$					-0.447	0.002	-0.197	0.002
					y_{98} -0.405		y_{98} -0.167	
$\hat{Y}_{BUP}$					-0.399	0.029	-0.150	0.017
$\hat{Y}_{CMP}$					-0.409	0.003	-0.165	0.002
$\hat{Y}_{CAP}$					-0.415	0.003	-0.170	0.002
					y_{99} -0.356		y_{99} -0.132	
$\hat{Y}_{BUP}$					-0.355	0.013	-0.132	0.017
$\hat{Y}_{CMP}$					-0.363	0.004	-0.130	0.002
$\hat{Y}_{CAP}$					-0.374	0.005	-0.138	0.002
					y_{100} -0.284		y_{100} -0.087	
$\hat{Y}_{BUP}$					-0.287	0.010	-0.087	0.002
$\hat{Y}_{CMP}$					-0.288	0.007	-0.082	0.002
$\hat{Y}_{CAP}$					-0.314	0.007	-0.097	0.002

its slight bias as a trade-off. CMP is an inferior method that should be avoided for this purpose. The choice between BUP and CAP in practice would depend on whether the priority is absolute accuracy of the average prediction (favoring BUP) or the min-



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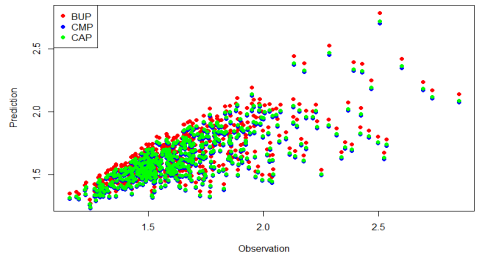

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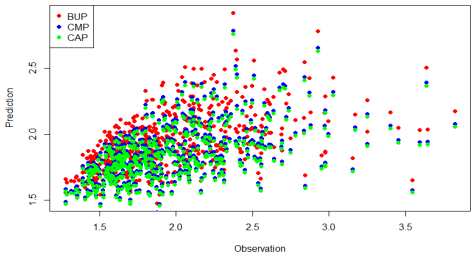

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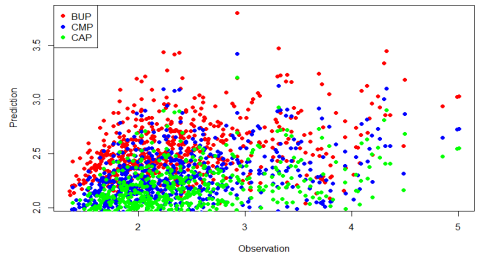

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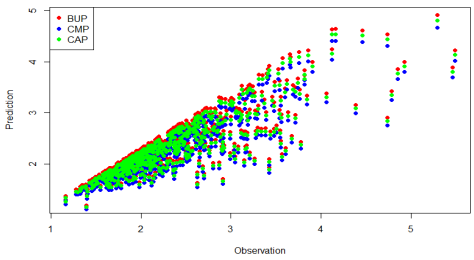

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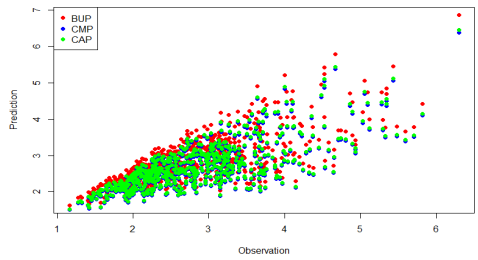

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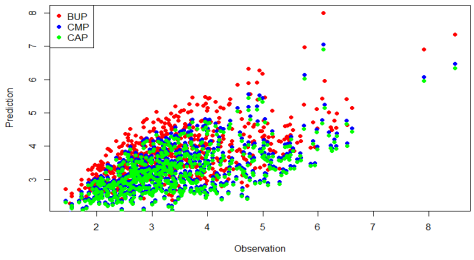

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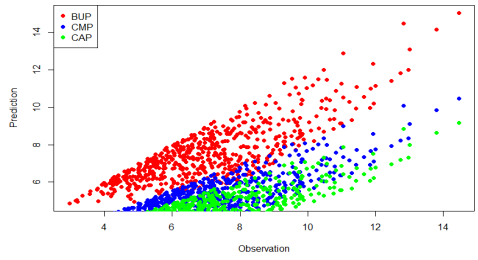
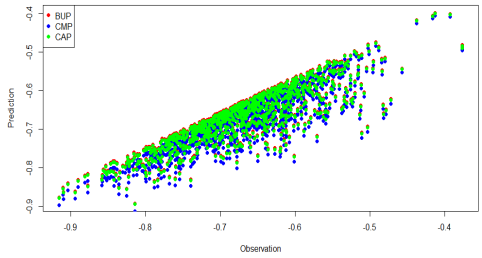


$t = 20, \gamma = 0.5$

Figure 2: Plots of predictions vs. observations for Fréchet distribution, $n = 20$ and $m = 16$.

imization of total imputation error (potentially favoring CAP in large samples).



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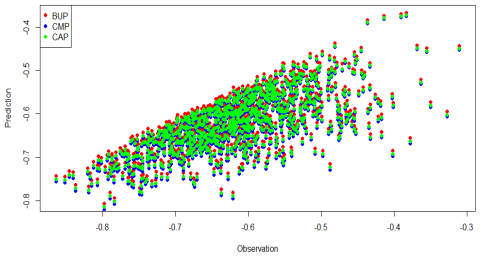

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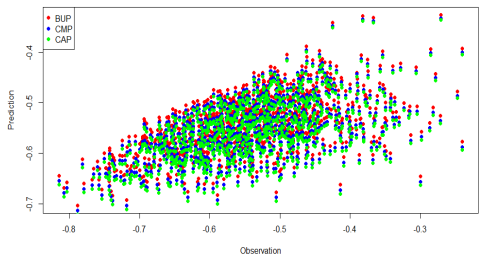

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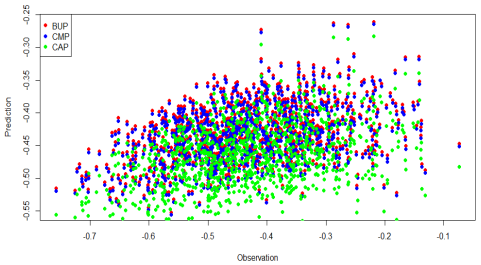

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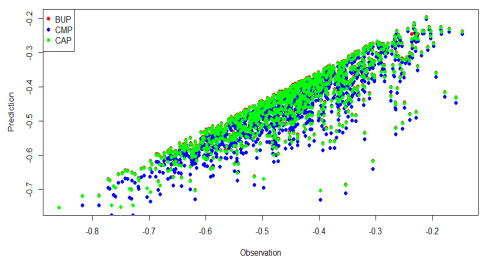

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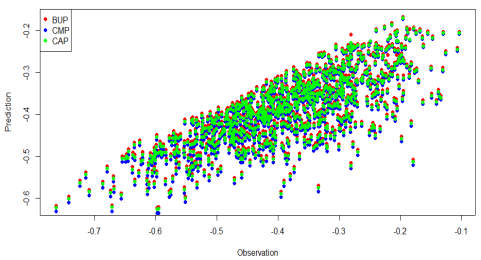

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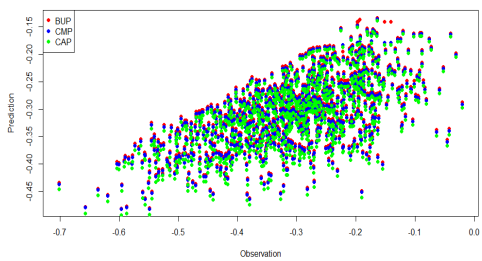

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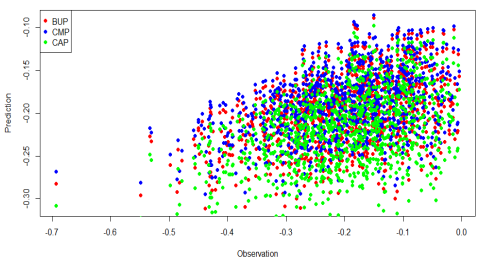

$t = 20, \gamma = -0.5$

Figure 3: Plots of predictions vs. observations for Weibull distribution, $n = 20$ and $m = 16$.

4.2 Simulation results for Fréchet

The values of $\gamma = 0.25$ and 0.5 are considered for Fréchet distribution. The results are presented in tables 2 and 3. The scatter-plots of predictions versus true observations

are presented in Figure 2. The BUP method maintains its dominance, particularly as the data complexity increases with higher positive γ values, while the CAP and CMP methods struggle significantly with bias and stability.

The BUP generally provides the closest mean estimates to the true y values across most scenarios, particularly for larger values of γ. The MSE tends to increase with both γ and the magnitude of the true value, indicating greater prediction difficulty for heavy-tailed distributions and extreme observations. For smaller sample sizes ($n = 10, 20$), BUP typically demonstrates lower MSE compared to CMP and CAP. However, in larger samples ($n = 50, 100$), CMP and CAP sometimes achieve competitive or even lower MSE for moderate observations, though BUP remains superior for the most extreme values. When $\gamma = 0.25$, all predictors perform reasonably well with relatively small errors. However, when $\gamma = 0.5$, prediction errors rise significantly. In particular, CMP and CAP exhibit substantial underestimation of large true values, which leads to considerably higher MSE for extreme observations such as $y_{100} = 16.950$. Overall, BUP shows the most consistent performance across varying conditions, while CMP and CAP offer viable alternatives for certain moderate-value predictions in larger samples.

4.3 Simulation results for Weibull

The values $\gamma = -0.25$ and -0.5 apply to Weibull distribution. The results are presented in Tables 4 and 5. The scatter-plots of predictions versus true observations are presented in Figure 3. The results reveal that the value of γ has a profound impact on the relative performance of the BUP and CAP imputation methods, effectively inverting the hierarchy observed in the previous analysis. The overarching pattern is that the performance of the BUP method is highly sensitive to the value of γ, while the CAP method demonstrates remarkable robustness.

When $\gamma = -0.25$, we observe a performance profile that is consistent with the first analysis. The BUP method generally serves as the gold standard, providing mean estimates that are closest to the true values across most scenarios. For instance, in the case of $n = 100, m = 90$, predicting $y_{100} = -0.284$, BUP's mean estimate is -0.287, which is significantly closer to the truth than CAP's -0.314. Its MSE is also typically the lowest or highly competitive, confirming its status as the Best Predictor under these conditions.

However, a dramatic shift occurs when $\gamma = -0.5$. In this regime, the BUP method becomes unstable and can produce severe outliers with very high MSE, indicating a breakdown in its predictive accuracy. A clear example is seen for $n = 50, m = 45$ when predicting $y_{48} = -0.243$; BUP's mean estimate is -0.194, which is a substantial deviation, and its MSE skyrockets to 0.142, which is an order of magnitude larger than the MSE of CAP and CMP for the same case. This pattern of BUP exhibiting high MSE for certain values when $\gamma = -0.5$ is repeated, for example, with y_{47} and y_{49}. In stark contrast, the CAP method remains stable and reliable. Its mean estimates stay consistently close to the true values, and its MSE remains low and uniform across all predictions for a given γ. This makes CAP the superior and safest choice when $\gamma = -0.5$, as it avoids the catastrophic errors that can affect the BUP method.

The CMP method continues its trend from the previous analysis as the weakest performer. It consistently shows the highest level of bias, typically underestimating the true values more than the other methods. While its MSE is sometimes similar to

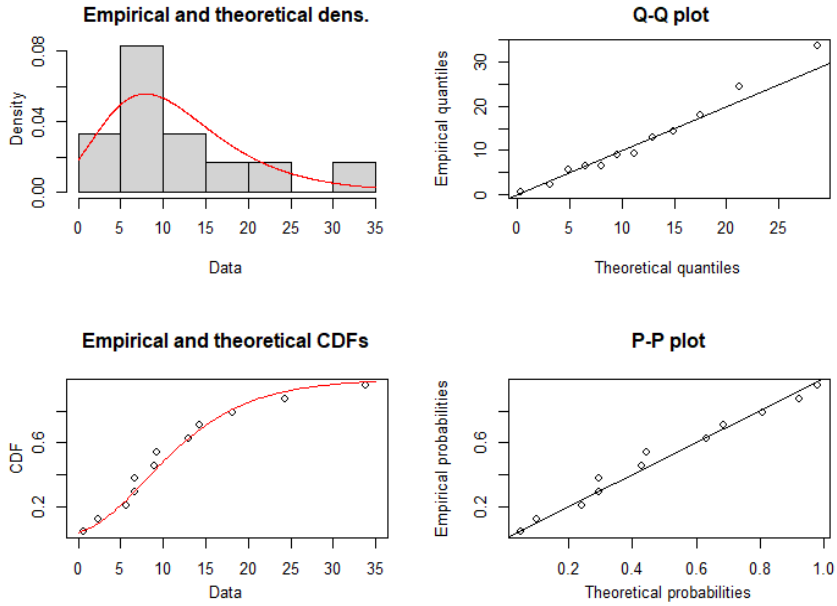

Figure 4: Goodness of fit Images for real data.

CAP's, it is almost never the best and is often the worst, solidifying its position as the least recommended method.

In conclusion, the choice between BUP and CAP is now critically dependent on the value of γ. For a milder parameter value of $\gamma = -0.25$, BUP is the best overall performer, offering high accuracy and reliability. However, for a stronger parameter value of $\gamma = -0.5$, the BUP method becomes unreliable due to its potential for large errors, and the CAP method emerges as the robust and superior choice due to its consistent precision and stability. Therefore, CAP should be preferred in contexts where the underlying data generating process is characterized by a stronger negative γ parameter, or in any situation where the value of γ is uncertain and a robust, fail-safe method is required.

5 Real example

In the following example, we examine a dataset from Dunsmore (1983) pertaining to a rock-crushing machine. In the original study, the observations were modeled using an exponential distribution. The machine requires resetting whenever the size of the rock currently being crushed exceeds that of the immediately preceding rock. The data used here consist of the ordered observations recorded up to the third reset event. To assess model fit, we applied the chi-square goodness-of-fit test, which yielded a p-value of 0.512, indicating no significant departure from the Gumbel distribution. This conclusion is further supported by the diagnostic plots shown in Figure 4, which reveal no systematic deviations from the theoretical distribution. Consequently, the Gumbel distribution was adopted for subsequent analysis.

Next, we evaluated the robustness of the prediction methods under missing data conditions. First, we omitted the final four observations (approximately 0.33 percent of the dataset)-though such a loss may be considered unrealistic in practice. The corresponding prediction results are summarized in the accompanying table and Figure 5. Among the methods, BUP produced the most accurate predictions, as reflected by the lowest mean squared error (MSE).

Real	14.3	18.1	24.4	33.8	MSE
BUP	14.98214	17.49199	21.08191	27.95340	11.5
CMP	14.36776	16.83507	20.27023	26.62340	17.54
CAP	14.81396	17.02532	19.99006	24.83274	25.32

We then repeated the prediction exercise with only the last observed value (33.8) removed. In this scenario, the predicted values were 31.13648 for BUP, 29.11193 for CMP, and 29.11193 for CAP.

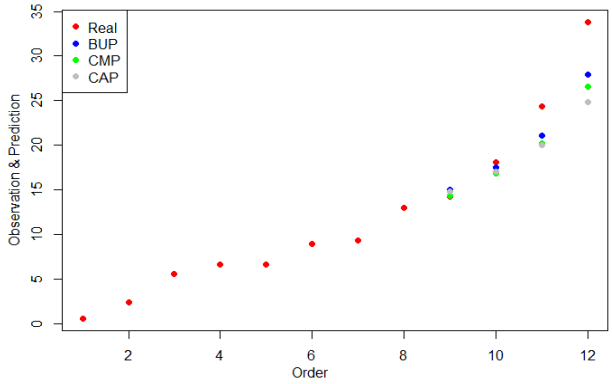

Figure 5: Plot of predictions and real data.

6 Conclusion

This study has addressed the problem of predicting missing order statistics within the generalized extreme value (GEV) family. A necessary and sufficient condition for the existence of the conditional moments of order statistics was established based on the domain of the extreme value index (EVI), γ. Three prediction methodologies-the best unbiased predictor (BUP), the conditional median predictor (CMP), and the conditional average predictor (CAP)-were formulated and derived for the Fréchet, Gumbel, and Weibull distributions. Comprehensive simulations were conducted to evaluate their performance. The results demonstrate that the optimal predictor is contingent upon the specific distribution within the GEV family: For the Gumbel distribution, the BUP method demonstrated superior overall performance, offering high predictive accuracy and robustness. The CAP method served as a viable alternative in large-sample scenarios where variance minimization is the primary objective, despite its inherent bias. The CMP method was found to be comparatively ineffective and is not recommended. For the Fréchet distribution, the BUP method was unequivocally optimal, providing

unbiased estimates and maintaining predictive accuracy. In contrast, the CAP and CMP methods exhibited fundamental limitations due to severe underestimation and instability. The superiority of BUP increased with larger values of γ and greater data extremity. For the Weibull distribution, the efficacy of BUP and CAP was critically dependent on γ. For a milder shape parameter ($\gamma = 0.25$), BUP provided high accuracy and reliability. However, for a stronger shape parameter ($\gamma = 0.5$), BUP became unreliable, while CAP emerged as the more robust and precise choice. Therefore, CAP is recommended in contexts involving a strongly negative γ. In summary, this research provides a clear framework and comparative analysis for selecting an appropriate prediction method for missing extreme order statistics, with recommendations tailored to the underlying distributional characteristics.

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