Journal of Statistical Modelling: Theory and Applications
Vol. 6, No. 2, 2025, pp. 45-67
Yazd University Press 2025

JSMTA

Research Paper

Multicomponent stress-strength parameter in exponentiated Kumaraswamy distribution and progressive censoring

Akram Kohansal*[1], Zahra Barikbin[2]
[1]Department of Statistics, Imam Khomeini International University, Qazvin, Iran
[2]Department of Applied Mathematics, Imam Khomeini International University, Qazvin, Iran

Received: May 11, 2025/ Revised: January 27, 2026/ Accepted: April 16, 2026

Abstract: This study conducts statistical inference on a Multicomponent reliability stress-strength system with non-identical component strengths, using the exponentiated Kumaraswamy distribution for progressively censored samples. We investigate estimation of the system reliability parameter via both classical and Bayesian inference frameworks, including maximum likelihood estimation, approximate Bayesian estimation, and highest posterior density interval construction. We compare the performance of different estimators using Monte Carlo simulation, with evaluation metrics including mean squared error and coverage probability. A real-data example is also provided to demonstrate the effectiveness of the proposed model, highlighting its practical utility and relevance for both reliability engineering and statistical analysis.

Keywords: Bayesian estimation; Classical estimation; Multicomponent stress-strength model; Progressive censoring.
Mathematics Subject Classification (2010): 62F10, 62F15, 62N02.

1 Introduction

In reliability analysis, statistical inference for stress–strength models plays a central role in evaluating system performance under uncertainty. In this paper, we study the Multicomponent stress-strength reliability parameter $R_{\boldsymbol{s},\boldsymbol{k}}$ under progressive Type-II censoring schemes, where both stress and strength variables follow the exponentiated

*Corresponding author: `kohansal@ikiu.ac.ir`

Kumaraswamy distribution. The main objective of this study is to develop classical and Bayesian inferential procedures for $R_{\boldsymbol{s},\boldsymbol{k}}$ in the presence of realistic censoring mechanisms. The novelty of this work lies in investigating a generalized Multicomponent stress-strength model with non-identical strength components under progressive censoring, which has not been previously explored for the exponentiated Kumaraswamy distribution.

Censoring schemes are commonly employed in life-testing experiments to reduce time and cost. Among them, progressive Type-II censoring provides greater flexibility by allowing the removal of surviving units at pre-specified failure times. In this scheme, N units are placed on test and the experiment continues until n failures are observed, with $R_1, \ldots, R_n$ units removed sequentially such that $R_1 + \ldots + R_n + n = N$. This scheme includes conventional Type-II censoring and complete sampling as special cases. Further details on progressive censoring schemes can be found in Balakrishnan and Aggarwala (2000).

In the field of reliability analysis, one of the key and foundational problems is inferring the stress-strength parameter. This parameter, denoted by $R = P(Y < X)$, where Y and X represent stress and strength respectively, are independent random variables. For a system to be reliable, the strength X must exceed the stress Y. The intersection of statistics and reliability theory allows for the estimation of the stress-strength parameter, which began with the groundbreaking work of Birnbaum (1956). Since then, numerous researchers have delved into inferring the reliability parameter using both classical and Bayesian methods. Various authors have examined this model for both complete and censored samples, including Kohansal and Nadarajah (2019), Shoaee and Khorram (2016), and Nadar et al. (2014).

Reliability engineers refer to a system with more than one component as a Multicomponent system. In such a system, there is a common stress component and k identical and independent strength components. The system is considered reliable as long as at least s out of the k strength components exceed the stress level. This model was first introduced by Bhattacharyya and Johnson (1974) as

$$R_{s,k} = \sum_{p=s}^{k} \binom{k}{p} \int_{-\infty}^{\infty} \big(1 - F_X(y)\big)^p \big(F_X(y)\big)^{k-p} dF_Y(y), \tag{1}$$

where the strength variables $(X_1, \ldots, X_k)$ are assumed to be independent and identically distributed with a cumulative distribution function $F_X(\cdot)$, and the stress variable Y has a cumulative distribution function $F_Y(\cdot)$. This model has since garnered significant attention and has been explored for both complete and censored samples by researchers such as Nadar and Kızılaslan (2016), Kohansal (2019), Kızılaslan and Nadar (2018), and Kohansal and Shoaee (2021).

As previously mentioned, it is important to consider the assumption that strengths are independent and identically distributed (i.i.d.) in various applications. However, this assumption may not hold in practical scenarios where system components have different structures. Therefore, in this study, our focus will be on Multicomponent stress-strength models with non-identical random strengths.

The study revolves around a system consisting of $\boldsymbol{k} = (k_1, \ldots, k_m)$ strength components. In such systems, the components follow non-identical distributions, where the i-th component, with $i = 1, \ldots, m$, has a cumulative distribution function (CDF) of $F_X(\cdot)$ and all strength variables are influenced by a common stress Y with a CDF of $F_Y(\cdot)$. This system's reliability depends on at least $\boldsymbol{s} = (s_1, \ldots, s_m)$ out of the $\boldsymbol{k}$ strength components exceeding the stress level. Building on previous work, Rasethuntsa and Nadar (2018) have enhanced the model described in (1) as

$$R_{\boldsymbol{s},\boldsymbol{k}} = \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \left(\prod_{l=1}^{m} \binom{k_l}{p_l} \right) \int_{-\infty}^{\infty} \prod_{l=1}^{m} \left(\left(1 - F_l(y)\right)^{p_l} \left(F_l(y)\right)^{k_l - p_l} \right) dF_Y(y). \quad (2)$$

In the recent studies by Kohansal et al. (2023) and Kohansal (2025), the model was examined for the modified Weibull extension and exponentiated Weibull distributions in the context of progressive censored data. However, we now investigate this model for progressive censoring samples, where the strengths and stress variables are modeled using the exponentiated Kumaraswamy distribution.

The Kumaraswamy distribution is widely utilized in reliability, engineering, finance, hydrology, physics, and environmental studies for modeling lifetime data. It offers flexibility in modeling failure time data, allowing for an increasing failure rate function. Recognizing its significance, various modifications have been introduced, including the exponentiated Kumaraswamy distribution (EKuD) proposed by Lemontea et al. (2013). Several classical lifetime distributions, such as the Weibull and Beta distributions, have been widely used in stress-strength reliability modeling. The Weibull distribution is particularly popular due to its flexibility in modeling monotone hazard rate functions, while the Beta distribution is well suited for bounded data. However, these models may be limited in capturing a wide range of hazard rate behaviors within a single framework. The exponentiated Kumaraswamy distribution provides greater flexibility through its additional shape parameters, allowing for more adaptable modeling of lifetime data with bounded support. This added flexibility makes the EKuD a suitable alternative for stress-strength models under progressive censoring, especially when complex failure rate patterns are observed. The PDF, CDF, and hazard failure rate function (HFR) of the EKuD are defined as

$$\begin{aligned} f(x) &= \alpha\beta\gamma x^{\alpha-1}(1-x^\alpha)^{\beta-1}\left(1-(1-x^\alpha)^\beta\right)^{\gamma-1}, \qquad 0<x<1,\ \alpha,\beta,\gamma>0, \quad (3)\\ F(x) &= \left(1-(1-x^\alpha)^\beta\right)^\gamma, \qquad 0<x<1,\ \alpha,\beta,\gamma>0, \quad (4)\\ H(x) &= \frac{\alpha\beta\gamma x^{\alpha-1}(1-x^\alpha)^{\beta-1}\left(1-(1-x^\alpha)^\beta\right)^{\gamma-1}}{1-\left(1-(1-x^\alpha)^\beta\right)^\gamma}, \qquad 0<x<1,\ \alpha,\beta,\gamma>0. \end{aligned}$$

We define a particular distribution, denoted as EKu(α, β, γ), with parameters α, β and γ. Some shapes of the PDF and HFR of EKuD are provided in Figure 1. The EKuD distribution exhibits an increasing failure rate function. Therefore, when empirical studies indicate that the failure rate function of the distribution is decreasing or bathtub-shaped, the EKuD distribution can be utilized for analyzing such datasets. This distribution is suitable for modeling various natural phenomena with bounded outcomes, such as individual heights, test scores, atmospheric temperatures, hydrological data, economic indicators like unemployment rates, and more.

We proceed with the paper as outlined below. Section 2 discusses the point statistical estimation of $R_{\boldsymbol{s},\boldsymbol{k}}$ in the general case, where we derive the MLE and Bayesian estimation. Due to the lack of an explicit form, we approximate the Bayesian estimation using the Markov Chain Monte Carlo (MCMC) method. Additionally, we

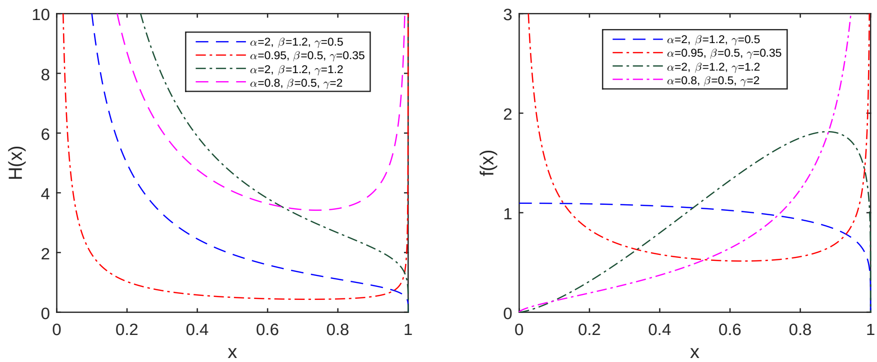

Figure 1: Shape of HFR (left) and PDF (right) functions of EKuD.

investigate the HPD intervals for interval estimation. In Section 3, we consider the statistical estimation of $R_{\boldsymbol{s},\boldsymbol{k}}$ when the common parameters are unknown. Here, we obtain the MLE, approximate Bayesian estimation and HPD intervals. In Section 4, we consider the statistical estimation of $R_{\boldsymbol{s},\boldsymbol{k}}$ when the common parameters are known. Here, we obtain the MLE, approximate Bayesian estimation by two methods MCMC and Lindley's approximation and HPD intervals. Section 5 presents a comparison of theoretical methods using numerical simulation results, and a real dataset is used to demonstrate the applicability of this new model. Finally, we provide the conclusions in Section 6.

2 General case

When analyzing a real data set, researchers often encounter a scenario where the common parameters vary. Therefore, studying this particular case is crucial. Furthermore, some formulas in Section 3 can be derived from this scenario.

We have $X_1 \sim EKu(\alpha_1, \beta_1, \gamma_1)$, $X_2 \sim EKu(\alpha_2, \beta_2, \gamma_2)$, and so on up to $X_m \sim EKu(\alpha_m, \beta_m, \gamma_m)$ as a collection of independent random variables as well as a random variable $Y \sim EKu(\alpha, \beta, \gamma)$. Through the use of (3) and (4), we are able to compute the Multicomponent reliability with non-identical component strengths in (2) as illustrated below

$$
\begin{aligned}
R_{\boldsymbol{s},\boldsymbol{k}} &= \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \binom{k_1}{p_1} \cdots \binom{k_m}{p_m} \int_0^1 \alpha\beta\gamma y^{\alpha-1}(1-y^{\alpha})^{\beta-1}\left(1-(1-y^{\alpha})^{\beta}\right)^{\gamma-1} \\
&\quad \times \prod_{l=1}^{m} \left(1-\left(1-(1-y^{\alpha_l})^{\beta_l}\right)^{\gamma_l}\right)^{p_l} \left(1-(1-y^{\alpha_l})^{\beta_l}\right)^{\gamma_l(k_l-p_l)} dy.
\end{aligned}
$$

2.1 MLE of $R_{\boldsymbol{s},\boldsymbol{k}}$

To derive the MLE of $R_{\boldsymbol{s},\boldsymbol{k}}$, we make use of the invariance property. We begin by obtaining the MLEs for the unknown parameters $\alpha_1, \ldots, \alpha_m, \alpha$, $\beta_1, \ldots, \beta_m, \beta$ and

$\gamma_1, \dots, \gamma_m, \gamma$. Next, we conduct a life-testing experiment with n systems and construct the likelihood function. Thus, the observed samples can be expressed as follows

$$\overset{\text{Observed stress variables}}{Y = \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}} \quad \text{and} \quad \overset{\text{Observed strength variables}}{X_l = \begin{bmatrix} X_{11}^{(l)} & \dots & X_{1k_l}^{(l)} \\ \vdots & \ddots & \vdots \\ X_{n1}^{(l)} & \dots & X_{nk_l}^{(l)} \end{bmatrix}}, \ l = 1, \dots, m.$$

In the following, we make the assumption that $Y_1, \dots, Y_n$ represents a progressive censoring sample from $EKu(\alpha, \beta, \gamma)$ with the censoring scheme $N, n, S_1, \dots, S_n$. Additionally, we have $X_{i91}^{(l)}, \dots, X_{ik_l}^{(l)}$ for $i = 1, \dots, n$, $l = 1, \dots, m$ as a progressive censoring sample from $EKu(\alpha_l, \beta_l, \gamma_l)$ with the censoring scheme $K_l, k_l, R_{i1}^{(l)}, \dots, R_{ik_l}^{(l)}$, where $i = 1, \dots, n$, $l = 1, \dots, m$. With this setup, we can express the likelihood function as a function of the parameters values, $\Theta = (\alpha_1, \dots, \alpha_m, \alpha, \beta_1, \dots, \beta_m, \beta, \gamma_1, \dots, \gamma_m, \gamma)$ by

$$L(\Theta|\text{data}) \propto \prod_{i=1}^{n} \left(\prod_{l=1}^{m} \Big(\prod_{j_l=1}^{k_l} f_l(x_{ij_l}^{(l)}) \big(1 - F_l(x_{ij_l}^{(l)})\big)^{R_{ij_l}^{(l)}} \Big) \right) f_Y(y_i) \big(1 - F_Y(y_i)\big)^{S_i}.$$

Regarding the benefit of this likelihood function, it can be considered a versatile function from which various other likelihood functions can be derived. Some examples include:

- When $R_{ij_l}^{(l)} = 0$ and $S_i = 0$, the likelihood function becomes $R_{\boldsymbol{s},\boldsymbol{k}}$ in the context of a complete sample.
- Setting $\boldsymbol{k} = (k_1, k_2, 0, \dots, 0)$ results in $R_{\boldsymbol{s},\boldsymbol{k}}$ with two non-identical components in the progressive censoring case.
- For $\boldsymbol{k} = (k, 0, \dots, 0)$, the likelihood function simplifies to $R_{s,k}$ in the progressive censoring scenario.
- Combining $\boldsymbol{k} = (k_1, k_2, 0, \dots, 0)$ with $R_{ij_l}^{(l)} = 0$ and $S_i = 0$ leads to $R_{\boldsymbol{s},\boldsymbol{k}}$ with two non-identical components in the complete sample case.
- Similarly, using $\boldsymbol{k} = (k, 0, \dots, 0)$ with $R_{ij_l}^{(l)} = 0$ and $S_i = 0$ gives $R_{s,k}$ in the complete sample case.
- If $\boldsymbol{k} = (1, 0, \dots, 0)$, the likelihood function represents $R = P(X < Y)$ in the progressive censoring case.
- Lastly, combining $\boldsymbol{k} = (1, 0, \dots, 0)$ with $R_{ij_l}^{(l)} = 0$ and $S_i = 0$ results in $R = P(X < Y)$ in the context of a complete sample.

To derive the likelihood function from the observed data, we can follow the steps below

$$\begin{aligned} L(\Theta|\text{data}) \quad \propto \quad & \Big(\prod_{l=1}^{m} (\alpha_l \beta_l \gamma_l)^{nk_l} \Big) (\alpha\beta\gamma)^n \prod_{l=1}^{m} \big(B_{n,k_l}^1(\alpha_l) B_{n,k_l}^2(\alpha_l, \beta_l) B_{n,k_l}^3(\alpha_l, \beta_l, \gamma_l) \big) \\ & \times C_n^1(\alpha) C_n^2(\alpha, \beta) C_n^3(\alpha, \beta, \gamma), \end{aligned}$$

where

$$B_{n,k_l}^1(\alpha_l) \quad = \quad \prod_{i=1}^{n} \prod_{j_l=1}^{k_l} (x_{ij_l}^{(l)})^{\alpha_l - 1}, \quad B_{n,k_l}^2(\alpha_l, \beta_l) = \prod_{i=1}^{n} \prod_{j_l=1}^{k_l} \Big(1 - (x_{ij_l}^{(l)})^{\alpha_l}\Big)^{\beta_l - 1},$$

$$B^3_{n,k_l}(\alpha_l,\beta_l,\gamma_l) = \prod_{i=1}^{n}\prod_{j_l=1}^{k_l}\left(A^{\gamma_l-1}_{ij_l}(\alpha_l,\beta_l)\times\left(1-A^{\gamma_l}_{ij_l}(\alpha_l,\beta_l)\right)^{R^{(l)}_{ij}}\right), \tag{5}$$

$$C^1_n(\alpha) = \prod_{i=1}^{n} y_i^{\alpha-1}(1-y_i^\alpha)^{\beta-1}, \quad C^2_n(\alpha,\beta)=\prod_{i=1}^{n}(1-y_i^\alpha)^{\beta-1},$$

$$C^3_n(\alpha,\beta,\gamma) = \prod_{i=1}^{n}\left(A^{\gamma-1}_i(\alpha,\beta)\times\left(1-A^\gamma_i(\alpha,\beta)\right)^{S_i}\right), \tag{6}$$

in which

$$A_{ij_l}(\alpha_l,\beta_l) = 1-\left(1-\left(x^{(l)}_{ij_l}\right)^{\alpha_l}\right)^{\beta_l}, \quad A_i(\alpha,\beta) = 1-(1-y_i^\alpha)^\beta. \tag{7}$$

To calculate the MLEs of unknown parameters, we must first derive the log-likelihood function from the provided function and then solve the following equations simultaneously.

$$\begin{aligned}
\frac{\partial\ell}{\partial\alpha_l} =& \frac{nk_l}{\alpha_l} + \sum_{i=1}^{n}\sum_{j_l=1}^{k_l}\log\left(x^{(l)}_{ij_l}\right) - \sum_{i=1}^{n}\sum_{j_l=1}^{k_l}(\beta_l-1)\frac{\left(x^{(l)}_{ij_l}\right)^{\alpha_l}\log\left(x^{(l)}_{ij_l}\right)}{1-\left(x^{(l)}_{ij_l}\right)^{\alpha_l}}\\
&+\sum_{i=1}^{n}\sum_{j_l=1}^{k_l}(\gamma_l-1)\frac{\beta_l\left(1-\left(x^{(l)}_{ij_l}\right)^{\alpha_l}\right)^{\beta_l-1}\left(x^{(l)}_{ij_l}\right)^{\alpha_l}\log\left(x^{(l)}_{ij_l}\right)}{A_{ij_l}(\alpha_l,\beta_l)}\\
&-\sum_{i=1}^{n}\sum_{j_l=1}^{k_l}R^{(l)}_{ij}\frac{\gamma_l\beta_l A^{\gamma_l-1}_{ij_l}(\alpha_l,\beta_l)\left(1-\left(x^{(l)}_{ij_l}\right)^{\alpha_l}\right)^{\beta_l-1}\left(x^{(l)}_{ij_l}\right)^{\alpha_l}\log\left(x^{(l)}_{ij_l}\right)}{1-A^{\gamma_l}_{ij_l}(\alpha_l,\beta_l)}, \; l=1,\ldots,m,\\
\frac{\partial\ell}{\partial\alpha} =& \frac{n}{\alpha} + \sum_{i=1}^{n}\log(y_i) - (\beta-1)\sum_{i=1}^{n}\frac{y_i^\alpha\log(y_i)}{1-y_i^\alpha} + (\gamma-1)\beta\sum_{i=1}^{n}\frac{\left(1-y_i^\alpha\right)^{\beta-1}y_i^\alpha\log(y_i)}{A_i(\alpha,\beta)}\\
&-\gamma\beta\sum_{i=1}^{n}S_i\frac{A^{\gamma-1}_i(\alpha,\beta)\left(1-y_i^\alpha\right)^{\beta-1}y_i^\alpha\log(y_i)}{1-A^\gamma_i(\alpha,\beta)},\\
\frac{\partial\ell}{\partial\beta_l} =& \frac{nk_l}{\beta_l} + \sum_{i=1}^{n}\sum_{j_l=1}^{k_l}\log\left(1-\left(x^{(l)}_{ij_l}\right)^{\alpha_l}\right)\\
&-\sum_{i=1}^{n}\sum_{j_l=1}^{k_l}(\gamma_l-1)\frac{\left(1-\left(x^{(l)}_{ij_l}\right)^{\alpha_l}\right)^{\beta_l}\log\left(1-\left(x^{(l)}_{ij_l}\right)^{\alpha_l}\right)}{A_{ij_l}(\alpha_l,\beta_l)}\\
&+\sum_{i=1}^{n}\sum_{j_l=1}^{k_l}R^{(l)}_{ij}\frac{\gamma_l A^{\gamma_l-1}_{ij_l}(\alpha_l,\beta_l)\left(1-\left(x^{(l)}_{ij_l}\right)^{\alpha_l}\right)^{\beta_l}\log\left(1-\left(x^{(l)}_{ij_l}\right)^{\alpha_l}\right)}{1-A^{\gamma_l}_{ij_l}(\alpha_l,\beta_l)}, \; l=1,\ldots,m,\\
\frac{\partial\ell}{\partial\beta} =& \frac{n}{\beta} + \sum_{i=1}^{n}\log(1-y_i^\alpha) - (\gamma-1)\sum_{i=1}^{n}\frac{\left(1-y_i^\alpha\right)^\beta\log(1-y_i^\alpha)}{A_i(\alpha,\beta)}\\
&-\gamma\sum_{i=1}^{n}S_i\frac{A^{\gamma-1}_i(\alpha,\beta)\left(1-y_i^\alpha\right)^\beta\log(1-y_i^\alpha)}{1-A^\gamma_i(\alpha,\beta)},
\end{aligned}$$

$$\frac{\partial \ell}{\partial \gamma_l} = \frac{nk_l}{\gamma_l} + \sum_{i=1}^{n}\sum_{j_l=1}^{k_l} \log\big(A_{ij_l}(\alpha_l,\beta_l)\big)$$
$$-\sum_{i=1}^{n}\sum_{j_l=1}^{k_l} R_{ij}^{(l)} \frac{A_{ij_l}^{\gamma_l}(\alpha_l,\beta_l)\log\big(A_{ij_l}(\alpha_l,\beta_l)\big)}{1-A_{ij_l}^{\gamma_l}(\alpha_l,\beta_l)}, \ l=1,\dots,m,$$
$$\frac{\partial \ell}{\partial \gamma} = \frac{n}{\gamma} + \sum_{i=1}^{n}\log\big(A_i(\alpha,\beta)\big) - \sum_{i=1}^{n} S_i \frac{A_i^{\gamma}(\alpha,\beta)\log\big(A_i(\alpha,\beta)\big)}{1-A_i^{\gamma}(\alpha,\beta)}.$$

The MLEs can be obtained by solving these equations together using a numerical method such as the Newton-Raphson algorithm. Lastly, the invariance property of MLEs dictates that the MLE of $R_{\boldsymbol{s},\boldsymbol{k}}$, denoted as $\widehat{R}_{\boldsymbol{s},\boldsymbol{k}}^{MLE}$, can be calculated as

$$\widehat{R}_{\boldsymbol{s},\boldsymbol{k}}^{MLE} = \sum_{p_1=s_1}^{k_1}\cdots\sum_{p_m=s_m}^{k_m}\binom{k_1}{p_1}\cdots\binom{k_m}{p_m}\int_0^1 \widehat{\alpha}\widehat{\beta}\widehat{\gamma}y^{\widehat{\alpha}-1}(1-y^{\widehat{\alpha}})^{\widehat{\beta}-1}\big(1-(1-y^{\widehat{\alpha}})^{\widehat{\beta}}\big)^{\widehat{\gamma}-1}$$
$$\times\prod_{l=1}^{m}\Big(1-\big(1-(1-y^{\widehat{\alpha}_l})^{\widehat{\beta}_l}\big)^{\widehat{\gamma}_l}\Big)^{p_l}\big(1-(1-y^{\widehat{\alpha}_l})^{\widehat{\beta}_l}\big)^{\widehat{\gamma}_l(k_l-p_l)}dy. \tag{8}$$

2.2 Bayes estimation of $R_{\boldsymbol{s},\boldsymbol{k}}$

In this part of the analysis, using the squared error loss function, our goal is to determine the estimation and credible intervals for $R_{\boldsymbol{s},\boldsymbol{k}}$. We are assuming that the unknown parameters follow independent gamma distributions. Thus, we must take into account the prior distributions of these parameters as:

$$\begin{aligned} &\alpha_l \sim \Gamma(a_l,b_l),\ l=1,\dots,m, \quad \alpha\sim\Gamma(a,b),\\ &\beta_l \sim \Gamma(c_l,d_l),\ l=1,\dots,m, \quad \beta\sim\Gamma(c,d),\\ &\gamma_l \sim \Gamma(e_l,f_l),\ l=1,\dots,m, \quad \gamma\sim\Gamma(e,f). \end{aligned}$$

By this selection, for $\Theta=(\alpha_1,\dots,\alpha_m,\alpha,\beta_1,\dots,\beta_m,\beta,\gamma_1,\dots,\gamma_m,\gamma)$, the joint posterior density function can be written by

$$\pi(\Theta|\text{data}) \propto L(\Theta|\text{data})\Big(\prod_{l=1}^{3m+3}\pi_l(\theta_l)\Big). \tag{9}$$

Upon conducting some calculations, it is determined that the Bayes estimations of the unknown parameters cannot be derived in closed forms from equation (9). Therefore, an approximation method is necessary to obtain the Bayesian estimations. To achieve this, we suggest using the MCMC method. By simplifying (9), we can express it as:

$$\begin{aligned}\pi(\Theta|\text{data}) \ \propto\ & \Big(\prod_{l=1}^{m}\alpha_l^{nk_l+a_l-1}e^{-\alpha_l b_l}\Big)\Big(\prod_{l=1}^{m}\beta_l^{nk_l+c_l-1}e^{-\beta_l d_l}\Big)\Big(\prod_{l=1}^{m}\gamma_l^{nk_l+e_l-1}e^{-\gamma_l f_l}\Big)\\ &\times\alpha^{n+a-1}e^{-\alpha b}\times\beta^{n+c-1}e^{-\beta d}\times\gamma^{n+e-1}e^{-\gamma f}\\ &\times\prod_{l=1}^{m}\big(B_{n,k_l}^{1}(\alpha_l)B_{n,k_l}^{2}(\alpha_l,\beta_l)B_{n,k_l}^{3}(\alpha_l,\beta_l,\gamma_l)\big)C_n^1(\alpha)C_n^2(\alpha,\beta)C_n^3(\alpha,\beta,\gamma),\end{aligned}$$

where $B^i_{n,k_l}(\cdot)$ and $C^i_n(\cdot)$, $i = 1,2,3$ are specified in (5) and (6). Consequently, the posterior probability density functions (PDFs) of the parameters can be computed.

$$\begin{aligned}
\pi(\alpha_l|\beta_l,\gamma_l,\text{data}) &\propto \alpha_l^{nk_l+a_l-1}e^{-\alpha_l b_l} \\
&\quad \times B^1_{n,k_l}(\alpha_l)B^2_{n,k_l}(\alpha_l,\beta_l)B^3_{n,k_l}(\alpha_l,\beta_l,\gamma_l),\ l=1,\ldots,m, \\
\pi(\alpha|\beta,\gamma,\text{data}) &\propto \alpha^{n+a-1}e^{-\alpha b}\times C^1_n(\alpha)C^2_n(\alpha,\beta)C^3_n(\alpha,\beta,\gamma), \\
\pi(\beta_l|\alpha_l,\gamma_l,\text{data}) &\propto \beta_l^{nk_l+c_l-1}e^{-\beta_l d_l}\times B^2_{n,k_l}(\alpha_l,\beta_l)B^3_{n,k_l}(\alpha_l,\beta_l,\gamma_l),\ l=1,\ldots,m, \\
\pi(\beta|\alpha,\gamma,\text{data}) &\propto \beta^{n+c-1}e^{-\beta d}\times C^2_n(\alpha,\beta)C^3_n(\alpha,\beta,\gamma), \\
\pi(\gamma_l|\alpha_l,\beta_l,\text{data}) &\propto \gamma_l^{nk_l+e_l-1}e^{-\gamma_l f_l}\times B^3_{n,k_l}(\alpha_l,\beta_l,\gamma_l),\ l=1,\ldots,m, \\
\pi(\gamma|\alpha,\beta,\text{data}) &\propto \gamma^{n+e-1}e^{-\gamma f}\times C^3_n(\alpha,\beta,\gamma).
\end{aligned}$$

As we see, the posterior probability density functions for whole parameters are not readily defined distributions. Hence, we employ the Metropolis-Hastings method to produce random samples from them, facilitating the implementation of the Gibbs sampling algorithm in the ensuing manner:

1. Start with $(\alpha_{1(0)},\ldots,\alpha_{m(0)},\alpha_{(0)},\beta_{1(0)},\ldots,\beta_{m(0)},\beta_{(0)},\gamma_{1(0)},\ldots,\gamma_{m(0)},\gamma_{(0)})$.
2. Initialize $t=1$.
3. Generate $\alpha_{1(t)}$ from $\pi(\alpha_1|\beta_{1(t-1)},\gamma_{1(t-1)},\text{data})$ using the Metropolis-Hastings method with a proposal distribution of $N(\alpha_{1(t-1)},1)$.

$\vdots$

$m+2$. Generate $\alpha_{m(t)}$ from $\pi(\alpha_m|\beta_{m(t-1)},\gamma_{m(t-1)},\text{data})$ using the Metropolis-Hastings method with a proposal distribution of $N(\alpha_{m(t-1)},1)$.
$m+3$. Generate $\alpha_{(t)}$ from $\pi(\alpha|\beta_{(t-1)},\gamma_{(t-1)},\text{data})$ using the Metropolis-Hastings method with a proposal distribution of $N(\alpha_{(t-1)},1)$.
$m+4$. Generate $\beta_{1(t)}$ from $\pi(\beta_1|\alpha_{1(t-1)},\gamma_{1(t-1)},\text{data})$ using the Metropolis-Hastings method with a proposal distribution of $N(\beta_{1(t-1)},1)$.

$\vdots$

$2m+3$. Generate $\beta_{m(t)}$ from $\pi(\beta_m|\alpha_{m(t-1)},\gamma_{m(t-1)},\text{data})$ using the Metropolis-Hastings method with a proposal distribution of $N(\beta_{m(t-1)},1)$.
$2m+4$. Generate $\beta_{(t)}$ from $\pi(\beta|\alpha_{(t-1)},\gamma_{(t-1)},\text{data})$ using the Metropolis-Hastings method with a proposal distribution of $N(\beta_{(t-1)},1)$.
$2m+5$. Generate $\gamma_{1(t)}$ from $\pi(\gamma_1|\alpha_{1(t-1)},\beta_{1(t-1)},\text{data})$ using the Metropolis-Hastings method with a proposal distribution of $N(\gamma_{1(t-1)},1)$.

$\vdots$

$3m+4$. Generate $\gamma_{m(t)}$ from $\pi(\gamma_m|\alpha_{m(t-1)},\beta_{m(t-1)},\text{data})$ using the Metropolis-Hastings method with a proposal distribution of $N(\gamma_{m(t-1)},1)$.
$3m+5$. Generate $\gamma_{(t)}$ from $\pi(\gamma|\alpha_{(t-1)},\beta_{(t-1)},\text{data})$ using the Metropolis-Hastings method with a proposal distribution of $N(\gamma_{(t-1)},1)$.
$3m+6$. Calculate the value

$$R_{(t)\boldsymbol{s},\boldsymbol{k}} = \sum_{p_1=s_1}^{k_1}\cdots\sum_{p_m=s_m}^{k_m}\binom{k_1}{p_1}\cdots\binom{k_m}{p_m}\int_0^1 \alpha_{(t)}\beta_{(t)}\gamma_{(t)}y^{\alpha_{(t)}-1}(1-y^{\alpha_{(t)}})^{\beta_{(t)}-1}$$

$$\times\big(1-(1-y^{\alpha_{(t)}})^{\beta_{(t)}}\big)^{\gamma_{(t)}-1}\prod_{l=1}^{m}\Big(1-\big(1-(1-y^{\alpha_{l(t)}})^{\beta_{l(t)}}\big)^{\gamma_{l(t)}}\Big)^{p_l}$$
$$\times\big(1-(1-y^{\alpha_{l(t)}})^{\beta_{l(t)}}\big)^{\gamma_{l(t)}(k_l-p_l)}dy.$$

$3m+7$. Update t by incrementing it by 1.
$3m+8$. Repeat Steps 3 to $3m+7$ a total of T times.

The Bayesian estimation of $R_{\boldsymbol{s},\boldsymbol{k}}$ can be obtained using the above process

$$\widehat{R}^{MC}_{\boldsymbol{s},\boldsymbol{k}}=\frac{1}{T}\sum_{t=1}^{T}R_{(t)\boldsymbol{s},\boldsymbol{k}}. \tag{10}$$

Additionally, a $100(1-\eta)\%$ HPD credible interval for $R_{\boldsymbol{s},\boldsymbol{k}}$, inspired by the method proposed by Chen and Shao (1999), can be obtained as follows. Initially, arrange $R_{(1)\boldsymbol{s},\boldsymbol{k}},\ldots,R_{(T)\boldsymbol{s},\boldsymbol{k}}$ in ascending order as $R_{((1)\boldsymbol{s},\boldsymbol{k})},\ldots,R_{((T)\boldsymbol{s},\boldsymbol{k})}$ and then generate all the $100(1-\eta)\%$ confidence intervals for $R_{\boldsymbol{s},\boldsymbol{k}}$ as

$$(R_{((1)\boldsymbol{s},\boldsymbol{k})},R_{(([T(1-\eta)])\boldsymbol{s},\boldsymbol{k})}),\ldots,(R_{(([T\eta])\boldsymbol{s},\boldsymbol{k})},R_{((T)\boldsymbol{s},\boldsymbol{k})}),$$

where $[T]$ denotes the largest integer less than or equal to T. The HPD credible interval for $R_{\boldsymbol{s},\boldsymbol{k}}$ corresponds to the interval with the shortest length.

3 Inference on $R_{\boldsymbol{s},\boldsymbol{k}}$ with unknown common parameters

In various empirical data analyses, the typical parameter values for strength and stress variables are often very similar. In these instances, it is common practice to assume that they are equal. This assumption allows for estimations to be made in situations where common parameters are known. Another advantage of this approach is the precision of the estimations that can be achieved.

Now, we have $X_1\sim EKu(\alpha,\beta,\gamma_1)$, $X_2\sim EKu(\alpha,\beta,\gamma_2)$, and so on up to $X_m\sim EKu(\alpha,\beta,\gamma_m)$ as a collection of independent random variables as well as a random variable $Y\sim EKu(\alpha,\beta,\gamma)$. Through the use of (3) and (4), we are able to compute the Multicomponent reliability with non-identical component strengths in (2) as illustrated below

$$\begin{aligned}
R_{\mathbf{s},\mathbf{k}} &= \sum_{p_1=s_1}^{k_1}\cdots\sum_{p_m=s_m}^{k_m}\Big(\prod_{l=1}^{m}\binom{k_l}{p_l}\Big)\int_0^1\prod_{l=1}^{m}\Big(1-\big(1-(1-y^{\alpha})^{\beta}\big)^{\gamma_l}\Big)^{p_l}\\
&\quad\times\big(1-(1-y^{\alpha})^{\beta}\big)^{\gamma_l(k_l-p_l)}\alpha\beta\gamma y^{\alpha-1}(1-y^{\alpha})^{\beta-1}\\
&\quad\times\big(1-(1-y^{\alpha})^{\beta}\big)^{\gamma-1}dy \qquad\qquad \text{Put: } t=1-(1-y^{\alpha})^{\beta}\\
&= \sum_{p_1=s_1}^{k_1}\cdots\sum_{p_m=s_m}^{k_m}\Big(\prod_{l=1}^{m}\binom{k_l}{p_l}\Big)\gamma\int_0^1 t^{\sum_{l=1}^{m}\gamma_l(k_l-p_l)+\gamma-1}\prod_{l=1}^{m}(1-t_l^{\gamma})^{p_l}dt\\
&= \sum_{p_1=s_1}^{k_1}\cdots\sum_{p_m=s_m}^{k_m}\Big(\prod_{l=1}^{m}\binom{k_l}{p_l}\Big)\gamma\int_0^1 t^{\sum_{l=1}^{m}\gamma_l(k_l-p_l)+\gamma-1}
\end{aligned}$$

$$
\begin{aligned}
&\times \prod_{l=1}^{m}\Big(\sum_{q_l=0}^{k_l-p_l}\binom{k_l-p_l}{q_l}(-1)^{q_l}t^{\gamma_l q_l}\Big)dt \\
=\;& \sum_{p_1=s_1}^{k_1}\cdots\sum_{p_m=s_m}^{k_m}\sum_{q_1=0}^{k_1-p_1}\cdots\sum_{q_m=0}^{k_m-p_m}\Big(\prod_{l=1}^{m}\binom{k_l}{p_l}\Big)\times\Big(\prod_{l=1}^{m}\binom{k_l-p_l}{q_l}\Big) \\
&\times(-1)^{\sum_{l=1}^{m}q_l}\gamma\int_0^1 t^{\sum_{l=1}^{m}\gamma_l(k_l-p_l+q_l)+\gamma-1}dt \\
=\;& \sum_{p_1=s_1}^{k_1}\cdots\sum_{p_m=s_m}^{k_m}\sum_{q_1=0}^{k_1-p_1}\cdots\sum_{q_m=0}^{k_m-p_m}\Big(\prod_{l=1}^{m}\binom{k_l}{p_l}\Big)\times\Big(\prod_{l=1}^{m}\binom{k_l-p_l}{q_l}\Big) \\
&\times(-1)^{\sum_{l=1}^{m}q_l}\frac{\gamma}{\sum_{l=1}^{m}\gamma_l(k_l-p_l+q_l)+\gamma}.
\end{aligned}
$$

3.1 MLE of $R_{\boldsymbol{s},\boldsymbol{k}}$

In this case, the method of utilizing the MLE approach is similar to what has been previously explained. To calculate the MLEs of unknown parameters, under the same conditions that we explained in Section 2.1 and setting $\Theta = (\alpha, \beta, \gamma_1, \dots, \gamma_m, \gamma)$, we must first derive the log-likelihood function from the provided function and then solve the following equations simultaneously.

$$
\begin{aligned}
\frac{\partial \ell}{\partial \alpha} =\;& \frac{n(\sum_{l=1}^{k}k_l+1)}{\alpha}+\sum_{i=1}^{n}\sum_{l=1}^{k}\sum_{j_l=1}^{k_l}\log\big(x_{ij_l}^{(l)}\big)-\sum_{i=1}^{n}\sum_{l=1}^{k}\sum_{j_l=1}^{k_l}(\beta-1)\frac{\big(x_{ij_l}^{(l)}\big)^{\alpha}\log\big(x_{ij_l}^{(l)}\big)}{1-\big(x_{ij_l}^{(l)}\big)^{\alpha}} \\
&+\sum_{i=1}^{n}\sum_{l=1}^{k}\sum_{j_l=1}^{k_l}(\gamma_l-1)\frac{\beta\Big(1-\big(x_{ij_l}^{(l)}\big)^{\alpha}\Big)^{\beta-1}\big(x_{ij_l}^{(l)}\big)^{\alpha}\log\big(x_{ij_l}^{(l)}\big)}{A_{ij_l}(\alpha,\beta)} \\
&-\sum_{i=1}^{n}\sum_{l=1}^{k}\sum_{j_l=1}^{k_l}R_{ij}^{(l)}\frac{\gamma_l\beta A_{ij_l}^{\gamma_l-1}(\alpha,\beta)\Big(1-\big(x_{ij_l}^{(l)}\big)^{\alpha}\Big)^{\beta-1}\big(x_{ij_l}^{(l)}\big)^{\alpha}\log\big(x_{ij_l}^{(l)}\big)}{1-A_{ij_l}^{\gamma_l}(\alpha,\beta)} \\
&+\sum_{i=1}^{n}\log(y_i)-(\beta-1)\sum_{i=1}^{n}\frac{y_i^{\alpha}\log(y_i)}{1-y_i^{\alpha}}+(\gamma-1)\beta\sum_{i=1}^{n}\frac{\big(1-y_i^{\alpha}\big)^{\beta-1}y_i^{\alpha}\log(y_i)}{A_i(\alpha,\beta)} \\
&-\gamma\beta\sum_{i=1}^{n}S_i\frac{A_i^{\gamma-1}(\alpha,\beta)\big(1-y_i^{\alpha}\big)^{\beta-1}y_i^{\alpha}\log(y_i)}{1-A_i^{\gamma}(\alpha,\beta)}, \\
\frac{\partial \ell}{\partial \beta} =\;& \frac{n(\sum_{l=1}^{k}k_l+1)}{\beta}+\sum_{i=1}^{n}\sum_{j_l=1}^{k_l}\log\Big(1-\big(x_{ij_l}^{(l)}\big)^{\alpha}\Big)
\end{aligned}
$$

$$
\begin{aligned}
& -\sum_{i=1}^{n}\sum_{j_l=1}^{k_l}(\gamma_l-1)\frac{\left(1-\left(x_{ij_l}^{(l)}\right)^{\alpha}\right)^{\beta}\log\left(1-\left(x_{ij_l}^{(l)}\right)^{\alpha}\right)}{A_{ij_l}(\alpha,\beta)} \\
& +\sum_{i=1}^{n}\sum_{j_l=1}^{k_l}R_{ij}^{(l)}\frac{\gamma_l A_{ij_l}^{\gamma_l-1}(\alpha,\beta)\left(1-\left(x_{ij_l}^{(l)}\right)^{\alpha}\right)^{\beta_l}\log\left(1-\left(x_{ij_l}^{(l)}\right)^{\alpha}\right)}{1-A_{ij_l}^{\gamma_l}(\alpha,\beta)} \\
& +\sum_{i=1}^{n}\log(1-y_i^{\alpha})-(\gamma-1)\sum_{i=1}^{n}\frac{\left(1-y_i^{\alpha}\right)^{\beta}\log(1-y_i^{\alpha})}{A_i(\alpha,\beta)} \\
& -\gamma\sum_{i=1}^{n}S_i\frac{A_i^{\gamma-1}(\alpha,\beta)\left(1-y_i^{\alpha}\right)^{\beta}\log(1-y_i^{\alpha})}{1-A_i^{\gamma}(\alpha,\beta)},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\ell}{\partial\gamma_l} &= \frac{nk_l}{\gamma_l}+\sum_{i=1}^{n}\sum_{j_l=1}^{k_l}\log\left(A_{ij_l}(\alpha,\beta)\right) \\
&\quad -\sum_{i=1}^{n}\sum_{j_l=1}^{k_l}R_{ij}^{(l)}\frac{A_{ij_l}^{\gamma_l}(\alpha,\beta)\log\left(A_{ij_l}(\alpha,\beta)\right)}{1-A_{ij_l}^{\gamma_l}(\alpha,\beta)},\ l=1,\ldots,m, \\
\frac{\partial\ell}{\partial\gamma} &= \frac{n}{\gamma}+\sum_{i=1}^{n}\log\left(A_i(\alpha,\beta)\right)-\sum_{i=1}^{n}S_i\frac{A_i^{\gamma}(\alpha,\beta)\log\left(A_i(\alpha,\beta)\right)}{1-A_i^{\gamma}(\alpha,\beta)}.
\end{aligned}
$$

The MLEs can be obtained by solving these equations together using a numerical method such as the Newton-Raphson algorithm. Lastly, the invariance property of MLEs dictates that the MLE of $R_{\boldsymbol{s},\boldsymbol{k}}$, denoted as $\widehat{R}_{\boldsymbol{s},\boldsymbol{k}}^{MLE}$, can be calculated as

$$
\begin{aligned}
\widehat{R}_{\boldsymbol{s},\boldsymbol{k}}^{MLE} &= \sum_{p_1=s_1}^{k_1}\cdots\sum_{p_m=s_m}^{k_m}\sum_{q_1=0}^{k_1-p_1}\cdots\sum_{q_m=0}^{k_m-p_m}\left(\prod_{l=1}^{m}\binom{k_l}{p_l}\right)\times\left(\prod_{l=1}^{m}\binom{k_l-p_l}{q_l}\right) \\
&\quad \times(-1)^{\sum_{l=1}^{m}q_l}\frac{\widehat{\gamma}}{\sum_{l=1}^{m}\widehat{\gamma}_l(k_l-p_l+q_l)+\widehat{\gamma}}.
\end{aligned}
\tag{11}
$$

3.2 Bayes estimation of $R_{\boldsymbol{s},\boldsymbol{k}}$

In this part of the analysis, using the squared error loss function, our goal is to determine the estimation and credible intervals for $R_{\boldsymbol{s},\boldsymbol{k}}$. We are assuming that the unknown parameters follow independent gamma distributions. Thus, we must take into account the prior distributions of these parameters as

$$
\begin{aligned}
& \alpha\sim\Gamma(a,b),\ \beta\sim\Gamma(c,d), \\
& \gamma_l\sim\Gamma(e_l,f_l),\ l=1,\ldots,m,\quad \gamma\sim\Gamma(e,f).
\end{aligned}
$$

Similar to Section 2.2 it is determined that the Bayes estimations of the unknown parameters cannot be derived in closed forms, so, an approximation method is necessary to obtain the Bayesian estimations. To achieve this, we suggest using the MCMC method. Now, by setting $\Theta = (\alpha, \beta, \gamma_1, \ldots, \gamma_m, \gamma)$, we can express joint posterior density function as

$$\begin{aligned}
\pi(\Theta|\text{data}) \quad \propto \quad & \alpha^{n(\sum\limits_{l=1}^{k} k_l+1)+a-1} e^{-\alpha b} \times \beta^{n(\sum\limits_{l=1}^{k} k_l+1)+c-1} e^{-\beta d} \times \big(\prod_{l=1}^{m} \gamma_l^{nk_l+e_l-1} e^{-\gamma_l f_l}\big) \\
& \times \gamma^{n+e-1} e^{-\gamma f} \prod_{l=1}^{m} \big(B^1_{n,k_l}(\alpha) B^2_{n,k_l}(\alpha,\beta) B^3_{n,k_l}(\alpha,\beta,\gamma_l)\big) \\
& \times C^1_n(\alpha) C^2_n(\alpha,\beta) C^3_n(\alpha,\beta,\gamma),
\end{aligned}$$

where $B^i_{n,k_l}(\cdot)$ and $C^i_n(\cdot)$, $i = 1, 2, 3$ are specified in (5) and (6), respectively. Consequently, the posterior PDFs of the parameters can be computed.

$$\begin{aligned}
\pi(\alpha|\beta,\gamma_l,\gamma,\text{data}) \quad \propto \quad & \alpha^{n(\sum\limits_{l=1}^{k} k_l+1)+a-1} e^{-\alpha b} \times B^1_{n,k_l}(\alpha) B^2_{n,k_l}(\alpha,\beta) B^3_{n,k_l}(\alpha,\beta,\gamma_l) \\
& \times C^1_n(\alpha) C^2_n(\alpha,\beta) C^3_n(\alpha,\beta,\gamma), \\
\pi(\beta|\alpha,\gamma_l,\gamma,\text{data}) \quad \propto \quad & \beta_l^{n(\sum\limits_{l=1}^{k} k_l+1)+c-1} e^{-\beta d} \times B^2_{n,k_l}(\alpha,\beta) B^3_{n,k_l}(\alpha,\beta,\gamma_l) \\
& \times C^2_n(\alpha,\beta) C^3_n(\alpha,\beta,\gamma), \\
\pi(\gamma_l|\alpha,\beta,\text{data}) \quad \propto \quad & \gamma_l^{nk_l+e_l-1} e^{-\gamma_l f_l} \times B^3_{n,k_l}(\alpha,\beta,\gamma_l),\ l = 1,\ldots,m, \\
\pi(\gamma|\alpha,\beta,\text{data}) \quad \propto \quad & \gamma^{n+e-1} e^{-\gamma f} \times C^3_n(\alpha,\beta,\gamma).
\end{aligned}$$

As we see, the posterior probability density functions for whole parameters are not readily defined distributions. Hence, we employ the Metropolis-Hastings method to produce random samples from them, facilitating the implementation of the Gibbs sampling algorithm in the ensuing manner:
1. Start with $(\alpha_{(0)}, \beta_{(0)}, \gamma_{1(0)}, \ldots, \gamma_{m(0)}, \gamma_{(0)})$.
2. Initialize $t = 1$.
3. Generate $\alpha_{(t)}$ from $\pi(\alpha|\beta_{(t-1)}, \gamma_{(t-1)}, \text{data})$ using the Metropolis-Hastings method with a proposal distribution of $N(\alpha_{(t-1)}, 1)$.
4. Generate $\beta_{(t)}$ from $\pi(\beta|\alpha_{(t-1)}, \gamma_{(t-1)}, \text{data})$ using the Metropolis-Hastings method with a proposal distribution of $N(\beta_{(t-1)}, 1)$.
5. Generate $\gamma_{1(t)}$ from $\pi(\gamma_1|\alpha_{1(t-1)}, \beta_{1(t-1)}, \text{data})$ using the Metropolis-Hastings method with a proposal distribution of $N(\gamma_{1(t-1)}, 1)$.

$\vdots$

$m+4$. Generate $\gamma_{m(t)}$ from $\pi(\gamma_m|\alpha_{m(t-1)}, \beta_{m(t-1)}, \text{data})$ using the Metropolis-Hastings method with a proposal distribution of $N(\gamma_{m(t-1)}, 1)$.
$m+5$. Generate $\gamma_{(t)}$ from $\pi(\gamma|\alpha_{(t-1)}, \beta_{(t-1)}, \text{data})$ using the Metropolis-Hastings method with a proposal distribution of $N(\gamma_{(t-1)}, 1)$.
$m+6$. Calculate the value

$$R_{(t)\boldsymbol{s},\boldsymbol{k}} = \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \sum_{q_1=0}^{k_1-p_1} \cdots \sum_{q_m=0}^{k_m-p_m} \Big(\prod_{l=1}^{m} \binom{k_l}{p_l}\Big) \times \Big(\prod_{l=1}^{m} \binom{k_l-p_l}{q_l}\Big)$$

$$\times(-1)^{\sum_{l=1}^{m} q_l} \frac{\gamma_{(t)}}{\sum_{l=1}^{m} \gamma_{l(t)}(k_l - p_l + q_l) + \gamma_{(t)}}.$$

$m+7$. Update t by incrementing it by 1.
$m+8$. Repeat Steps 3 to $m+7$ a total of T times.

The Bayesian estimation of $R_{\boldsymbol{s},\boldsymbol{k}}$ can be obtained using the above process:

$$\widehat{R}^{MC}_{\boldsymbol{s},\boldsymbol{k}} = \frac{1}{T}\sum_{t=1}^{T} R_{(t)\boldsymbol{s},\boldsymbol{k}}. \tag{12}$$

Additionally, a $100(1-\eta)\%$ HPD credible interval for $R_{\boldsymbol{s},\boldsymbol{k}}$, inspired by the method proposed by Chen and Shao (1999) as we explained in Section 2.2.

4 Inference on $R_{\boldsymbol{s},\boldsymbol{k}}$ with known common parameters

In this section, we explore the MLE and Bayesian inference of $R_{\boldsymbol{s},\boldsymbol{k}}$ under squared error loss functions, assuming that the common parameters α and β are known. About MLE, we can follow the same manner as we explained in Section 3.1. So, we skip mentioning it again. About the Bayesian inference, given that the random variables $\gamma_1, \ldots, \gamma_m$ and γ follow independent gamma distributions, we can derive the posterior PDFs for $\gamma_1, \ldots, \gamma_m$ as well as for γ. This approach parallels the methodology outlined in Section 2 and allows us to analyze the behavior of $R_{\boldsymbol{s},\boldsymbol{k}}$ effectively. So, the posterior PDFs of these parameters can be derived as follows

$$\begin{aligned}
\pi(\gamma_l|\alpha,\beta,\text{data}) &\propto \gamma_l^{nk_l+e_l-1} e^{-\gamma_l f_l} \Big(\prod_{i=1}^{n}\prod_{j_l=1}^{k_l} A_{ij_l}^{\gamma_l-1}(\alpha,\beta)\Big) \\
&\quad \times\Bigg(\prod_{i=1}^{n}\prod_{j_l=1}^{k_l}\Big(1-A_{ij_l}^{\gamma_l}(\alpha,\beta)\Big)^{R_{ij}^{(l)}}\Bigg),\ l=1,\ldots,m, \\
\pi(\gamma|\alpha,\beta,\text{data}) &\propto \gamma^{n+e-1} e^{-\gamma f}\Big(\prod_{i=1}^{n} A_i^{\gamma-1}(\alpha,\beta)\Big)\Big(\prod_{i=1}^{n}\big(1-A_i^{\gamma}(\alpha,\beta)\big)^{S_i}\Big).
\end{aligned}$$

Similar to previous section, we use the Gibbs sampling and obtain

$$\widehat{R}^{MC}_{\boldsymbol{s},\boldsymbol{k}} = \frac{1}{T}\sum_{t=1}^{T} R_{(t)\boldsymbol{s},\boldsymbol{k}}.$$

We utilize the approach described in Chen and Shao (1999) to construct the highest posterior density (HPD) credible interval for $R_{\boldsymbol{s},\boldsymbol{k}}$ at the confidence level of $100(1-\eta)\%$.

4.1 Lindley's approximation

One of the key numerical methods for obtaining approximate Bayesian estimations is presented by Lindley (1980). This method can be described as follows. Under squared error loss, the Bayesian estimation of $U(\psi)$ is given by

$$\mathbb{E}(u(\psi)|\text{data}) = \frac{\int u(\psi)e^{Q(\psi)}d\psi}{\int e^{Q(\psi)}d\psi}, \tag{13}$$

where $Q(\psi) = \rho(\psi) + \ell(\psi)$, with $\rho(\psi)$ representing the logarithm of the prior density of ψ and $\ell(\psi)$ being the log-likelihood function. The expression $\mathbb{E}(u(\psi)|\text{data})$ in (13) is approximated by Lindley (1980) as follows

$$\mathbb{E}(u(\psi)|\text{data}) = u + \frac{1}{2}\sum_i\sum_j(u_{i,j} + 2u_i\rho_j)\sigma_{i,j} + \frac{1}{2}\sum_i\sum_j\sum_k\sum_p \ell_{i,j,k}\sigma_{i,j}\sigma_{k,p}u_p\Big|_{\psi=\widehat{\psi}}, \tag{14}$$

where $\psi = (\psi_1, \ldots, \psi_m)$, $i, j, k, p = 1, \ldots, m$, and $\widehat{\psi}$ denotes the MLE of ψ. Here, $u = u(\psi)$, $u_i = \frac{\partial u}{\partial\psi_i}$, $u_{i,j} = \frac{\partial^2 u}{\partial\psi_i\partial\psi_j}$, $\ell_{i,j,k} = \frac{\partial^3\ell}{\partial\psi_i\partial\psi_j\partial\psi_k}$, $\rho_j = \frac{\partial\rho}{\partial\psi_j}$, and $\sigma_{i,j}$ is the (i, j)-th element of the inverse of the matrix $[-\ell_{i,j}]$, all evaluated at the MLE of the parameters. The (14) can be reformulated for $m + 1$ parameters as follows

$$\widehat{u}^{Lin} = u + (\sum_{i=1}^{m+1} u_i d_i + d_{m+2} + d_{m+3}) + \frac{1}{2}\sum_{i=1}^{m+1} A_i(\sum_{j=1}^{m+1} u_j\sigma_{i,j}), \tag{15}$$

where

$$\begin{aligned}
d_i &= \sum_{j=1}^{m+1}\rho_j\sigma_{i,j},\ i = 1, \ldots, m+1,\ d_{m+2} = \sum_{\substack{i=1\\ i<j}}^{m+1}\sum_{j=1}^{m+1} u_{i,j}\sigma_{i,j},\ d_{m+3} = \frac{1}{2}\sum_{i=1}^{m+1} u_{i,i}\sigma_{i,i},\\
A_i &= \sum_{\substack{j=1\\ j\le k}}^{m+1}\sum_{k=1}^{m+1}\ell_{j,k,i}\times\begin{cases}\sigma_{j,k} & j = k,\\ 2\sigma_{j,k} & j < k,\end{cases}\quad i = 1, \ldots, m+1.
\end{aligned}$$

For $(\psi_1, \ldots, \psi_m, \psi_{m+1}) \equiv (\gamma_1, \ldots, \gamma_m, \gamma)$ and $u \equiv R_{\boldsymbol{s},\boldsymbol{k}}$, we have

$$\begin{aligned}
\rho_l &= \frac{e_l - 1}{\gamma_l} - f_l,\ l = 1, \ldots, m,\quad \rho_{m+1} = \frac{e-1}{\lambda} - f,\\
\ell_{l,l} &= -\frac{nk_l}{\gamma_l^2} - \sum_{i=1}^{n}\sum_{j_l=1}^{k_l} R_{ij}^{(l)}\frac{A_{ij_l}^{\gamma_l}(\alpha,\beta)\log^2\left(A_{ij_l}(\alpha,\beta)\right)}{\left(1 - A_{ij_l}^{\gamma_l}(\alpha,\beta)\right)^2}, \qquad l = 1, \ldots, m,\\
\ell_{m+1,m+1} &= -\frac{n}{\gamma^2} - \sum_{i=1}^{n} S_i\frac{A_i^{\gamma}(\alpha,\beta)\log^2\left(A_i(\alpha,\beta)\right)}{\left(1 - A_i^{\gamma}(\alpha,\beta)\right)^2},\\
\ell_{l,k} &= 0,\ l = 1, \ldots, m+1, l \neq k.
\end{aligned}$$

The values of $\sigma_{i,j}, i, j = 1, \ldots, m+1$ can be obtained from $\ell_{i,j}, i, j = 1, \ldots, m+1$. Also,

$$\ell_{l,l,l} = \frac{2nk_l}{\gamma_l^3} - \sum_{i=1}^{n}\sum_{j_l=1}^{k_l} R_{ij}^{(l)}\frac{\left(1 + A_{ij_l}^{\gamma_l}(\alpha,\beta)\right)\log^3\left(A_{ij_l}(\alpha,\beta)\right)}{\left(1 - A_{ij_l}^{\gamma_l}(\alpha,\beta)\right)^3},\ l = 1, \ldots, m,$$

$$\ell_{m+1,m+1,m+1} = \frac{2n}{\gamma^3} - \sum_{i=1}^{n} S_i \frac{\left(1 + A_i^{\gamma}(\alpha,\beta)\right)\log^3\left(A_i(\alpha,\beta)\right)}{\left(1 - A_i^{\gamma}(\alpha,\beta)\right)^3},$$

and other $\ell_{i,j,k} = 0$ and $A_{ij_l}(\alpha,\beta)$ and $A_i(\alpha,\beta)$ are specified in (7). Also, we have

$$u_l = \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \sum_{q_1=0}^{k_1-p_1} \cdots \sum_{q_m=0}^{k_m-p_m} \left(\prod_{l=1}^{m} \binom{k_l}{p_l}\right) \times \left(\prod_{l=1}^{m} \binom{k_l - p_l}{q_l}\right)(-1)^{\sum_{l=1}^{m} q_l} \times A,$$

$$u_{l,k} = \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \sum_{q_1=0}^{k_1-p_1} \cdots \sum_{q_m=0}^{k_m-p_m} \left(\prod_{l=1}^{m} \binom{k_l}{p_l}\right) \times \left(\prod_{l=1}^{m} \binom{k_l - p_l}{q_l}\right)(-1)^{\sum_{l=1}^{m} q_l} \times B,$$

where

$$A = \begin{cases} \dfrac{\gamma(k_l - p_l + q_l)}{\left(\sum_{l=1}^{m} \gamma_l(k_l - p_l + q_l) + \gamma\right)^2} & l = 1, \ldots, m, \\[2ex] \dfrac{\sum_{l=1}^{m} \gamma_l(k_l - p_l + q_l)}{\left(\sum_{l=1}^{m} \gamma_l(k_l - p_l + q_l) + \gamma\right)^2} & l = m+1, \end{cases}$$

$$B = \begin{cases} \dfrac{2\gamma(k_l - p_l + q_l)(k_k - p_k + q_k)}{\left(\sum_{l=1}^{m} \gamma_l(k_l - p_l + q_l) + \gamma\right)^3} & l, k = 1, \ldots, m, \\[2ex] -\dfrac{2\sum_{l=1}^{m} \gamma_l(k_l - p_l + q_l)}{\left(\sum_{l=1}^{m} \gamma_l(k_l - p_l + q_l) + \gamma\right)^3} & l = m+1, \\[2ex] -\dfrac{(k_l - p_l + q_l)\left(\sum_{l=1}^{m} \gamma_l(k_l - p_l + q_l) - \gamma\right)}{\left(\sum_{l=1}^{m} \gamma_l(k_l - p_l + q_l) + \gamma\right)^3} & l = 1, \ldots, m,\ k = m+1. \end{cases}$$

Once the aforementioned values are determined, Lindley's estimation of $R_{\boldsymbol{s},\boldsymbol{k}}$, denoted as $\widehat{R}^{Lin}_{\boldsymbol{s},\boldsymbol{k}}$, can be derived from (15). It is important to note that all parameters must be evaluated at the MLEs $(\widehat{\gamma}_1, \ldots, \widehat{\gamma}_m, \widehat{\gamma})$ for $(\gamma_1, \ldots, \gamma_m, \gamma)$.

5 Numerical study and data analysis

5.1 Simulation experiments

In this section, our analysis involves Monte Carlo simulation studies to compare different estimations. For point, we assess their performance using mean square errors (MSEs), while for interval estimates, we evaluate them based on average confidence lengths (AL) and coverage percentages (CP). We make the assumption that the simulated system comprises two strength components. To obtain the simulation results we consider the system with 3 components for strength variables. What I mean is that we set $l = 1, 2, 3$ and $m = 3$. It is important to note that we perform the study for different censoring schemes as outlined in Table 1. In order to obtain the simulation, we generate 2000 samples. We notice that in the simulation results tables $\boldsymbol{k} = (k_1, k_2, k_3)$ and $\boldsymbol{s} = (s_1, s_2, s_3)$, $\boldsymbol{R_i} = (R_i, R_i, R_i)$, $i = 1, \ldots, 6$. In the general case, we obtain simulation results by using $(\alpha_1, \alpha_2, \alpha_3, \alpha, \beta_1, \beta_2, \beta_3, \beta, \gamma_1, \gamma_2, \gamma_3, \gamma) =$

Table 1: Different censoring schemes.

(k_i, K_i)	C.S.		(n, N)	C.S.	
	R_1	(0,0,0,0,5)		S_1	(0,0,0,0,5)
(5,10)	R_2	(5,0,0,0,0)	(5,10)	S_2	(5,0,0,0,0)
	R_3	(1,1,1,1,1)		S_3	(1,1,1,1,1)
	R_4	$(0^{*9},10)$		S_4	$(0^{*9},10)$
(10,20)	R_5	$(10,0^{*9})$	(10,20)	S_5	$(10,0^{*9})$
	R_6	(1^{*10})		S_6	(1^{*10})

$(2, 2.5, 3, 4, 1, 1.5, 1, 2.5, 2, 3, 4, 3.5)$. Furthermore, we perform $T = 3000$ repetitions of the Gibbs sampling algorithm. Additionally, we employ two priors for our analysis.
Prior 1: $a_l = c_l = e_l = a = c = d = 0, \quad b_l = d_l = f_l = b = d = f = 0,$
Prior 2: $a_l = c_l = e_l = a = c = d = 0.5,\ b_l = d_l = f_l = b = d = f = 0.35,\ l = 1, \ldots, 3.$
In this scenario, we calculate the MLE and Bayes estimate of $R_{\boldsymbol{s},\boldsymbol{k}}$ using equations (8) and (10), respectively. Additionally, we compute the 95% asymptotic and highest posterior density HPD intervals for $R_{\boldsymbol{s},\boldsymbol{k}}$. The results are presented in Table 2.

In the case where the common parameters are unknown, we generate simulation results based on $(\alpha, \beta, \gamma_1, \gamma_2, \gamma_3, \gamma) = (2, 3, 2.5, 1, 3.5, 3)$. In this scenario, we set the repetition numbers in the Gibbs sampling algorithm to $T = 3000$. Furthermore, we utilize two distinct priors for analysis as
Prior 3: $a = c = e_l = e = 0, \qquad b = d = f_l = f = 0,$
Prior 4: $a = c = e_l = e = 0.45,\ b = d = f_l = f = 0.55,\ l = 1, \ldots, 3.$
In this case, we determine the estimates for $R_{\boldsymbol{s},\boldsymbol{k}}$ using both the MLE and MCMC Bayes approach using equations (11) and (12), respectively. Furthermore, we calculate the 95% HPD intervals for $R_{\boldsymbol{s},\boldsymbol{k}}$. The resulting can be found in Table 3.

In the case where the common parameters are known, we generate simulation results based on $(\alpha, \beta, \gamma_1, \gamma_2, \gamma_3, \gamma) = (2, 1.5, 2.5, 3, 1, 1.5)$. In this scenario, we set the repetition numbers in the Gibbs sampling algorithm to $T = 3000$. Furthermore, we utilize two distinct priors for analysis as
Prior 5: $e_l = e = 0, \qquad f_l = f = 0,$
Prior 6: $e_l = e = 0.30,\ f_l = f = 0.5,\ l = 1, \ldots, 3.$
In this case, we determine the estimates for $R_{\boldsymbol{s},\boldsymbol{k}}$ using both the MLE and MCMC and Lindley's approximation Bayes approach. Furthermore, we calculate the 95% HPD intervals for $R_{\boldsymbol{s},\boldsymbol{k}}$. The resulting can be found in Table 4.

The findings of the simulation study presented in Tables 2 and 4 can be summarized as follows:
• Point estimations: Bayes estimations outperform other methods, with informative priors yielding better results than non-informative priors in terms of MSEs.
• Interval estimations: Bayes estimations show superior performance, with informative priors outperforming non-informative priors based on ALs and CPs. • Increasing sample size n for fixed values of $\boldsymbol{s}$ and $\boldsymbol{k}$ results in decreased MSEs and ALs, and increased CPs.
• Increasing $\boldsymbol{k}$ for fixed values of $\boldsymbol{s}$ and n leads to decreased MSEs and ALs, and increased CPs.

The observed trends may be attributed to the fact that larger sample sizes provide more information for estimation.

Table 2: Results of the point and interval estimates of $R_{\boldsymbol{s},\boldsymbol{k}}$, in general case.

$(\boldsymbol{k}, n, \boldsymbol{s})$	C.S	MLE	MCMC					
			Prior 1			Prior 2		
		MSE	MSE	AL	CP	MSE	AL	CP
$(\mathbf{5}, 5, \mathbf{2})$	$(\boldsymbol{R_1}, S_1)$	0.0756	0.0700	0.7235	0.942	0.0601	0.6845	0.944
	$(\boldsymbol{R_2}, S_2)$	0.0750	0.0710	0.7251	0.940	0.0600	0.6819	0.945
	$(\boldsymbol{R_3}, S_3)$	0.0743	0.0708	0.7239	0.941	0.0609	0.6861	0.944
$(\mathbf{5}, 10, \mathbf{2})$	$(\boldsymbol{R_1}, S_4)$	0.0605	0.0512	0.6720	0.944	0.0455	0.6152	0.949
	$(\boldsymbol{R_2}, S_5)$	0.0600	0.0510	0.6795	0.945	0.0450	0.6137	0.948
	$(\boldsymbol{R_3}, S_6)$	0.0607	0.0519	0.6725	0.944	0.0457	0.6142	0.949
$(\mathbf{10}, 5, \mathbf{2})$	$(\boldsymbol{R_4}, S_1)$	0.0450	0.0400	0.6275	0.948	0.0364	0.5736	0.950
	$(\boldsymbol{R_5}, S_2)$	0.0459	0.0412	0.6281	0.949	0.0360	0.5748	0.951
	$(\boldsymbol{R_6}, S_3)$	0.0443	0.0409	0.6294	0.948	0.0369	0.5736	0.950
$(\mathbf{10}, 10, \mathbf{2})$	$(\boldsymbol{R_4}, S_4)$	0.0356	0.0302	0.5834	0.950	0.0274	0.5221	0.953
	$(\boldsymbol{R_5}, S_5)$	0.0350	0.0309	0.5837	0.951	0.0270	0.5291	0.953
	$(\boldsymbol{R_6}, S_6)$	0.0361	0.0300	0.5816	0.950	0.0279	0.5371	0.952
$(\mathbf{5}, 5, \mathbf{4})$	$(\boldsymbol{R_1}, S_1)$	0.0764	0.0711	0.7332	0.940	0.0625	0.6792	0.945
	$(\boldsymbol{R_2}, S_2)$	0.0769	0.0719	0.7354	0.941	0.0620	0.6782	0.945
	$(\boldsymbol{R_3}, S_3)$	0.0760	0.0705	0.7364	0.940	0.0627	0.6799	0.944
$(\mathbf{5}, 10, \mathbf{4})$	$(\boldsymbol{R_1}, S_4)$	0.0600	0.0524	0.6812	0.944	0.0467	0.6528	0.948
	$(\boldsymbol{R_2}, S_5)$	0.0608	0.0520	0.6891	0.945	0.0460	0.6528	0.949
	$(\boldsymbol{R_3}, S_6)$	0.0610	0.0518	0.6828	0.944	0.0469	0.6549	0.948
$(\mathbf{10}, 5, \mathbf{4})$	$(\boldsymbol{R_4}, S_1)$	0.0450	0.0405	0.6492	0.948	0.0360	0.6035	0.951
	$(\boldsymbol{R_5}, S_2)$	0.0459	0.0400	0.6497	0.949	0.0369	0.6075	0.951
	$(\boldsymbol{R_6}, S_3)$	0.0462	0.0407	0.6490	0.948	0.0361	0.6091	0.950
$(\mathbf{10}, 10, \mathbf{4})$	$(\boldsymbol{R_4}, S_4)$	0.0340	0.0300	0.5934	0.950	0.0269	0.5319	0.952
	$(\boldsymbol{R_5}, S_5)$	0.0349	0.0307	0.5974	0.951	0.0260	0.5346	0.952
	$(\boldsymbol{R_6}, S_6)$	0.0351	0.0310	0.5960	0.951	0.0264	0.5377	0.954

To assess the convergence of the MCMC algorithm used in the Bayesian estimation, trace plots were generated for selected model parameters under different censoring schemes. The trace plots indicate stable behavior and satisfactory mixing of the chains, suggesting convergence of the MCMC algorithm in all cases. Representative trace plots are provided in Figures 2-4 to illustrate these findings. These diagnostics ensure that the posterior summaries and HPD intervals reported in the paper are based on converged MCMC samples and can be considered reliable.

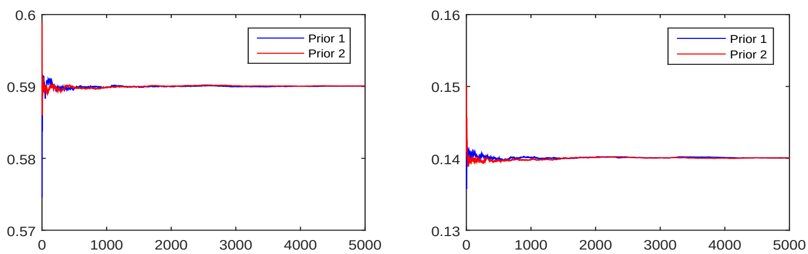

Figure 2: Trace plot in general case with censoring scheme (R_3, R_3, R_3, S_3), $(\boldsymbol{k}, \boldsymbol{s}) = (\mathbf{5}, \mathbf{2})$ (left) and (R_3, R_3, R_3, S_6), $(\boldsymbol{k}, \boldsymbol{s}) = (\mathbf{5}, \mathbf{4})$ (right).

Table 3: Results of the point and interval estimates of $R_{\boldsymbol{s},\boldsymbol{k}}$, when common parameters are unknown.

$(\boldsymbol{k}, n, \boldsymbol{s})$	C.S	MLE	MCMC					
			Prior 3			Prior 4		
		MSE	MSE	AL	CP	MSE	AL	CP
$(\mathbf{5}, 5, \mathbf{2})$	$(\boldsymbol{R_1}, S_1)$	0.0684	0.0623	0.6524	0.940	0.0555	0.6070	0.944
	$(\boldsymbol{R_2}, S_2)$	0.0677	0.0620	0.6530	0.942	0.0560	0.6042	0.945
	$(\boldsymbol{R_3}, S_3)$	0.0680	0.0635	0.6519	0.941	0.0553	0.6039	0.944
$(\mathbf{5}, 10, \mathbf{2})$	$(\boldsymbol{R_1}, S_4)$	0.0550	0.0486	0.6113	0.944	0.0402	0.5527	0.948
	$(\boldsymbol{R_2}, S_5)$	0.0552	0.0480	0.3182	0.945	0.0400	0.5591	0.949
	$(\boldsymbol{R_3}, S_6)$	0.0542	0.0471	0.6143	0.945	0.0410	0.5580	0.948
$(\mathbf{10}, 5, \mathbf{2})$	$(\boldsymbol{R_4}, S_1)$	0.0400	0.0364	0.5633	0.948	0.0315	0.5112	0.950
	$(\boldsymbol{R_5}, S_2)$	0.0408	0.0360	0.5627	0.948	0.0310	0.5120	0.950
	$(\boldsymbol{R_6}, S_3)$	0.0403	0.0369	0.5620	0.949	0.0319	0.5197	0.951
$(\mathbf{10}, 10, \mathbf{2})$	$(\boldsymbol{R_4}, S_4)$	0.0308	0.0277	0.5037	0.950	0.0205	0.4677	0.952
	$(\boldsymbol{R_5}, S_5)$	0.0300	0.0270	0.5081	0.950	0.0204	0.4619	0.951
	$(\boldsymbol{R_6}, S_6)$	0.0304	0.0273	0.5046	0.951	0.0200	0.4633	0.952
$(\mathbf{5}, 5, \mathbf{4})$	$(\boldsymbol{R_1}, S_1)$	0.0675	0.0610	0.6666	0.942	0.0553	0.6074	0.944
	$(\boldsymbol{R_2}, S_2)$	0.0670	0.0615	0.6615	0.941	0.0559	0.6050	0.943
	$(\boldsymbol{R_3}, S_3)$	0.0679	0.0608	0.6690	0.940	0.0550	0.6038	0.944
$(\mathbf{5}, 10, \mathbf{4})$	$(\boldsymbol{R_1}, S_4)$	0.0543	0.0460	0.6047	0.944	0.0400	0.5449	0.948
	$(\boldsymbol{R_2}, S_5)$	0.0540	0.0469	0.6028	0.945	0.0409	0.5462	0.949
	$(\boldsymbol{R_3}, S_6)$	0.0549	0.0463	0.6093	0.944	0.0415	0.5454	0.948
$(\mathbf{10}, 5, \mathbf{4})$	$(\boldsymbol{R_4}, S_1)$	0.0405	0.0351	0.5537	0.949	0.0310	0.5199	0.950
	$(\boldsymbol{R_5}, S_2)$	0.0408	0.0359	0.5520	0.948	0.0315	0.5180	0.951
	$(\boldsymbol{R_6}, S_3)$	0.0400	0.0354	0.5580	0.949	0.0318	0.5107	0.951
$(\mathbf{10}, 10, \mathbf{4})$	$(\boldsymbol{R_4}, S_4)$	0.0307	0.0269	0.5078	0.951	0.0200	0.4519	0.952
	$(\boldsymbol{R_5}, S_5)$	0.0300	0.0260	0.5081	0.951	0.0209	0.4572	0.952
	$(\boldsymbol{R_6}, S_6)$	0.0302	0.0265	0.5067	0.950	0.0210	0.4568	0.955

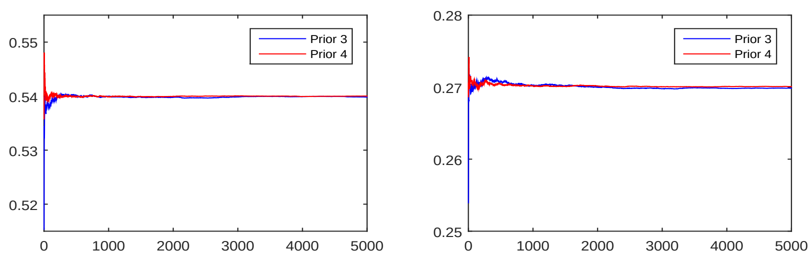

Figure 3: Trace plot with unknown common parameters with censoring scheme (R_4, R_4, R_4, S_1), $(\boldsymbol{k}, \boldsymbol{s}) = (\mathbf{10}, \mathbf{2})$ (left) and (R_4, R_4, R_4, S_4), $(\boldsymbol{k}, \boldsymbol{s}) = (\mathbf{10}, \mathbf{4})$ (right).

5.2 Real data analysis

In this section, we examine the monthly water capacity of the Shasta Reservoir in California, as detailed in https://cdec.water.ca.gov/dynamicapp/QueryMonthly?s=SHA. Recent studies, such as Kohansal (2019) and Kohansal (2025), have also investigated this dataset. Understanding the probability of drought occurrence is crucial for agricultural planning. One potential scenario to consider is as follows: if the water capacity of a reservoir during August, July, and September exceeds the volume recorded in

Table 4: Results of the point and interval estimates of $R_{\boldsymbol{s},\boldsymbol{k}}$, when common parameters are known.

$(\boldsymbol{k}, n, \boldsymbol{s})$	C.S	MLE	MCMC						Lindley	
			Prior 5			Prior 6			Prior 5	Prior 6
		MSE	MSE	AL	CP	MSE	AL	CP	MSE	MSE
$(\mathbf{5}, 5, \mathbf{2})$	$(\boldsymbol{R_1}, S_1)$	0.0532	0.0500	0.5024	0.940	0.0463	0.4627	0.944	0.0512	0.0493
	$(\boldsymbol{R_2}, S_2)$	0.0530	0.0508	0.5034	0.942	0.0460	0.4651	0.943	0.0501	0.0490
	$(\boldsymbol{R_3}, S_3)$	0.0538	0.0506	0.5043	0.941	0.0467	0.4682	0.945	0.0519	0.0495
$(\mathbf{5}, 10, \mathbf{2})$	$(\boldsymbol{R_1}, S_4)$	0.0519	0.0460	0.4537	0.945	0.0436	0.4069	0.949	0.0485	0.0454
	$(\boldsymbol{R_2}, S_5)$	0.0515	0.468	0.4572	0.945	0.0430	0.4035	0.948	0.0480	0.0450
	$(\boldsymbol{R_3}, S_6)$	0.0510	0.0463	0.4591	0.944	0.0438	0.4075	0.949	0.0483	0.0453
$(\mathbf{10}, 5, \mathbf{2})$	$(\boldsymbol{R_4}, S_1)$	0.0484	0.0430	0.4132	0.948	0.0400	0.3594	0.950	0.0450	0.0422
	$(\boldsymbol{R_5}, S_2)$	0.0480	0.0439	0.4152	0.949	0.0405	0.3581	0.950	0.0459	0.0420
	$(\boldsymbol{R_6}, S_3)$	0.0489	0.0430	0.4167	0.948	0.0409	0.3599	0.951	0.0451	0.0429
$(\mathbf{10}, 10, \mathbf{2})$	$(\boldsymbol{R_4}, S_4)$	0.0461	0.0408	0.3428	0.950	0.0366	0.3053	0.952	0.0420	0.0382
	$(\boldsymbol{R_5}, S_5)$	0.0468	0.0409	0.3482	0.950	0.0360	0.3050	0.951	0.0428	0.0381
	$(\boldsymbol{R_6}, S_6)$	0.0460	0.0410	0.3467	0.951	0.0364	0.3094	0.952	0.0427	0.0389
$(\mathbf{5}, 5, \mathbf{4})$	$(\boldsymbol{R_1}, S_1)$	0.0530	0.0505	0.5034	0.942	0.0460	0.4651	0.945	0.0510	0.0495
	$(\boldsymbol{R_2}, S_2)$	0.0536	0.0507	0.5022	0.941	0.0463	0.4692	0.944	0.0517	0.0493
	$(\boldsymbol{R_3}, S_3)$	0.0532	0.0500	0.5061	0.941	0.0469	0.4667	0.945	0.0513	0.0490
$(\mathbf{5}, 10, \mathbf{4})$	$(\boldsymbol{R_1}, S_4)$	0.0513	0.0469	0.4527	0.945	0.0438	0.4062	0.949	0.0487	0.0450
	$(\boldsymbol{R_2}, S_5)$	0.0518	0.0463	0.4533	0.945	0.0432	0.4085	0.949	0.0483	0.0457
	$(\boldsymbol{R_3}, S_6)$	0.0517	0.0467	0.4582	0.945	0.0430	0.4015	0.948	0.0480	0.0459
$(\mathbf{10}, 5, \mathbf{4})$	$(\boldsymbol{R_4}, S_1)$	0.0483	0.0430	0.4175	0.949	0.0408	0.3577	0.950	0.0453	0.0428
	$(\boldsymbol{R_5}, S_2)$	0.0487	0.0439	0.4167	0.948	0.0403	0.3591	0.951	0.0457	0.0426
	$(\boldsymbol{R_6}, S_3)$	0.0481	0.0435	0.4195	0.948	0.0407	0.3564	0.950	0.0459	0.0425
$(\mathbf{10}, 10, \mathbf{4})$	$(\boldsymbol{R_4}, S_4)$	0.0469	0.0408	0.3462	0.950	0.0369	0.3022	0.952	0.0420	0.0380
	$(\boldsymbol{R_5}, S_5)$	0.0463	0.0412	0.3485	0.949	0.0360	0.3054	0.952	0.0428	0.0389
	$(\boldsymbol{R_6}, S_6)$	0.0461	0.0403	0.3471	0.950	0.0364	0.3066	0.951	0.0423	0.0385

December of the previous year for at least one out of the next five years, we can infer that a drought is unlikely. Thus, the probability of non-drought occurrence can be represented by $R_{\boldsymbol{s},\boldsymbol{k}}$.

According to aforementioned scenario, variables $X_{11}^{(1)}, \ldots, X_{15}^{(1)}$, $X_{11}^{(2)}, \ldots, X_{15}^{(2)}$, and $X_{11}^{(3)}, \ldots, X_{15}^{(3)}$ represent the water capacities for July, August, and September from 1976 to 1980. Also, similarly, $X_{21}^{(1)}, \ldots, X_{25}^{(1)}$, $X_{21}^{(2)}, \ldots, X_{25}^{(2)}$, and $X_{21}^{(3)}, \ldots, X_{25}^{(3)}$ correspond to the capacities for the same months from 1982 to 1986, and so forth, with $X_{81}^{(1)}, \ldots, X_{85}^{(1)}$, $X_{81}^{(2)}, \ldots, X_{85}^{(2)}$, and $X_{81}^{(3)}, \ldots, X_{85}^{(3)}$ representing the capacities from 2018 to 2022. Additionally, $Y_1, Y_2, \ldots, Y_8$ denote the capacities for December from 1975 and 1981 to 2017. To simplify calculations, all data have been normalized by dividing by 4,552,000 acre-feet, which is the total capacity of the reservoir. It is important to note that this normalization does not affect the statistical inference.

Now, we fit the EKuD to these datasets separately. The results are as follows: For X_1, the estimated parameters are $(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}) = (8.4599, 3.5989, 0.3078)$ with a p-value of 0.9827. For X_2, the estimated parameters are $(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}) = (5.1951, 4.9099, 0.5085)$ and the p-value is 0.7245. For X_3, $(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}) = (1.0088, 2.5921, 4.2072)$ with a p-value of 0.3561. For Y, the estimated parameters are $(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}) = (2.6529, 8.4702, 46.5731)$ and the p-value is 0.8317. These p-values indicate that the EKu distribution provides

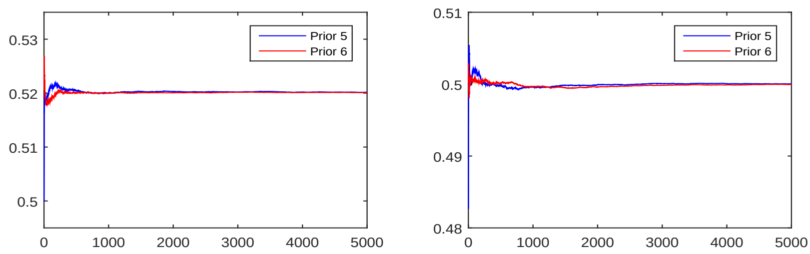

Figure 4: Trace plot with known common parameters with censoring scheme (R_2, R_2, R_2, S_2), $(\boldsymbol{k}, \boldsymbol{s}) = (\mathbf{5}, \mathbf{2})$ (left) and (R_5, R_5, R_5, S_5), $(\boldsymbol{k}, \boldsymbol{s}) = (\mathbf{10}, \mathbf{4})$ (right).

adequate fits for the datasets. Furthermore, based on the estimated parameters, we can consider the general case. The empirical distribution functions and PP-plots for these datasets are presented in Figure 5.

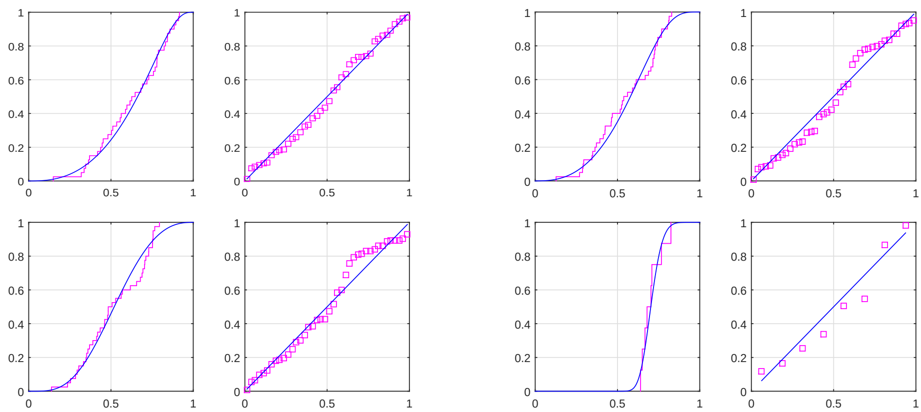

Figure 5: Empirical distribution function (left) and the PP-plot (right) for X_1 (left, top), X_2 (right, top), X_3 (left, bottom) and Y (right, bottom).

For the complete dataset, using $\boldsymbol{s} = (1, 1)$ and $\boldsymbol{k} = (5, 5)$ with non-informative priors, the estimated value $\widehat{R}^{MC}_{\boldsymbol{s},\boldsymbol{k}}$ and the corresponding 95% HPD interval are 0.4149 and $(0.2364, 0.6988)$, respectively. Next, we generate two distinct progressive censoring schemes as follows:
Scheme 1: $R^{(1)} = R^{(2)} = R^{(3)} = [1, 0, 0, 0]$, $\quad S = [0, 0, 0, 0, 0, 0, 1]$, $\quad (\boldsymbol{k} = (4, 4), \boldsymbol{s} = (2, 2))$.
Scheme 2: $R^{(1)} = R^{(2)} = R^{(3)} = [1, 1, 0]$, $\quad S = [1, 0, 0, 0, 0, 1]$, $\quad (\boldsymbol{k} = (3, 3), \boldsymbol{s} = (2, 2))$.
For Scheme 1, using non-informative priors, the estimated value $\widehat{R}^{MC}_{\boldsymbol{s},\boldsymbol{k}}$ and the corresponding 95% HPD interval are 0.3022 and $(0.1025, 0.6174)$, respectively. For Scheme 2, also with non-informative priors, the estimated value $\widehat{R}^{MC}_{\boldsymbol{s},\boldsymbol{k}}$ and the corresponding 95% HPD interval are 0.2039 and $(0.0925, 0.6532)$, respectively. By comparing the point and interval estimates, we observe that Scheme 1 performs better than Scheme 2, as anticipated.

MCMC diagnostics based on the real data analysis are provided to assess the convergence of the Bayesian estimation procedure. For this purpose, trace plots, autocorrelation plots of the MCMC chains, the Gelman-Rubin diagnostic plot illustrating the evolution of the potential scale reduction factor over iterations, and the posterior density plot of the reliability parameter $R_{\boldsymbol{s},\boldsymbol{k}}$ are presented in Figure 6. The trace plots indicate stable behavior and satisfactory mixing of the chains after the burn-in period. The autocorrelation plots show a rapid decay, suggesting good sampling efficiency. In addition, the Gelman–Rubin shrink factors remain below the commonly accepted threshold of 1.1, providing further evidence of convergence. The posterior density of $R_{\boldsymbol{s},\boldsymbol{k}}$ appears unimodal and well behaved. These diagnostics collectively support the convergence of the MCMC algorithm and the reliability of the posterior estimates. Similar convergence behavior was observed for other real data scenarios and parameter configurations; these results are omitted for brevity.

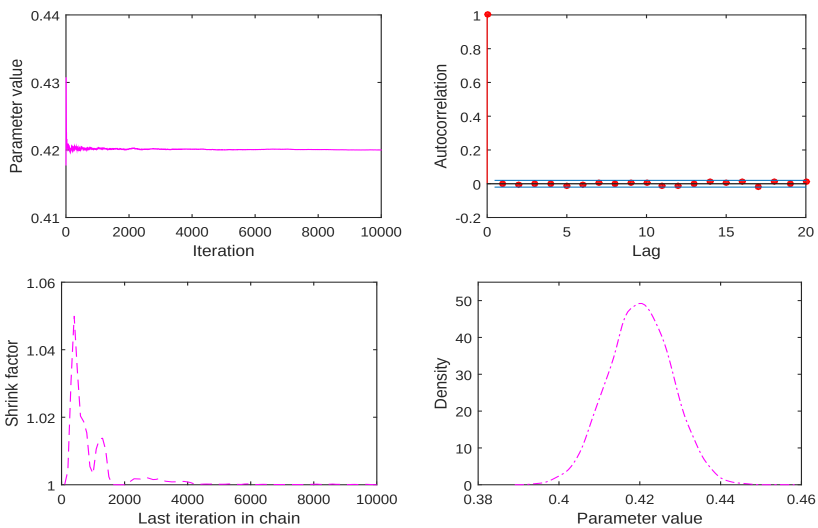

Figure 6: MCMC diagnostic plots of $R_{\mathbf{s},\mathbf{k}}$, in real data set.

6 Discussion and conclusions

In this study, we investigate the statistical inference of a multicomponent stress-strength system characterized by non-identical component strengths, specifically focusing on the EKu distribution under a progressive censoring scheme. To achieve this, we derive several point and interval estimations within both classical and Bayesian frameworks, including MLE, different Bayesian estimation, and HPD intervals. We also explore these estimations across different scenarios, considering cases where common parameters are either unknown, known, or in a general context. The theoretical approaches are evaluated through a Monte Carlo simulation study. Key findings indicate that Bayesian estimates outperform classical methods. Furthermore, within the

Bayesian framework, the use of informative priors yields superior performance compared to non-informative priors regarding both point and interval estimates.

Acknowledgement

The authors are grateful to the Editor in Chief and referees for the helpful comments and suggestions to improve this manuscript.

References

Balakrishnan, N. and Aggarwala, R. (2000). *Progressive Censoring: Theory, Methods and Applications*. Boston: Birkhäuser.

Bhattacharyya, G.K. and Johnson, R.A. (1974). Estimation of reliability in multicomponent stress-strength model. *Journal of the American Statistical Association*, **69**(348):966–970.

Birnbaum, Z.W. (1956). *On a use of Mann-Whitney statistics*. Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, **3**:13–17.

Chen, M.H. and Shao, Q.M. (1999). Monte Carlo estimation of Bayesian Credible and HPD intervals. *Journal of Computational and Graphical Statistics,* **8**(1):69–92.

Kızılaslan, F. and Nadar, M. (2018). Estimation of reliability in a multicomponent stress-strength model based on a bivariate Kumaraswamy distribution. *Statistical Papers,* **59**:307–340.

Kohansal, A. (2019). On estimation of reliability in a multicomponent stress-strength model for a Kumaraswamy distribution based on progressively censored sample. *Statistical Papers,* **60**:2185–2224.

Kohansal, A. (2025). Stress-strength model of m-components system in censored data. *TWMS Journal of Applied and Engineering Mathematics,* **15**(4):924–940.

Kohansal, A. and Nadarajah, S. (2019). Stress-strength parameter estimation based on Type-II hybrid progressive censored samples for a Kumaraswamy distribution. *IEEE Transactions on Reliability,* **68**(4):1296–1310.

Kohansal, A., Pérez-González, C.J. and Fernández, A.J. (2023). Multi-component reliability inference in modified Weibull extension distribution and progressive censoring scheme. *Bulletin of the Malaysian Mathematical Sciences Society,* **46**:Article Number 61.

Kohansal, A. and Shoaee, S. (2021). Bayesian and classical estimation of reliability in a multicomponent stress-strength model under adaptive hybrid progressive censored data. *Statistical Papers,* **62**:309–359.

Lemontea, A.J., Barreto-Souzaa, W. and Cordeiro, G.M. (2013). The exponentiated Kumaraswamy distribution and its log-transform. *Brazilian Journal of Probability and Statistics,* **27**:31–53.

Lindley, D.V. (1980). Approximate Bayesian methods. *Trabajos de Estadistica Y de Investigacion Operativa,* **31**:281–288.

Nadar, M. and Kızılaslan, F. (2016). Estimation of reliability in a multicomponent stress–strength model based on a Marshall-Olkin bivariate Weibull distribution. *IEEE Transactions on Reliability,* **65**:370–380.

Nadar, M., Kızılaslan, F. and Papadopoulos, A. (2014). Classical and Bayesian estimation of $P(Y < X)$ for Kumaraswamy's distribution. *Journal of Statistical Computation and Simulation,* **84**:1505–1529.

Rasethuntsa, T.R. and Nadar, M. (2018). Stress-strength reliability of a non-identical-component strengths system based on upper record values from the family of Kumaraswamy generalized distributions. *Statistics,* **52**(3):684–716.

Shoaee, Sh. and Khorram, E. (2016). Statistical inference of $\mathcal{R} = P(Y < X)$ for Weibull distribution under Type-II progressively hybrid censored data. *Journal of Statistical Computation and Simulation,* **86**(18):3815–3834.